Ausbreitung und Streuung in einer holographischen Quantenfeldtheorie

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I hereby declare that this thesis is my own work, and that I have not used any sources and aids other than those stated in the thesis.

Propagation and Scattering in a Holographic Quantum Field Theory

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Abstract

In this work we present a slight modification of the model presented by Friedrich et al. in [32] and the consequences of these modifications, particularly in terms of scattering amplitudes and particle propagation. Briefly, these models consider reducing the Hilbert space of a lattice-regularized quantum field, $\mathcal{H}$, by overlapping different Fourier modes, in order to achieve area-scaling of the dimension of $\mathcal{H}$. This area scaling of degrees of freedom is motivated by the holographic principle. These overlaps cause a multitude of changes in the theory, such as a modification of the anti-commutation relation of the fields.

In the original model, the overlaps were achieved by defining the creation/annihilation operators of the theory to be linear combinations of other creation/annihilation operators that followed the canonical commutation relations. In the version explored here, the creation operators are made by linear combinations of only other creation operators, instead of linear combinations of creation *and* annihilation operators; the opposite is true for annihilation operators. Consequently, we can understand the fields of the "overlapped theory" as being linear combinations of the fields of an un-overlapped theory. Because there is a large number of linear combinations possible, the resulting "overlapped theory" appears to have more degrees of freedom than the original one, but the dimension of the Hilbert space remains small.

By adapting the Hamiltonian correctly, it is shown that the resulting particle and multi-particle states have the correct *expectation value* of the energy, even though they are not energy eigenstates. The "life-time" of the particle states is calculated, and shown to differ from the one in [32]. Next, it is shown that the theory can also be explored in the path integral formalism. This, then, allows us to derive changes to the Feynman rules for the calculation of scattering amplitudes in quantum field theories with interactions.

Chapter 1

Introduction

100 years ago from the writing of this thesis (up to the month!) Werner Heisenberg published his breakthrough paper "*Über quantentheoretische Umdeutung kinematischer und mechanischer Beziehungen*", or "*On the quantum-theoretical reinterpretation of kinematical and mechanical relationships*" [39]. Although terms like "linear operator", "non-commuting", or "matrix" are not explicitly present in the text, glancing at the equations reveals that these concepts are indeed present in the equations. Not more than a few months later [31], they were recognized as such, and the now famous relation was first written:

$$\left[\hat{\boldsymbol{Q}}^i, \hat{\mathbf{P}}^j\right] = i\delta^{ij}. \tag{1.1}$$

This would later become the basis of what is known today as canonical quantization [25]. Although this process is effective — its predictions have been verified for a century — its origins remain mysterious. Moreover, it, by itself, isn't of much use, one needs a classical theory to quantize[1], i.e. one *must identify the degrees of freedom in nature* as a starting point.

In the task of finding the degrees of freedom in nature, the first steps were obvious, the theory should be Lorentz invariant, in accordance to Einstein's special relativity. In practice, this meant that, while the degrees of freedom one quantizes in Equation 1.1 may be relative to one observer, at least they should agree with the quantized degrees of freedom of another observer, when preforming the appropriate Lorentz transformation. Quantum field theory was found to be amenable to these demands, but one can go beyond special relativity. After Lorentz invariance, the next step was the more general diffeomorphism invariance, something required by the theory of general relativity. This proved to be a much tougher challenge, and it remains a tough challenge to this day.

More recently, however, another way of constraining the degrees of freedom of nature was found, perhaps relating to diffeomorphism invariance, or gravity, or something entirely new, the holographic principle.

[1] Recent developments challenge this way of thinking, see subsection 2.3.5

The holographic principle was discovered by exploring matter's effect on space-time via gravity, but, as we will see, it also makes appearances in other instances that do not involve gravity at all, thus shrouding it's exact origins in mystery. The story of canonical quantization, however, shows us that this unexplained origin should not stop us from using the holographic principle to arrive at new theories of nature. After all, it pairs nicely with the idea of quantization; although the holographic principle does not betray what degrees of freedom we should quantize, it does tell us how many of them there are, which may provide a big step in the task of finding a theory of quantum gravity or beyond.

The main goal of this work is to explore some aspects of a version of the model presented in [32]. This is a model that seeks to combine quantum field theory with the reduction of degrees of freedom one might expect from application of the holographic principle. The focus on the present work will be on the propagation and scattering of particle analogues of said model. Although explicit calculations of scattering matrices in an interacting quantum field theory lie outside of this work, the basis of said calculations will be laid out.

1.1 Summary of the Results

- Modification of the overlapping process of [32] is introduced, different and convenient perspectives of said process are also discussed (*chapter 4*).
- New definitions of particle and multi-particle states are introduced that are in close analogy to their standard definitions in normal quantum field theory (*chapter 4*).
- New life-time values for these new particle states are calculated, they are found to differ from the original calculation (*chapter 5*).
- The theory is translated from the Schrödigner/Heisenberg picture to the path integral formalism (*chapter 6*).
- The theory is solved in the path integral formalism and changes to the Feynman rules for the calculation of scattering amplitudes are discussed (*chapters 7,8*).

1.2 Outline of the Chapters

In chapter 2, we will explore what holography means and where it comes from. First, a couple of estimations of the amount of degrees of freedom in quantum field theory and general relativity are presented. Then, we will move on to define the non-covariant holographic principle. We will not explore the covariant formulation, as it's not of too much relevance for this specific work. Finally in the chapter, we will look at instances where holography or holographic-like properties appear in other theories, as well as theories building up on holography as the starting point.

In chapter 3, we will provide a quick review of the free Weyl field in normal quantum field theory, with some special notation and perspective that will allow us to better understand the overlapping process to be undertaken in chapter 4. It is also worth pointing out that the quantum field used here will include both UV and IR cutoffs, thus we explore a quantum field theory in a finite volume with lattice regularization. This is done so that the Hilbert space of the quantum field is of finite dimension, and thus the amount of degrees of freedom can be properly calculated.

In chapter 4, we will apply an overlapping procedure in the style of [32]. Though our definition of said process will look superficially different at first, it will be clear that the one presented here is a special case of the model from said other paper. We will first do general overlapping and then continue with shell-wise overlapping. After that, we will introduce an analog to a particle or plane wave state in the overlapped theory, as well as an analog to multi-particle states. Notice that these differ from [32]. The biggest advantage of narrowing the scope to this specific model is the presence of a vacuum state, which allows the analytic calculation of many properties, such as the expected value of the number operator or Hamiltonian for multi-particle states.

In chapter 5, we explore some of the consequences of the construction, including a finite life-time for the particle states, as well as their propagation in more detail. Our analysis of these properties will be analytic, backed up by some numerical experiments.

In chapter 6, we derive the path integral for the overlapped Weyl field, we do so by defining transformations between overlapped and un-overlapped fields, and then applying the usual derivation of the path integral formalism to the un-overlapped fields. In this chapter we also explain how to calculate correlation functions from the path integral, and in the last part of the chapter a way of introducing interaction terms to the path integral is sketched out. This explored in detail would allow for the introduction of an interacting overlapped quantum field theory in the future.

In chapter 7, we solve the path integral by "completing the square", as one would for a normal free quantum field. The difficulties caused by the now overlapping fields are addressed by expressing overlapped fields as linear combinations of un-overlapped fields. The end result is a new kind of propagator for the theory.

In chapter 8, we derive the LSZ-reduction formula for the overlapped field. One can then use the methods of the previous chapter to calculate some scatting matrices in an overlapped non-interacting quantum field theory. A set of modifications of the Feynman rules for the calculation of scattering matrices is the result of this chapter.

In chapter 9, we discuss our findings and propose some further ways of testing this theory and some similarities between it and other work, as well as potential future directions.

1.3 Notation

1.3.1 Vectors

Throughout this work we use Planck units, such that $\hbar = c = k_B = G = 1$, and we use the mostly negative sign convention for the metric (in this work we only investigate minkowski space-time):

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2. \tag{1.2}$$

Euclidean vectors, also called **3**-vectors in this work are written in **bold**, $\mathit{4}$-vectors are written as all other *mathematical variables* and identified by context. Einstein notation is used, such that Greek letters signify space and time components and Latin letters indicate spatial components:

$$x = \sum_{\mu=0}^{3} x^\mu e_\mu = x^\mu e_\mu = x^0 e_0 - x^i \boldsymbol{e}_i = \begin{pmatrix} x^0 \\ \boldsymbol{x} \end{pmatrix}. \tag{1.3}$$

1.3.2 (Linear) Functions and Functional Integration

We will also often distinguish between a function f and a function evaluated at a point $f(x)$. In the case of functional integration:

$$\int df(0) \int df(\delta t) \int df(2\delta t) \cdots \int df(t)\ F(f(0), f(\delta t) \dots f(t)) \overset{\delta t \to 0}{\longrightarrow} \int \mathcal{D}[f(t)]\ F[f], \tag{1.4}$$

where f may be a function of t and $\boldsymbol{x}$, but $\mathcal{D}[f(t)]$ means we treat f as a function only of t for the purposes of functional integration. We take $\mathcal{D}[f]$ to mean we integrate over all parameters. In analogy to functions, we will often distinguish linear operators M from their components $M_{\boldsymbol{qp}}$. Though if the linear operator affords representation by matrix, it will be called as such.

Chapter 2

Holography

2.1 The Amount of Degrees of Freedom in Nature

2.1.1 Quantum Field Theory

As mentioned in the introduction, after the quantum mechanics began to be formalized, the search for relativistic quantum mechanics was also undertaken. A natural solution to the demands of relativity was found in Quantum Field Theory (QFT). In the years following its introduction, QFT in the form of quantum electrodynamics and then the standard model made some of the most accurately verified predictions in the history of physics, like the anomalous magnetic momenta of particles and their masses [1][10][30][23]. To add to this, arguments based on symmetries and symmetry braking in the quantum fields led to the successful predictions of entirely new particles and interactions[34][40][29][65].

On the other hand, QFT has mechanisms to change and add its own parameters in accordance with energy scales via renormalization[1]. This, then, hints at the possibility that some of the QFTs we hold as accurate now may be the result of deeper, simpler, yet unknown high energy QFTs. Famous examples of such theories hidden in high energy scales include string theory and super-symmetry, both of which are forms of QFTs. If that is not the case, it is also possible that in the high UV-limit the parameters of the theory converge to finite values, and that the most fundamental theory of nature ends up being similar to the current starting point, a QFT, this is the idea behind the asymptotic safety program [28][27].

Given this great track record and even greater promise, it is no surprise that it is often assumed that the fundamental theory of nature must be a quantum field theory of some kind. We will also keep this assumption in the back of our minds for the following chapters, and in particular in the next subsection.

[1]In some sense, this flexibility has also caused great criticism of the theory, as contributions to the parameters calculated this way often diverge and need to be regularized to finite, empirically measured quantities.

2.1.2 Adding Degrees of Freedom Up

The object of study in quantum field theory is, of course, the quantum field. Like any other field, this is a function from points in space to some field space. In the case of quantum field theory this field space is represented — as with the observables in non-relativistic, quantum theory — with a linear operator acting some Hilbert space, $\mathcal{H}$:

$$\begin{array}{ccc} \psi : \mathcal{M} & \longrightarrow & \mathcal{L}(\mathcal{H}). \\ x & \longrightarrow & \hat{\psi}(x) \end{array} \tag{2.1}$$

These operators then evolve in accordance to the Heisenberg picture, like [67]:

$$\hat{\psi}(\boldsymbol{x}, t) = e^{i\hat{H}t}\hat{\psi}(\boldsymbol{x}, 0)e^{-i\hat{H}t}. \tag{2.2}$$

Our intuition now suggests that the amount of such operators in a given region of space will grow with the volume of said region. However, do note that there are infinitely many points in any region of space-time, even if that region is of finite volume. In order to get meaningfully count the amount of points in a volume, we would like some sort of "smallest volume cell" or voxel to discretize space, and then count these voxels up. This is where the effects of gravity may come into the picture, one can expect that scales smaller than the Planck scale cannot be resolved due to a quantum theory of gravity [12]. Moreover, at the Planck scale, energies larger than the Planck energy result in a black hole, thus there is a natural energy cut off for the Hamiltonian dictating the dynamics of $\hat{\psi}(x)$. If, with these conditions, $\hat{\psi}(x)$ ends up having n possible distinct states, then the amount of states possible for the entire field in a region of space V is:

$$\Omega \propto n^V = \dim(\mathcal{H}). \tag{2.3}$$

Where we introduced the Hilbert space of the entire field in region V, $\mathcal{H}$. This then helps us motivate the definition of *number of degrees of freedom of a quantum system* as [12]:

$$\mathcal{N} = \ln(\dim(\mathcal{H})). \tag{2.4}$$

This now shows that — given the above assumptions — the amount of degrees of freedom in a region of space is, according to quantum field theory, proportional to V.

Note that $\mathcal{N}$ is also proportional to the maximum entropy a quantum system can have, $S = -\operatorname{tr}(\mathbf{1}/d \ln(\mathbf{1}/d))$, if $d = \dim \mathcal{H}$. This can be explained by recalling that entropy is a measure of our (willful) ignorance of the specific micro-state that is responsible for a given macro-state. If we now think of *the quantum field in a region of space V, ignoring everything about it, other than it being a quantum field in a volume V* as a macro-state, then the number of possible micro states will be given by the amount of degrees of freedom a quantum field can have. To put it in another way, our ignorance of all possible micro states will now be the sum of our ignorance of each individual degree of freedom. This immediately implies $S \propto \mathcal{N}$

2.1.3 Breaking Degrees of Freedom Down

As stated before, gravity puts an upper bound on the amount of stuff that can exist in a region of space before that region collapses into a black hole. Moreover, according to the *No-hair theorem* [20][42][43] a black hole is only characterized by only 3 parameters: mass, angular momentum and charge. This seems to suggest that all information and complexity that falls into a black hole will be erased, although, according to unitarity in quantum mechanics, this cannot happen. This is the crux of the *black hole information paradox* [38]. Furthermore, this process would be irreversible, as the surface area of a black hole horizon should only grow with time, this is the so called *area law* or *second law of black hole thermodynamics* [36][2]:

$$\frac{dA}{dt} \geq 0. \tag{2.5}$$

This being an irreversible process, where a quantity only increases over time, motivated Bekenstein to propose a connection between this area and entropy. After trying some different options he came to the conclusion that the area of a black hole's horizon and its entropy ought to be proportional to one another [4]. However, with nonzero entropy and energy black holes should have a finite temperature and emit blackbody radiation. Hawking showed that this is indeed the case, and this allowed him to fix the proportionality between entropy and area [37]:

$$S = \frac{A}{4}. \tag{2.6}$$

So far this analysis only holds for black holes, although similar analysis can be done for Rindler [75][24] and cosmological horizons [33]; what about normal matter? To explore the relationship between matter entropy and black hole entropy, consider the Susskind process as described by [12]: A clump of matter inside a sphere of surface area A with less mass than the required to collapse this enclosed region of space to a black hole, $M > m$. Now adding more mass $M - m$ to the same region would create a black hole with the above entropy. As the entropy of the original clump and the added mass must both be positive, the original entropy of m could not have been bigger than the one given above, if the entropy of black holes and matter added together is not allowed to decrease over time. With this thought experiment and it's assumptions, Susskind showed that the above equality constitutes an upper bound for the amount of normal matter entropy that is localized inside a sphere of surface area A[70]. Given that the entropy of a system is the sum of the entropy of each individual degree of freedom, the number of them in a region V of space apparently grows with said region's surface area, ∂V, which is far less than what one would intuitively expect and less than what local field theory was suggesting.

2.2 The Holographic Principle

We have seen that naively counting the degrees of freedom in a QFT inside a volume V results in more degrees of freedom than what gravity seems to allow. In fact, there seems to be a lot less degrees of freedom in nature than what one would initially imagine. It is unclear where exactly this pattern comes from, but we can hypothesize that it has its origins in a deeper, more fundamental theory of nature. This doesn't mean that local QFT is wrong, one *can* have degrees of freedom that are redundant or non-dynamical. But the point is that this would not be a *fundamental* theory of nature, as it is implied that a simpler theory with less degrees of freedom exists. To be concrete, we can take the logarithm of the dimensionality of the Hilbert space describing a system to mean its number of fundamental degrees of freedom, as we did in section 2.1.2. If we do so, and this system unitarily evolves into a black hole, then the analysis from before demands that the original Hilbert space must have had a dimension given by the holographic bound, $e^{A/4}$. To take this to be true, is to follow *the holographic principle* [12][41][70]:[2]

> *A region with boundary of area A is fully described by no more than A/4 degrees of freedom, or about 1 bit of information per Planck area. A fundamental theory, unlike local field theory, should incorporate this counterintuitive result.*

The holographic principle then favors unitarity over locality, or at least over local QFT. This is not the first time we have to rescue unitarity at the expense of another concept from a black hole; in thought experiments regarding the black hole information paradox, we are forced to choose between unitarity and the idea of a "global" quantum state that describes the system inside and outside the event horizon [38]. Here, once again, unitarity is usually the favored one and black hole complementarity is adopted [74][73][71][48]. It states that the unitary evolution of quantum states may be different for observers inside and outside the event horizon, but since these observers can never meet and compare their descriptions this is not a problem. However, assuming black hole complementarity may put the holographic principle into question. From the point of view of an observer part of a system that collapses into a black hole, they and the rest of the system will end up in the singularity, when they do, the entropy of the system would be compressed to a vanishingly small area, clearly breaking the holographic bound.

The problem here lies in the fact that concepts like unitary evolution and area are presupposing a fixed time axis and space hypersurfaces, things we know to be relative. Thus, it is not a trivial task to define unitarity or holographic boundaries, even though we have been using these concepts carelessly in the above reasoning. Helpfully, there are formulations of these concepts that are Lorentz covariant. The covariant formulation of the holographic principle is called the covariant entropy bound [11], nonetheless, we won't be using this version of the holographic principle, as for flat space-times the above given version will suffice. There are many indications that the holographic principle is an avenue worth exploring; over the past years since it was first proposed, it has been found in various forms in the many more theories and observations briefly summarized in the next section.

[2] this is a (non-covariant) preliminary version of the principle in the cited work

2.3 Holography Beyond the Horizon

In this section we present some theories and observations of other theories that show some aspects of holography outside the context of black holes or Rindler-/cosmological- horizons. These theories, then, present evidence that the holographic principle may have deep origins, origins that perhaps go beyond General Relativity (GR).

2.3.1 Holographic Aspects Already in Quantum Field Theory

Note that the given derivation of $N \propto V$ given in section 2.1.2 is rather crude, and, although intuitive, may not really yield the correct scaling of degrees of freedom in QFT. Indeed, having every operator $\hat{\psi}(\boldsymbol{x})$ have it's own energy cutoff still allows for the creation of a black hole by considering many $\hat{\psi}(\boldsymbol{x})$ together. if we saturate the individual energy bound of all the $\hat{\psi}(\boldsymbol{x})$ inside a sphere of radius R. The total mass-energy in this region of space is $M \propto R^3$, but Schwarzschild radius goes linearly with energy, $M = r_s$. Thus, there will be a tipping point when the region is now encompassed by the black hole, which follows area scaling of degrees of freedom. This indicates that the UV cut-off given in section 2.1.2 may be too lenient [12].

There are other more involved calculations that show that the relative entropy between the inside of a volume and the outside, does seem to follow area scaling for a field of coupled quantum harmonic oscillators, [66][8] and some more recent work by Casini and Huerta, [21][22] show that QFT does appear to follow some form of Bekenstein's bound, which is a stronger version of the above holographic bound [5]. In this latter case they manage to achieve this by calculating the relative entropy between a vacuum state and another state, both reduced to a region of space time V, i.e. defining density matrix $\rho = |\psi\rangle\langle\psi|$ or $\rho = |0\rangle\langle 0|$ and then taking the trace over the Hilbert subspace $\mathcal{H}_{-V}$, $\rho_V = \mathrm{tr}_{-V}(\rho)$. Where we assume the total Hilbert space decomposes as $\mathcal{H} = \mathcal{H}_V \otimes \mathcal{H}_{-V}$. As the authors point out, however, there are a multitude of problems in trying to localize states and observables in quantum field theory [13][35][63]. Another smaller contention one could have with this approach is that the vacuum in QFT seems to also has physical effects on other states, examples of this include the vacuum energy, the Casimir effect or density fluctuations in the cosmic microwave background. These examples put into question whether we should neglect the entropy of the vacuum. The task of finding which states are physical is the focus of the work briefly presented in the next subsection.

