Local Symmetry is a Lie!

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In this document I try to find an intuitive explanation for how to understand gauge theory. The biggest difference here with the standard literature is that I put emphasis on understanding local gauge transformations as passive transformations. Because this differs from the standard way Yang-Mills Theory is presented, a 0-th section is included with a review on passive and active transformations and then showing the connection to the standard picture.

0 Prologue: Not a Lie, but an Omission

0.1 Symmetries from Transformations

When one mentions transformations in physics one should differentiate between the 2 types of transformations, they are very different in their nature:

- **Active Transformations:** Changing the directions/positions/vectors of the objects in the theory.
- **Passive Transformations:** Changing the components/coordinates/basis of the objects in the theory, while leaving the actual directions/positions/vectors unchanged.

We can easily see that there is always a set of passive transformations that is allowed, as the actual physical vectors that describe the dynamics are left invariant under these transformations. In some sense *passive transformations are not true transformations, they are merely a change in our description of the physical objects of the theory.* We should not be able to infer any new behavior or symmetry from these kinds of transformation, as the physical objects of the transformed system are *exactly the same* as those of the untransformed system.

On the other hand, *active transformations are actual transformations on the system and they reveal the symmetries of the physical systems in question*, for example: rotations around a Coulomb potential, translations inside a homogeneous field, or time translation for (most) isolated systems. In all of these examples, we compare *different points* of the systems in question and find that the dynamics are the same (not merely equivalent). What follows next, however, confuses things as we pick a coordinate system that conforms to said symmetry. In a system charted out by a “smart” coordinate system, certain active transformations become indistinguishable from passive ones when looking only at the coordinates. Consequently, no distinction is made between the two, even though physically they are completely different![1] In this document, I want to make this distinction clear.

But how do we know what kind of passive transformations are allowed? This isn't as clear as one might first think; How do we know we can passively transform a velocity vector by $\mathrm{SO}(n)$ and not $\mathrm{SU}(n)$? Well this actually comes from active transformations! We see that we can actively turn velocity vectors with $R(\theta, \phi) \in \mathrm{SO}(3)$ and likewise we can turn the basis vectors that build the vector space with $R(\theta, \phi)^\dagger \in \mathrm{SO}(3)$, thus passively:

$$\vec{v} = \vec{e}_i v^i \xrightarrow[\text{Passive Trafo.}]{} \vec{e}_i (R(\theta, \phi)^\dagger)^i_j R(\theta, \phi)^j_k v^k = \vec{e}_i v^i$$

In other words to find out if we live in $\mathbb{R}^3$ or $\mathbb{C}^3$ we need to look at the experiments, the active transformations.

0.2 From $F = ma$ to the Standard Model: Covariant Derivatives all the way!

In $\vec{F} = m\vec{a}$ both $\vec{F}$ and $\vec{a}$ are so called euclidean or galilean vectors. We can apply active galilean transformations to $\vec{F}$ and $\vec{a}$ without changing the dynamics. If we actively turn all the acceleration vectors with some matrix and all the forces with the same matrix we have the same dynamics

But this puts in a dilemma: If we can actively turn all vectors on all points in space (for instance with $R(\theta, \phi) \in \mathrm{SO}(3)$) then we know to transform these vectors, and we should be able to preform the same kind of passive transformations on the basis vectors/coordinates in all points in space. However, these would be passive transformations and they, as we previously saw, should never change the dynamics in question, even if, say we preform *different passive transformations* on *different points in space*. In other words, the global active transformation points to *local passive transformations*. Let us make sure that this is the case:

$$\begin{aligned}
\vec{e}_i &\xrightarrow[\text{Passive Trafo.}]{} \vec{e}_i (R(x(t))^\dagger \equiv \vec{e}_i(t) \\
v^i(t) &\xrightarrow[\text{Passive Trafo.}]{} R(x(t)) v^i(t) \equiv v^i(t) \\
\vec{v} &\xrightarrow[\text{Passive Trafo.}]{} \vec{v}
\end{aligned}$$

[1]for example, does the condition $F(\theta) = \tilde{F}(\tilde{\theta})$ mean that the F is the same at 2 different points or that we can represent the same point $\tilde{\theta}$ with θ if we modify F appropriately?

thus in $\vec{F} = m\dot{\vec{v}}$ we get:

$$\begin{aligned}
F^k = m\,(\dot{v})^k &= m(\frac{d}{dt}\vec{e}_i v^i)^k \\
&= m\left((\partial_t \vec{e}_i)v^i + \vec{e}_i(\partial_t v^i)\right)^k \\
&= m(\vec{e})^{\dagger k}\left((\partial_t \vec{e}_i)v^i + \vec{e}_i(\partial_t v^i)\right) \\
&= m\big(\underbrace{(\vec{e})^{\dagger k}(\partial_t \vec{e}_i)}_{\equiv \Gamma_{t}{}^{k}_{i}} v^i + \delta^k_i(\dot{v})^i\big) \\
&= m\left(\Gamma^k_{ti} v^i + (\partial_t v)^k\right) = m D_t v^k
\end{aligned}$$

where we introduced the 1-form $(\vec{e})^{\dagger k}$ to get the components of the vector $(\vec{e})^{\dagger k}\vec{v} = v^k$ via $(\vec{e}_i)^{\dagger}(\vec{e}_j) = \delta^i_j$ and where we defined the christoffel symbol in euclidean space $(\vec{e})^{\dagger k}(\partial_t \vec{e_i}) = \Gamma^k_{ti}$ and along with it the covariant derivative D_t. Some further remarks from the above:

- We only *need* to keep track of the change in vector coordinates v^i and we can forget about what happens to the basis vectors, because the covariant derivative subsumes any transformation made to them (this is what is usually done when Yang-Mills theory is presented).
- The local transformations are merely passive transformations, no real dynamics comes from them, and the covariant derivative guarantees that.
- The global active transformations guide how we can transform the basis vectors and thus how we *can* preform (local) passive transformations (we can turn the system upside-down, but we *can't* "multiply it by i ").
- The fact that the system is invariant under these local passive transformation is then interpreted as a symmetry in the system, though more accurately it is only a symmetry in the coordinate description of the system (the physicial system is agnostic to any passive transformations).
  - Extending the above calculation for 4 dimensional space time with the above interpretation of invariance under passive transformations, we get that the system is "invariant under diffeomorphisms" as we can make the v^i whatever we like, as long as we can get it there via basis transformation $v^i = \frac{\partial y_j}{\partial x_i} v'^j$

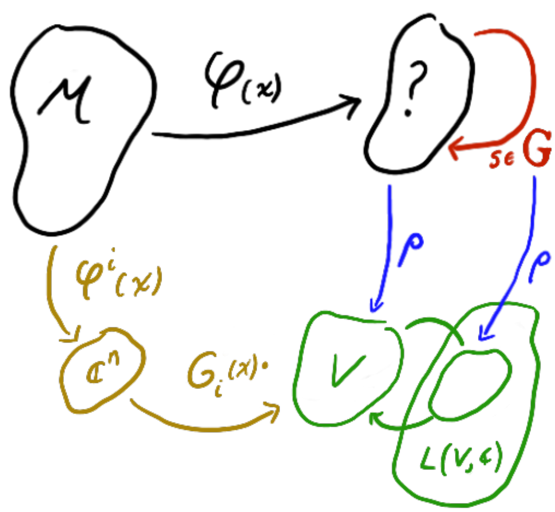
1 Classical Yang-Mills

1.1 What is a Particle Field, anyways?

A field, whether -scalar, -vector, -tensor or otherwise is a function from space-time, $\mathbb{R}^4$ or more generally, $\mathcal{M}$ to some other set like $\mathbb{C}$, or $\mathbb{C}^n$, a particle field $\varphi(x)$ should be no different. Thus we know what the domain of $\varphi(x)$ is ($x \in \mathcal{M}$), but don't know much about the co-domain, the "field space". Let's suppose that through experiment some automorphisms "s" of the co-domain have been found and let us further suppose that these global active transformations leave the dynamics unchanged. We have then found a global symmetry group for the field, $\mathrm{G} \ni s$. However, to study the dynamics of this field we don't have to deal with the abstract group G or the mysterious co-domain of $\varphi(x)$, instead we use the representations of the group of G, $\rho(\mathrm{G})$. The representation of the group is a group homomorphism to matrices so that we can represent the group operation with matrix multiplication:

$$\rho : \mathrm{G} \to \mathrm{GL}(V), \qquad \rho(\mathrm{g}_1 \circ \mathrm{g}_2) = \rho(\mathrm{g}_1)\rho(\mathrm{g}_2) \quad \text{for } \mathrm{g}_1, \mathrm{g}_2 \in \mathrm{G}$$

In such a representation we promote φ to a vector field $\varphi = G_i \varphi^i$, defining the basis vectors G_i of the vector space that gets acted on by the representation of the symmetry group V [2]. All of the maps mentioned can then be organized in a little map of maps:



[2]We will deal through multiple vector spaces in this document. Greek indices (μ, ν, etc.) are, as usual, space-time indices ($e_\mu v^\mu \in \mathrm{T}_p\mathcal{M}$). Latin indices in the middle of the alphabet (i, j, etc.) indicate *vectors that are acted on by representation of the group* ($G_i v^i \in V$, assuming $\rho(\mathrm{G}) = \mathrm{GL}(V)$). Finally, Latin indices at the beginning of the alphabet (a, b, etc.) indicate elements of the representation of the Lie algebra ($A^a T_a \in \mathfrak{gl}(\mathfrak{g})$). The only exception to these guidelines is section 0.

