Documentation for entanglement entropy code

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1 Entropy calculation

Let $|\Psi\rangle$ be the given normalized wave function with L qubits. The subsystems A and B are of size $L/2$. Then $\rho_{AB} = |\Psi\rangle\langle\Psi|$ is the density matrix of the subsystem A and B, and by definition$\rho_A = Tr_B(\rho_{AB})$.

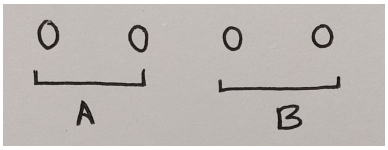

Figure 1: Example of a system with $L = 4$ qubits with subsystems A and B of size $L/2 = 4$.

The $(s, s^{'})$ element of the reduced density matrix ρ_A is given by

$$\langle s|\,\rho_A\,\left|s^{'}\right\rangle = \sum_{s^{''}} \left\langle ss^{''}\right|\,\rho_{AB}\,\left|s^{'}s^{''}\right\rangle = \sum_{s^{''}} \left\langle ss^{''}\right|\left.\Psi\right\rangle\left\langle\Psi\right|\left.s^{'}s^{''}\right\rangle$$

Here we sum over the subsystem B with the index $s^{''}$. For example, for $L = 4$ qubits case, the wavefunction is

$$|\Psi\rangle = \frac{1}{\sqrt{N}}(c_{0000}\,|0000\rangle + c_{0001}\,|0001\rangle + c_{0010}\,|0010\rangle + ... + c_{1111}\,|1111\rangle) \quad (1)$$

We can also label the states as

$$|\Psi\rangle \quad = \quad \frac{1}{\sqrt{N}}(c_0\,|0\rangle + c_1\,|1\rangle + c_2\,|2\rangle + ... + c_{15}\,|15\rangle) \quad (2)$$

Let us say we are looking for the element $s = 00$ (binary) or $s = 0$ (decimal) and $s^{'} = 10$ (binary) or $s = 2$, (decimal) the sum looks like the following

$$\langle 00|\,\rho_{AB}\,|10\rangle \quad = \quad \langle 0000|\,\rho\,|1000\rangle + \langle 0001|\,\rho\,|1001\rangle + \langle 0010|\,\rho\,|1010\rangle + \langle 0011|\,\rho\,|1011\rangle$$

in decimal notation

$$\begin{aligned}
\langle 0|\,\rho_A\,|2\rangle &= \langle 0|\,\rho_{AB}\,|8\rangle + \langle 1|\,\rho_{AB}\,|9\rangle + \langle 2|\,\rho_{AB}\,|10\rangle + \langle 3|\,\rho_{AB}\,|11\rangle && (3)\\
&= \langle 0|\psi\rangle\langle\psi|8\rangle + \langle 1|\psi\rangle\langle\psi|9\rangle + \langle 2|\psi\rangle\langle\psi|10\rangle + \langle 3|\psi\rangle\langle\psi|11\rangle && (4)\\
&= \frac{1}{N}(c_0^* c_8 + c_1^* c_9 + c_2^* c_{10} + c_3^* c_{11}) && (5)\\
&= \Psi^\dagger[0:3]\Psi[8:11] && (6)
\end{aligned}$$

In general, we can write

$$\psi_{s'} = \Psi[2^L s^{'} : 2^L s^{'} + 2^L - 1] \tag{7}$$

Then, (6) gives ($L = 4$ in this case)

$$\langle 0|\,\rho_A\,|2\rangle = \Psi^\dagger_{0'}\Psi_{2'} \tag{8}$$

In general, we can write (L is the total system size)

$$\langle s|\,\rho_A\,\left|s^{'}\right\rangle = \Psi^\dagger_s \Psi_{s'} \tag{9}$$

Since ρ_A is Hermitian, we only need to find the elements above and including the diagonal, we then can replace them elements below the diagonal by the complex conjugate of the elements above the diagonal.