2.3.2 Overcounting of States due to Gauge Symmetry

One of the difficulties that arise when quantizing a gauge theory is the many gauge equivalent states that appear when doing so. Given that general relativity can be thought of as a gauge theory [6], this opens the door to an intriguing possibility; perhaps the over-counting of states in normal QFT or in perturbative canonical quantum gravity is due to different states which are actually made equivalent by quantizing general relativity. This would mean a reduction in the dimension of the Hilbert space, and consequently a reduction in

the amount of states that can be simultaneously orthogonal to one another. This is an idea explored by Maxfield in [52]. There he replaces the perturbative states of QFT, $|i\rangle \in \mathcal{H}_{\text{pert}}$, with physical states $|j\rangle\rangle \in \mathcal{H}_{\text{phys}}$. The inner product is accordingly modified:

$$\langle\langle i|j\rangle\rangle = \langle i|\eta|j\rangle = \eta_{ij} = \delta_{ij} + R_{ij}, \tag{2.7}$$

where we expect the components R_{ij} to be small, yet the rank of the positive semi-definite hermitian operator η would then be equal to $\dim \mathcal{H}_{\text{phys}}$ which we expect to be much less than $\dim \mathcal{H}_{\text{pert}}$. To calculate this rank, an explicit form of the inner product is presented; two motivations are used for this definition: a sum over geometries approach and from a non-perturbative gauge theory approach. Then, which states belong to the kernel of η are identified, and finally, these states are ignored when counting up the states that span $\mathcal{H}_{\text{phys}}$. The resulting dimensionality matches what we expect from holography.

Maxfield's work is particularly relevant for what follows in the later chapters of this thesis. While his focus lies on how to motivate the operator η and calculating the rank of said operator, here, as in [32], η will be treated as a random matrix with appropriate rank from the start, and the focus will be on exploring the consequences of working in a much reduced Hilbert space. Reducing the dimensions of the Hilbert space so directly isn't a common practice in the literature in the context of holography, but as stated previously, it is indispensable in the context of gauge theories. If a GR-related gauge symmetry turns out to be responsible for the holographic principle, this may lead one to ask whether holography is present in some form in gauge theories more broadly. Indeed, this is something the program presented in the next subsection has confirmed.

2.3.3 Celestial Conformal Field Theory

A very different approach to holography is the research program of celestial conformal field theory (CCFT) [60][69][54]. This program explores a connection between soft theorems, memory effects, and asymptotic symmetries. The former of these is a result from from the study of scattering amplitudes, while the latter two refer to the symmetries the fields have very far away from the origin and are often discussed with relation to general relativity, but they are present in other gauge theories as well; as an example of this connection, consider the soft photon theorem in quantum electrodynamics. In this theory, the soft photon theorem helps regulate divergencies in some scattering matrices. It turns out that the scattering matrices that vanish and where the soft photon theorem cannot give them finite value also break the asymptotic symmetries electromagnetism has. One can show that this symmetry is, in fact, equivalent to the soft photon theorem. To be more precise, one can interpret these low energy photons as transformations on the electromagnetic field that act like a member of this symmetry group [69][49][68].

These asymptotic symmetries and memory effects are defined on the so called "celestial sphere", this corresponds to a sphere around the origin with $r \to \infty$ and $t \to \pm\infty$, such that $u = t - r$ and $v = t + r$ get mapped to the same point on said sphere. A key aspect in the derivation of the soft theorems and these asymptotic symmetries is Lorentz invariance, which acts as a global conformal transformation at the celestial sphere [56]. Field theories with this type of symmetry are called conformal field theories (CFT) and they have been studied extensively, particularly in the context of string theory.

A goal of the program is to find a dual pair between a 4D quantum gravity theory for asymptotically flat space-times inside the sphere and a 2D holographic conformal field theory in this celestial sphere. The first steps to achieve this would be finding symmetries between these two descriptions. Some progress towards this was laid out long ago in the form of the soft graviton theorem [77], the Bondi-van der Burg-Metzner-Sachs (BMS) group [9][64], and the gravitational memory effect [15][14][82]. This program also hopes to leverage the advances made by string theorists with the AdS/CFT correspondence.

2.3.4 String Theory and AdS/CFT

Among the possible UV theories mentioned in section 2.1 the most famous is string theory. Though, string theory is not (or at least was not thought to be) a monolith. There have been several types of string theory that throughout the years have been shown to be equivalent to each other, or to other non-stringy theories, under the appropriate transformations. These equivalences are called dualities. For the holographic principle, the most relevant duality is the one between Type IIB string theory in an asymptotically $\mathrm{AdS}_5 \times \mathbf{S}^5$ background and super-symmetric $U(n)$-Yang-Mills theory with 16 real supercharges in 3+1 space-time as found by Maldacena in [50]. This latter theory is what is meant by the conformal field theory (CFT). It can be shown easily enough that the conformal boundary of $\mathrm{AdS}_5 \times \mathbf{S}^5$ is given by $\mathbb{R} \times \mathbf{S}^3$. What is not trivial, however, is that the fields in the Type IIB string theory in the bulk are defined by those of the super-symmetric gauge theory in the boundary, and that the size of the gauge group, n, grows with the curvature of the space-time in the bulk, and thus with it its 8 dimensional area. One can then preform an analysis analogous to the one given in section 2.1.2 and see that the degrees of freedom in the bulk grows linearly with the degrees of freedom in the boundary, as given by the Yang-Mills theory [12][72].

Given that our best, proven theories are gauge theories, it is no surprise that there has been great interest in theories that reproduce gauge theories from extra space-time dimensions since Maldacena first showed the duality [61][62], though little progress has been made to extend this duality to general space-times. These efforts are, however, best described as "inverse holography" as they try to explain what we would usually consider to be the bulk space-time as the boundary of an even higher dimensional space-time [12].

2.3.5 Emergent Space-Time & Gravity

So far in this section we have presented theories that happen to be holographic or aspects of theories that show holography. One can also turn this on its head and explore the kind of theories that follow from the holographic principle along with a few other axioms. For example, Jacobson in [44] used the relationship between area and entropy along with the laws of thermodynamics to arrive at the Einstein field equations, this was followed up by [58][76]. Later ideas in a similar vein started exploring the possibility of constructing space-time from quantum state factors [18][17]. In these works, geometric notions like distance and area are functions of properties like entanglement entropy and the mutual information. Thus, in these latter cases the relationship between geometry and entropy is the starting point, and the space-time manifold is an emergent property in the theory.

These more recent radical ideas represent a final departure from classical physics, as even today the vast majority of quantum theories explored are quantized versions of classical theories, including all of the above. To cast doubt into this philosophy is to go back 100 years and ask whether Heisenberg's quantization of classical observables (Equation 1.1) is the right way of doing things.

Chapter 3

The Weyl Field

3.1 The Hamiltonian & Lagrangian

In this chapter we will provide a brief review of the Weyl-field [78][59]. We won't go into details of how the Weyl field behaves under Lorentz transformations, suffice it to say that $\hat{\psi}$ is a two component, anti-commuting operator field. Do note that, although no particles in the standard model are Weyl particles, any of the fermions in it can be built by combining Weyl fields, thus they represent a natural starting point to test the overlapping procedure.

We will start with the Lagrangian of the Weyl field on a volume V:

$$L = \int_V dx^3 \ \hat{\psi}^\dagger(x) i\sigma^\mu \partial_\mu \hat{\psi}(x). \tag{3.1}$$

The canonical momentum of this field is given by [55]:

$$\hat{\pi}(x) = \frac{\delta L}{\delta \partial_0 \hat{\psi}} = \hat{\psi}^\dagger(x) i\sigma^0. \tag{3.2}$$

As opposed to Equation 1.1, we are dealing here with an *anti*-commuting, fermionic field, as such we have the equal-time *anti*-commutation relations:

$$\{\hat{\psi}(t, \boldsymbol{x}), \hat{\pi}(t, \boldsymbol{y})\} = i\delta(\boldsymbol{x} - \boldsymbol{y}) \mathbf{1}_{2\times 2}, \tag{3.3}$$

where the matrix $\mathbf{1}_{2\times 2}$ acts on the spinor space. Before going any further, it is convenient at this point to introduce new, slightly unorthodox notation. We treat $\hat{\pi}$ and $\hat{\psi}$ as vectors on the Hilbert space of (an equivalence class of) all measurable functions defined on V, $\mathcal{H}_V$. Importantly, this is a different Hilbert space from the "quantum Hilbert space", that groups together quantum-mechanical vector-states, $\mathcal{H}$. This former Hilbert space merely groups together the space of classical field configuration in V, in the spirit of an L^2 space. Thinking about the fields this way, we re-write Equation 3.1 using the cannonical inner product of said Hilbert space, $\langle \cdot, \cdot \rangle_V$ as:

$$L = \langle \hat{\psi}^\dagger, (i\sigma^\mu \partial_\mu \hat{\psi})^\dagger \rangle_V. \tag{3.4}$$

We can now transition to Hamiltonian mechanics by applying the Legendre transform:

$$H = \langle \hat{\pi}, (\partial_0 \hat{\psi})^\dagger \rangle_V - L = \langle \hat{\psi}^\dagger i\sigma^0, (\partial_0 \hat{\psi})^\dagger \rangle_V - \langle \hat{\psi}^\dagger, (i\sigma^\mu \partial_\mu \hat{\psi})^\dagger \rangle_V. \tag{3.5}$$

Applying the properties of the adjoint, our convention for the metric, and the linearity of the inner product we get:

$$H = \langle \hat{\psi}^\dagger, (i\sigma^j \partial_j \hat{\psi})^\dagger \rangle_V \tag{3.6}$$

We will be ultimately interested in the dimension of $\mathcal{H}$; if we think about the quantum Hilbert space as being the tensor product of the quantum Hilbert spaces of all the points in volume V, it is clear that the dimension will diverge. To fix this we can introduce lattice regularization — equivalent to a UV-cutoff — such that we only need to preform the tensor multiplication over finitely many lattice points, the amount of points will naturally be proportional to V. We can go further, and instead of thinking about "the amount of points in V", we can express everything in terms of the dimension of $\mathcal{H}_V$, which encompasses more clearly both; the view that we introduce a UV-cutoff and the view that we introduced a shortest scale.[1] This also means that we can think of the degrees of freedom of the field and $\dim \mathcal{H}$ as being ultimately dependent on $\dim \mathcal{H}_V$. Furthermore, since $\dim \mathcal{H}_V$ is now finite, we can replace the integral in the inner product $\langle \cdot, \cdot \rangle_V$ with a finite sum, like the dot product. Then, Like the dot product, we evaluate the inner product as the sum of the products of the components of the vectors. It is most advantageous to preform this summing up of vector components in the fourier basis, defined as:

$$\hat{\psi}(\boldsymbol{x}, t) = \sum_{\boldsymbol{p}} \frac{1}{\sqrt{|\boldsymbol{p}|V}} (\hat{a}_{\boldsymbol{p}}(t) u(\boldsymbol{p}) e^{i\boldsymbol{p}\boldsymbol{x}} + \hat{b}^\dagger_{\boldsymbol{p}}(t) u(\boldsymbol{p}) e^{-i\boldsymbol{p}\boldsymbol{x}}) \tag{3.7}$$

or equivalently:

$$\hat{\psi}_{\boldsymbol{p}} = \frac{1}{\sqrt{|\boldsymbol{p}|}} (\hat{a}_{\boldsymbol{p}} u(\boldsymbol{p}) + \hat{b}^\dagger_{-\boldsymbol{p}} u(-\boldsymbol{p})). \tag{3.8}$$

We have also introduced the eigenvectors of operator $\sigma^j \boldsymbol{p}_j$:

$$\sigma^j p_j u(\pm \boldsymbol{p}) = \pm |\boldsymbol{p}| u(\pm \boldsymbol{p}) \qquad u(\boldsymbol{p})^\dagger u(\boldsymbol{p}) = |\boldsymbol{p}|. \tag{3.9}$$

Noting that the self adjoint-ness of $\sigma^j \boldsymbol{p}_j$ means that the two eigenvectors are orthogonal. Moreover, we get the following algebra for the annihilation operators, $\hat{a}_{\boldsymbol{p}}$ and $\hat{b}_{\boldsymbol{p}}$ from the anti-commutation relations of $\hat{\psi}(x)$ and $\hat{\pi}(x)$:

$$\{\hat{a}_{\boldsymbol{p}}, \hat{a}_{\boldsymbol{q}}\} = \{\hat{b}_{\boldsymbol{p}}, \hat{b}_{\boldsymbol{q}}\} = \{\hat{a}_{\boldsymbol{p}}, \hat{b}_{\boldsymbol{q}}\} = \{\hat{a}_{\boldsymbol{p}}, \hat{b}^\dagger_{\boldsymbol{q}}\} = 0 \tag{3.10}$$

$$\{\hat{a}_{\boldsymbol{p}}, \hat{a}^\dagger_{\boldsymbol{q}}\} = \{\hat{b}_{\boldsymbol{p}}, \hat{b}^\dagger_{\boldsymbol{q}}\} = \delta_{\boldsymbol{q}\boldsymbol{p}}, \tag{3.11}$$

[1] When we eventually overlap this field, we will also have the advantage that we do not need to be explicit on which sub-space of $\mathcal{H}_V$ we get rid of to achieve holographic scaling, thus allowing for a large set of models.

where $\hat{a}^\dagger_{\boldsymbol{p}}$ and $\hat{b}^\dagger_{\boldsymbol{p}}$ are called creation operators. This basis also diagonalizes operator $i\sigma^j\partial_j$ and allows one to write:

$$\hat{H} = \sum_{\boldsymbol{p}} \hat{\psi}^\dagger_{\boldsymbol{p}}(\sigma^j \boldsymbol{p}_j)\hat{\psi}_{\boldsymbol{p}} = \sum_{\boldsymbol{p}} |\boldsymbol{p}|(\hat{a}^\dagger_{\boldsymbol{p}}\hat{a}_{\boldsymbol{p}} + \hat{b}^\dagger_{\boldsymbol{p}}\hat{b}_{\boldsymbol{p}} - \{\hat{b}_{\boldsymbol{p}}, \hat{b}^\dagger_{\boldsymbol{p}}\}). \tag{3.12}$$

Note that the anti-commutator produces a constant term in the Hamiltonian, this is often referred to as the vacuum energy of the field. Here it has a finite value, dependent on the number of $\boldsymbol{p}$ modes we have in the theory; it diverges in most other contexts. However, its interpretation is an open question: On the one hand, constant contributions to the energy density of space should have a measurable effect when considering a theory of gravity. On the other hand, divergent quantities such as these are often regularized in interacting QFT to empirically measured values, the original values are suppressed with regulators and counter-terms. Regardless, for the phenomenology we are interested in this work this constant value should have little to no effect. Because of this, we re-normalize to the empirically measured value of ≈ 0, such that the Hamiltonian now reads:

$$\hat{H} = \sum_{\boldsymbol{p}} |\boldsymbol{p}|(\hat{a}^\dagger_{\boldsymbol{p}}\hat{a}_{\boldsymbol{p}} + \hat{b}^\dagger_{\boldsymbol{p}}\hat{b}_{\boldsymbol{p}}). \tag{3.13}$$

3.2 Particle States

A sensible task at this point is to look for the eigenstates of the above Hamiltonian. This can be easily done by assuming the existence of a lowest energy state, called $|0\rangle$ that gets annihilated by the annihilation operators, $0 = \hat{a}_{\boldsymbol{p}}|0\rangle$. Using the anti-commutation relations Equation 3.10 one can show that:

$$\hat{H}\hat{a}^\dagger_{\boldsymbol{p}_1}\hat{a}^\dagger_{\boldsymbol{p}_2}\dots\hat{a}^\dagger_{\boldsymbol{p}_n}|0\rangle = (|\boldsymbol{p}_1| + |\boldsymbol{p}_2| + \dots|\boldsymbol{p}_n|)\hat{a}^\dagger_{\boldsymbol{p}_1}\hat{a}^\dagger_{\boldsymbol{p}_2}\dots\hat{a}^\dagger_{\boldsymbol{p}_n}|0\rangle \tag{3.14}$$

and a similar thing holds true using b-type creation operators. Since we expect for a massless particle $E = |\boldsymbol{p}|$, the state $\hat{a}^\dagger_{\boldsymbol{p}}|0\rangle$ with energy eigenvalue $|\boldsymbol{p}|$ is readily given the interpretation of a single particle state with momentum $\boldsymbol{p}$, and the above eigenstate is said to be a multi-particle state. However, it is also possible to go the other way; one can *define* particle states to be like the one above and then look for a Hamiltonian with the correct energy eigenvalue, such that $E = |\boldsymbol{p}|$. Doing so one must arrive at the Hamiltonian in Equation 3.13, as this is nothing else than defining a Hamiltonian on it's eigen-basis. This highlights the fact that the definition of particle-states is strongly linked with the Hamiltonian in question. This also explains why, in curved space-times —where the Hamiltonian density may be different in different space-time regions— particle creation may occur.

3.3 The Hilbert Space

At this point we can construct the Hilbert space $\mathcal{H}$ more explicitly using Jordan-Wigner [46] factors as done in [32]:

$$\mathcal{H} = \bigotimes_{\boldsymbol{p}}^{\text{JW}} \mathcal{H}_{\boldsymbol{p}}, \tag{3.15}$$

each factor being:

$$\mathcal{H}_{\boldsymbol{p}} = \mathcal{H}^a_{\boldsymbol{p}} \otimes^{\text{JW}} \mathcal{H}^b_{\boldsymbol{p}}, \tag{3.16}$$

where the Jordan-Wigner factors where used again, and $\mathcal{H}^a_{\boldsymbol{p}}$ and $\mathcal{H}^b_{\boldsymbol{p}}$ are the Hilbert spaces the (creation)/annihilation operators $\hat{a}^{(\dagger)}_{\boldsymbol{p}}$ and $\hat{b}^{(\dagger)}_{\boldsymbol{p}}$ act on non-trivially respectably. The Jordan-Wigner factors are needed here to insure the right anti-commutation relations are achieved. For details see the appendix D of [32].

For the theory explored here, as in [32], we choose V to be a box of side length l. This means that we sum over a finite sets of fourier modes $\boldsymbol{p} \in \{2\pi(n_1, n_2, n_3)/l | n_i \in \mathbb{Z}, n_i < l/\Lambda_{UV}\}$, we call the cardinality of this set N and thus it is implied that the vector space $\mathcal{H}_V$ is of dimension N. This is because, we are assuming, we can write the vector ψ as a linear combination of N components, the $\psi_{\boldsymbol{p}}$. Clearly, N will scale with the volume, $l^3 = V$. We get from Equation 3.15:

$$\dim \mathcal{H} = \dim \left(\bigotimes_{\boldsymbol{p}}^{\text{JW}} \mathcal{H}_{\boldsymbol{p}} \right) \tag{3.17}$$

$$= \prod_{\boldsymbol{p}} \dim(\mathcal{H}_{\boldsymbol{p}}) \tag{3.18}$$

$$= (\dim(\mathcal{H}_{\boldsymbol{p}}))^N = (\dim(\mathcal{H}_{\boldsymbol{p}}))^{\dim \mathcal{H}_V}, \tag{3.19}$$

so this means that the degrees of freedom as defined by [12] will scale as:

$$\mathcal{N} = \ln(\dim \mathcal{H}) \propto N \,\widetilde{\propto}\, V. \tag{3.20}$$

Chapter 4

Overlapped Weyl Field

We have seen how the dimension of the Hilbert space of a regularized Weyl field in a region of space V, $\mathcal{H}$, grows with the volume of V. In this chapter, the goal will be to reduce the dimension of the Hilbert space from N to n, in order to achieve holographic scaling. As mentioned previously, we use a special case of the overlapping procedure introduced in [32] to achieve this, even though the overlapping procedure presented here looks superficially different.

4.1 Overlapping the Weyl field

If the Hilbert space $\mathcal{H}$ is of dimension $\ldots^n$ instead of $\ldots^N$, then a collection of $\ldots^N$ supposed basis vectors that span the Hilbert space actually represent an over-complete basis. By "over-complete basis" we mean a set of vectors that *do* span the Hilbert space, however, they are not all linearly independent[1]. Taking this to be the case, we are able to write the vectors of the over-complete basis by linear combination of non-over-complete basis vectors, a proper basis of linearly independent vectors.