Another way of looking at this is that we start out with a function living on a manifold and we add to each point on this manifold some more structure in the form of so called fibers, the resulting structure is called a “Fiber Bundle”. Now we are studying functions living on these so called fiber bundles. In mathematics, gauge theory is the study of principle bundles, which are fiber bundles where the fibers carry group structure.

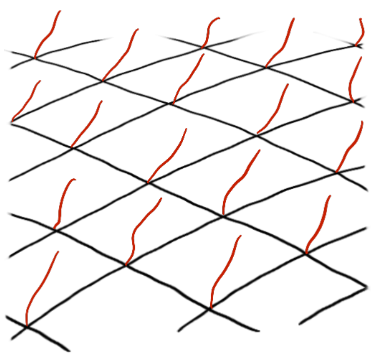


Figure 1: Generic manifold with fibers (in red) attached

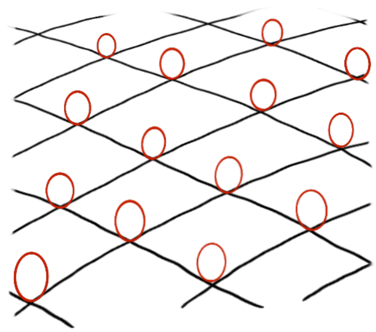


Figure 2: Generic manifold with closed fibers (in red) attached

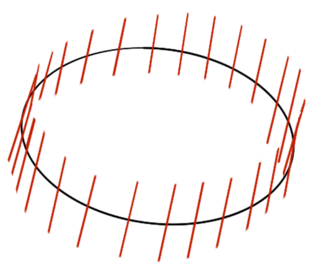


Figure 3: a Circle with simple fibers attached via the Cartesian product (a trivial fiber bundle)

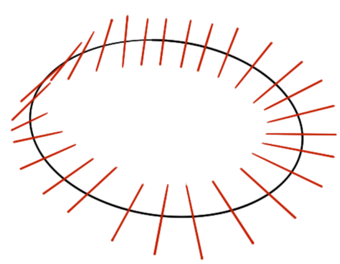


Figure 4: A Möbius strip, not a trivial fiber bundle

1.2 Gauge Transformations as Coordinate Transformations

Let us start with a Dirac field to change things up a bit [3]:

$$\mathcal{L} = i\overline{\psi}\partial\!\!\!/\psi - m\overline{\psi}\psi \tag{1}$$

Let us suppose that this Dirac field is found to (actively) transform under some symmetry group G. First let us define U as an the output of the representation ρ. Then we upgrade ψ to a vector on the vector space that gets acted on by U, so that it gets transformed under the symmetry group $\rho(G)$[3]:

$$U \in \rho(\mathrm{G}), \qquad\qquad \psi \to G_i\psi^i \equiv \psi$$

and we now If the field transforms under global active transformation as $\psi \to U\psi$, then the adjoint transforms as:

$$\overline{\psi} = \psi^\dagger\gamma^0 \xrightarrow[\text{Active Trafo.}]{} (U\psi)^\dagger\,\gamma^0 = \psi^\dagger\gamma^0 U^\dagger = \overline{\psi}U^\dagger$$

Where U and γ^0 commute as they act on different vector spaces. This transformation rule is to be expected as it more-or-less defines the concept of an adjoint in more general contexts[4]. Since we first take U to be a global transformation it commutes with $\partial\!\!\!/$ and obviously $\mathcal{L}$ is left invariant.

What about passive transformations, though? These leave the vector ψ unchanged, but they transform both coordinates and basis vectors:

$$\psi = G_i\psi^i \xrightarrow[\text{Passive Trafo.}]{} G_j U^{\dagger}{}^j_i U^i_k\psi^k = \tilde{G}_l\tilde{\psi}^l = \psi$$

However, because these keep the actual vectors unchanged, they are not true transformations. What happens when we make a change of basis/coordinates at different points differently?:

$$U \to U(x) \qquad \tilde{G}_i \to \tilde{G}_i(x) \qquad \tilde{\psi}^i \to \tilde{\psi}^i(x) \qquad \psi \to \psi$$

The vector ψ is still left unchanged so we expect (1) to remain unchanged, but let's make sure! The only trouble from an x dependent U comes from the

[3]Even though I don't distinguish it with notation, keep in mind that from now on until subsection 1.3, ψ (without indices) is a vector on this new vector space, after 1.3 we will only deal with the vector components with suppressed indices.

[4]For example, the Dirac adjoint is actually defined this way, replacing U with λ and using $\gamma^0\lambda^\dagger\gamma^0 = \lambda^{-1}$ so that $\overline{\psi}\psi$ acts as a lorentz scalar.

derivative. As always in physics let's look at the i-th component:

$$
\begin{aligned}
(\partial_\mu \psi)^i &= \left(\partial_\mu \left(G_j \psi^j\right)\right)^i \\
&= \left(\left(\partial_\mu G_j\right) \psi^j + G_j \left(\partial_\mu \psi^j\right)\right)^i \\
&= G^{\dagger i} \left(\left(\partial_\mu G_j\right) \psi^j + G_j \left(\partial_\mu \psi^j\right)\right) \\
&= \underbrace{G^{\dagger i} \left(\partial_\mu G_j\right)}_{=A_{\mu}{}^i_j} \psi^j + G^{\dagger i} G_j \left(\partial_\mu \psi^j\right) \\
&= A_{\mu}{}^i_j \psi^j + \delta^i_j \partial_\mu \psi^j = D_{\mu}{}^i_j \psi^j \equiv D_\mu \psi^i
\end{aligned} \tag{2}
$$

Where, to get the i-th component we introduced the 1-form $G^{\dagger i}$ that has the property $(G_i)^\dagger G_j = G^{\dagger i} G_j = \delta^i_j$. We are picking an orthonormal basis. We have also introduced the (mathematical) gauge potential or connection A_μ[5]. Notice that the covariant derivative merely gives an index to ψ, so that we take the derivative of vector components rather than a vector, but we do so as if we were taking the derivative of a vector (covariantly!).[6] A_μ for it's part, keeps track of how basis vectors change from one point to the next.

Let us see how A_μ transforms under passive local transformation:

$$
\begin{aligned}
A_{\mu}{}^j_i = G^{\dagger i} \left(\partial_\mu G_j\right) \xrightarrow[\text{Passive Trafo.}]{} & \left(G_k U^{\dagger k}_i\right)^\dagger \left(\partial_\mu \left(G_m U^{\dagger m}_j\right)\right) \\
&= U^i_k G^{\dagger k} \left(\partial_\mu G_m\right) U^{\dagger m}_{\ j} + U^i_k G^{\dagger k} G_m \partial_\mu U^{\dagger m}_{\ j} \\
&= U^i_k A_{\mu}{}^k_m U^{\dagger m}_{\ j} + U^i_k \delta^k_m \partial_\mu U^{\dagger m}_{\ j} \\
&= \left(U A_\mu U^\dagger + U \partial_\mu U^\dagger\right)^i_j \\
&= \left(U A_\mu U^\dagger - \left(\partial_\mu U\right) U^\dagger\right)^i_j
\end{aligned} \tag{3}
$$

The last equality follows from $\partial_\mu \left(U U^\dagger\right) = 0$. From here one would usually show that if the gauge potential transforms like this "under local gauge transformation", then the covariant derivative varies covariantly. This is a fairly standard calculation and we will not repeat it here, especially since we can get a much better understanding as to why it might already be true. Under our interpretation here we are dealing with a passive transformation that only transforms components and basis, and it does so contravariantly. Consequently $\psi \to \psi$, $(\partial_\mu \psi)^i \to U^i_j (\partial_\mu \psi)^j$, and comparing with (3), we have: $(D_\mu \psi)^i \to U^i_j (D_\mu \psi)^j$. Then from $\psi^j \to U^j_i \psi^i$, the aptly named covariant derivative varies covariantly:

$$
(D_\mu)^i_j = U^i_k \left(D_\mu\right)^k_l U^{\dagger l}_{\ j} \tag{4}
$$

[5] the definition given above will only be valid until curvature is introduced.