We then diagonalize the matrix ρ_A then find the entropy as

$$S = -\sum_i \lambda_i \log \lambda_i \tag{10}$$

Where λ_i are the eigenvalues of ρ_A.

2 Rolling operator

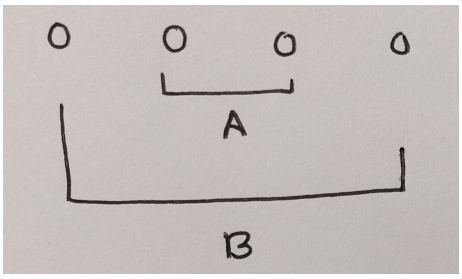

Figure 2: The qubits after one roll

Let the initial wavefunction be

$$\begin{aligned}|\Psi_i\rangle &= \frac{1}{\sqrt{N}}(c_{0000}\,|0000\rangle + c_{0001}\,|0001\rangle + c_{0010}\,|0010\rangle + c_{0011}\,|0011\rangle \\ &+ c_{0100}\,|0100\rangle + c_{0101}\,|0101\rangle + c_{0110}\,|0110\rangle + c_{0111}\,|0111\rangle \\ &+ c_{1000}\,|1000\rangle + c_{1001}\,|1001\rangle + c_{1010}\,|1010\rangle + c_{1011}\,|1011\rangle \\ &+ c_{1100}\,|1100\rangle + c_{1101}\,|1101\rangle + c_{1110}\,|1110\rangle + c_{1111}\,|1111\rangle)\end{aligned}$$

After one roll of the qubits the wavefunction becomes

$$\begin{aligned}|\Psi_f\rangle &= \frac{1}{\sqrt{N}}(c_{0000}\,|0000\rangle + c_{0001}\,|1000\rangle + c_{0010}\,|0001\rangle + c_{0011}\,|1001\rangle \\ &+ c_{0100}\,|0010\rangle + c_{0101}\,|1010\rangle + c_{0110}\,|0011\rangle + c_{0111}\,|1011\rangle \\ &+ c_{1000}\,|0100\rangle + c_{1001}\,|1100\rangle + c_{1010}\,|0101\rangle + c_{1011}\,|1101\rangle \\ &+ c_{1100}\,|0110\rangle + c_{1101}\,|1110\rangle + c_{1110}\,|0111\rangle + c_{1111}\,|1111\rangle)\end{aligned}$$

It is easier to see the effect of rolling in decimal notation. The initial wavefunction is

$$|\Psi_i\rangle = \frac{1}{\sqrt{N}}\sum_{k=0}^{15} c_k\,|k\rangle \tag{11}$$

we get, after one roll

$$\begin{aligned}|\Psi_f\rangle &= \frac{1}{\sqrt{N}}(c_0\,|0\rangle + c_1\,|8\rangle + c_2\,|1\rangle + c_3\,|9\rangle + c_4\,|2\rangle + c_5\,|10\rangle + c_6\,|3\rangle \\ &+ c_7\,|11\rangle + c_8\,|4\rangle + c_9\,|12\rangle + c_{10}\,|5\rangle + c_{11}\,|13\rangle + c_{12}\,|6\rangle + c_{13}\,|14\rangle \\ &+ c_{14}\,|7\rangle + c_{15}\,|15\rangle)\end{aligned}$$

Reordering the kets, we get

$$|\Psi_f\rangle = \frac{1}{\sqrt{N}}[c_0 \quad c_4 \quad c_9 \quad c_8 \quad c_{10} \quad c_{12} \quad c_{14} \quad c_1 \quad c_3 \quad c_5 \quad c_7 \quad c_9 \quad c_{11} \quad c_{13} \quad c_{15}]^\dagger$$

This can be summarized as

$$|\Psi_f\rangle = R\,|\Psi_i\rangle \tag{12}$$

Where R is the rolling operator that roll the operator once. The wavefunction after any number of rotations can be determined using the steps above.