We won't *explicitly* write the $\ldots^N$ over-complete basis vectors as linear combinations of $\ldots^n$ basis vectors[2]. Instead we will provide the same effect by assuming that the N basis vectors of the Hilbert space $\mathcal{H}_V$ actually represent an over-complete basis and can be written by n basis vectors. This is because the last calculation of the previous chapter showed that $\dim \mathcal{H} = \ldots^{\dim \mathcal{H}_V}$. Thus it is enough to reduce $\dim \mathcal{H}_V$ to n. To this end, we write the N (over-complete) vector components $\hat{\psi}_{\boldsymbol{p}}$ as linear combinations of the n (non-over-complete) components $\hat{\psi}_{\boldsymbol{i}}$:

$$\hat{\psi}_{\boldsymbol{p}} = \sum_{\boldsymbol{i}} F_{\boldsymbol{pi}} \hat{\psi}_{\boldsymbol{i}}, \tag{4.1}$$

[1] It is worth noting that, strictly speaking, an over-complete basis as defined here is not a basis, as all the vectors composing a basis must be linearly independent according to the usual definition of basis.

[2] To see reasons we should overlap the vector space $\mathcal{H}_V$, we can refer to the reasons given in [32]. Some alternative interpretations are given in chapter 9 of this work

where F is a $N \times n$ Matrix. Equivalently, if the $\hat{\psi}_{\boldsymbol{i}}$ lets itself to be written as a sum of annihilation/creation operators similar to Equation 3.8, with operators $\hat{A}_{\boldsymbol{i}}$ and $\hat{B}_{\boldsymbol{i}}$ (instead of operators $\hat{a}_{\boldsymbol{i}}$ and $\hat{b}_{\boldsymbol{i}}$), we can replace these creation and annihilation operators with the linear combinations:

$$\sum_{\boldsymbol{i}} F_{\boldsymbol{pi}} \hat{A}_{\boldsymbol{i}} =: \hat{c}_{\boldsymbol{p}} \qquad\qquad \sum_{\boldsymbol{i}} \overline{F}_{\boldsymbol{pi}} \hat{B}_{-\boldsymbol{i}} =: \hat{d}_{-\boldsymbol{p}}, \tag{4.2}$$

where $\overline{z}$ is the complex conjugate of $z \in \mathbb{C}$, so that:

$$\hat{\psi}_{\boldsymbol{p}} = \frac{1}{\sqrt{|\boldsymbol{p}|}} (\hat{c}_{\boldsymbol{p}} u(\boldsymbol{p}) + \hat{d}^{\dagger}_{-\boldsymbol{p}} u(-\boldsymbol{p})). \tag{4.3}$$

This way of expressing the overlaps is closer to that of [32], and thus we can more easily spot the difference. In said referenced work the linear combinations involve creation and annihilation from both positive and negative fourier modes, as opposed to here, where the creation operators are linear combinations of only other creation operators and no annihilation operators, and the opposite is true for their adjoints. There are several advantages to this, for instance, if $\hat{A}_{\boldsymbol{p}}$ annihilates the vacuum (which we will assume to be the case) so will $\hat{c}_{\boldsymbol{p}}$. If we impose the canonical anti-commutation relations on $\hat{\psi}_{\boldsymbol{i}}$ and $\hat{\pi}_{\boldsymbol{i}} = i\hat{\psi}^{\dagger}_{\boldsymbol{i}}$, then the anti-commutation relations that vanish (Equation 3.10) remain the same, but those that don't (Equation 3.11) are now:

$$\{\hat{c}_{\boldsymbol{p}}, \hat{c}^{\dagger}_{\boldsymbol{q}}\} = \sum_{\boldsymbol{l},\boldsymbol{k}} F_{\boldsymbol{pk}} \overline{F}_{\boldsymbol{ql}} \{\hat{A}_{\boldsymbol{k}}, \hat{A}^{\dagger}_{\boldsymbol{l}}\} = \sum_{\boldsymbol{l},\boldsymbol{k}} F_{\boldsymbol{pk}} \overline{F}_{\boldsymbol{ql}} \delta_{\boldsymbol{kl}} = (FF^{\dagger})_{\boldsymbol{pq}} =: M_{\boldsymbol{pq}} \tag{4.4}$$

$$\{\hat{d}_{-\boldsymbol{p}}, \hat{d}^{\dagger}_{-\boldsymbol{q}}\} = \sum_{\boldsymbol{l},\boldsymbol{k}} \overline{F}_{\boldsymbol{pk}} F_{\boldsymbol{ql}} \{\hat{B}_{-\boldsymbol{k}}, \hat{B}^{\dagger}_{-\boldsymbol{l}}\} = \sum_{\boldsymbol{l},\boldsymbol{k}} \overline{F}_{\boldsymbol{pk}} F_{\boldsymbol{ql}} \delta_{\boldsymbol{kl}} = (FF^{\dagger})_{\boldsymbol{qp}} = M_{\boldsymbol{qp}}. \tag{4.5}$$

If this theory is supposed to mimic standard QFT closely, we expect something like $(FF^{\dagger}) = M \approx \mathbf{1}_{N\times N}$. Yet, because F has rank n, M must also be of rank n and not N. This means we can't have $M = \mathbf{1}_{N\times N}$. To try and bring together these incompatible properties [32] chose the column vectors of F, v at random from a complex unit n-sphere, consequently the components of the matrix M are given by dot products of said vectors, $M_{\boldsymbol{qp}} = v_{\boldsymbol{q}} \cdot v_{\boldsymbol{p}}$. This, then, implies $M_{\boldsymbol{pp}} = 1$ for all $\boldsymbol{p}$, while the expectation value $\mathbb{E}(M_{\boldsymbol{pq}}) = \delta_{\boldsymbol{pq}}$. We will do something similar here, but we chose real vectors instead of complex ones. Note that in the rest of this work the expectation value $\mathbb{E}(\dots)$ is taken with respect to said distributions of v. Having now a new operator algebra, it is tempting now to make the replacements $\hat{\psi}_{\boldsymbol{p}} \to \sum_{\boldsymbol{i}} F_{\boldsymbol{pi}} \hat{\psi}_{\boldsymbol{i}}$ in Equation 3.12 and study that theory, however, note that this will drastically change the Hamiltonian:

$$\hat{H} = \sum_{\boldsymbol{p},\boldsymbol{i},\boldsymbol{j}} \hat{\psi}^{\dagger}_{\boldsymbol{j}} F_{\boldsymbol{pj}} (\sigma^k \boldsymbol{p}_k) F_{\boldsymbol{pi}} \hat{\psi}_{\boldsymbol{i}} \neq \langle \hat{\psi}^{\dagger}, (i\sigma^j \partial_j \hat{\psi})^{\dagger} \rangle_V. \tag{4.6}$$

Leading to inconvenient expectation values. To remedy this we preventively add a scalar factor, $D = \mathbf{1}_{n\times n}/\lambda_{\boldsymbol{p}}$. This $\lambda_{\boldsymbol{p}}$ will then be fixed in section 4.2.

One can also justify this scalar factor by noting that we are interpreting the fourier basis to be over-complete, thus a lot of those fourier components are redundant and misrepresent the length of vector $\hat{\psi}$. An example of this in $\mathcal{H}_V = \mathbb{R}^2$ would be the vector $\boldsymbol{v} = \boldsymbol{e}_1$ with $\langle \boldsymbol{v}, \boldsymbol{v} \rangle = 1$ and the over-complete basis $\boldsymbol{e}_1$,$\boldsymbol{e}_2$ and $\boldsymbol{e}_3 = (\boldsymbol{e}_1 + \boldsymbol{e}_2)/\sqrt{2}$. If we define $\boldsymbol{v}^i = \langle \boldsymbol{v}, \boldsymbol{e}_i \rangle$ then we get $\sum_i \boldsymbol{v}^i \boldsymbol{v}^i > \langle \boldsymbol{v}, \boldsymbol{v} \rangle$. In order to account for such artifacts, we add a factor of $1/\lambda_{\boldsymbol{p}}$ to Equation 3.12 to be fixed shortly. With this factor we can now rewrite Equation 3.12 with the new creation/annihilation operators:

$$\hat{H} = \sum_{\boldsymbol{p}} \frac{|\boldsymbol{p}|}{\lambda_{\boldsymbol{p}}} (\hat{c}^\dagger_{\boldsymbol{p}} \hat{c}_{\boldsymbol{p}} + \hat{d}^\dagger_{\boldsymbol{p}} \hat{d}_{\boldsymbol{p}} - \{\hat{d}_{\boldsymbol{p}}, \hat{d}^\dagger_{\boldsymbol{p}}\}). \tag{4.7}$$

It's also possible to rescale the operators $\hat{c}_{\boldsymbol{p}}$ and $\hat{d}_{\boldsymbol{p}}$ by a factor of $\sqrt{\lambda_{\boldsymbol{p}}}$ to get rid of the $1/\lambda_{\boldsymbol{p}}$ factor in front of the above Hamiltonian, at the price of changing the anti-commutation relations. In any case, the resulting theory tries to mimic a typical QFT (with a harmonic oscillator on every point in space), but the Hilbert space is much reduced, and with it the algebra of the usual operators is modified. As a consequence of the reduced dimension, the vacuum energy, $\langle 0|\hat{H}|0\rangle = -\sum_{\boldsymbol{p}} \langle 0|\{\hat{d}_{\boldsymbol{p}}, \hat{d}^\dagger_{\boldsymbol{p}}\}|0\rangle$, will be somewhat suppressed, but for the reasons explained in the previous chapter we will still ignore it:

$$\hat{H} \to \sum_{\boldsymbol{p}} \frac{|\boldsymbol{p}|}{\lambda_{\boldsymbol{p}}} (\hat{c}^\dagger_{\boldsymbol{p}} \hat{c}_{\boldsymbol{p}} + \hat{d}^\dagger_{\boldsymbol{p}} \hat{d}_{\boldsymbol{p}}). \tag{4.8}$$

4.1.1 Shells

The above takes a QFT with a Hilbert space of dimension $\ldots^N$ and maps it to a Hilbert space of dimension $\ldots^n$. However, in order to accomplish holographic *scaling*, the amount of degrees of freedom should scale with the amount of momenta available, $n \propto k$ [19]. To achieve this [32] split the system in multiple fourier shells of width Δ_s and they repeat the above overlapping process within each shell. As a result, the N_s creation/annihilation operators of shell s only act non-trivially on a Hilbert space of dimension $\ldots^{n_s}$, $\mathcal{H}_s$. The total Hilbert space can then be written with the JW-embedding of the different $\mathcal{H}_s$ as explained in [32]. Each of these shells and the total Hilbert space have then the Hamiltonians:

$$H_s = k_s \sum_{\boldsymbol{p} \in s} \frac{1}{\lambda_{\boldsymbol{p}}} (\hat{c}^\dagger_{\boldsymbol{p}} \hat{c}_{\boldsymbol{p}} + \hat{d}^\dagger_{\boldsymbol{p}} \hat{d}_{\boldsymbol{p}}) \qquad\qquad H \approx \sum_s H_s, \tag{4.9}$$

where $k_s - \Delta_s/2 \leq |\boldsymbol{p}| \leq k_s + \Delta_s/2$ for all $\boldsymbol{p} \in s$ and $|s| = N_s$. As a consequence of this shell-wise overlapping, we get the usual anti-commutation relations for operators in different shells, $\{\hat{c}_{\boldsymbol{p}}, \hat{c}^\dagger_{\boldsymbol{q}}\} = 0$ if $\boldsymbol{q} \in s \neq s' \ni \boldsymbol{p}$. This also means that M (with components going over all shells) is now a block diagonal matrix, i.e. for $\boldsymbol{p} \in s, \boldsymbol{q} \notin s$ we get $M_{\boldsymbol{pq}} = 0$. If we call the block matrix corresponding to shell s M_s, to achieve holography, we then must have:

$$\operatorname{rank}(M_s) = n_s \propto 2\pi k_s \Delta_s \Big/ \frac{(2\pi)^2}{L^2}. \tag{4.10}$$

4.2 Particle States

Given that we have removed some points in the lattice by the overlapping procedure, usual translational invariance is lost and consequently plane waves, $\hat{c}^\dagger_{\boldsymbol{p}}|0\rangle = |\boldsymbol{p}\rangle$, are not eigenstates of the Hamiltonian as pointed out by [32]. They can, however, be somewhat close to an eigenstate if $(FF^\dagger) \approx \mathbf{1}_{N\times N}$:

$$\hat{H}\hat{c}^\dagger_{\boldsymbol{p}}|0\rangle = \sum_s k_s \sum_{\boldsymbol{q}\in s} \frac{1}{\lambda_{\boldsymbol{p}}}\{\hat{c}_{\boldsymbol{q}}, \hat{c}^\dagger_{\boldsymbol{p}}\}\hat{c}^\dagger_{\boldsymbol{q}}|0\rangle = k_s \sum_{\boldsymbol{q}\in s} \frac{1}{\lambda_{\boldsymbol{p}}} M_{\boldsymbol{qp}}\hat{c}^\dagger_{\boldsymbol{q}}|0\rangle. \tag{4.11}$$

Acting with $\langle\boldsymbol{p}|$ in Equation 4.11 and taking the expectation value we get:

$$\mathbb{E}[\langle 0|\hat{c}_{\boldsymbol{p}}\hat{H}\hat{c}^\dagger_{\boldsymbol{p}}|0\rangle] = \frac{1}{\lambda_{\boldsymbol{p}}} k_s \sum_{\boldsymbol{q}\in s} \mathbb{E}\left[M^2_{qp}\right] = \frac{1}{\lambda_{\boldsymbol{p}}} k_s \sum_{\boldsymbol{q}\in s/\{\boldsymbol{p}\}} \frac{1}{n} \approx \frac{k_s}{\lambda_{\boldsymbol{p}}}\frac{N_s}{n_s}, \tag{4.12}$$

where we ignored the term $\boldsymbol{q} = \boldsymbol{p}$ in the sum as it is of order $\mathcal{O}(1/N_s)$. The expectation value of $M^2_{\boldsymbol{qp}}$ is calculated in appendix C. It is clear that setting $\lambda_{\boldsymbol{p}} \to N_s/n_s$ will provide the correct expectation value for the Hamiltonian, so we will keep this value for the rest of this work. Note that $M_{\boldsymbol{pp}} = 1$ implies $\langle\boldsymbol{p}|\boldsymbol{p}\rangle = 1$. We can also define multi-particle states as:

$$|\boldsymbol{p}_1, \boldsymbol{p}_2, \ldots \boldsymbol{p}_m\rangle = \hat{c}^\dagger_{\boldsymbol{p}_1}\hat{c}^\dagger_{\boldsymbol{p}_2}\ldots\hat{c}^\dagger_{\boldsymbol{p}_m}|0\rangle. \tag{4.13}$$

Because the overlapped creation operators anti-commute, overlapped multi-particle states also follow fermionic statistics in the sense that $|\boldsymbol{p}_1, \boldsymbol{p}_2\rangle = -|\boldsymbol{p}_2, \boldsymbol{p}_1\rangle$. Defining particle and multi-particle states this way also means that we need to change the definition of the number particle operator as well:

$$\hat{N}_{\boldsymbol{p}} = \hat{c}^\dagger_{\boldsymbol{p}}\hat{c}_{\boldsymbol{p}} \qquad \hat{N}_s = \frac{n_s}{N_s}\sum_{\boldsymbol{p}\in s}\hat{c}^\dagger_{\boldsymbol{p}}\hat{c}_{\boldsymbol{p}} \qquad \hat{N} = \sum_s \hat{N}_{\boldsymbol{s}}. \tag{4.14}$$

Using the above derivation of the expectation value of $\hat{H}$ as a guide, it's easy to see that $\hat{N}$ gives the right expectation value under $|\boldsymbol{p}\rangle$. To calculate the expectations value of multi-particle states and their norm we need Wick's theorem [79]:

$$\begin{aligned}
\hat{c}_{\boldsymbol{p}_1}\hat{c}_{\boldsymbol{p}_2}\ldots\hat{c}_{\boldsymbol{p}_m}\hat{c}^\dagger_{\boldsymbol{p}_m}\hat{c}^\dagger_{\boldsymbol{p}_{m-1}}\ldots\hat{c}^\dagger_{\boldsymbol{p}_1} =:\ & \hat{c}_{\boldsymbol{p}_1}\hat{c}_{\boldsymbol{p}_2}\ldots\hat{c}_{\boldsymbol{p}_m}\hat{c}^\dagger_{\boldsymbol{p}_m}\hat{c}^\dagger_{\boldsymbol{p}_{m-1}}\ldots\hat{c}^\dagger_{\boldsymbol{p}_1}: \\
&+ \sum_{\text{single}} :\hat{c}^\bullet_{\boldsymbol{p}_1}\hat{c}_{\boldsymbol{p}_2}\ldots\hat{c}_{\boldsymbol{p}_m}\hat{c}^{\dagger\bullet}_{\boldsymbol{p}_m}\hat{c}^\dagger_{\boldsymbol{p}_{m-1}}\ldots\hat{c}^\dagger_{\boldsymbol{p}_1}: \\
&+ \sum_{\text{double}} :\hat{c}^\bullet_{\boldsymbol{p}_1}\hat{c}^{\bullet\bullet}_{\boldsymbol{p}_2}\ldots\hat{c}_{\boldsymbol{p}_m}\hat{c}^{\dagger\bullet}_{\boldsymbol{p}_m}\hat{c}^{\dagger\bullet\bullet}_{\boldsymbol{p}_{m-1}}\ldots\hat{c}^\dagger_{\boldsymbol{p}_1}: \\
&\vdots \\
&+ \sum_{\text{all}} :\hat{c}^\bullet_{\boldsymbol{p}_1}\hat{c}^{\bullet\bullet}_{\boldsymbol{p}_2}\hat{c}^{\bullet\bullet\bullet}_{\boldsymbol{p}_2}\ldots\hat{c}^{\dagger\bullet}_{\boldsymbol{p}_m}\hat{c}^{\dagger\bullet\bullet}_{\boldsymbol{p}_{m-1}}\hat{c}^{\dagger\bullet\bullet\bullet}_{\boldsymbol{p}_{m-1}}\cdots: .
\end{aligned} \tag{4.15}$$

The "single", "double", etc. Indicate how many different contractions ($\bullet, \bullet\bullet, \dots$) there are, and all combinations of contractions are in the sum. We have also only explicitly written down the non-zero contractions, and in general these are:

$$\hat{c}_q^{\bullet}\hat{c}_p^{\dagger\bullet} = \hat{c}_q\hat{c}_p^{\dagger} - :\hat{c}_q\hat{c}_p^{\dagger} := \{\hat{c}_q, \hat{c}_p^{\dagger}\} = M_{qp}, \tag{4.16}$$

which also shows that the contractions of the type $\hat{c}_q^{(\dagger)\bullet}\hat{c}_p^{(\dagger)\bullet}$ vanish. When taking the expectation value with respect to the vacuum all terms with remaining creation and annihilation operators vanish, as they are all normal ordered. In other words, only the last sum, fully made up of contractions, remains. After taking into account Equation 4.16 and the appropriate sign, we get:

$$\begin{aligned}
\langle 0|\hat{c}_{\boldsymbol{p}_m}\dots\hat{c}_{\boldsymbol{p}_2}\hat{c}_{\boldsymbol{p}_1}\hat{c}_{\boldsymbol{p}_1}^{\dagger}\hat{c}_{\boldsymbol{p}_2}^{\dagger}\dots\hat{c}_{\boldsymbol{p}_m}^{\dagger}|0\rangle &= \sum_{\pi\in\mathcal{S}(\{\boldsymbol{p}_i\})}\operatorname{sign}(\pi)\prod_{j=1}^{m}M_{\boldsymbol{p}_j\boldsymbol{p}_{\pi(j)}} && (4.17)\\
&= \det\left(M_{\mathbb{C}(\{\boldsymbol{p}_i\}_{i=1}^{i=m})^2}\right) && (4.18)\\
&= \prod_s \det\left(M_{\mathbb{C}(\{\boldsymbol{p}_i\}_{i=1}^{i=m}\cap\ s)^2}\right). && (4.19)
\end{aligned}$$