[6] Most textbooks then suppress ψ^i's index and this subtlety is often missed

Going back to A_μ, notice that the (mathematical) A_μ is anti-hermitian:

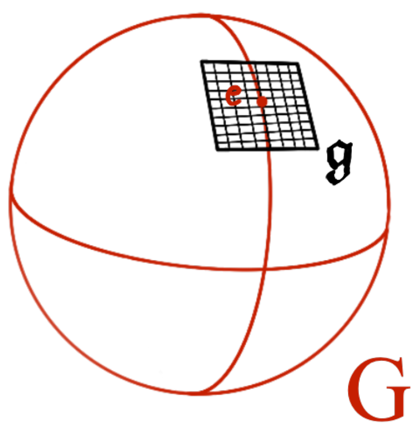
$$\left({A_\mu}^\dagger\right)^i_j = \left({A_\mu}^j_i\right)^\dagger = \left({G^\dagger}^j\left(\partial_\mu G_i\right)\right)^\dagger = \left(\partial_\mu {G^\dagger}^i\right)G_j = -{G^\dagger}^i\left(\partial_\mu G_j\right) = -\left(A_\mu\right)^i_j$$

where the fourth equality comes from $\partial_\mu\left({G^\dagger}^i G_j\right) = 0$. To understand better what A_μ is, let us evaluate the transformation law (3) at the limit where U approaches **1** (suppressing representation indices):

$$A_\mu \to UA_\mu U^\dagger + U\partial_\mu U^\dagger \xrightarrow[U\to\mathbf{1}]{} A_\mu + \left[\partial_\mu U^\dagger\right]_{U=\mathbf{1}}$$

We see that A_μ is acts like a vector on an affine space on a small region around the **1** matrix. This roughly means the group is also a manifold. Groups that are also manifolds are called lie groups G and their tangent space around the identity is called their lie algebra $\mathfrak{g}$. We see also that A_μ is a vector and a member of $\mathrm{T}_e\mathrm{G}$ and thus also a member of the representation of the lie algebra:

$$\left(A_\mu\right)^i_j = \left(T_a\right)^i_j A^a_\mu \in \rho(\mathfrak{g})$$



The basis vectors of this vector space T_a are also called the generators of the lie algebra. We can then write the representations of the lie group as:

$$U(x) = e^{\left(T_a\right)^i_j \theta^a(x)}$$

Notice that for U to be unitary, T_a has to be traceless. To get the full structure of the group from the lie algebra we need to be able to multiply different members of the the lie group, we can use the classic Baker-Campbell-Hausdorff formula to do so (suppressing representation indices for the remainder of the subsection):

$$U\tilde{U} = e^{T_a\theta^a}e^{T_b\tilde{\theta}^b} = e^{T_a\theta^a + T_b\tilde{\theta}^b + [T_a,T_b]\theta^a\tilde{\theta}^b + \frac{1}{2!}[T_a,[T_a,T_b]]{\theta^a}^2\tilde{\theta}^b + \ldots}$$

We see that the commutators of the generators of the (representation of the) lie algebra define the structure of the group, with this in mind we define the structure constants f^{abc}:

$$[T_a, T_b] = f_{ab}{}^{c} T_c$$

Note that we can raise or lower indices for free[7] In the case of an Abelian group, the structure constants are all 0. There is also the expansions of the transformations to linear order in small θ:

$$U(x) = \mathbb{1} + T_a \theta^a(x) + \mathcal{O}(\theta^2)$$

which then implies for the passive transformation/gauge transformation of A_μ:

$$A_\mu \to A_\mu + [T_a, A_\mu]\theta^a(x) + \partial_\mu \theta^a T^a + \mathcal{O}(\theta^2)$$

1.3 A Gauge Invariant Lagrangian

If we want to work with the vector components ψ^i rather than the abstract vector ψ while keeping everything covariant, all we need to do is replace derivatives with covariant derivatives, so that (1) becomes:

$$\mathcal{L} = i\overline{\psi}_i \not{D} \psi^i - m\overline{\psi}_i \psi^i \tag{5}$$

One could argue nothing has changed we are now merely looking at the vector components of ψ rather than the abstract notion of a vector; indeed if we look at the equations of motions (EOM)s, no new dynamics seem present[8]:

$$\begin{aligned}
\delta S &= \int \delta\overline{\psi}_i \left(i\not{D}\psi^i - m\psi^i\right) + i\overline{\psi}_i \left(\delta\not{D}\psi^i\right) - m\overline{\psi}_i \delta\psi^i \sqrt{-g} dx^4 \\
&\overset{(6)}{=} \int \sqrt{-g} dx^4 \delta\overline{\psi}_i \left(i\not{D}\psi^i - m\psi_i\right) - i\overline{D_\mu \psi}_i \gamma^\mu \left(\delta\psi^i\right) - m\overline{\psi}_i \delta\psi^i \sqrt{-g} dx^4 \\
&= \int \sqrt{-g} dx^4 \delta\overline{\psi}_i \left(i\not{D}\psi^i - m\psi^i\right) - \left(i\overline{\not{D}\psi}_i + m\overline{\psi}_i\right) \delta\psi^i \sqrt{-g} dx^4 \overset{!}{=} 0
\end{aligned}$$

where we used the following product rule and got a boundary term $\partial_\mu(\overline{\psi}_i \psi^i)$ that we then ignored:

$$\begin{aligned}
\partial_\mu \left(\phi_i^\dagger \varphi^i\right) &= \left(\partial^\mu \phi_i^\dagger\right) \varphi^i + \phi_i^\dagger \left(\partial_\mu \varphi^i\right) + \phi_i^\dagger A_{\mu}{}^{i}_{j} \varphi^j - \phi_j^\dagger A_{\mu}{}^{j}_{i} \varphi^i \\
&= \left(\partial_\mu \phi^i\right)^\dagger \varphi^i + \phi_i^\dagger D_\mu \varphi^i + \left(A_{\mu}{}^{i}_{j} \phi^j\right)^\dagger \varphi^i \\
&= \left(D_\mu \phi^i\right)^\dagger \varphi^i + \phi_i^\dagger D_\mu \varphi^i \qquad = (D_\mu \phi)_i^\dagger \varphi^i + \phi_i^\dagger D_\mu \varphi^i
\end{aligned} \tag{6}$$

We recognize from the variation of the action the same old Dirac equation:

$$(i\not{D} - m)\psi^i = 0 \implies (i\not{\partial} - m)\psi = 0$$

[7] this also means in most of the literature the Einstein summation notation becomes repeated index summation convention.

[8] see appendix A for the calculation with a complex scalar field

The only thing that has changed is that now ψ is now a vector in the vector space that gets acted on by the representations of G and that's about it. One might think we need to add more terms in the Lagrangian (5), but there seems no good reason to do it for now, specially since they wouldn't change the EOMs... Who needs fields?

We will proceed with this Lagrangian and see what we find, Let's derive the stress energy tensor (SEM) [2]:

$$T^{\mu\nu} = \frac{\partial}{\partial\left(\partial_\mu \psi^i\right)} D^\nu \psi^i + D^\nu \overline{\psi^i} \frac{\partial}{\partial\left(\partial_\mu \overline{\psi}_i\right)} - g_{\mu\nu}\mathcal{L} \quad = \quad i\overline{\psi}_i \gamma^\mu D^\nu \psi^i - g_{\mu\nu}\mathcal{L}$$

We'll ignore the problems with this SEM, such as it being complex and non-symmetric, after all $T_{\mu\nu}$ can sometimes be ambiguous... Nevertheless this is what you might get by upgrading the normal SEM from normal to covariant derivatives.[9] Despite these issues, we expect it to be conserved[10]:

$$\begin{aligned}
\partial_\mu T^{\mu\nu} &= \partial_\mu \left(i\overline{\psi}_i \gamma^\mu D^\nu \psi^i\right) - \partial^\nu \left(i\overline{\psi}_i \not{D}\psi^i - m\overline{\psi}_i \psi^i\right) \\
&\overset{(6)}{=} \left(i\overline{D_\mu \psi_i}\right) \gamma^\mu D^\nu \psi^i + i\overline{\psi}_i \gamma^\mu D_\mu D^\nu \psi^i \\
&- \left(i\overline{D^\nu \psi_i}\right) \not{D}\psi^i - i\overline{\psi}_i D^\nu \not{D}\psi^i + m\left(\left(\overline{D^\nu \psi_i}\right)\psi^i + \overline{\psi}_i D^\nu \psi^i\right) \\
&= \cancel{\left(i\overline{\not{D}\psi_i}\right) D^\nu \psi^i} + i\overline{\psi}_i \gamma_\mu D^\mu D^\nu \psi^i \\
&- \cancel{\left(i\overline{D^\nu \psi_i}\right) \not{D}\psi^i} - i\overline{\psi}_i \gamma_\mu D^\nu D^\mu \psi^i + m\left(\cancel{\left(\overline{D^\nu \psi_i}\right)\psi^i} + \cancel{\overline{\psi}_i D^\nu \psi^i}\right) \\
&\overset{(EOM)}{=} i\overline{\psi}_i \gamma_\mu D^\mu D^\nu \psi^i - i\overline{\psi}_i \gamma_\mu D^\nu D^\mu \psi^i \\
&= i\overline{\psi}_i \gamma_\mu \left[D^\mu, D^\nu\right] \psi^i = i\overline{\psi}_i \gamma_\mu F^{\mu\nu} \psi^i
\end{aligned} \tag{7}$$

Surprisingly the SEM isn't conserved in this theory! the divergence is proportional to a new, covariant object $F^{\mu\nu} = [D^\mu, D^\nu]$. We can also further expand this object and get (suppressing representation indices):

$$\begin{aligned}
F_{\mu\nu} &= \left[\partial_\mu + A_\mu, \partial_\nu + A_\nu\right] = \left[\partial_\mu, A_\nu\right] + \left[\partial_\nu, A_\mu\right] + \left[A_\mu, A_\nu\right] \\
&= \partial_\mu A_\nu - \partial_\nu A_\mu + \left[A_\mu, A_\nu\right] = \left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^a{}_{bc} A^b_\mu, A^c_\nu\right) T_a
\end{aligned} \tag{8}$$

Since no partial derivatives fall on the fields, we can shift things around a bit on (7) if we compensate by being explicit about the indices:

$$\partial_\mu T^{\mu\nu} = i\overline{\psi}_i \gamma_\mu \left(F^{\mu\nu}\right)^i_j \psi^j = i\left(F^{\mu\nu}\right)^i_j \overline{\psi}_i \gamma_\mu \psi^j \equiv \left(F^{\mu\nu}\right)^i_j \mathcal{J}_{\mu}{}^j_i = \mathrm{Tr}\left(F^{\mu\nu}\mathcal{J}_\mu\right)$$

Where we defined $\mathcal{J}_{\mu}{}^j_i = \overline{\psi}_i \gamma_\mu \psi^j$, the following holds for it:

$$\mathcal{J}_{\mu}{}^j_i = \frac{\partial \mathcal{L}}{\partial A^\mu{}^i_j}$$

[9] see appendix B for an alternative SEM and how it relates to this one.