In the second line we introduced the minor of M, $\det(M_{\mathbb{C}(\{\boldsymbol{p}_i\}_{i=1}^{i=m})^2})$, which is the determinant of the square matrix formed by removing all columns and rows but those indexed by $\boldsymbol{p}_i \in \{\boldsymbol{p}_i\}_{i=1}^{i=m} = \{\boldsymbol{p}_1, \boldsymbol{p}_2 \dots \boldsymbol{p}_m\}$. We also used the fact that M is a block diagonal for the last line. Similarly, using now the fact that the $M_{\boldsymbol{p}_j\boldsymbol{p}_{\pi(j)}}$ from different shells are independent and the series expansion for these determinants given in Appendix A we can calculate the expectation value of the norm to be:

$$\begin{aligned}
\mathbb{E}\left[\langle \boldsymbol{p}_1, \boldsymbol{p}_2, \dots \boldsymbol{p}_m|\boldsymbol{p}_1, \boldsymbol{p}_2, \dots \boldsymbol{p}_m\rangle\right] &= \mathbb{E}\left[\langle 0|\hat{c}_{\boldsymbol{p}_m}\dots\hat{c}_{\boldsymbol{p}_2}\hat{c}_{\boldsymbol{p}_1}\hat{c}_{\boldsymbol{p}_1}^{\dagger}\hat{c}_{\boldsymbol{p}_2}^{\dagger}\dots\hat{c}_{\boldsymbol{p}_m}^{\dagger}|0\rangle\right] && (4.20)\\
&= \prod_s \mathbb{E}\left[\det\left(M_{\mathbb{C}(\{\boldsymbol{p}_i\}_{i=1}^{i=m}\cap\ s)^2}\right)\right] && (4.21)\\
&= \prod_s \left(1 + \mathcal{O}\left(\frac{1}{n_s}\right)\right) \approx 1. && (4.22)
\end{aligned}$$

For the expectation value of the number operator for shell s w.r.t. state $|\boldsymbol{p}_1, \boldsymbol{p}_2, \dots \boldsymbol{p}_m\rangle =: |\{\boldsymbol{p}_i\}_{i=1}^{i=m}\rangle$ we have then:

$$\begin{aligned}
\langle\{\boldsymbol{p}_i\}_{i=1}^{i=m}|\hat{N}_s|\{\boldsymbol{p}_i\}_{i=1}^{i=m}\rangle &= \frac{n_s}{N_s}\sum_{\boldsymbol{p}_0\in s}\langle 0|\hat{c}_{\boldsymbol{p}_m}\dots\hat{c}_{\boldsymbol{p}_1}\hat{c}_{\boldsymbol{p}_0}^{\dagger}\hat{c}_{\boldsymbol{p}_0}\hat{c}_{\boldsymbol{p}_1}^{\dagger}\dots\hat{c}_{\boldsymbol{p}_m}^{\dagger}|0\rangle && (4.23)\\
&= \frac{n_s}{N_s}\sum_{\boldsymbol{p}_0\in s}\det M_{\mathbb{C}(\{\boldsymbol{p}_i\}_{i=1}^{i=m})^2} - \det M_{\mathbb{C}(\{\boldsymbol{p}_i\}_{i=0}^{i=m})^2} && (4.24)\\
&= \frac{n_s}{N_s}\Bigg(\sum_{\boldsymbol{p}_0\in s}\det\left(M_{\mathbb{C}(\{\boldsymbol{p}_i\}_{i=1}^{i=m}\cap\ s)^2}\right) && (4.25)\\
&\quad - \det\left(M_{\mathbb{C}(\{\boldsymbol{p}_i\}_{i=0}^{i=m}\cap s)^2}\right)\Bigg)\prod_{\tilde{s}\neq s}\det\left(M_{\mathbb{C}(\{\boldsymbol{p}_i\}_{i=1}^{i=m}\cap\tilde{s})^2}\right),
\end{aligned}$$

where we simply used the anti commutator of $\hat{c}_{\boldsymbol{p}_0}$ and its adjoint and then we factorized out the determinants of the submatricies corresponding to momenta that aren't part of s using Equation 4.19. When taking the expectation value of the above it is convenient to ignore the terms where $\boldsymbol{p}_0 = \boldsymbol{p}_i$, as there are only $m \ll N_s$ such terms at most. This also means that the expectation value is now independent of the specific value of $\boldsymbol{p}_0$ and the sum simply gives a factor of N_s. Similarly, the minors of different shells are independent:

$$\begin{aligned}
\mathbb{E}\left[\langle\{\boldsymbol{p}_i\}_{i=1}^{i=m}|\hat{N}_s|\{\boldsymbol{p}_i\}_{i=1}^{i=m}\rangle\right] &= n_s\Bigg(\mathbb{E}\det\left(M_{\mathbb{C}(\{\boldsymbol{p}_i\}_{i=1}^{i=m}\cap s)^2}\right) && (4.26)\\
&\quad -\mathbb{E}\left[\det\left(M_{\mathbb{C}(\{\boldsymbol{p}_i\}_{i=0}^{i=m}\cap s)^2}\right)\right] + \mathcal{O}\left(\frac{1}{N_s}\right)\Bigg)\times\\
&\quad \times\prod_{\tilde{s}\neq s}\mathbb{E}\det\left(M_{\mathbb{C}(\{\boldsymbol{p}_i\}_{i=1}^{i=m}\cap\tilde{s})^2}\right)\\
&= n_s\Bigg(\left(1-\binom{m}{2}\frac{1}{n_s}+\mathcal{O}\left(\frac{1}{n_s^2}\right)\right) && (4.27)\\
&\quad -\left(1-\binom{m+1}{2}\frac{1}{n_s}+\mathcal{O}\left(\frac{1}{n_s^2}\right)\right)\Bigg)\times\\
&\quad \times\prod_{\tilde{s}\neq s}\left(1+\mathcal{O}\left(\frac{1}{n_{\tilde{s}}}\right)\right)+\mathcal{O}\left(\frac{n_s}{N_s}\right)\\
&= m+\mathcal{O}\left(\frac{1}{n_s}\right)+\sum_{s\neq\tilde{s}}\mathcal{O}\left(\frac{1}{n_{\tilde{s}}}\right)+\mathcal{O}\left(\frac{n_s}{N_s}\right), && (4.28)
\end{aligned}$$

where we used the series expansion from appendix A in the second step to get the expectation value of the minors. As we see, the multi-particle states have the correct expectation value for the number operator in the limit $n_s \to \infty, n_s/N_s \to 0$ and consequently they will have the right expectation value for the Hamiltonian (up to the resolutions provided by finite thickness of the momentum shell). The numerical experiments plotted in the next page seem to confirm this.

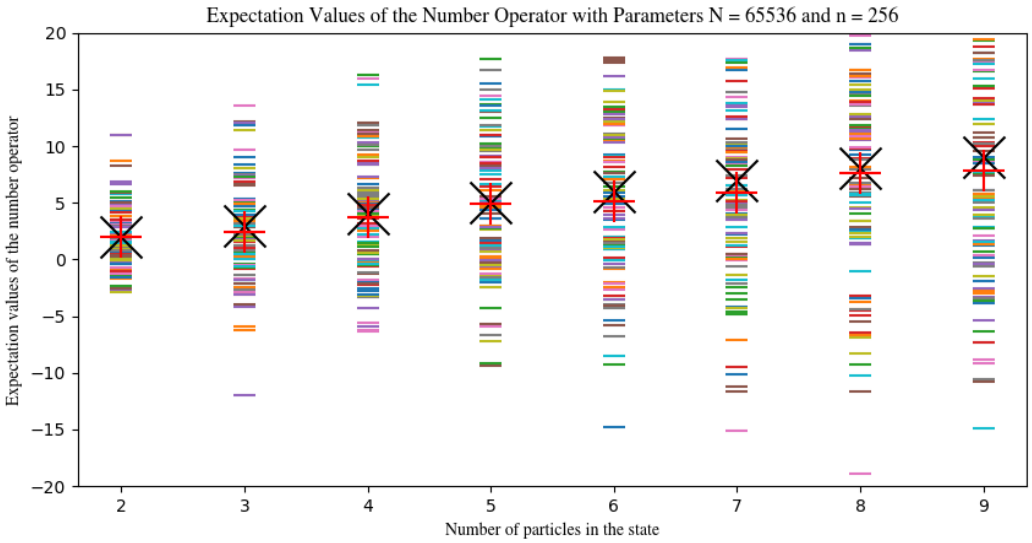


Figure 4.1: Numerical Calculation of the expected number of particles in an m-particle state as defined above. The parameters of the shell are $n_s = 256$ and $N_s = 256^2$. Lines of different colors indicate different combinations of different momenta, here 100 are plotted. The red "+"s indicate the average of all combinations, or in other words, the numerically observed left hand side of Equation 4.28. The black "X"s are m.

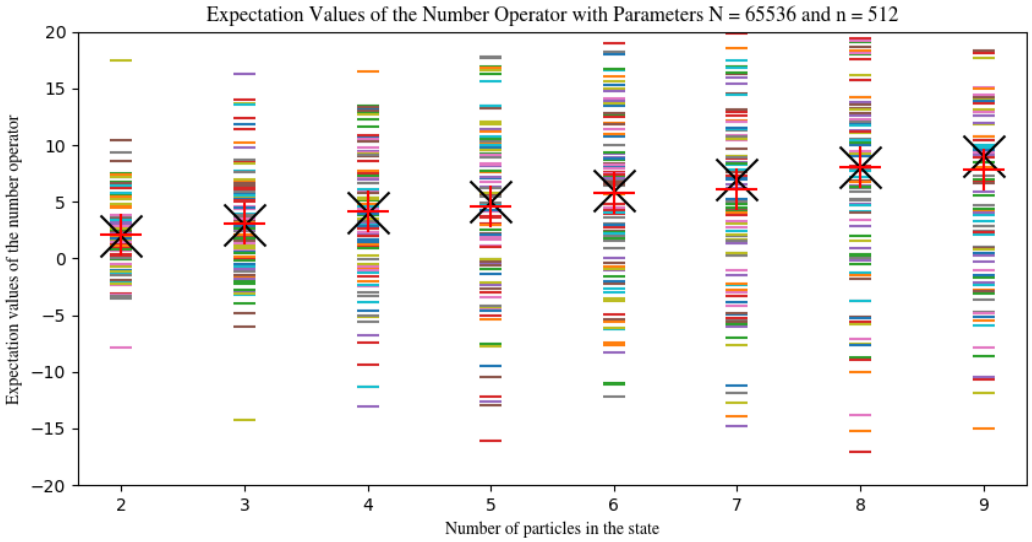


Figure 4.2: Same graph as the above, now with a larger n_s, $n_s = 512$. Notice that the expected value of the number operator aligns slightly better with m in an m-particle state.

Chapter 5

Propagation and Life-time

In this chapter we start exploring the phenomenology of the overlapped Weyl field. We will do so with a particular emphasis in how the field propagates, so that later on we examine what a theory with scattering would imply. These observations will necessitate some changes in the usual understanding of scattering experiments.

5.1 Life-time

Particle states, $|\boldsymbol{p}\rangle$, not being eigenstates of the theory is a very clear departure from typical QFT. These states not being eigenstates of the Hamiltonian, means that they won't be eigenstates of the evolution operator, $\hat{U}(t) = e^{i\hat{H}t}$, either. As a result of this, a particle that starts with momentum $\boldsymbol{p}$ may end up measured as having momentum $\boldsymbol{q}$, we call this "scattering off the vacuum" in this work. Given this odd behavior we are very interested in knowing how fast a state $|\boldsymbol{p}\rangle$ can evolve into other states, this is what we mean with the "life-time" of particles. Note that according to the definition of the Shell Hamiltonian, we have:

$$\hat{H}_s \propto \hat{N}_s \quad \Longrightarrow \quad \frac{d}{dt}\hat{N}_s = [\hat{H}_s, \hat{N}_s] = 0, \tag{5.1}$$

which means that the total number of particles in a shell is actually conserved.

5.1.1 Original Measure for Life-time

To get a measure for this scrambling of the particles [32] consider the expectation value of the 2nd time derivative of the particle number operator for a single mode under their 1 particle states. Since the definition of a 1 particle state and Hamiltonian differs slightly here, we will re-do this calculation in the first part of this section and in the second part we will talk about more general measures of life-time.

The value calculated here corresponds to:

$$\frac{1}{T^2} = \mathbb{E}\left(\langle p| \left[\left[\hat{N}_{\boldsymbol{p}}, \hat{H}\right] \hat{H}\right] |p\rangle\right) \approx 2k_s^2 \frac{n_s}{N_s}. \tag{5.2}$$

This differs from the original result of [32]. The derivation of the result is:

$$\langle p|\left[\hat{H}_s,\left[\hat{H}_s,\hat{N}_{\boldsymbol{p}}\right]\right]|p\rangle = \tfrac{n_s^2}{N_s^2}k_s^2\textstyle\sum_{\boldsymbol{k},\boldsymbol{q}\in s} \langle p|\left[\hat{c}_{\boldsymbol{q}}^\dagger\hat{c}_{\boldsymbol{q}},\left[\hat{c}_{\boldsymbol{k}}^\dagger\hat{c}_{\boldsymbol{k}},\hat{c}_{\boldsymbol{p}}^\dagger\hat{c}_{\boldsymbol{p}}\right]\right]|p\rangle \tag{5.3}$$

$$\begin{aligned}= \tfrac{n_s^2}{N_s^2}k_s^2\textstyle\sum_{\boldsymbol{k},\boldsymbol{q}\in s} & +\langle 0|\hat{c}_{\boldsymbol{p}}\hat{c}_{\boldsymbol{q}}^\dagger\hat{c}_{\boldsymbol{q}}\hat{c}_{\boldsymbol{q}}^\dagger\hat{c}_{\boldsymbol{k}}\hat{c}_{\boldsymbol{k}}^\dagger\hat{c}_{\boldsymbol{p}}\hat{c}_{\boldsymbol{p}}\hat{c}_p^\dagger|0\rangle \\ & -\langle 0|\hat{c}_{\boldsymbol{p}}\hat{c}_{\boldsymbol{q}}^\dagger\hat{c}_{\boldsymbol{q}}\hat{c}_{\boldsymbol{q}}^\dagger\hat{c}_{\boldsymbol{p}}\hat{c}_{\boldsymbol{p}}\hat{c}_{\boldsymbol{k}}^\dagger\hat{c}_{\boldsymbol{k}}\hat{c}_p^\dagger|0\rangle \\ & +\langle 0|\hat{c}_{\boldsymbol{p}}\hat{c}_{\boldsymbol{p}}^\dagger\hat{c}_{\boldsymbol{p}}\hat{c}_{\boldsymbol{p}}^\dagger\hat{c}_{\boldsymbol{k}}\hat{c}_{\boldsymbol{k}}^\dagger\hat{c}_{\boldsymbol{q}}\hat{c}_{\boldsymbol{q}}\hat{c}_p^\dagger|0\rangle \\ & -\langle 0|\hat{c}_{\boldsymbol{p}}\hat{c}_{\boldsymbol{k}}^\dagger\hat{c}_{\boldsymbol{k}}\hat{c}_{\boldsymbol{p}}^\dagger\hat{c}_{\boldsymbol{p}}\hat{c}_{\boldsymbol{q}}^\dagger\hat{c}_{\boldsymbol{q}}\hat{c}_p^\dagger|0\rangle\end{aligned} \tag{5.4}$$

$$\begin{aligned}= \tfrac{n_s^2}{N_s^2}k_s^2\textstyle\sum_{\boldsymbol{k},\boldsymbol{q}\in s} & +M_{\boldsymbol{pq}}M_{\boldsymbol{qk}}M_{\boldsymbol{kp}} \\ & -M_{\boldsymbol{pq}}M_{\boldsymbol{qp}}M_{\boldsymbol{pk}}M_{\boldsymbol{kp}} \\ & +M_{\boldsymbol{pk}}M_{\boldsymbol{kq}}M_{\boldsymbol{qp}} \\ & -M_{\boldsymbol{pk}}M_{\boldsymbol{kp}}M_{\boldsymbol{pq}}M_{\boldsymbol{qp}}\end{aligned} \tag{5.5}$$

$$\begin{aligned}= 2\tfrac{n_s^2}{N_s^2}k_s^2\Re\textstyle\sum_{\boldsymbol{k},\boldsymbol{q}\in s} & M_{\boldsymbol{pq}}M_{\boldsymbol{qk}}M_{\boldsymbol{kp}} \\ & -|M_{\boldsymbol{pq}}|^2|M_{\boldsymbol{pk}}|^2.\end{aligned} \tag{5.6}$$

Taking the expectation value:

$$\begin{aligned}\mathbb{E}\left(\langle p|\frac{d^2}{dt^2}\hat{N}_{\boldsymbol{p}}|p\rangle\right) = 2\frac{n_s^2}{N_s^2}k_s^2\Re\mathbb{E}\Big(& \textstyle\sum_{\boldsymbol{q}=\boldsymbol{k}=\boldsymbol{p}} 1-1 \\ & \textstyle\sum_{\boldsymbol{q}=k\neq\boldsymbol{p}} |M_{\boldsymbol{pk}}|^2-|M_{\boldsymbol{pk}}|^4 \\ & \textstyle\sum_{\boldsymbol{q}\neq\boldsymbol{k}} M_{\boldsymbol{pq}}M_{\boldsymbol{qk}}M_{\boldsymbol{kp}}-|M_{\boldsymbol{pq}}|^2|M_{\boldsymbol{pk}}|^2\Big)\end{aligned} \tag{5.7}$$

$$= 2\frac{n_s^2}{N_s^2}k_s^2\Re\left(0+\frac{(N_s-1)}{n_s}-\frac{3(N_s-1)}{n_s(n_s+2)}+0\right) \tag{5.8}$$

$$= 2k_s^2\frac{n_s(n_s-1)(N_s-1)}{N_s^2(n_s+2)} \approx 2k_s^2\frac{n_s}{N_s}. \tag{5.9}$$

Here we used the expectation values of appendix C, and the fact that if we have $\boldsymbol{k}\neq\boldsymbol{q}$, then it follows that $\mathbb{E}(M_{\boldsymbol{pq}}M_{\boldsymbol{qk}}M_{\boldsymbol{kp}}-|M_{\boldsymbol{pq}}|^2|M_{\boldsymbol{pk}}|^2)=0$. This can be seen by explicitly writing out the matrix components M as the result of dot products $M_{\boldsymbol{qk}}=v_{\boldsymbol{q}}\cdot v_{\boldsymbol{k}}$. For convenience we chose to evaluate said dot products using an orthonormal basis that has its first basis vector be $v_{\boldsymbol{p}}$, meaning $v_{\boldsymbol{p}}^i=\delta_1^i$, $M_{\boldsymbol{qk}}=\sum_{i=1}^{n_s}v_{\boldsymbol{q}}^iv_{\boldsymbol{k}}^i$ and $M_{\boldsymbol{kp}}=v_{\boldsymbol{k}}^1$ for any $\boldsymbol{k}$, thus:

$$\mathbb{E}(M_{\boldsymbol{pq}}M_{\boldsymbol{qk}}M_{\boldsymbol{kp}}) = \mathbb{E}\left(v_{\boldsymbol{q}}^1\left(\sum_{i=1}^{n_s}v_{\boldsymbol{q}}^iv_{\boldsymbol{k}}^i\right)v_{\boldsymbol{k}}^1\right) \tag{5.10}$$

$$= \mathbb{E}\left(v_{\boldsymbol{q}}^1\left(v_{\boldsymbol{q}}^1v_{\boldsymbol{k}}^1+\sum_{i=2}^{n_s}v_{\boldsymbol{q}}^iv_{\boldsymbol{k}}^i\right)v_{\boldsymbol{k}}^1\right) \tag{5.11}$$

$$= \mathbb{E}\left(v_{\boldsymbol{q}}^1v_{\boldsymbol{q}}^1v_{\boldsymbol{k}}^1v_{\boldsymbol{k}}^1\right)+\mathbb{E}\left(v_{\boldsymbol{q}}^1\right)\sum_{i=2}^{n_s}\mathbb{E}\left(v_{\boldsymbol{q}}^i\right)\mathbb{E}\left(v_{\boldsymbol{k}}^i\right)\mathbb{E}\left(v_{\boldsymbol{k}}^1\right) \tag{5.12}$$

$$= \mathbb{E}\left(|M_{\boldsymbol{pq}}|^2|M_{\boldsymbol{pk}}|^2\right). \tag{5.13}$$

Where in the second to last step we used $\boldsymbol{k}\neq\boldsymbol{q}$ and the fact that the different components of the vectors are not correlated.

5.1.2 General Measures of Life-time

Consider the measurement of the state of the field to determine if a previously excited particle $|k(t)\rangle$ has decayed or not. We don't need to specify how this observable looks like for the following. Call the corresponding self adjoint operator $\hat{D} = \sum_i |D_i\rangle D_i \langle D_i|$. The probability of measuring a particle that was created at time t_i having decayed by the time t_f is:

$$\begin{aligned} |\langle D_{\rm y}|k(t_f)\rangle|^2 &= |\langle D_{\rm y}|e^{i\hat{H}(t_f-t_i)}|k(t_i)\rangle|^2 \qquad (5.14) \\ &= |\langle D_{\rm y}|e^{i\hat{H}(t_f-t_i)}|E\rangle|^2 + |\langle D_{\rm y}|e^{i\hat{H}(t_f-t_i)}|k_\perp\rangle|^2 \\ &\quad + \langle D_{\rm y}|e^{i\hat{H}(t_f-t_i)}|E\rangle\langle k_\perp|e^{i\hat{H}(t_f-t_i)}|D_{\rm y}\rangle \\ &\quad + \langle D_{\rm y}|e^{i\hat{H}(t_f-t_i)}|k_\perp\rangle\langle E|e^{i\hat{H}(t_f-t_i)}|D_{\rm y}\rangle, \qquad (5.15) \end{aligned}$$

where we introduced a normalized eigen-state of the Hamiltonian $|E\rangle$ and the orthogonal vector to the eigen-state $|k_\perp\rangle = |E\rangle - |k(t_i)\rangle$. Ideally, we pick an eigen-state that minimizes the norm of $|k_\perp\rangle$. Given that $|\langle D_{\rm y}|e^{i\hat{H}t}|E\rangle|^2 = |\langle D_{\rm y}|E\rangle|^2$ remains constant, the change in the probability from $t = t_i$ to t_f of measuring mode $|k(t)\rangle$ is partly due to the magnitude of $|k_\perp\rangle$; The smaller this vector is, the less is the likelihood that $|k\rangle$ has decayed, regardless of the specifics of observable $\hat{D}$.

One can show that the magnitude of $|k_\perp\rangle$ is proportional to the quantity $1/T^2$ of the previous subsection. The idea is to calculate the magnitude of component of $H|k\rangle$ orthogonal to $|k\rangle$. In order to get $1/T^2 \propto \langle k_\perp|\hat{H}^\dagger\hat{H}|k_\perp\rangle \propto \langle k_\perp|k_\perp\rangle$.

5.2 Scattering & Time Evolution

5.2.1 Scattering in General

Before we start exploring the details of scattering matrices, or S-matrices, it's worth pointing out that the idea of what scattering experiments are and what is it that they are designed to do. The goal of scattering experiments is to isolate particles in time and space to carefully control what interactions can occur. The key assumption being that particles will only interact with other particles that they are close to. As such, scattering experiments —by their very nature— are designed to probe the possible local interactions of a theory. This posses a challenge if one is to explore the effects of non-local interactions, like the overlaps of the present theory.

At the same time, however, isolating particles and the causal influences on them means that the level of precision attained in scattering experiments is extremely high. Moreover, in interacting QFTs, particles will interact with all the fields they couple to, even if there are no particles of said fields present. This could mean that modifications to QFT, that are in-principle non-local, as the explored overlaps, could, indeed, have big contributions to scattering matrices. In order to explore these latter, more indirect effects, we require a QFT with interactions. This is something we leave to the last chapters of this work; in

this section we will focus on the theory as described in chapter 4, that is, an overlapped theory with no interactions, or overlapped "free theory".

5.2.2 Scattering in Normal QFT

In scattering experiments, we are interested in the probability of finding a set of particles with possibly different momenta after some initial set of particles has evolved. Usually this initial set of particles needs to come together for interactions to occur and for the probabilities to not be trivial. We denote the state of the field with particles coming in as $|i\rangle$ and the state of the field with particles coming out as $|f\rangle$. Because in typical QFT interactions are short ranged and far away particles act as eigenstates of a free field Hamiltonian, we pick an initial set of particles that are localized at $t = 0$, but were further apart at $t \ll 0$. Similarly, we are interested in particles that move further apart as $0 \ll t$, for example the wave packet centered around $\boldsymbol{q}$[67]:

$$\hat{a}^\dagger_{\text{pack}}(\boldsymbol{q})|0\rangle \propto \int d^3k e^{-(\boldsymbol{q}-\boldsymbol{k})^2/4\sigma^2}\hat{a}^\dagger_{\boldsymbol{k}}|0\rangle \tag{5.16}$$

will be at the origin at $t = 0$, but will propagate away (and spread out a little) as $t \to -\infty$. So long as $\boldsymbol{q}_1 \neq \boldsymbol{q}_2$ we can write down a state where the particles are separated in the far past, but come together at $t = 0$:

$$|i\rangle = \lim_{t\to-\infty} \hat{a}^\dagger_{\text{pack}}(t, \boldsymbol{q}_1)\hat{a}^\dagger_{\text{pack}}(t, \boldsymbol{q}_2)|0\rangle. \tag{5.17}$$

Doing a similar thing for outgoing particles $|f\rangle$, we can define the S-matrix elements of normal QFT as:

$$S_{if} = \langle i|f\rangle \approx \lim_{t_i\to\infty}\lim_{t_f\to-\infty}\langle \boldsymbol{q}_1, \boldsymbol{q}_2 \dots \boldsymbol{q}_n|U(t_f - t_i)|\boldsymbol{q}'_1, \boldsymbol{q}'_2 \dots \boldsymbol{q}'_{n'}\rangle. \tag{5.18}$$

5.2.3 Scattering & Time evolution in the Overlapped QFT

It is not straight forward to apply this line of thought to the overlapped Weyl field, because single particle states are not eigenstates of the overlapped Hamiltonian and there will be some scattering between single particles and the vacuum even in the absence of extra interaction terms in the Hamiltonian. Taking the above equation for the S-matrix and replacing the usual creation operators with the overlapped creation operators we get (without yet taking the limit of $t \to \pm\infty$):

$$\langle \boldsymbol{p}|U(t_f - t_i)|\boldsymbol{p}'\rangle = \langle 0|\hat{c}_{\boldsymbol{p}} \exp\Big(i(t_f - t_i)\sum_s \frac{n_s}{N_s}k_s \sum_{q\in s}\hat{c}^\dagger_{\boldsymbol{q}}\hat{c}_{\boldsymbol{q}}\Big)\hat{c}^\dagger_{\boldsymbol{p}'}|0\rangle \tag{5.19}$$

$$= \sum_{m=0}^{\infty}\frac{(i(t_f - t_i))^m}{m!}\langle 0|\hat{c}_{\boldsymbol{p}}\Big(\sum_s \frac{n_s}{N_s}k_s \sum_{q\in s}\hat{c}^\dagger_{\boldsymbol{q}}\hat{c}_{\boldsymbol{q}}\Big)^m \hat{c}^\dagger_{\boldsymbol{p}'}|0\rangle, \tag{5.20}$$

which won't result in $\propto \delta_{\boldsymbol{p},\boldsymbol{p}'}$ as it would in normal Free QFT, unless $\boldsymbol{p}$ and $\boldsymbol{p}'$ belong to different shells. If this is not the case and $\boldsymbol{p}, \boldsymbol{p}' \in s$ we get:

$$\begin{aligned}
\langle \boldsymbol{p}|U(t_f - t_i)|\boldsymbol{p}'\rangle &= \sum_{m=0}^{\infty} \frac{1}{m!}\Big(i(t_f - t_i)\frac{n_s}{N_s}k_s\Big)^m \cdot \\
&\quad \cdot \sum_{q_1 \in s}\sum_{q_2 \in s}\cdots\sum_{q_m \in s} \langle 0|\hat{c}_{\boldsymbol{p}}\hat{c}^\dagger_{\boldsymbol{q}_1}\hat{c}_{\boldsymbol{q}_1}\hat{c}^\dagger_{\boldsymbol{q}_2}\hat{c}_{\boldsymbol{q}_2}\ldots\hat{c}^\dagger_{\boldsymbol{q}_m}\hat{c}_{\boldsymbol{q}_m}\hat{c}^\dagger_{\boldsymbol{p}'}|0\rangle \qquad (5.21)\\
&= \sum_{m=0}^{\infty} \frac{i^m}{m!}\Big((t_f - t_i)\frac{n_s}{N_s}k_s\Big)^m \cdot \\
&\quad \cdot \sum_{q_1 \in s}\sum_{q_2 \in s}\cdots\sum_{q_m \in s} M_{\boldsymbol{p}\boldsymbol{q}_1}M_{\boldsymbol{q}_1\boldsymbol{q}_2}\ldots M_{\boldsymbol{q}_m\boldsymbol{p}'} \qquad (5.22)\\
&= \sum_{m=0}^{\infty} \frac{i^m}{m!}\Big((t_f - t_i)\frac{n_s}{N_s}k_s\Big)^m \left(M_s^{m+1}\right)_{\boldsymbol{p}\boldsymbol{p}'} \qquad (5.23)\\
&= \left(M\exp\left(i(t_f - t_i)\frac{n_s}{N_s}k_s M\right)\right)_{\boldsymbol{p}\boldsymbol{p}'}. \qquad (5.24)
\end{aligned}$$

The expression get far more complicated when looking at scattering of multi-particle states; The reason they look so simple in the case of single particle states is that replacing $\hat{A}_{\boldsymbol{p}}|0\rangle$ with $\sum_{\boldsymbol{i}} F_{\boldsymbol{p}\boldsymbol{i}}\hat{A}_{\boldsymbol{i}}|0\rangle = \hat{c}_{\boldsymbol{p}}|0\rangle$ is essentially nothing more than a linear endomorphism in the one particle subspace of the Hilbert space. On the other hand n-particle states get mapped to n-particle states, but the linear map is more complicated to write down in terms of $F_{\boldsymbol{p}\boldsymbol{i}}$. Yet because this is still an endomorphism, this also demands that n-particle states only evolve to other n-particle states, even in the overlapped theory. Replacing $\hat{c}^\dagger_{\boldsymbol{p}'}|0\rangle$ in Equation 5.21 with a multi-particle state and doing the same with the bra in the other side, we see that this is indeed the case, as there will be too many creation operators or annihilation operators, one always remain after anti-commuting. A big advantage of Equation 5.24 is that it allows for the computation of $\langle \boldsymbol{p}|U(t_f - t_i)|\boldsymbol{p}'\rangle$ without having to deal with the Hilbert space of 2^{n_s} dimension that $U(t_f - t_i)$ belongs to. Instead, we only need to deal with the vector space of $M_{\boldsymbol{p}\boldsymbol{p}'}$ of dimension N_s. Examples of explicit computations are graphed in figures 5.1 to 5.3. We can take the expectation value of Equation 5.24 to get approximately (for short time scales and low energies $|t_f - t_i|k_s \ll \sqrt{N_s/n_s}$):

$$\begin{aligned}
\mathbb{E}\left[\langle \boldsymbol{p}|U(t_f - t_i)|\boldsymbol{p}'\rangle\right] &= \sum_{m=0}^{\infty} \frac{i^m}{m!}\Big((t_f - t_i)\frac{n_s}{N_s}k_s\Big)^m \mathbb{E}\left[\left(M_s^{m+1}\right)_{\boldsymbol{p}\boldsymbol{p}'}\right] \qquad (5.25)\\
&\approx \sum_{m=0}^{\infty} \frac{i^m}{m!}\Big((t_f - t_i)\frac{n_s}{N_s}k_s\Big)^m \left(\left(\frac{N_s}{n_s}\right)^m \delta_{\boldsymbol{p}\boldsymbol{p}'} \qquad (5.26)\right.\\
&\quad \left. + \delta_{\boldsymbol{p}\boldsymbol{p}'}\frac{m(m+1)}{2}\mathcal{O}\left(\frac{N_s}{n_s}\right)^{m-1}\right)\\
&\approx \delta_{\boldsymbol{p}\boldsymbol{p}'}\left(1 - \frac{(t_f - t_i)^2 k_s^2}{2}\left(\frac{n_s}{N_s}\right)\right)e^{i(t_f - t_i)k_s} \qquad (5.27)
\end{aligned}$$

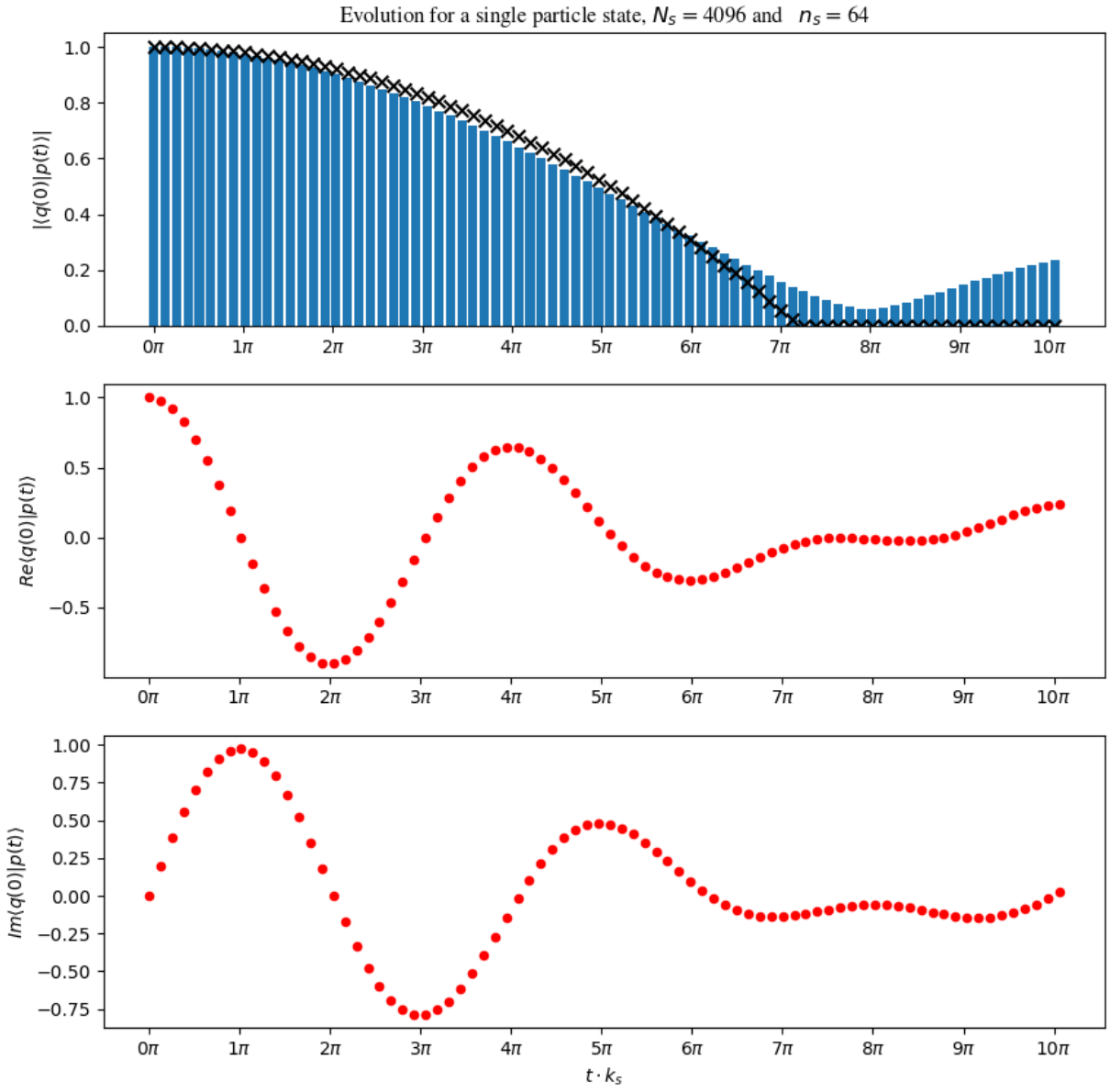


Figure 5.1: The Time evolution for a single particle state, $|\boldsymbol{p}(t)\rangle$, w.r.t. the original state , $|\boldsymbol{p}(0)\rangle$. Marked in black is the inverted parabola we expect on the magnitude of the state according to Equation 5.25. Notice that the wave seems to decay somewhat fast, this could be due to the low dimension $n_s = 64$, $N_s = 64^2$. The number $\langle \boldsymbol{p}(0)|\boldsymbol{p}(t)\rangle$ is calculated every time from the start according to Equation 5.21, with steps of $t \cdot k_s$ of 0.2.

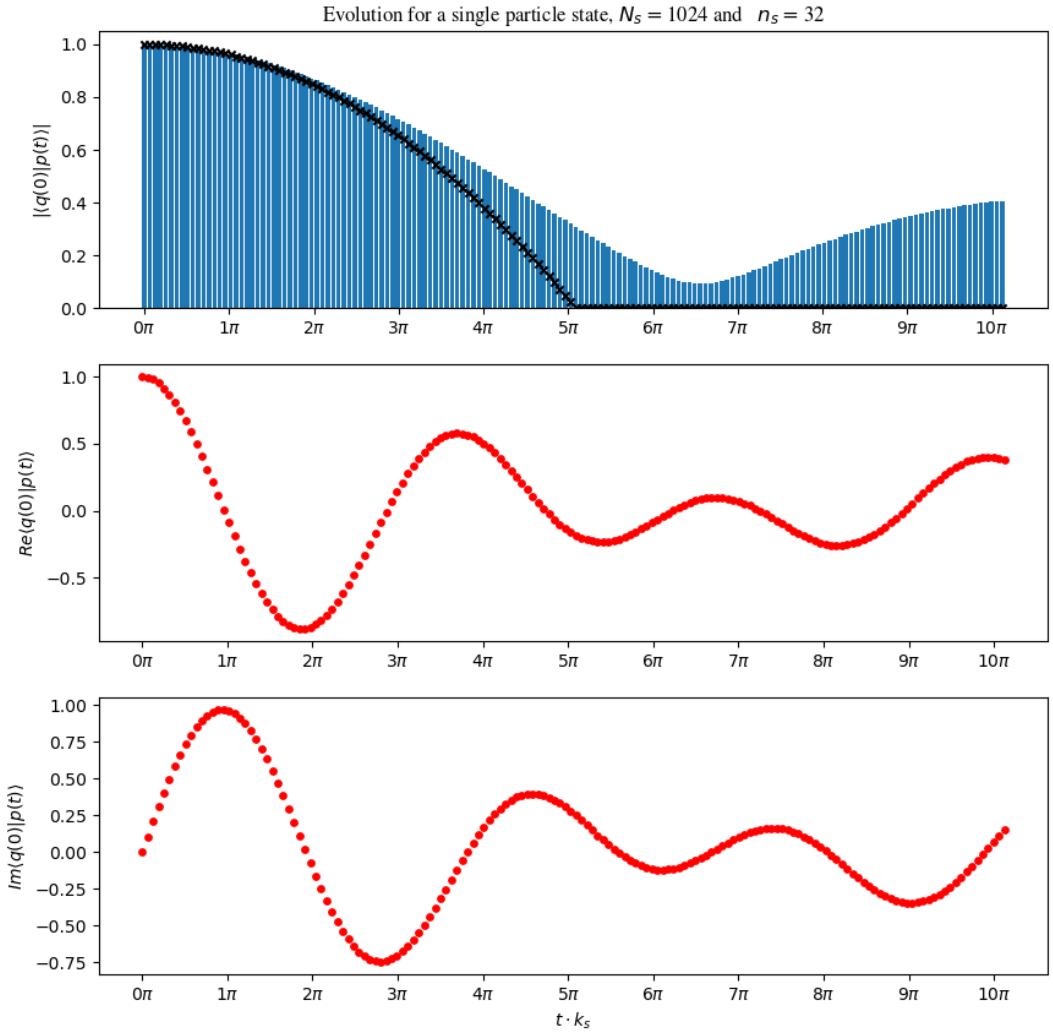


Figure 5.2: The Time evolution for a single particle state, $|\boldsymbol{p}(t)\rangle$, w.r.t. the original state , $|\boldsymbol{p}(0)\rangle$. Marked in black is the inverted parabola we expect on the magnitude of the state according to Equation 5.25. Notice that the even lower dimension of $n_s = 32$, $N_s = 32^2$ means that the wave decays even faster. The number $\langle \boldsymbol{p}(0)|\boldsymbol{p}(t)\rangle$ is calculated as before, now with steps of $t \cdot k_s$ of 0.1.