[10] see appendix A for the calculation with a complex scalar field with the SEM from General relativity

After some thought this result isn't too surprising, in the Abelian case for the group G, the above result reduces to the Lorentz force, and (5) becomes the Lagrangian for a particle in an external EM field! But why is there a force at all? The EOM seem oblivious to any new dynamics, and all we have done is added some internal indices to our fields, which the Lagrangian explicitly tries to ignore. How does this brake the homogeneity of space-time? The answer is curvature!

1.4 Keeping Track of Curvature

So far we have dealt with our fiber bundle as if it were a simple Cartesian product of a manifold and the fibers, Homogeneous! every point having the same tangent space as it's neighbors, but $F^{\mu\nu} \neq 0$ suggest the possibility of something more: Tangent spaces that change along different directions! This we wouldn't see in the EOM, those are only concerned with the very small neighborhood around a single point.[11] The Lagrangian (density), however, is already a function of the tangent space around a point (it has derivatives of fields), shifting it around reveals the possibility of some regions contributing more to the action than others!

This isn't the effect of a particular choice of coordinates either. The Curvature is covariant under passive transformations [4] (suppressing contracted indices):

$$\begin{aligned}(F^{\mu\nu})^i_j = [D^\mu, D^\nu]^i_j \xrightarrow[\text{Passive Trafo.}]{} & U^i D^\mu U^\dagger U D^\nu U^\dagger_j - U^i D^\nu U^\dagger U D^\mu U^\dagger_j \\ &= U^i F^{\mu\nu} U^\dagger_j\end{aligned}$$

coordinates on the fiber bundle really only describe points on the fiber bundle, but at the end of the day the points themselves are never observables all what matters are the relative positions of particles in space[12]. More important than points are actually the velocities (the tangent space) and how those change along different directions.

This is a nice story, but how do we fix (7)!? Well to restore homogeneity of space-time with respect to the action, all we need to do is keep track of the curvature in the Lagrangian by adding a $\mathcal{L}_{gf}$ term to the $\mathcal{L}_{matter}$ term we've had so far:

$$\mathcal{L} \to \mathcal{L} = \mathcal{L}_{gf} + \mathcal{L}_{matter}$$

and we will also need to promote the A_μ to a dynamic variable. To see why, let's think about what happens if we shift the action by some ε while putting

[11] A very similar thing happens in General Relativity, recall that the geodesic equation is agnostic to any curvature, particles simply fly in what locally looks like straight lines.

[12] Look up Einstein's hole argument to see if you really understand Curvature! Einstein thought he did, then he didn't, then he did again.

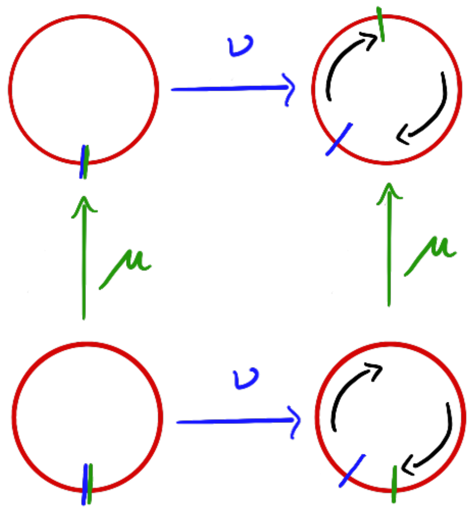


Figure 5: A schematic representation of curvature: on the left hand side the fibers (red) don't rotate when we move on the μ direction, on the right hand side *the fibers themselves rotate as we move in the* μ *direction*. Comparing the movement in the μ direction (D_μ) as we move in the ν direction (D_ν), $D_\nu D_\mu > 0$ to the movement in the ν direction (D_ν) as we move in the μ direction (D_μ), $D_\mu D_\nu \approx 0$ reveals the lack of homogeneity in space (curvature!)

the matter fields on shell:

$$\delta_\varepsilon S\big|_{\psi=\psi_0} = \int \frac{\delta S}{\delta A_\mu}\Big|_{\psi=\psi_0} \delta_\varepsilon A_\mu + \underbrace{\frac{\delta S}{\delta \psi}\Big|_{\psi=\psi_0}}_{=0} \delta_\varepsilon \psi \cdots = \int \frac{\delta S}{\delta A_\mu}\Big|_{\psi=\psi_0} \delta_\varepsilon A_\mu \dots$$

For this to vanish (if we don't want to fix A_μ) we need to put A_μ on shell. Then, if we also want to demand renormalizability in the quantum field theory, it and invariance under passive transformations/gauge invariance are enough to fix $\mathcal{L}_{gf}$ (with the possible addition of a θ-term). If we don't demand it, we cannot fully fix $\mathcal{L}_{gf}$, as the calculations in appendix C show. In Appendix C it will also be shown that, that term allowed by renormalizability and gauge-invariance is enough to restore homogeneity of space-time (wrt. $\mathcal{L}$), so that the full SEM is conserved. That term is:

$$\mathcal{L}_{gf} = -\frac{1}{2}\,\mathrm{Tr}\left(F^{\mu\nu}F_{\mu\nu}\right) \tag{9}$$

1.5 Yang-Mills Theory

With all of the above in mind the Lagrangian for a Dirac field that is "gauge invariant" and shows homogeneity throughout space-time (even when the Fiber bundle is curved) is [4]:

$$\mathcal{L} = -\frac{1}{2}\mathrm{Tr}\left(F_{\mu\nu}F^{\mu\nu}\right) + i\overline{\psi}_i \not{D}\psi^i - m\overline{\psi}_i\psi^i$$

The analog for complex scalar field is:

$$\mathcal{L} = -\frac{1}{2}\mathrm{Tr}\left(F_{\mu\nu}F^{\mu\nu}\right) + \left(D^\mu\varphi\right)_i^\dagger D_\mu\varphi^i - m^2\varphi_i^\dagger\varphi^i$$

So far we have actually been using something closer to the mathematicians notation, mostly because I wanted to write down less is. In what follows we introduce the physicist notation for Yang-Mills:

Physicist Notation	Mathematical Notation (here)
$A_\mu^\dagger = A_\mu$	$A_\mu^\dagger = -A_\mu$
$U(x) = e^{i\theta(x)^a T^a}$	$U(x) = e^{-\theta(x)^a (T^a)^i_j}$
$[T^a, T^b] = if^{abc}T^c$	$[T_a, T_b] = f_{ab}{}^c T_c$
$F^{\mu\nu} \to UF^{\mu\nu}U^\dagger$	$F^{\mu\nu} \to UF^{\mu\nu}U^\dagger$
$A_\mu \to UA_\mu U^\dagger - \frac{i}{g}(\partial_\mu U)U^\dagger$	$A_\mu \to UA_\mu U^\dagger - (\partial_\mu U)U^\dagger$
$D_\mu\phi_i = \partial_\mu\phi_i - ig(A_\mu)_{ij}\phi^j$	$D_\mu\phi^i = \partial_\mu\phi^i + (A_\mu)^i_j\phi^j$
$A^a_\mu \xrightarrow{\delta} A^a_\mu + \theta^b(x)A^c_\mu f^{abc} + \frac{1}{g}\partial_\mu\theta^a$	$A^a_\mu \xrightarrow{\delta} A^a_\mu - \theta^b(x)A^c_\mu f^a{}_{bc} + \partial_\mu\theta^a$
$F_{\mu\nu} = \frac{1}{ig}[D_\mu, D_\nu]$	$F_{\mu\nu} = [D_\mu, D_\nu]$
$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu$	$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^a{}_{bc}A^b_\mu A^c_\nu$
$F^a_{\mu\nu} \xrightarrow{\delta} F^a_{\mu\nu} - \theta^b F^c_{\mu\nu} f^{abc}$	$F^a_{\mu\nu} \xrightarrow{\delta} F^a_{\mu\nu} - \theta^b F^c_{\mu\nu} f^{abc}$

2 Quantum Yang-Mills

2.1 Faddev-Poppov Ghosts

Quantzing a theory ought to be easy just write down your classical action in:

$$\mathcal{Z} = \int \mathcal{D}\psi \mathcal{D}\overline{\psi} e^{iS} \tag{10}$$

and be done with it! However there are some obvious problems and some which are not so obvious from looking at (10), like states with negative norm or anomalous symmetries. Already however we might feel that something is missing from the above. From the earlier discussion we know that we should treat A_μ like a dynamical field and so we feel inclined to also include it in the measure of the path integral. At the same time we know A_μ varies with passive transformations which should not change the observables or to put it in another way, there is gauge redundancy in A_μ. The most logical thing is to fix the gauge with a gauge fixing function f and then integrate over it:

$$\mathcal{Z} = \int \mathcal{D}\psi \mathcal{D}\overline{\psi} \mathcal{D}A_\mu e^{iS} \delta\left(f\left(A_\mu\right)\right) \tag{11}$$

But remember that a fixed A_μ doesn't fix our coordinate system, A_μ merely keeps track of how basis vectors change from one point to the next.

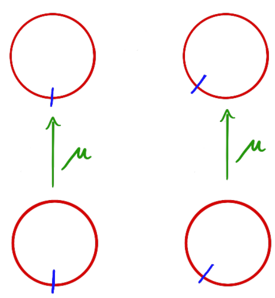


Figure 6: Two sets of basis vectors have the same A_μ, but are not the same.

So how many basis vector configurations fit in a fixed A_μ? Well that would look something like [5]:

$$\frac{1}{\Delta(A_\mu)} = \int_{U(x)\in\rho(G)} \mathcal{D}\left(U(x)\right)\delta(f(A_\mu)) \tag{12}$$

we need to divide out this term from the delta function in (11) to remove the excess degrees of freedom:

$$\mathcal{Z} = \int \mathcal{D}\psi \mathcal{D}\overline{\psi} \mathcal{D}A_\mu e^{iS} \delta\left(f\left(A_\mu\right)\right)\Delta(A_\mu) \tag{13}$$

Now we just need to calculate $\Delta(A_\mu)$, it turns out we can actually do so with a convenient term:

$$\delta\left(f\left(A_\mu\right)\right)\Delta(A_\mu) = \int \mathcal{D}c\mathcal{D}c^\dagger\mathcal{D}A_\mu e^{\int(-i/2\xi\partial A)d^4x + iS_{\text{ghost}}(c,c^\dagger)}$$

The ghost action looks like the typical action with a Lagrangian density $\mathcal{L} = \partial_\mu c_a^\dagger \partial^\mu c_a - \partial_\mu c_a^\dagger f^{abc} A^{\mu c} c_b$ So all we need to do is add these particles to our Feynman diagrams as well and that's it! Note that the c_a field follows fermionic statistics even though they have a bosonic Lagrangian, they must not appear in any external legs. Of course this is only part of the story, there is a lot more to talk about, like BRST symmetry, but I have to stop somewhere.

2.2 via Feynman Diagrams

Now we can apply the machinery of quantum field theory to solve the path integral! The standard procedure is to treat this integral like a Gaussian integral with some perturbations on it. Let's expand $\mathrm{Tr}(F^{\mu\nu}F_{\mu\nu})$ and see what we find choosing $\mathrm{Tr}(T_aT_b) = \frac{1}{2}\delta_{ab}$ [4]:

$$\frac{-1}{2}\mathrm{Tr}F^2 = \frac{1}{4}\left(\partial_\nu A^a_\mu - \partial_\mu A^a_\nu\right)^2 - \frac{gf_{abc}}{2}A^a_\mu A^b_\nu\left(\partial^\mu A^{c\nu} - \partial^\nu A^{c\mu}\right) - \frac{g^2}{4}\left(f_{abc}A^b_\mu A^c_\nu\right)^2$$

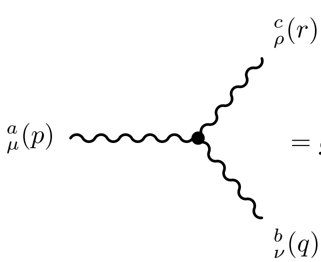
After using the anti-symmetry of f_{abc} and some partial integration , we get[1]:

$$\frac{-1}{2}\mathrm{Tr}F^2 = \frac{1}{2}A^{a\mu}\left(g_{\mu\nu}\partial^2 - \partial_\nu\partial_\mu\right)A^{a\nu} - gf_{abc}A^a_\mu A^b_\nu\partial^\mu A^{c\nu} - \frac{g^2}{4}\left(f_{abc}A^b_\mu A^c_\nu\right)^2$$

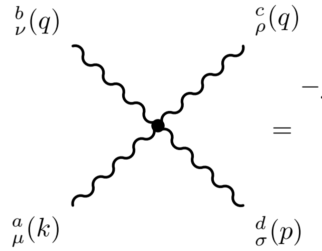
The first term (after gauge fixing) can be integrated over as a Gaussian integral and is the only one present in the Abelian theory ($f^{abc} = 0$): The corresponding Feynman Diagram (in the R_ξ gauge) is [1]:

$$= \frac{\delta^{ab}}{k^2 - i\epsilon}\left(g_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right)$$

But added to this we have two more terms that we have to treat as perturbations of the first [1]:



$$= gf^{abc}\left((q-r)_\mu g_{\nu\rho} + (r-p)_\nu g_{\mu\rho} + (p-q)_\rho g_{\mu\nu}\right)$$



$$= \begin{array}{l} -g^2\big(f^{abe}f^{cd}{}_e(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}) \\ \qquad + f^{ace}f^{db}{}_e(g_{\mu\sigma}g_{\rho\nu} - g_{\mu\nu}g_{\rho\sigma}) \\ \qquad + f^{ade}f^{bc}{}_e(g_{\mu\nu}g_{\sigma\rho} - g_{\mu\rho}g_{\sigma\nu})\big) \end{array}$$

This means that the non-Abelian gauge bosons interact with each other! Notice that they couple to each other with the structure constants f^{abc}, This is what people mean when they say the gauge boson transforms under the adjoint representation[13]. The gauge bosons interacting with each other also means that there is no true free field solution, even without the presence of other fields. Yet, all of this will work so long as g, the coupling constant, is small, but even then it ain't pretty; We started out with a Lagrangian rich with symmetry and now we are forced to brake it and put it back together diagram by diagram.

2.3 via Lattice Field Theory

In the words of Anthony Zee [5]: Wilson proposed a way out: Do violence to Lorentz invariance rather to gauge invariance. The idea is to do quantum field theory in a hyper-cubic lattice of spacing $a \to 0$ in Euclidean space-time, instead of a continuum of points. Hopefully, in this limit we get back proper Lorentz invariance. Be warned that the notation on this subsection clashes strongly with the previous one! The idea is the same, however. Take a point on the lattice $\mathbf{i}$, we are interested in a vector on that position $x_{\mathbf{i}}$ (NOT the i-th component of the vector). $x_{\mathbf{i}}$ is again the vector that gets transformed under global gauge transformations. We take 4 of these $x_{\mathbf{i}}$,$x_{\mathbf{j}}$, $x_{\mathbf{k}}$, $x_{\mathbf{l}}$ and build a square of side length a called a plaquette (see Figure 7). Next we associate a member of the representation of the group $U \in \rho(G)$ to one of the sides, like $U_{\mathbf{i-j}}$. We can then build a gauge/passive transformation invariant quantity:

$$S(P_{\mathbf{ijkl}}) = \mathrm{ReTr}\,(U_{\mathbf{i-j}}\,U_{\mathbf{j-k}}\,U_{\mathbf{k-l}}\,U_{\mathbf{l-i}})$$

Next we can define pure Yang-Mills theory by[5]:

$$\mathcal{Z} = \int \Pi dU e^{\frac{1}{2g^2}\sum_P S(P)} \tag{14}$$

Where we sum over all plaquettes. For small coupling constants, large values of $S(P)$ are better tolerated, given by $U_{\mathbf{i-j}} \approx \mathbb{1}$. It can be shown that the above

[13] recall that the representation of an algebra maps an algebra element A_μ to a linear map, in this case that linear map is the commutator $[A_\mu, \cdot]$ that acts on other members of the algebra. All of this is a fancy way of saying that the different A_μs act on each other via the commutator

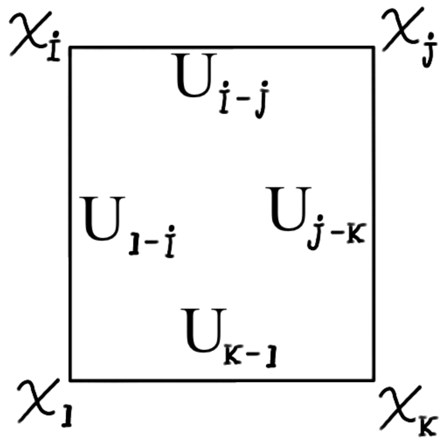


Figure 7: Plaquette.