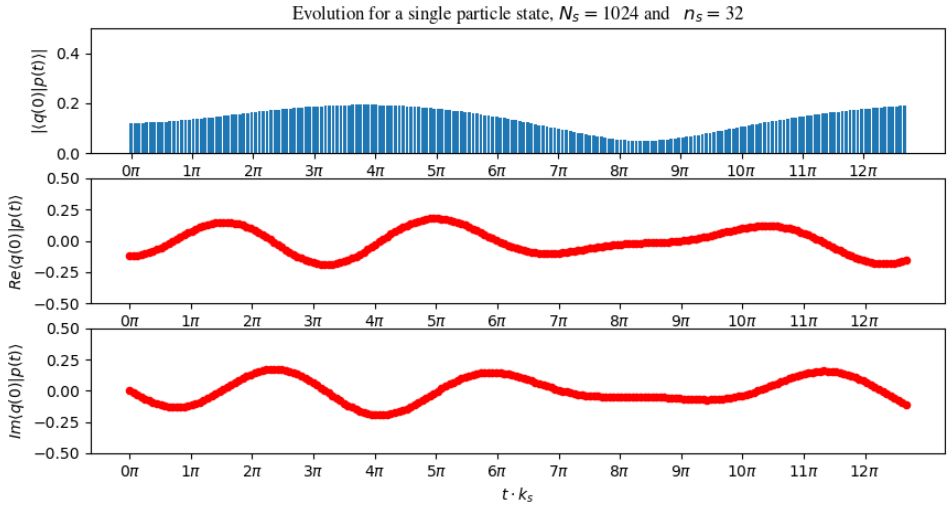


Figure 5.3: The Time evolution for a single particle state, $|\boldsymbol{p}(t)\rangle$, now w.r.t. to some other random state, $|\boldsymbol{q}(0)\rangle$. Notice that this number is small, in particular for small $t \cdot k_s$, the parameters for the M matrix are, $n_s = 32$, $N_s = 1024$, like before.

For the exact expressions and the exact conditions of their validity and their derivation, see appendix B. Equation 5.27 agrees nicely with the calculation for the life-time, as the deviation from the standard result only happens once the second term in the parenthesis becomes of the order of one. In general, the more as time goes on the larger the effects of the overlap become. Yet these results show that particles may maintain their state and evolve as usual for some time before transitioning to another state. So it makes sense to postulate some time scale $\tau < \infty$ such that $U(\tau)|\boldsymbol{k}\rangle$ is approximately equal to $|\boldsymbol{k}\rangle$ up to a phase factor, but at the same time τ should take particles far enough apart so that a scattering experiment may be prepared. This means:

$$\tau \ll \frac{1}{k_s}\sqrt{\frac{N_s}{n_s}} \qquad\qquad \tau \gg d, \tag{5.28}$$

where d is some distance the particles should propagate apart, given they move at a speed of 1 (in natural units). With these restrictions in place we now define the scattering matrix for the overlapped Weyl fields as:

$$S_{if} = \langle i|f\rangle \approx \lim_{t_i \to -\tau} \lim_{t_f \to \tau} \langle \boldsymbol{p}_1, \boldsymbol{p}_2 \dots \boldsymbol{p}_n | U(t_f - t_i) | \boldsymbol{p}'_1, \boldsymbol{p}'_2 \dots \boldsymbol{p}'_{n'} \rangle. \tag{5.29}$$

as opposed to Equation 5.18. Since the multi-particle states have the correct expectation value for the Hamiltonian, they will match the usual case up to linear order in τ. If the particles come from different shells and/or the time evolution is not long enough for the overlaps to take effect, we can leverage results Equation 5.27 and Equation 4.28 to get (defining $s_i \ni p_i$):

$$\mathbb{E}(\langle \boldsymbol{p}_1, \boldsymbol{p}_2 \dots \boldsymbol{p}_n | U(\tau + \tau) | \boldsymbol{p}'_1, \boldsymbol{p}'_2 \dots \boldsymbol{p}'_n \rangle) \approx e^{2\tau i k_{s_1} + 2\tau i k_{s_2} \dots 2\tau i k_{s_n}} \delta_{\boldsymbol{p}_1, \boldsymbol{p}'_1} \dots. \tag{5.30}$$

The full expression is more complicated as stated previously, we get terms like:

$$\sum_{q_2 \in s} \cdots \sum_{q_m \in s} \langle 0 | \hat{c}_{\boldsymbol{p}_1} \hat{c}_{\boldsymbol{p}_2} \ldots \hat{c}_{\boldsymbol{p}_n} \hat{c}^\dagger_{\boldsymbol{q}_1} \hat{c}_{\boldsymbol{q}_1} \hat{c}^\dagger_{\boldsymbol{q}_2} \hat{c}_{\boldsymbol{q}_2} \ldots \hat{c}^\dagger_{\boldsymbol{q}_m} \hat{c}_{\boldsymbol{q}_m} \hat{c}^\dagger_{\boldsymbol{p}'_1} \hat{c}^\dagger_{\boldsymbol{p}'_2} \ldots \hat{c}^\dagger_{\boldsymbol{p}'_n} | 0 \rangle, \tag{5.31}$$

which means that, not only do we have matrix exponentiation, which is a function of m, but also we get terms for every possible permutation, which is a function of n. Arguably the most significant contribution comes from the m, as it gets taken to infinity in the sum in the definition of the exponent, while n remains constant. Either case, we see that the deviations to standard QFT occur when studying the propagation of many-particle states for long periods of time $t > \tau$.

Chapter 6

The Path Integral Formalism for an Overlapping Quantum Field

Most of the time, quantum field theories with interactions between multiple fields are explored through perturbative expansions of the path integral. In order to perform a similar analysis for the overlapped theory we need a path integral for this model, so this will be the goal of this chapter. The procedure to achieve this will be to first find transformations that bring us closer to a normal QFT with the usual anti-commutation relations, and then follow the usual derivation. Because the derivation presented here is very close to the standard one, (once appropriate transformations have been made) we won't separately review the usual approach in this work.

6.1 The Free Field Path Integral

6.1.1 Naive Formulation of the Holographic Path Integral

The idea behind the path integral formulation of quantum mechanics is to brake up the evolution of a quantum state in many small steps:

$$\langle \psi(t_i), t_i | \psi(t_f), t_f \rangle = \langle \psi(t_i) | e^{i\hat{H}(t_f - t_i)} | \psi(t_f) \rangle \tag{6.1}$$

$$= \langle \psi(t_i) | e^{i\hat{H}\delta t} e^{i\hat{H}\delta t} \dots e^{i\hat{H}\delta t} | \psi(t_f) \rangle. \tag{6.2}$$

Then, normally, we would evaluate the evolution after each time step δt in a set of basis vectors:

$$\int \mathcal{D}[\psi(\boldsymbol{x})] \; |\psi\rangle\langle\psi|. \tag{6.3}$$

However, in this model, the vectors $|\psi(\boldsymbol{x})\rangle$ define an over-complete set of basis vectors. To make sure we are properly expressing our states, we should write everything in terms of non-over-complete basis vectors; that is to say, in non-overlapping basis vectors, and this we will do in the next subsection.

6.1.2 Correct(-ish) Formulation of the Holographic Path Integral

The previous subsection showed that we should perform the transition to path integral formalism with non-overlapping degrees of freedom. Since we need to change basis anyway, we should also use the opportunity to find a non-overlapping basis that also diagonalizes the Hamiltonian. Having transitioned to the path integral, we can then perform an inverse transformation to express everything in terms of the overlapped Weyl field. Note that these transformation are nothing more than linear transformations on the Hilbert-space $\mathcal{H}_V$. Lets start with the Hamiltonian written in terms of the non-overlapped creation and annihilation operators, i.e. we plug Equation 4.2 into Equation 4.9 (taking now the F matrices to be real):

$$H = \sum_{\boldsymbol{ijkl}} \hat{A}^\dagger_{\boldsymbol{i}}(F^\dagger DF)_{\boldsymbol{ij}}\hat{A}_{\boldsymbol{j}} + B^\dagger_{\boldsymbol{k}}(F^\dagger DF)_{\boldsymbol{kl}}\hat{B}_{\boldsymbol{l}}, \tag{6.4}$$

where the Matrix D is a diagonal matrix that incorporates the appropriate k_sN_s/n_s coefficients for the different shells. Note that the matrix $F^\dagger DF$ is a $(\sum_s n_s) \times (\sum_s n_s)$ matrix and thus it has a chance at being diagonalized. Indeed, [32] showed this is possible for each shell and thus all of them together, and they called the corresponding basis transformation $R_{\boldsymbol{ij}}$ such that $\hat{B}^a_{\boldsymbol{i}} = \sum_{\boldsymbol{j}} R_{\boldsymbol{ij}}\hat{A}_{\boldsymbol{j}}$ is an eigenvector of said matrix. We define similarly $\hat{B}^b_{\boldsymbol{i}} = \sum_{\boldsymbol{j}} R_{\boldsymbol{ij}}\hat{B}_{\boldsymbol{j}}$ for the B-like particles and we can combine these diagonalized creation/annihilation operators in a single field as:

$$\hat{\beta}_{\boldsymbol{i}} = \frac{u(\boldsymbol{i})}{\sqrt{|\boldsymbol{i}|}}\hat{B}^a_{\boldsymbol{i}} + \frac{u(-\boldsymbol{i})}{\sqrt{|\boldsymbol{i}|}}\left(\hat{B}^b_{-\boldsymbol{i}}\right)^\dagger \tag{6.5}$$

in analogy to $\psi_{\boldsymbol{p}}$. We have decided to express the β fields as functions of a $\mathbf{3}$-vector, $\boldsymbol{i}$, also in analogy to $\boldsymbol{p}$. The only important property of the $\boldsymbol{i}$ vectors is that there are far less of them than there are $\boldsymbol{p}$ vectors; their magnitude will be rescaled shortly, and their direction only serves to distinguish them. This makes it hard to assign a unique interpretation to the $\boldsymbol{i}$ vectors, and thus we will mostly treat them as multi-indices. Writing β in terms of the original creation/annihilation operators, we get:

$$\hat{\beta}_{\boldsymbol{i}} = \frac{1}{\sqrt{|\boldsymbol{i}|}}\sum_{\boldsymbol{kj}} u(\boldsymbol{i})R_{\boldsymbol{ik}}\hat{A}_{\boldsymbol{k}} + u(-\boldsymbol{i})R_{-\boldsymbol{ij}}\hat{B}^\dagger_{\boldsymbol{j}}. \tag{6.6}$$

We have added the vectors $u(\pm\boldsymbol{i})$ so that the A-type particles and B-type particles don't mix in products of the form $\beta^\dagger\beta$, and so that we can write (up to constant vacuum energy):

$$H = \sum_s k_s\frac{N_s}{n_s}\sum_{\boldsymbol{i}\in s^{-1}}\hat{\beta}^\dagger_{\boldsymbol{i}}\lambda_{\boldsymbol{i}}\sigma^j\left(\frac{\boldsymbol{i}}{|\boldsymbol{i}|}\right)_j\hat{\beta}_{\boldsymbol{i}} = \sum_{\boldsymbol{i}}\hat{\beta}^\dagger_{\boldsymbol{i}}\sigma^j(\tilde{\mathbf{k}}_{\boldsymbol{i}})_j\hat{\beta}_{\boldsymbol{i}}\ , \tag{6.7}$$

for compactness we introduced the overlapped momentum $\tilde{\mathbf{k}}_{\boldsymbol{i}} = (k_sN_s\lambda_i/(n_s))(\boldsymbol{i}/|i|)$, which gives $\boldsymbol{i}$ the correct magnitude. The $\lambda_{\boldsymbol{i}}$ are the eigenvalues of the matrix $F^\dagger F$, the ones

explored in [32]. We are also defining $s^{-1} = \{\boldsymbol{i} : \exists \boldsymbol{p} \in s, F_{\boldsymbol{pi}} \neq 0\}$. It is also clear that the $\hat{\beta}_{\boldsymbol{i}}$ and $i\hat{\beta}^\dagger_{\boldsymbol{i}}$ follow the canonical commutation relations:

$$\begin{aligned}
\left\{\hat{\beta}_{\boldsymbol{i}}, i\hat{\beta}^\dagger_{\boldsymbol{j}}\right\} &= \frac{i}{\sqrt{|\boldsymbol{i}||\boldsymbol{j}|}} \sum_{\boldsymbol{kq}} R_{\boldsymbol{ik}} R_{\boldsymbol{jq}} \left\{\hat{A}_{\boldsymbol{k}}, \hat{A}^\dagger_{\boldsymbol{q}}\right\} u(\boldsymbol{i}) u^\dagger(\boldsymbol{j}) \qquad (6.8)\\
&\quad + R_{-\boldsymbol{ik}} R_{-\boldsymbol{jq}} \left\{\hat{B}^\dagger_{\boldsymbol{k}}, \hat{B}_{\boldsymbol{q}}\right\} u(-\boldsymbol{i}) u^\dagger(-\boldsymbol{j})\\
&= \frac{i}{|\boldsymbol{i}|} \sum_{\boldsymbol{k}} R_{\boldsymbol{ik}} R_{\boldsymbol{jk}} u(\boldsymbol{i}) u^\dagger(\boldsymbol{j}) + R_{-i\boldsymbol{k}} R_{-j\boldsymbol{k}} u(-\boldsymbol{i}) u^\dagger(-\boldsymbol{j}) \qquad (6.9)\\
&= i\delta_{\boldsymbol{ij}} \mathbf{1}_{2\times 2}\ , \qquad (6.10)
\end{aligned}$$

where we used the fact that R is an orthogonal matrix at the last step. It is also possible to express the un-overlapped creation and annihilation operators in terms of the $\hat{\beta}_{\boldsymbol{i}}$:

$$\hat{A}_{\boldsymbol{k}} = \sum_{\boldsymbol{i}} R^T_{\boldsymbol{ki}} \frac{u^\dagger(\boldsymbol{i})}{\sqrt{|\boldsymbol{i}|}} \hat{\beta}_{\boldsymbol{i}} \qquad \hat{B}^\dagger_{\boldsymbol{k}} = \sum_{\boldsymbol{i}} R^T_{\boldsymbol{k}-\boldsymbol{i}} \frac{u^\dagger(-\boldsymbol{i})}{\sqrt{|-\boldsymbol{i}|}} \hat{\beta}_{\boldsymbol{i}}\ . \qquad (6.11)$$

At this point, it is worth mentioning that, because we now have fields that follow the canonical commutation relations, we can follow the standard derivation of the path integral from the Hamiltonian formalism. What follows is nothing more than the usual derivation with new names for the fermionic fields, β and $i\beta^\dagger$. We will mostly follow [67] and [3], with modifications to get a *field* theory version, for a more formal approach see [57] or [16]. Although the following treatment is fairly standard, we should point out it is hardly rigorous, as the task of properly defining functional integration is a topic of active research.

The Eigenvalues of anti-commuting operators, Grassmann numbers

In order to proceed we need to first explore the eigenvalues of these anti-commuting operators. Equation 6.5 shows us we can split the field operator into a creation operator and an annihilation operator, each acting in a different sub-space, so if we want to diagonalize the field operator, it makes sense to diagonalize these operators first. Suppose we have an operator $\hat{\theta}$ that anti-commutes with itself, and take $|\theta\rangle$ to be an eigenvector with eigenvalue θ:

$$\hat{\theta}\hat{\theta} = -\hat{\theta}\hat{\theta} \quad \Longrightarrow \quad \hat{\theta}^2 = 0 \quad \Longrightarrow \quad 0 = \hat{\theta}^2|\theta\rangle = \theta^2|\theta\rangle. \qquad (6.12)$$

This then suggests that θ is an anti-commuting number, this is indeed the case. Such numbers are called Grassmann numbers, and they have the following defining property:

$$\theta_1\theta_2 = -\theta_2\theta_1. \qquad (6.13)$$

One can also have functions of Grassmann numbers, but as we have seen, any power of a Grassmann number greater than 2 must vanish. This means that for the polynomial expansion of any function of a Grassmann number must have [3][7]:

$$f(\theta) = f_0 + f_1\theta, \qquad (6.14)$$

where f_0 and f_1 are two real or complex numbers. If we want to define integration over a Grassmann number, one can show that the only way of doing it so that integration is linear, invariant under shifts of the argument of the integrand, i.e. $f(\theta_1) \to f(\theta_1 + \theta_2)$, and allows partial integration is by defining it as[7]:

$$\int d\theta \; f(\theta) = \int d\theta \; (f_0 + f_1\theta) = f_1. \tag{6.15}$$

While there are many more properties of Grassmann numbers, these are enough to proceed.

Coherent state basis

In the following we tackle the Hilbert space $\mathcal{H}$ one factor at a time. As such, we split our vectors and operators to define:

$$|0\rangle = \bigotimes_{\boldsymbol{i}}^{\text{JW}} (|0_{\boldsymbol{i}}^b\rangle \otimes^{\text{JW}} |0_{\boldsymbol{i}}^a\rangle) \qquad \mathbf{1} \otimes \mathbf{1} \otimes \cdots \otimes \hat{a}_{\boldsymbol{i}} \otimes \cdots \otimes \mathbf{1}|v\rangle = \hat{B}_{\boldsymbol{i}}^a|v\rangle, \tag{6.16}$$

and we do the same with the operator $\hat{B}_{\boldsymbol{i}}^b$ and their adjoints, such that *the operator* $\hat{a}_{\boldsymbol{i}}$, $\hat{b}_{\boldsymbol{i}}$ *and their adjoints act only on* $\text{span}\{|0\rangle_{\boldsymbol{i}}^a, \hat{a}_{\boldsymbol{i}}^\dagger|0\rangle_i^a\}$, *or* $\text{span}\{|0\rangle_{\boldsymbol{i}}^b, \hat{b}_{\boldsymbol{i}}^\dagger|0\rangle_i^b\}$, *respectively.*[1] We also demand that $\hat{a}_{\boldsymbol{i}}$ follows the usual anti-commutation relations with its adjoint, but note that, as $\hat{a}_{\boldsymbol{i}}$ and $\hat{b}_{\boldsymbol{i}}$ are not defined in the same Hilbert space, we get for $\boldsymbol{j} \neq \boldsymbol{i}$:

$$\left\{\hat{a}_{\boldsymbol{i}}, \hat{a}_{\boldsymbol{i}}^\dagger\right\} = 1 \qquad \left\{\hat{a}_{\boldsymbol{i}}, \hat{b}_{\boldsymbol{i}}^\dagger\right\} = \left\{\hat{a}_{\boldsymbol{i}}, \hat{a}_{\boldsymbol{j}}^\dagger\right\} = \text{undef.} \tag{6.17}$$

and the same should hold true for the adjoints.

The eigenvectors of these creation and annihilation operators are called coherent states, and they are:

$$|a_{\boldsymbol{i}}\rangle = \left(1 - a_{\boldsymbol{i}}\hat{a}_{\boldsymbol{i}}^\dagger\right)|0_{\boldsymbol{i}}^a\rangle \qquad |b_{\boldsymbol{i}}\rangle = \left(1 - b_{\boldsymbol{i}}\hat{b}_{\boldsymbol{i}}^\dagger\right)|0_{\boldsymbol{i}}^b\rangle. \tag{6.18}$$

This simply follows from the anti-commutation of the annihilation operators, their adjoints and their eigenvalues, $a_{\boldsymbol{i}}$:

$$\begin{aligned}
\hat{a}_{\boldsymbol{i}}|a_{\boldsymbol{i}}\rangle &= \hat{a}_{\boldsymbol{i}}\left(1 - a_{\boldsymbol{i}}\hat{a}_{\boldsymbol{i}}^\dagger\right)|0_{\boldsymbol{i}}^a\rangle && (6.19)\\
&= a_{\boldsymbol{i}}\hat{a}_{\boldsymbol{i}}\hat{a}_{\boldsymbol{i}}^\dagger|0_{\boldsymbol{i}}^a\rangle && (6.20)\\
&= a_{\boldsymbol{i}}|0_{\boldsymbol{i}}^a\rangle && (6.21)\\
&= a_{\boldsymbol{i}}\left(1 - a_{\boldsymbol{i}}\hat{a}_{\boldsymbol{i}}\right)^\dagger|0_{\boldsymbol{i}}^a\rangle = a_{\boldsymbol{i}}|a_{\boldsymbol{i}}\rangle, && (6.22)
\end{aligned}$$

where we used $a_{\boldsymbol{i}}^2 = 0$.