formula reduces to pure Yang-Mills theory in the $a \to 0$ limit. To see this we define:

$$U_{\mathbf{i-j}} = e^{iaA_{\mathbf{i-j}}}$$

where $A_{\mathbf{i-j}}$, lives in the mid point between $\mathbf{i}$ and $\mathbf{j}$ and points in the direction $\mathbf{i} \to \mathbf{j}$, this then means $A_{\mathbf{i-j}} = -A_{\mathbf{j-i}}$. Using the BCH formula we get:

$$\begin{aligned}
\mathrm{Tr}(\dots) &= \mathrm{Tr}\left(e^{i\left(aA_{\mathbf{i-j}}+aA_{\mathbf{j-k}}+a^2\frac{i}{2}[A_{\mathbf{i-j}},A_{\mathbf{j-k}}]\right)\cdots}e^{i\left(aA_{\mathbf{k-l}}+aA_{\mathbf{l-i}}+a^2\frac{i}{2}[A_{\mathbf{k-l}},A_{\mathbf{l-i}}]\right)\cdots}\right) \\
&= \mathrm{Tr}\left(e^{i\left(aA_{\mathbf{i-j}}+aA_{\mathbf{j-k}}+a^2\frac{i}{2}[A_{\mathbf{i-j}},A_{\mathbf{j-k}}]+aA_{\mathbf{k-l}}+aA_{\mathbf{l-i}}+a^2\frac{i}{2}[A_{\mathbf{k-l}},A_{\mathbf{l-i}}]\right)\cdots}\right) \\
&= \mathrm{Tr}\left(e^{ia^2\left(a^{-1}((A_{\mathbf{l-i}}-A_{\mathbf{k-j}})-(A_{\mathbf{l-k}}-A_{\mathbf{i-j}}))+\frac{i}{2}[A_{\mathbf{i-j}},A_{\mathbf{j-k}}]+\frac{i}{2}[A_{\mathbf{k-l}},A_{\mathbf{l-i}}]\right)\cdots}\right) \\
&\xrightarrow{a\to 0} \mathrm{Tr}\left(e^{ia^2(\partial_\mu A_\nu-\partial_\nu A_\mu+i[A_\mu,A_\nu])+\dots}\right) = \mathrm{Tr}\left(e^{ia^2F_{\mu\nu}-\frac{a^4}{2}F_{\mu\nu}F_{\mu\nu}\cdots}\right)
\end{aligned}$$

Where we defined e_μ to be the direction parallel to $\mathbf{i} \to \mathbf{j}$ and e_ν to be parallel to $\mathbf{i} \to \mathbf{l}$. Lastly:

$$S(P_{\mathbf{ijkl}}) = \mathrm{ReTr}\left(e^{ia^2F_{\mu\nu}-\frac{a^4}{2}F_{\mu\nu}F_{\mu\nu}\cdots}\right) \approx \mathrm{Tr}\,\mathbb{1} - \mathrm{Tr}\frac{a^4}{2}F_{\mu\nu}F_{\mu\nu}\dots$$

Here we bypass any troubles with gauge fixing or Faddeev-Poppov determinant. It also complements nicely the standard QFT procedures, as here the focus is on Yang-Mills symmetry and the dynamics that come from it, not on the relativistic scattering of particles. Lattice Gauge Theory also allows for numerical calculations, for instance the masses of hadrons can be computed.

Along with the above, Wilson also introduced another, more general gauge invariant object called a Wilson loop [5]:

$$W(\gamma) \equiv \mathrm{Tr}\left(U_{\gamma_1\gamma_2}U_{\gamma_2\gamma_3}U_{\gamma_3\gamma_4}\dots U_{\gamma_n\gamma_1}\right) \quad \xrightarrow{a\to 0} \quad \mathrm{Tr}\,\mathrm{P}\,e^{\oint_\gamma dx^\mu A_\mu}$$

where P denotes the path ordering. To evaluate it you simply put it in the partition function:

$$\langle W(\gamma)\rangle = \int \Pi dU e^{\frac{1}{2g^2}\sum_P S(P)} W(\gamma)$$

One nice Wilson loop is a very long lived rectangle C that stretches some distance through space R and a long time T [5]:

$$\langle \mathrm{Tr}\,\mathrm{P}\, e^{\oint_C dx^\mu A_\mu}\rangle \approx e^{E(R)T} \implies \ln\langle \mathrm{Tr}\,\mathrm{P}\, e^{\oint_C dx^\mu A_\mu}\rangle \approx E(R)T$$

For Abelian theory, the above is essentially the perimeter of the rectangle. For non-Abelian theory, the conjecture is that $E(R) \propto R$ and that would mean that the logarithm of the expectation value of the loop goes as the area of the enclosed rectangle. This is called the area law.

2.4 Asymptotic Freedom and Infrared Slavery

Gauge theory for arbitrary groups is a nice story, but in the day to day, most people deal with one gauge theory (maybe two if you call General Relativity a gauge theory), Electromagnetism. Why do we only see the Abelian case? Why do we only see $U(1)$ on large scales? This phenomenon has a name, it's called confinement. In essence we never see non-Abelian "charge" in nature. For the $SU(3)$ of quarks this means all particles we observe are a colorless combinations of particles, not the individual "colored" quarks.

On the other hand, there is the work of Particle physicist, for whom a non-Abelian may be very relevant, one may think too relevant! Just think of all the Feynman diagrams that become possible as soon as photons start interacting with each other! It is nightmare! For some time this was a big worry for gauge theory, it's a nice, symmetric theory, but can one calculate anything useful with it? Paradoxically, experiments show that at high energies quarks behave as if they were free, i.e. At high energies the strong force seems to vanish, while at low energies it keeps quarks held tightly together. This weird behavior may not after all be all that weird for QFT. As here the coupling constants depend on the energy scale. To test this one just has to compute a few loop-diagrams and see how the coupling constant flows. Alternatively one can also use functional methods, either way one would find something like [6]:

$$\frac{dg}{d\ln(\mu)} = \beta(g) = \frac{g^3}{16\pi^2}\left[-\frac{11}{3}f^{acd}f^{acd} + \frac{4}{3}\mathrm{Tr}(T_aT_a)\right]$$

From it we can immediately tell that the minus sign in front of those structure constants that the non-Abelian gauge theory has the potential to behave very differently from the Abelian case. Solving the differential equation and writing out the solution in terms of $\alpha_S = \frac{g^2}{4\pi}$ you get [6]:

$$\alpha_s(Q) = \frac{\alpha_s(\mu)}{1 + (1/4\pi)\left(11 - \frac{2}{3}n_f\right)\ln\left(\frac{Q^2}{\mu^2}\right)}$$

n_f here is the number of fermions in the theory. This wonderful behavior means that the non-Abelian gauge theory is manageable at high energies despite the large number of diagrams involved, it is called *Asymptotic freedom*. The opposite side of the coin is *Infrared Slavery* and it means that perturbative calculations are no longer possible as the coupling constant approaches 1.

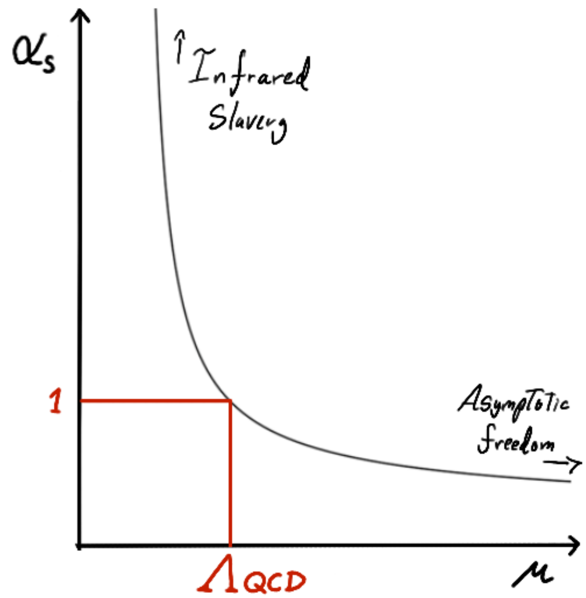


Figure 8: The coupling constant with respect to the energy scale.

This growing strength of the non-Abelian gauge theory at low energies/longer distances, is sometimes called vacuum anti-polarization. It goes some way to explain why we never see colored particles on their own. Though this is no proof of it, all of this reasoning rests upon perturbations of weak interactions,[14] all that goes down the drain as soon as $\alpha_S \approx 1$. What holds very true at all times is that curvature, the field, interacts with itself in non-Abelian gauge theory. Qualitatively, this has the effect that when separating particles that couple to the non-Abelian gauge field, tubes or strings of field get built between the charged particles, concentrating the field lines. When the energy density is high enough, new particles get exited into existence, these then shield the "non-Abelian charge".

[14]weak as in not strong.

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A Classical Yang-Mills for Bosons

A.1 Equations of Motion

The action of a gauge invariant bosonic field is:

$$S = \int (D^\mu \varphi)_i^\dagger D_\mu \varphi^i - m^2 \varphi_i^\dagger \varphi^i dx^4$$

The variation of the action then gives:

$$\begin{aligned}
\delta S &= \int \left(\delta (D^\mu \varphi)_i^\dagger\right) D_\mu \varphi^i + (D^\mu \varphi)_i^\dagger \left(\delta D_\mu \varphi^i\right) - m^2 \left(\delta \varphi_i^\dagger \varphi^i + \varphi_i^\dagger \delta \varphi^i\right) dx^4 \\
&\overset{(6)}{=} \int -\delta \varphi_i^\dagger D^\mu D_\mu \varphi^i - (D^\mu D_\mu \varphi)_i^\dagger \delta \varphi^i - m^2 \left(\delta \varphi_i^\dagger \varphi^i + \varphi_i^\dagger \delta \varphi^i\right) dx^4 \\
&= \int -\delta \varphi_i^\dagger \left(D^\mu D_\mu \varphi^i + m^2 \varphi^i\right) - \left((D^\mu D_\mu \varphi)_i^\dagger + m^2 \varphi_i^\dagger\right) \delta \varphi^i dx^4 \overset{!}{=} 0
\end{aligned}$$

A.2 Non-Abelian Lorentz Force

Plugging the action for bosons in the formula for the Stress energy tensor (16), we get:

$$T^{\mu\nu} = (D^\mu \varphi)_i^\dagger D^\nu \varphi^i + (D^\nu \varphi)_i^\dagger D^\mu \varphi^i - \eta^{\mu\nu} \mathcal{L}$$

Taking then the divergence and using (6) we get:

$$\begin{aligned}
\partial_\mu T^{\mu\nu} &= \cancel{(D_\mu D^\mu \varphi)_i^\dagger D^\nu \varphi^i} + (D^\mu \varphi)_i^\dagger D_\mu D^\nu \varphi^i + (D_\mu D^\nu \varphi)_i^\dagger D^\mu \varphi^i + \cancel{(D^\nu \varphi)_i^\dagger D_\mu D^\mu \varphi^i} \\
&- (D^\nu D_\mu \varphi)_i^\dagger D^\mu \varphi^i - (D_\mu \varphi)_i^\dagger D^\nu D^\mu \varphi^i - \cancel{m^2 (D^\nu \varphi)_i^\dagger \varphi^i} - \cancel{m^2 \varphi_i^\dagger D^\nu \varphi^i} \\
&\overset{EOM}{=} (D^\mu \varphi)_i^\dagger D_\mu D^\nu \varphi^i + (D_\mu D^\nu \varphi)_i^\dagger D^\mu \varphi^i \\
&- (D^\nu D_\mu \varphi)_i^\dagger D^\mu \varphi^i - (D_\mu \varphi)_i^\dagger D^\nu D^\mu \varphi^i \\
&= (D_\mu \varphi)_i^\dagger \left([D^\mu, D^\nu] \varphi^i\right) + ([D^\mu, D^\nu] \varphi)_i^\dagger D_\mu \varphi^i \\
&= (D_\mu \varphi)_i^\dagger (F^{\mu\nu})^i_j \varphi^j - (F^{\mu\nu})^j_i \varphi_j^\dagger D_\mu \varphi^i = \mathrm{Tr}\left(F^{\mu\nu} \mathcal{J}_\mu\right)
\end{aligned}$$

Where we defined:

$$(\mathcal{J}_\mu)^i_j = (D_\mu \varphi)_j^\dagger \varphi^i - \varphi_j^\dagger D_\mu \varphi^i$$

B Stress Energy Tensor Shenanigans

The Stress Energy Tensor on subsection 1.3:

$$T^{\mu\nu} = \frac{\partial}{\partial\left(\partial_\mu \psi^i\right)} D^\nu \psi^i + D^\nu \overline{\psi^i} \frac{\partial}{\partial\left(\partial_\mu \overline{\psi}_i\right)} - g_{\mu\nu}\mathcal{L} \quad = \quad i\overline{\psi}_i \gamma^\mu D^\nu \psi^i - g_{\mu\nu}\mathcal{L}$$

Has many issues, besides how do we know we should upgrade the partial derivatives to covariant ones in the prescription for the SEM? We can abandon this prescription all together and use the definitive formula for the SEM, the one from GR:

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}} \overset{(16)}{=} 2\frac{\partial \mathcal{L}}{\partial g^{\mu\nu}} - g_{\mu\nu}\mathcal{L}$$

plugging in (5) we get (remembering we have to symmetrize the $\frac{\partial \mathcal{L}}{\partial g^{\mu\nu}}$):

$$T^{\mu\nu} = i\overline{\psi}_i \gamma^\mu D^\nu \psi^i + i\overline{\psi}_i \gamma^\nu D^\mu \psi^i - g^{\mu\nu}\mathcal{L} \tag{15}$$

Which is different from what we previously got. What gives? To be precise, this tensor is actually the Belinfante–Rosenfeld tensor, if we define the SEM tensor to be:

$$\delta S = \frac{1}{2} \int d^4x \sqrt{-g}\, T^{\mu\nu} \delta g_{\mu\nu}$$

Then we can define a new tensor with a total derivative without impacting the variation of the action or the above definition:

$$\tilde{T}^{\mu\nu} = T^{\mu\nu} + \partial_\rho h^{\rho\mu\nu} \qquad \text{where} \qquad h^{\rho\nu\mu} = -h^{\rho\mu\nu}$$

Thus

$$2\delta S = \int d^4x \sqrt{-g}\, \tilde{T}^{\mu\nu} \delta g_{\mu\nu} = \int d^4x \sqrt{-g} \left(T^{\mu\nu} + \partial_\rho h^{\rho\mu\nu}\right) \delta g_{\mu\nu}$$

In the case of the SEM in section 1.3, the h in question is the Belinfante–Rosenfeld modification term, which is related to the spin current:

$$h^{\rho\mu\nu} = \frac{i}{2} \overline{\psi} \gamma^\rho \gamma^\mu \gamma^\nu \psi$$

we get then (using the product rule of covariant derivatives):

$$\partial_\rho h^{\rho\mu\nu} = \frac{i}{2} \overline{\not{D}\psi} \gamma^\mu \gamma^\nu \psi + \frac{i}{2} \overline{\psi} \gamma^\rho \gamma^\mu \gamma^\nu D_\rho \psi$$

Next we use the anti-commutation rules of the gamma matrices:

$$\begin{aligned}
\partial_\rho h^{\rho\mu\nu} &= \frac{i}{2} \overline{\not{D}\psi} \gamma^\mu \gamma^\nu \psi + \frac{i}{2} \overline{\psi} \left(-\gamma^\mu \gamma^\rho \gamma^\nu - 2g^{\mu\rho}\gamma^\nu\right) D_\rho \psi \\
&= \frac{i}{2} \overline{\not{D}\psi} \gamma^\mu \gamma^\nu \psi + \frac{i}{2} \overline{\psi} \left(\gamma^\mu \gamma^\nu \gamma^\rho + 2g^{\rho\nu}\gamma^\mu - 2g^{\mu\rho}\gamma^\nu\right) D_\rho \psi \\
&= \frac{i}{2} \left(\overline{\not{D}\psi} \gamma^\mu \gamma^\nu \psi + \overline{\psi} \gamma^\mu \gamma^\nu \not{D}\psi\right) + i\overline{\psi}_i \gamma^\mu D^\nu \psi^i - i\overline{\psi}_i \gamma^\nu D^\mu \psi^i
\end{aligned}$$

plugging in the equations of motion makes the term in parenthesis vanish. Adding then this to (15) we get back the SEM that we calculated previously (after some slight rescaling).

C Conserving The Stress Energy Tensor in a Classical Gauge Invariant Theory

Starting from Noether's Theorem:

$$\delta_\varepsilon S = \int \partial_\mu T^{\mu\nu}_{new} \varepsilon_\nu \sqrt{-g} dx^4 = \int \left(\partial_\mu T^{\mu\nu}_{old} + \partial_\mu T^{\mu\nu}_{gf} \right) \varepsilon_\nu \sqrt{-g} dx^4$$

We can run the Noether theorem in reverse, plugging in what we already know to get:

$$\delta_\varepsilon S = \int \left[\mathrm{Tr} \left(F^{\mu\nu} \frac{\partial \mathcal{L}}{\partial A^\mu} \right) + \partial_\mu T^{\mu\nu}_{gf} \right] \varepsilon_\nu \sqrt{-g} dx^4$$

For this to vanish, the term in brackets must be 0. We can also completely decouple the matter Lagrangian from the calculations by assuming that it contains no terms like $\partial_\rho A_\mu$ and by using the fact that A_μ solves the Euler Lagrange equations:

$$0 = \mathrm{Tr} \left(F^{\mu\nu} \left(\partial_\rho \frac{\partial \mathcal{L}_{gf}}{\partial \left(\partial_\rho A^\mu \right)} - \frac{\partial \mathcal{L}_{gf}}{\partial A^\mu} \right) \right) + \partial_\mu T^{\mu\nu}_{gf}$$

To make further progress, let's assume that $\mathcal{L}_{gf}$ is a function of the gauge field A_μ only via the field strength tensor $F_{\mu\nu}$:

$$\mathrm{Tr} \left(F^{\mu\nu} \left(\partial^\rho \left(\frac{\partial \mathcal{L}_{gf}}{\partial F^{\alpha\beta}} \frac{\partial F^{\alpha\beta}}{\partial \left(\partial^\rho A^\mu \right)} \right) - \frac{\partial \mathcal{L}_{gf}}{\partial F^{\alpha\beta}} \frac{\partial F^{\alpha\beta}}{\partial A^\mu} \right) \right) = -\partial_\mu T^{\mu\nu}_{gf}$$

Comparing with (8), for the first partial derivative involving A_μ we get an antisymmetric term $\delta^\alpha_\rho \delta^\beta_\mu - \delta^\alpha_\mu \delta^\beta_\rho$ this term gets quickly contracted with the antisymmetric partial derivative next to it, giving a factor of 2. The second partial derivative involving A_μ doesn't actually contribute. We get:

$$2\mathrm{Tr} \left(F^{\mu\nu} \partial^\rho \frac{\partial \mathcal{L}_{gf}}{\partial F^{\rho\mu}} \right) = -\partial_\mu T^{\mu\nu}_{gf}$$

Then We need a way of defining the SEM of the field, let's use the definitive definition for it, the one from general relativity [2]:

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g_{\mu\nu}} = \frac{2}{\sqrt{-g}} \left(\frac{\partial \left(\sqrt{-g} \mathcal{L} \right)}{\partial g_{\mu\nu}} - \partial_\delta \frac{\partial \left(\sqrt{-g} \mathcal{L} \right)}{\partial \left(\partial_\delta g_{\mu\nu} \right)} \right) = 2 \frac{\partial \mathcal{L}}{\partial g_{\mu\nu}} - g^{\mu\nu} \mathcal{L} \tag{16}$$