[1] Note that these operators are not the same as those of chapter 3

Notice that these coherent states are not normalized:

$$\langle a_{\boldsymbol{i}}|a_{\boldsymbol{i}}\rangle = \langle 0^a_{\boldsymbol{i}}|\left(1 - a_{\boldsymbol{i}}\hat{a}^\dagger_{\boldsymbol{i}}\right)^\dagger \left(1 - a_{\boldsymbol{i}}\hat{a}^\dagger_{\boldsymbol{i}}\right)|0^a_{\boldsymbol{i}}\rangle \tag{6.23}$$

$$= \langle 0^a_{\boldsymbol{i}}|0^a_{\boldsymbol{i}}\rangle - a^\dagger_{\boldsymbol{i}}a_{\boldsymbol{i}}\langle 0^a_{\boldsymbol{i}}|\hat{a}_{\boldsymbol{i}}\hat{a}^\dagger_{\boldsymbol{i}}|0^a_{\boldsymbol{i}}\rangle \tag{6.24}$$

$$= 1 - a^\dagger_{\boldsymbol{i}}a_{\boldsymbol{i}} = e^{-a^\dagger_i a_j}. \tag{6.25}$$

In the last step we used the polynomial expansion for the exponential of Grassmann numbers, Equation 6.14. The analogous of the above can be easily shown for $|b_{\boldsymbol{i}}\rangle$. Using these eigenvectors, we can see that:

$$\int da^\dagger_{\boldsymbol{i}} da_{\boldsymbol{i}}\ e^{a^\dagger_i a_i}|a_{\boldsymbol{i}}\rangle\langle a_{\boldsymbol{i}}| = \int da^\dagger_{\boldsymbol{i}} da_{\boldsymbol{i}}\ (1 + a^\dagger_{\boldsymbol{i}}a_{\boldsymbol{i}})\left(1 - a_{\boldsymbol{i}}\hat{a}^\dagger_{\boldsymbol{i}}\right)|0^a_{\boldsymbol{i}}\rangle\langle 0^a_{\boldsymbol{i}}|\left(1 - \hat{a}_{\boldsymbol{i}}a^\dagger_{\boldsymbol{i}}\right) \tag{6.26}$$

$$= \int da^\dagger_{\boldsymbol{i}} da_{\boldsymbol{i}}\ \cdots + a^\dagger_{\boldsymbol{i}}a_{\boldsymbol{i}}\left(\hat{a}^\dagger_{\boldsymbol{i}}|0^a_{\boldsymbol{i}}\rangle\langle 0^a_{\boldsymbol{i}}|\hat{a}_{\boldsymbol{i}} + |0^a_{\boldsymbol{i}}\rangle\langle 0^a_{\boldsymbol{i}}|\right) + \ldots \tag{6.27}$$

$$= \hat{a}^\dagger_{\boldsymbol{i}}|0^a_{\boldsymbol{i}}\rangle\langle 0^a_{\boldsymbol{i}}|\hat{a}_{\boldsymbol{i}} + |0^a_{\boldsymbol{i}}\rangle\langle 0^a_{\boldsymbol{i}}| = \mathbf{1}^a_{\boldsymbol{i}}. \tag{6.28}$$

The third line follows from the definition of the integral over Grassmann numbers, which allows us to ignore all higher and lower powers of $a_{\boldsymbol{i}}$ and $a^\dagger_{\boldsymbol{i}}$. We can do a similar thing for the Hilbert space of $\hat{b}_{-\boldsymbol{i}}$, and if we assume that $a_{\boldsymbol{i}}$ and $b_{-\boldsymbol{i}}$ commute with each other, we get:

$$\mathbf{1}_{\boldsymbol{i}} = \mathbf{1}^a_{\boldsymbol{i}} \otimes \mathbf{1}^b_{-\boldsymbol{i}} = \int da^\dagger_{\boldsymbol{i}} da_{\boldsymbol{i}}\ e^{a^\dagger_i a_i}|a_{\boldsymbol{i}}\rangle\langle a_{\boldsymbol{i}}| \otimes \int db^\dagger_{-\boldsymbol{i}} db_{-\boldsymbol{i}}\ e^{b^\dagger_{-i} b_{-i}}|b_{-\boldsymbol{i}}\rangle\langle b_{-\boldsymbol{i}}| \tag{6.29}$$

$$= \int da^\dagger_{\boldsymbol{i}} da_{\boldsymbol{i}} \int db^\dagger_{-\boldsymbol{i}} db_{-\boldsymbol{i}}\ e^{a^\dagger_i a_i + b^\dagger_{-i} b_{-i}}|a_{\boldsymbol{i}}\rangle\langle a_{\boldsymbol{i}}| \otimes |b_{-\boldsymbol{i}}\rangle\langle b_{-\boldsymbol{i}}| \tag{6.30}$$

$$= \int d\beta^\dagger_{\boldsymbol{i}} d\beta_{\boldsymbol{i}}\ e^{a^\dagger_i a_i + b^\dagger_{-i} b_{-i}}|\beta_{\boldsymbol{i}}\rangle\langle\beta_{\boldsymbol{i}}|, \tag{6.31}$$

where we have defined $|\beta_{\boldsymbol{i}}\rangle = |a_{\boldsymbol{i}}\rangle \otimes |b_{-\boldsymbol{i}}\rangle$. This makes sense, as the field β is just a linear combination of creation and annihilation operators. Finally, we multiply over all $\boldsymbol{i}$s in order to get the unit operator of the whole Hilbert space:

$$\mathbf{1} = \bigotimes_{\boldsymbol{i}} \mathbf{1}_{\boldsymbol{i}} = \int \mathcal{D}[\beta^\dagger_{\boldsymbol{i}}]\mathcal{D}[\beta_{\boldsymbol{i}}]\ e^{\sum_i a^\dagger_i a_i + b^\dagger_{-i} b_{-i}}|\beta\rangle\langle\beta|, \tag{6.32}$$

where we have defined:

$$|\beta\rangle = \bigotimes_{\boldsymbol{i}} |\beta_{\boldsymbol{i}}\rangle = \bigotimes_{\boldsymbol{i}} |a_{\boldsymbol{i}}\rangle \otimes |b_{-\boldsymbol{i}}\rangle. \tag{6.33}$$

Time slicing

We are interested in calculating the probability amplitude of finding the field in state $\beta(t_n)$ at time t_n, given it was in $\beta(t_0)$ at time t_0:

$$\langle\beta(t_0),t_0|\beta(t_n),t_n\rangle = \langle\beta(t_0)|e^{i\hat{H}(t_0-t_n)}|\beta(t_n)\rangle = \langle\beta(t_0)|e^{-i\hat{H}\delta t}\mathbf{1}e^{-i\hat{H}\delta t}\mathbf{1}\dots\mathbf{1}e^{-i\hat{H}\delta t}|\beta(t_n)\rangle \tag{6.34}$$

$$= \langle\beta(t_0)|e^{-i\hat{H}\delta t}\left[\prod_{i=1}^{n-1}\mathbf{1}e^{-i\hat{H}\delta t}\right]|\beta(t_n)\rangle \tag{6.35}$$

$$= \langle\beta(t_0)|e^{-i\hat{H}\delta t}\left[\prod_{i=1}^{n-1}\int\mathcal{D}[\beta_{\boldsymbol{i}}^\dagger]\mathcal{D}[\beta_{\boldsymbol{i}}]\; e^{\sum_{\boldsymbol{i}} a^\dagger(t_i)_{\boldsymbol{i}}a_{\boldsymbol{i}}(t_i)+b^\dagger_{-\boldsymbol{i}}(t_i)b_{-\boldsymbol{i}}(t_i)}\times\right. \tag{6.36}$$

$$\left.\times\,|\beta(t_i)\rangle\langle\beta(t_i)|e^{-i\hat{H}\delta t}\right]|\beta(t_0)\rangle. \tag{6.37}$$

$$= \prod_{i=1}^{n}\int\mathcal{D}[\beta_{\boldsymbol{i}}^\dagger]\mathcal{D}[\beta_{\boldsymbol{i}}]\; e^{\sum_{\boldsymbol{i}} a^\dagger(t_i)_{\boldsymbol{i}}a_{\boldsymbol{i}}(t_i)+b^\dagger_{-\boldsymbol{i}}(t_i)b_{-\boldsymbol{i}}(t_i)}\langle\beta(t_{i-1})|e^{-i\hat{H}\delta t}|\beta(t_i)\rangle, \tag{6.38}$$

where we have split up the time interval $t_n - t_0$ into n intervals, each of length δt. We have also introduced the unit operator, written in the coherent states basis (Equation 6.31). In the product above we get terms like:

$$\langle\beta(t_j)|e^{-i\hat{H}\delta t}|\beta(t_{j+1})\rangle \approx \langle\beta(t_j)|1-i\hat{H}\delta t|\beta(t_{j+1})\rangle \tag{6.39}$$

$$= \langle\beta(t_j)|\beta(t_{j+1})\rangle - iH_W\delta t\langle\beta(t_j)|\beta(t_{j+1})\rangle \tag{6.40}$$

$$\overset{(6.5)}{=} (1-iH_W\delta t)\prod_{\boldsymbol{i}}\langle a_{\boldsymbol{i}}(t_j)|a_{\boldsymbol{i}}(t_{j+1})\rangle\langle b_{-\boldsymbol{i}}(t_j)|b_{-\boldsymbol{i}}(t_{j+1})\rangle \tag{6.41}$$

$$\approx e^{-iH_W\delta t-\sum_{\boldsymbol{i}} a^\dagger_{\boldsymbol{i}}(t_j)a_{\boldsymbol{i}}(t_{j+1})+b^\dagger_{-\boldsymbol{i}}(t_j)b_{-\boldsymbol{i}}(t_{j+1})}\,. \tag{6.42}$$

In the second step we defined the Weyl ordered Hamiltonian $H_W = \langle\beta|\hat{H}|\beta\rangle/\langle\beta|\beta\rangle$, then on the fourth step we used Equation 6.25. Throughout we ignored terms of $\mathcal{O}(\delta t^2)$. Expressing the β fields with creation/annihilation operators, (Equation 6.5) we note that:

$$\beta^\dagger_{\boldsymbol{i}}(t)\beta_{\boldsymbol{i}}(t') = \frac{1}{\langle\beta(t)|\beta(t')\rangle}\langle\beta(t)|\hat{\beta}^\dagger_{\boldsymbol{i}}\hat{\beta}_{\boldsymbol{i}}|\beta(t')\rangle \tag{6.43}$$

$$= \frac{1}{\langle\beta(t)|\beta(t')\rangle}\langle\beta(t)|\left(\hat{B}^a_{\boldsymbol{i}}\right)^\dagger\hat{B}^a_{\boldsymbol{i}} + \left(\hat{B}^b_{-\boldsymbol{i}}\right)^\dagger\hat{B}^b_{-\boldsymbol{i}}|\beta(t')\rangle + \mathcal{E}_0 \tag{6.44}$$

$$= a^\dagger_{\boldsymbol{i}}(t)a_{\boldsymbol{i}}(t') + b^\dagger_{-\boldsymbol{i}}(t)b_{-\boldsymbol{i}}(t') + \mathcal{E}_0. \tag{6.45}$$

Writing the last step more explicitly:

$$\langle\beta(t)|\left(\hat{B}^a_{\boldsymbol{i}}\right)^\dagger \hat{B}^a_{\boldsymbol{i}} + \left(\hat{B}^b_{-\boldsymbol{i}}\right)^\dagger \hat{B}^b_{-\boldsymbol{i}}|\beta(t')\rangle \tag{6.46}$$

$$\overset{(6.33)}{=} \left(\bigotimes_{\boldsymbol{j}}\langle a_{\boldsymbol{j}}(t)|\otimes\langle b_{-\boldsymbol{j}}(t)|\right)\left(\hat{B}^a_{\boldsymbol{i}}\right)^\dagger \hat{B}^a_{\boldsymbol{i}} + \left(\hat{B}^b_{-\boldsymbol{i}}\right)^\dagger \hat{B}^b_{-\boldsymbol{i}}\left(\bigotimes_{\boldsymbol{k}}|a_{\boldsymbol{k}}(t')\rangle\otimes|b_{-\boldsymbol{k}}(t')\rangle\right) \tag{6.47}$$

$$= \left(\bigotimes_{\boldsymbol{j}}\langle a_{\boldsymbol{j}}(t)|\otimes\langle b_{-\boldsymbol{j}}(t)|\right)\left(\bigotimes_{\boldsymbol{k}}|a_{\boldsymbol{k}}(t')\rangle\otimes|b_{-\boldsymbol{k}}(t')\rangle\right)\left(a^\dagger_{\boldsymbol{i}}(t)a_{\boldsymbol{i}}(t') + b^\dagger_{-\boldsymbol{i}}(t)b_{-\boldsymbol{i}}(t')\right), \tag{6.48}$$

where we used the defining property of the $\hat{a}_{\boldsymbol{i}}$, $\hat{b}^\dagger_{\boldsymbol{i}}$, $\hat{b}_{\boldsymbol{i}}$ and $\hat{a}^\dagger_{\boldsymbol{i}}$ operators (Equation 6.16). Plugging in Equation 6.42 into Equation 6.38 and Equation 6.45 in the exponents of the same equation, we get:

$$\langle\beta(t_0),t_0|\beta(t_n),t_n\rangle \propto \prod_{i=1}^{n}\int \mathcal{D}[\beta^\dagger_{\boldsymbol{i}}]\mathcal{D}[\beta_{\boldsymbol{i}}]\; e^{\sum_{\boldsymbol{i}}\beta^\dagger_{\boldsymbol{i}}(t_i)\beta_{\boldsymbol{i}}(t_i)}\langle\beta(t_{i-1})|e^{-i\hat{H}\delta t}|\beta(t_i)\rangle \tag{6.49}$$

$$\propto \prod_{i=1}^{n}\int \mathcal{D}[\beta^\dagger_{\boldsymbol{i}}]\mathcal{D}[\beta_{\boldsymbol{i}}]\; e^{\sum_{\boldsymbol{i}}\beta^\dagger_{\boldsymbol{i}}(t_i)\beta_{\boldsymbol{i}}(t_i)-\sum_{\boldsymbol{j}}\beta^\dagger_{\boldsymbol{j}}(t_i)\beta_{\boldsymbol{j}}(t_{i+1})-iH_W\delta t} \tag{6.50}$$

$$\propto \int\left[\prod_{i=1}^{n}\mathcal{D}[i\beta^\dagger_{\boldsymbol{i}}]\mathcal{D}[\beta_{\boldsymbol{i}}]\right]\; e^{\sum_i\left(i^2\sum_{\boldsymbol{j}}\beta^\dagger_{\boldsymbol{j}}(t_i)\left(\beta_{\boldsymbol{j}}(t_{i+1})-\beta_{\boldsymbol{j}}(t_i)\right)\delta t/\delta t - iH_W\delta t\right)} \tag{6.51}$$

$$\overset{\delta t\to 0}{\longrightarrow} \int \mathcal{D}[i\beta^\dagger]\mathcal{D}[\beta]\; e^{i\int dt L[i\beta^\dagger,\beta]}. \tag{6.52}$$

Throughout we have ignored the constant proportionality $e^{\mathcal{E}_0}$ and multiples thereof. Note that the functional integrals now treat β and $i\beta^\dagger$ as functions of time and (fourier-)space. Finally, we have also reintroduced the Lagrangian:

$$i\sum_{\boldsymbol{j}} i\beta^\dagger_{\boldsymbol{j}}(t_i)(\beta_{\boldsymbol{j}}(t_{i+1})-\beta_{\boldsymbol{j}}(t_i))/\delta t \;\; - iH_W \overset{\delta t\to 0}{\longrightarrow} i\sum_{\boldsymbol{j}}(i\beta^\dagger_{\boldsymbol{j}})\partial_0\beta_{\boldsymbol{j}} - iH_W = iL. \tag{6.53}$$

We can also write the path integral in terms of the action, S:

$$\langle\beta(t_0),t_0|\beta(t_n),t_n\rangle \propto \int_{t_0}^{t_n}\mathcal{D}[\beta]\mathcal{D}[i\beta^\dagger]e^{iS[\beta,i\beta^\dagger]} = Z_0. \tag{6.54}$$

The action is then given by:

$$S = \int_{t_0}^{t_n}\sum_s\sum_{\boldsymbol{j}\in s^{-1}}\beta^\dagger_{\boldsymbol{j}}(t)\left(i\partial_0 - \sigma^i(\tilde{\mathbf{k}}_{\boldsymbol{j}})_i\right)\beta_{\boldsymbol{j}}(t)dt \tag{6.55}$$

$$= |t_n-t_0|\sum_{s,\omega}\sum_{\boldsymbol{j}\in s^{-1}} -\beta^\dagger_{\boldsymbol{j}}(\omega)\left(\sigma^i(\tilde{\mathbf{k}}_{\boldsymbol{j}})_i+\omega\right)\beta_{\boldsymbol{j}}(\omega), \tag{6.56}$$

where we introduced the fourier coefficients:

$$\beta_{\boldsymbol{i}}(t) = \sum_{\omega} \beta_{\boldsymbol{i}}(\omega) e^{i\omega t} \quad \overset{|t_0 - t_n| \to \infty}{\longrightarrow} \quad \int \frac{d\omega}{2\pi} e^{i\omega t} \beta_{\boldsymbol{i}}(\omega), \tag{6.57}$$

and in the same limit when $|t_0 - t_n|$ is taken to infinity (and $\beta(\omega)$ is normalized properly), the action becomes:

$$S = \frac{1}{2\pi} \int d\omega \sum_{s} \sum_{\boldsymbol{j} \in s^{-1}} -\beta^{\dagger}_{\boldsymbol{j}}(\omega) \left(\sigma^i(\tilde{\mathbf{k}}_{\boldsymbol{j}})_i + \omega \right) \beta_{\boldsymbol{j}}(\omega). \tag{6.58}$$

6.1.3 Calculating the Expectation Value of Observables

The expectation value of a time ordered series of operators can also be given with a path integral as:

$$\langle T\hat{O}(t_1)\hat{O}(t_2) \rangle \propto \int_{t_i}^{t_f} \mathcal{D}[\beta]\mathcal{D}[i\beta^{\dagger}] O(t_1) O(t_2) e^{iS[\beta, i\beta^{\dagger}]}. \tag{6.59}$$

This can be seen by expanding the left hand side in the $|\beta\rangle$ basis and using the path integral version of the propagator, i.e. Equation 6.54. A particularly important example of the above is the case when $\hat{O} = \hat{\psi}(x)$, in this case we can add a source term to the action to be able to express such correlation functions as:

$$i^2 \langle 0 | T\hat{\psi}^{\dagger}(x_1)\hat{\psi}(x_2) | 0 \rangle \propto \frac{-\delta}{\delta h(x_1)} \frac{\delta}{\delta h^{\dagger}(x_2)} \bigg|_{h, h^{\dagger}=0} \int \mathcal{D}[\beta]\mathcal{D}[i\beta^{\dagger}] e^{iS + i\int \psi^{\dagger}(x)h(x) + ih^{\dagger}(x)\psi(x) dx^4} \tag{6.60}$$

$$:= \frac{-\delta}{\delta h(x_1)} \frac{\delta}{\delta h(x_2)} Z_0[h, h^{\dagger}] \bigg|_{h, h^{\dagger}=0}. \tag{6.61}$$

We fix the proportionality by demanding $Z_0[0, 0] = 1$ and we also add a minus sign to one of the source terms as they too are Grassmann variables, like in normal QFT. Implicitly we have also taken the limit $t_i \to -\infty$ and $t_f \to \infty$, and assumed that in this limit the field must be in its ground state. One can achieve this, for instance, by multiplying the Hamiltonian by a factor $(1 + i\epsilon)$ so that $\hat{H} \to (1 + i\epsilon)\hat{H}$, and taking the limit $\epsilon \to 0$ after taking time to infinity. For a more detailed discussion on this, see chapter 6 of [67].