Where we used the Euler Lagrange equations under the assumption that $\mathcal{L}$ is independent of $\partial_\delta g_{\mu\nu}$. So in our case get:

$$2\mathrm{Tr} \left(F^{\mu\nu} \partial^\rho \frac{\partial \mathcal{L}_{gf}}{\partial F^{\rho\mu}} \right) = -2 \partial^\rho g_{\rho\mu} \frac{\partial \mathcal{L}}{\partial g_{\mu\nu}} + \partial^\nu \mathcal{L} \tag{17}$$

If we further hone in our Ansatz as a polynomial in the field strength tensors somehow contracted with metric tensors under the trace of the representation vector space:

$$\mathcal{L}_{gf} = \text{Tr}\left(\sum_{n\geq 2} c_n \left(g^n\right)_{\alpha_1\alpha_2\ldots\alpha_{2n}} \left(F^n\right)^{\alpha_1\alpha_2\ldots\alpha_{2n}}\right) \tag{18}$$

noting that the metric must always symmetrically contract two different $F^{\mu\nu}$ or the term is 0, we can re-write the partial derivative with respect to the metric:

$$\begin{aligned}
g_{\rho\sigma}\frac{\partial \mathcal{L}_{gf}}{\partial g_{\sigma\nu}} &= \text{Tr}\left(\sum_{n\geq 2} c_n g_{\rho\sigma}\frac{\partial g_{\beta\gamma}}{\partial g_{\sigma\nu}} n \left(g^{n-1}\right)_{\ldots\mu\ldots\delta\ldots} F^{\mu\gamma}\left(F^{n-1}\right)^{\ldots\beta\ldots\delta\ldots}\right)\\
&= \text{Tr}\left(\sum_{n\geq 2} c_n g_{\rho\sigma}\delta^\sigma_\beta\delta^\nu_\gamma n \left(g^{n-1}\right)_{\ldots\mu\ldots\delta\ldots} F^{\mu\gamma}\left(F^{n-1}\right)^{\ldots\beta\ldots\delta\ldots}\right)\\
&= \text{Tr}\left(\sum_{n\geq 2} c_n g_{\rho\beta} n \left(g^{n-1}\right)_{\ldots\mu\ldots\delta\ldots} F^{\mu\nu}\left(F^{n-1}\right)^{\ldots\beta\ldots\delta\ldots}\right)
\end{aligned}$$

where we used the cyclicity of the trace to bring one of the $F^{\mu\nu}$ with recently freed indices to the front.

$$\begin{aligned}
g_{\rho\sigma}\frac{\partial \mathcal{L}_{gf}}{\partial g_{\sigma\nu}} &= \text{Tr}\left(\sum_{n\geq 2} c_n g_{\rho\beta} n \left(g^{n-1}\right)_{\ldots\mu\ldots\delta\ldots} F^{\mu\nu}\left(F^{n-1}\right)^{\ldots\beta\ldots\delta\ldots}\right)\\
&= \text{Tr}\left(\sum_{n\geq 2} c_n g_{\alpha\beta} \left(g^{n-1}\right)_{\ldots\gamma\ldots\delta\ldots} F^{\mu\nu} n\delta^\alpha_\rho\delta^\gamma_\mu \left(F^{n-1}\right)^{\ldots\beta\ldots\delta\ldots}\right)\\
&= \text{Tr}\left(F^{\mu\nu}\sum_{n\geq 2} c_n g_{\alpha\beta} \left(g^{n-1}\right)_{\ldots\gamma\ldots\delta\ldots} n\frac{\partial F^{\alpha\gamma}}{F^{\rho\mu}} \left(F^{n-1}\right)^{\ldots\beta\ldots\delta\ldots}\right)\\
&= -\,\text{Tr}\left(F^{\mu\nu}\frac{\partial \mathcal{L}_{gf}}{\partial F^{\rho\mu}}\right)
\end{aligned} \tag{19}$$

Plugging this in (17), we get:

$$2\text{Tr}\left(F^{\mu\nu}\partial^\rho\frac{\partial \mathcal{L}_{gf}}{\partial F^{\rho\mu}}\right) = 2\,\text{Tr}\left(\partial^\rho\left(F^{\mu\nu}\frac{\partial \mathcal{L}_{gf}}{\partial F^{\rho\mu}}\right)\right) + \partial^\nu\mathcal{L}_{gf}$$

Using the product rule we get:

$$2\text{Tr}\left(F^{\mu\nu}\partial^\rho\frac{\partial \mathcal{L}_{gf}}{\partial F^{\rho\mu}}\right) = \text{Tr}\left(\left(2\partial^\rho F^{\mu\nu}\right)\frac{\partial \mathcal{L}_{gf}}{\partial F^{\rho\mu}} + 2F^{\mu\nu}\partial^\rho\frac{\partial \mathcal{L}_{gf}}{\partial F^{\rho\mu}}\right) + \partial^\nu\mathcal{L}_{gf}$$

after using the chain rule of matrix calculus for $\partial^\nu \mathcal{L}_{gf}$ and assuming that $\mathcal{L}_{gf}$ depends on space-time only via the field strength, this is equivalent to:

$$
\begin{aligned}
0 &= \mathrm{Tr}\left((2\partial^\rho F^{\mu\nu}) \frac{\partial \mathcal{L}_{gf}}{\partial F^{\rho\mu}} + \frac{\partial \mathcal{L}_{gf}}{\partial F^{\rho\mu}} \partial^\nu F^{\rho\mu} \right) \\
&= \mathrm{Tr}\left((2\partial^\rho F^{\mu\nu} + \partial^\nu F^{\rho\mu}) \frac{\partial \mathcal{L}_{gf}}{\partial F^{\rho\mu}} \right) \\
&= \left(2D^\rho (F^{\mu\nu})^i_j + D^\nu (F^{\rho\mu})^i_j\right) \left(\frac{\partial \mathcal{L}_{gf}}{\partial F^{\rho\mu}} \right)^j_i \\
&= \left(2D^\rho (F^{\mu\nu})^i_j - D^\mu (F^{\nu\rho})^i_j - D^\rho (F^{\mu\nu})^i_j\right) \left(\frac{\partial \mathcal{L}_{gf}}{\partial F^{\rho\mu}} \right)^j_i \\
&= \mathrm{Tr}\left(2D^\rho (F^{\mu\nu})^i_j - 2D^\rho (F^{\mu\nu})^i_j\right) \left(\frac{\partial \mathcal{L}_{gf}}{\partial F^{\rho\mu}} \right)^j_i = 0
\end{aligned}
$$

where the second to last equality follows from the Bianchi identity. In essence, we see that any $\mathcal{L}_{gf}$ that complies with (19) and depends in spacetime only via the curvature is a suitable candidate for tracking the curvature. When quantizing the theory, a $\mathcal{L}_{gf}$ containing high orders of A_μ won't be re-normalizable, so a natural choice is to include only the first term in the polynomial Ansatz.

C.1 The Bianchi Identity

The Bianchi Identity is equivalent to the Leibniz rule for commutators, since $F^{\mu\nu}$ transforms in the adjoint representation:

$$
\begin{aligned}
(\partial_\rho F^{\mu\nu})^i_j &= \left(\partial_\rho \left(G_k \, (F^{\mu\nu})^k_m \, G^{\dagger m} \right) \right)^i_j \\
&= \left((\partial_\rho G_k) \, (F^{\mu\nu})^k_m \, G^{\dagger m} + G_k \left(\partial_\rho \, (F^{\mu\nu})^k_m \right) G^{\dagger m} + G_k \, (F^{\mu\nu})^k_m \, \partial_\rho G^{\dagger m} \right)^i_j \\
&= G^{\dagger i} \, (\partial_\rho G_k) \, (F^{\mu\nu})^k_m \, \delta^m_j + \partial_\rho \, (F^{\mu\nu})^k_m \, \delta^m_j \delta^i_k + \delta^i_k \, (F^{\mu\nu})^k_m \left(\partial_\rho G^{\dagger m} \right) G_j \\
&= \partial_\rho \, (F^{\mu\nu})^k_m \, \delta^m_j \delta^i_k + G^{\dagger i} \, (\partial_\rho G_k) \, (F^{\mu\nu})^k_m \, \delta^m_j - (F^{\mu\nu})^k_m \, G^{\dagger m} \partial_\rho G_j \delta^i_k \\
&= \partial_\rho \, (F^{\mu\nu})^k_m \, \delta^m_j \delta^i_k + (A_\mu)^i_k \, (F^{\mu\nu})^k_m \, \delta^m_j - (F^{\mu\nu})^k_m \, (A_\mu)^m_j \, \delta^j_k \\
&= \partial_\rho \, (F^{\mu\nu})^k_m \, \delta^m_j \delta^i_k + ([A_\mu, F^{\mu\nu}])^i_j \equiv D_\rho \, (F^{\mu\nu})^i_j
\end{aligned}
$$

We can also replace $(F^{\mu\nu})^i_j$ in the above with $(A^\mu)^i_j$ and we will see that the covariant derivative acts on it also like this (it transforms under the adjoint representation). The Bianchi Identity then reduces to the Jacobi Identity:

$$
\sum_{\mu,\nu,\rho \text{ cycl.}} [D_\mu, [D_\nu, D_\rho]] = 0
$$