6.2 Adding Interaction Terms to the Action

The biggest advantage of using the path integral formalism is being able to perturbatively introduce interaction terms into the Lagrangian. For instance, one can choose an interaction term of the form:

$$S_{int} = \int dx^4 g\phi(x)\psi^{\dagger}(x)\psi(x). \tag{6.62}$$

If on the left hand side of this functional we have another functional of the form $F[h, h^\dagger, J] = \exp[i \int d^4x(J\phi + h^\dagger\psi + \psi^\dagger h)]$ as before, we can write this interaction in terms of functional derivatives[67]:

$$S_{int} = \int dx^4 g \left(\frac{1}{i}\frac{\delta}{\delta J}\right)\left(\frac{-1}{i}\frac{\delta}{\delta h}\right)\left(\frac{1}{i}\frac{\delta}{\delta h^\dagger}\right) F[h, h^\dagger, J]\Bigg|_{h,h^\dagger,J=0}. \tag{6.63}$$

With this in hand we can also write the path integral for the interacting theory as[67]:

$$Z[J, h^\dagger, h] \propto \exp\left[i \int dx^4 g \left(\frac{1}{i}\frac{\delta}{\delta J}\right)\left(\frac{-1}{i}\frac{\delta}{\delta h}\right)\left(\frac{1}{i}\frac{\delta}{\delta h^\dagger}\right)\right] Z_0[J, h^\dagger, h]\Bigg|_{h,h^\dagger,J=0}, \tag{6.64}$$

where Z_0 is the free field partition function. In the next chapter, we will show that Z_0 can be written as

$$Z_0[h^\dagger, h] = \exp\left[i \int d^4x d^4y \; h^\dagger(x)\Delta_\psi(x, y)h(y)\right] \tag{6.65}$$

once we have solved the path integral (Equation 7.10). We can then expand this exponential function and the one in Equation 6.64 in powers of g,J, h and $h^\dagger$:

$$\begin{aligned} Z = \sum_V \frac{1}{V!}\left[ig \int d^4z \left(\frac{1}{i}\frac{\delta}{\delta J}\right)\left(\frac{-1}{i}\frac{\delta}{\delta h}\right)\left(\frac{1}{i}\frac{\delta}{\delta h^\dagger}\right)\right]^V \times \\ \times \sum_{P_1} \frac{1}{P_1!}\left[i \int d^4x d^4y \; h^\dagger(x)\Delta_\psi(x, y)h(y)\right]^{P_1} \times \ldots. \end{aligned} \tag{6.66}$$

Organizing the different ways of mixing functional derivatives is a task best performed with Feynman diagrams, as in standard quantum field theory. The only difference at this point, is the new form of the propagator, which, as we will see, does no longer conserve momentum.

Chapter 7

Solving the Path Integral

The free field path integral derived in the previous chapter is nothing more than a Gaussian integral, as such it can be computed by "completing the square in the exponential", this is in essence what we will do in the second part of this chapter, following [67]. First, we need to deal with the fact that we are interested in correlation functions in terms of the ψ fields, while the action is written in terms of the β fields, to address this issue we will need the appropriate transformations.

7.1 Transformations Between the β and ψ Fields

Our first task is to define an un-overlapped source term, $\tilde{h}$, to do so we demand:

$$\begin{aligned}
\int \frac{d\omega}{2\pi} \sum_{\boldsymbol{i}} \tilde{h}^{\dagger}_{\boldsymbol{i}} \beta_{\boldsymbol{i}} &\stackrel{!}{=} \int dx^4 h^{\dagger}(x)\psi(x) && (7.1)\\
&= \int dx^4 h^{\dagger}(x) \sum_{\boldsymbol{p}} \frac{1}{\sqrt{E_{\boldsymbol{p}} V}} (u(\boldsymbol{p})\hat{c}_{\boldsymbol{p}} e^{i\boldsymbol{px}} + u(\boldsymbol{p})\hat{d}^{\dagger}_{\boldsymbol{p}} e^{-i\boldsymbol{px}}) && (7.2)\\
&= \int dx^4 h^{\dagger}(x) \sum_{\boldsymbol{p}} \frac{1}{\sqrt{E_{\boldsymbol{p}} V}} \left(\sum_{\boldsymbol{ik}} u(\boldsymbol{p}) F_{\boldsymbol{pk}} R^{\dagger}_{\boldsymbol{ki}} \frac{u^{\dagger}(\boldsymbol{i})}{\sqrt{|\boldsymbol{i}|}} e^{i\boldsymbol{px}} \right. && (7.3)\\
&\quad \left. + u(\boldsymbol{p}) \sum_{\boldsymbol{ik}} F_{-\boldsymbol{pk}} R^{\dagger}_{\boldsymbol{k}-\boldsymbol{i}} \frac{u^{\dagger}(-\boldsymbol{i})}{\sqrt{|-\boldsymbol{i}|}} e^{-i\boldsymbol{px}} \right) \frac{\beta_{\boldsymbol{i}}(\omega)}{2\pi} e^{i\omega t}, && (7.4)
\end{aligned}$$

where we used the representation of $\hat{\psi}(x)$ in terms of overlapped creation and annihilation operators, i.e. the fourier transform of Equation 4.3, and then did the opposite for $\beta_{\boldsymbol{i}}$, Equation 6.6. Thus we define the functional:

$$\tilde{h}^{\dagger}_{\boldsymbol{i}}[h^{\dagger}(x)] = \int h^{\dagger}(x)\mathcal{R}^{\dagger}_{\boldsymbol{i}}(x)d^4x, \tag{7.5}$$

with:

$$\mathcal{R}^\dagger_{\boldsymbol{i}}(x) = \sum_{\boldsymbol{p}} \frac{e^{ip^\mu x_\mu}}{\sqrt{E_{\boldsymbol{p}} V}} \left((FR^\dagger)_{-\boldsymbol{p}\boldsymbol{i}} \frac{u(-\boldsymbol{p})u^\dagger(\boldsymbol{i})}{\sqrt{|\boldsymbol{i}|}} + (FR^\dagger)_{-\boldsymbol{p}-\boldsymbol{i}} \frac{u(\boldsymbol{p})u^\dagger(-\boldsymbol{i})}{\sqrt{|-\boldsymbol{i}|}} \right), \tag{7.6}$$

and we can do a similar thing with a $\beta^\dagger \tilde{h}$ term.

7.2 Solving the Path Integral

As mentioned previously the theory can be solved exactly in terms of the $\beta_{\boldsymbol{i}}$ fields, as the path integral is a simple gaussian. Interestingly, this means that the theory is still basically a free field theory despite its non-local behavior. We will now "complete the square" to absorb the source term into the β field, to do so we shift the field $\beta_{\boldsymbol{i}}(\omega)$ by the a constant term:

$$\beta_{\boldsymbol{i}}(\omega) \to \beta_{\boldsymbol{i}}(\omega) - \tilde{h}_{\boldsymbol{i}} \frac{\sigma^j(\tilde{\mathbf{k}}_{\boldsymbol{i}})_j - \omega}{(\tilde{\mathbf{k}}_{\boldsymbol{i}})^2 - \omega^2}, \tag{7.7}$$

so that the action plus source terms becomes:

$$S + i \int \frac{d\omega}{2\pi} \sum_{\boldsymbol{i}} \tilde{h}^\dagger_{\boldsymbol{i}} \beta_{\boldsymbol{i}} + \tilde{h}_{\boldsymbol{i}} \beta^\dagger_{\boldsymbol{i}} \to S + i \int \frac{d\omega}{2\pi} \sum_{\boldsymbol{i}} \tilde{h}^\dagger_{\boldsymbol{i}} \frac{\sigma^j(\tilde{\mathbf{k}}_{\boldsymbol{i}})_j - \omega}{(\tilde{\mathbf{k}}_{\boldsymbol{i}})^2 - \omega^2} \tilde{h}_{\boldsymbol{i}}. \tag{7.8}$$

We recognize the Weyl propagator (with $m = 0$) in fourier space[1]:

$$\Delta_\beta(\tilde{k}_{\boldsymbol{i}}) = \frac{-\sigma^\mu(\tilde{k}_{\boldsymbol{i}}(\omega))_\mu}{(k_{\boldsymbol{i}}(\omega))^2 + i\epsilon} \qquad \tilde{k}_{\boldsymbol{i}}(\omega) = \begin{pmatrix} \omega \\ k_s \lambda_i \frac{N_s}{n_s |\boldsymbol{i}|} \boldsymbol{i} \end{pmatrix} \quad \text{for } \boldsymbol{i} \in s. \tag{7.9}$$

Having separated the β field and the source term $\tilde{h}$, we can preform the gaussian integral over β to get some proportionality factor, which we had already defined to be $Z_0[0,0] = 1$:

$$Z_0[h^\dagger, h] = \exp\left[i \int \frac{d\omega}{2\pi} \sum_{\boldsymbol{i}} \tilde{h}^\dagger_{\boldsymbol{i}} \Delta_\beta(\tilde{k}_{\boldsymbol{i}}) \tilde{h}_{\boldsymbol{i}} \right]. \tag{7.10}$$

Thus according to Equation 6.61 the correlation $\langle 0|T\beta^\dagger(\tilde{\omega})\beta(\omega)|0\rangle$ function is (introducing the spinor indicies α, β):

$$\langle 0|T\hat{\beta}^\dagger_{\boldsymbol{i}\alpha}(\tilde{\omega})\hat{\beta}_{\boldsymbol{j}\beta}(\omega)|0\rangle \propto \frac{-1(2\pi)^2}{i^2} \frac{\delta}{\delta \tilde{h}_{\boldsymbol{i}\alpha}} \frac{\delta}{\delta \tilde{h}^\dagger_{\boldsymbol{j}\beta}} Z_0[h, h^\dagger] \Big|_{\tilde{h}, \tilde{h}^\dagger = 0} \tag{7.11}$$

$$= \delta_{\boldsymbol{i}\boldsymbol{j}} 2\pi\delta(\omega - \tilde{\omega}) \frac{-(\sigma^j(\tilde{\mathbf{k}}_{\boldsymbol{i}})_j - \omega)_{\alpha\beta}}{(\tilde{\mathbf{k}}_{\boldsymbol{i}})^2 - \omega^2 + i\epsilon}. \tag{7.12}$$

[1] the $+i\epsilon$ in the denominator is the Feynman pole prescription, we will later take $\epsilon \to 0^+$

Although the theory is simple in these un-overlapped degrees of freedom, we are more interested in the overlapped field, to go from one to another we can use the transformations given before (Equation 7.4):

$$\int \frac{d\omega}{2\pi} \sum_{\boldsymbol{i}} \tilde{h}^{\dagger}_{\boldsymbol{i}} \Delta_{\beta}(\tilde{k}_{\boldsymbol{i}}) \tilde{h}_{\boldsymbol{i}} = \int d^4x d^4y \; h^{\dagger}(x) \Delta_{\psi}(x,y) h(y). \tag{7.13}$$

Where we are defining the overlapped propagator:

$$\Delta_{\psi}(x,y) = \sum_{\boldsymbol{i}} \int \frac{d\omega}{2\pi} \mathcal{R}^{\dagger}_{\boldsymbol{i}}(x) \Delta_{\beta}(\tilde{k}_{\boldsymbol{i}}) \mathcal{R}_{\boldsymbol{i}}(y) \tag{7.14}$$

$$= \int \frac{d\omega}{2\pi V} \sum_{\boldsymbol{pqi}} (FR^{\dagger})_{\boldsymbol{pi}} \frac{-\Sigma(\boldsymbol{q},\boldsymbol{p},\tilde{k}_{\boldsymbol{i}}(\omega)) e^{i\omega(x^0-y^0)-i\boldsymbol{qy}+i\boldsymbol{px}}}{(\tilde{k}_{\boldsymbol{i}}(\omega))^2 + i\epsilon} (RF^{\dagger})_{\boldsymbol{iq}}. \tag{7.15}$$

We also defined:

$$\Sigma(\boldsymbol{q},\boldsymbol{p},\tilde{k}_{\boldsymbol{i}}(\omega))_{\alpha\beta} = (|\tilde{\mathbf{k}}_{\boldsymbol{i}}| - \omega) \frac{u(\boldsymbol{p}) u^{\dagger}(\boldsymbol{q})}{\sqrt{|\boldsymbol{p}||\boldsymbol{q}|}} - (|\tilde{\mathbf{k}}_{\boldsymbol{i}}| + \omega) \frac{u(-\boldsymbol{p}) u^{\dagger}(-\boldsymbol{q})}{\sqrt{|\boldsymbol{p}||\boldsymbol{q}|}}. \tag{7.16}$$

If the matrices $F^{\dagger}$ and F are chosen as in [32], the eigenvalues of the matrix $F^{\dagger}F$, $\lambda_i \approx N_s/n_s$, for large n_s, N_s, we discuss this in the second part of appendix C, but show no proof. We also have $|\tilde{\mathbf{k}}_{\boldsymbol{i}}| = k_s$ for $\boldsymbol{i} \in s^{-1}$. With these approximations the propagator simplifies considerably:

$$\Delta_{\psi}(x,y) = \int \frac{d\omega}{2\pi V} \sum_{s} \sum_{\boldsymbol{p},\boldsymbol{q} \in s} \frac{-\Sigma(\boldsymbol{q},\boldsymbol{p},k_s)}{\omega^2 - k_s^2 + i\epsilon} M_{\boldsymbol{pq}} e^{i\omega(x^0-y^0)-i\boldsymbol{qy}+i\boldsymbol{px}}. \tag{7.17}$$

Notice the following useful properties:

$$\Sigma(\boldsymbol{q},\boldsymbol{p},\tilde{k}_{\boldsymbol{i}}(\omega)) = \Sigma^{\dagger}(\boldsymbol{p},\boldsymbol{q},\tilde{k}_{\boldsymbol{i}}(\omega)) \qquad \Delta_{\psi}(x,y) = \Delta^{\dagger}_{\psi}(y,x). \tag{7.18}$$

Also note that in the case that $M_{\boldsymbol{pq}} = \delta_{\boldsymbol{pq}}$, i.e. when there are no overlaps, the propagator neatly collapses to the usual Weyl propagator with discretized **3**-momenta, but continuous energy, ω.

There are now two ways of calculating the correlation function $\langle 0|T\psi^{\dagger}(x)\psi(y)|0\rangle$ from the path integral; we can read it off directly from the above or we can equivalently use the

rules of functional calculus to get:

$$\langle T\hat{\psi}^\dagger_\alpha(x)\hat{\psi}_{\alpha'}(y)\rangle = \frac{-1}{i^2}\frac{\delta}{\delta h_\alpha(x)}\frac{\delta}{\delta h^\dagger_{\alpha'}(y)} Z_0[h^\dagger,h]\Big|_{h,h^\dagger=0} \tag{7.19}$$

$$= \int \frac{d\omega}{2\pi}\sum_{\boldsymbol{i}\gamma}\frac{\delta\tilde{h}_{\gamma\boldsymbol{i}}}{\delta h_\alpha(x)}\frac{\delta}{\delta\tilde{h}_{\gamma\boldsymbol{i}}}\int\frac{d\tilde{\omega}}{2\pi}\sum_{\boldsymbol{j}\gamma'}\frac{\delta\tilde{h}^\dagger_{\gamma'\boldsymbol{j}}}{\delta h^\dagger_{\alpha'}(y)}\frac{\delta}{\delta h^\dagger_{\gamma'\boldsymbol{j}}}Z_0\Big|_{h,h^\dagger=0} \tag{7.20}$$

$$= \int \frac{d\omega}{2\pi}\frac{d\tilde{\omega}}{2\pi}\sum_{\boldsymbol{i}\gamma\boldsymbol{j}\gamma'}\underbrace{\frac{\delta\tilde{h}_{\gamma\boldsymbol{i}}}{\delta h_\alpha(x)}}_{(\mathcal{R}_{\boldsymbol{i}}(x))_{\gamma\alpha}}\underbrace{\frac{\delta\tilde{h}^\dagger_{\gamma'\boldsymbol{j}}}{\delta h^\dagger_{\alpha'}(y)}}_{(\mathcal{R}^\dagger_{\boldsymbol{j}}(y))_{\alpha'\gamma'}}\underbrace{\frac{\delta}{\delta\tilde{h}_{\gamma\boldsymbol{i}}}\frac{\delta}{\delta h^\dagger_{\gamma'\boldsymbol{j}}}Z_0\Big|_{h,h^\dagger=0}}_{\delta_{\boldsymbol{ji}}\frac{\delta(\omega-\omega')}{2\pi}(\Delta_\beta(\tilde{k}_{\boldsymbol{i}}))_{\gamma'\gamma}} \tag{7.21}$$

$$= (\Delta_\psi(y,x))_{\alpha'\alpha}. \tag{7.22}$$

7.3 Testing the Path Integral Result

One can then compare the results of this chapter to what one would get by calculating the same 2 point function using the creation/annihilation mode expansion of $\hat{\psi}(x)$. To do so, first, we need to compute the following integral:

$$I = \int\frac{d\omega}{2\pi}\frac{(|\tilde{\mathbf{k}}_{\boldsymbol{i}}|-\omega)u(\boldsymbol{p})u^\dagger(\boldsymbol{q})-(|\tilde{\mathbf{k}}_{\boldsymbol{i}}|+\omega)u(-\boldsymbol{p})u^\dagger(-\boldsymbol{q})}{\omega^2-|\tilde{\mathbf{k}}_{\boldsymbol{i}}|^2+i\epsilon}e^{i\omega(y^0-x^0)}. \tag{7.23}$$

For this computation we can use contour integration and the residue theorem, we will choose the "upper" contour that goes towards $i\infty$ and encloses the pole at $\omega\approx-|\tilde{\mathbf{k}}_{\boldsymbol{i}}|$ *if* $y^0-x^0>0$, and we will choose the "lower" contour that goes towards $-i\infty$ and encloses the pole at $\omega\approx|\tilde{\mathbf{k}}_{\boldsymbol{i}}|$ in the case that $y^0-x^0<0$. Note that this latter pole goes in the mathematical negative direction. Using theta functions to organize the cases, the end result is:

$$I=\theta(y^0-x^0)u(\boldsymbol{p})u^\dagger(\boldsymbol{q})e^{-i|\tilde{\mathbf{k}}_s|(y^0-x^0)}+\theta(x^0-y^0)u(-\boldsymbol{p})u^\dagger(-\boldsymbol{q})e^{i|\tilde{\mathbf{k}}_s|(y^0-x^0)}. \tag{7.24}$$

Next, note that $|\tilde{\mathbf{k}}_i|=k_s\lambda_i n_s/N_s$ and that $\sum_{\boldsymbol{ij}}R^\dagger_{\boldsymbol{pi}}\lambda_{\boldsymbol{i}}\delta_{\boldsymbol{ij}}R_{\boldsymbol{jq}}=(F^\dagger F)_{\boldsymbol{pq}}$ thus:

$$\sum_{\boldsymbol{nmi}}F_{\boldsymbol{pn}}R^\dagger_{\boldsymbol{ni}}e^{i|\tilde{\mathbf{k}}_{\boldsymbol{i}}|(x^0-y^0)}R_{\boldsymbol{im}}F^\dagger_{\boldsymbol{mq}}=\sum_{\boldsymbol{nm}}F_{\boldsymbol{pn}}\left(e^{ik_sF^\dagger F\frac{n_s}{N_s}(x^0-y^0)}\right)_{\boldsymbol{nm}}F^\dagger_{\boldsymbol{mq}}. \tag{7.25}$$

Finally, we shift all matrix multiplications in the polynomial expansion of the exponential function and write $FF^\dagger=M$ to get a familiar expression (Equation 5.21):

$$\sum_{\boldsymbol{nm}}F_{\boldsymbol{pn}}\left(e^{ik_sF^\dagger F\frac{n_s}{N_s}(x^0-y^0)}\right)_{\boldsymbol{nm}}F^\dagger_{\boldsymbol{mq}}=\left(Me^{ik_sM\frac{n_s}{N_s}(x^0-y^0)}\right)_{\boldsymbol{pq}} \tag{7.26}$$

$$=\langle 0|\hat{c}_{\boldsymbol{p}}e^{i\hat{H}(x^0-y^0)}\hat{c}^\dagger_{\boldsymbol{q}}|0\rangle. \tag{7.27}$$

Thus putting everything together we get for $y^0 - x^0 > 0$:

$$\langle 0|T\hat{\psi}^\dagger_\alpha(x)\hat{\psi}_\beta(y)|0\rangle = (\Delta_\psi(y,x))_{\beta\alpha} = \sum_{\boldsymbol{pq}} \frac{-u_\beta(\boldsymbol{p})u^\dagger_\alpha(\boldsymbol{q})}{V\sqrt{|\boldsymbol{p}||\boldsymbol{q}|}} e^{-i\boldsymbol{qx}+i\boldsymbol{py}} \langle 0|\hat{c}_{\boldsymbol{p}} e^{i\hat{H}(x^0-y^0)}\hat{c}^\dagger_{\boldsymbol{q}}|0\rangle. \tag{7.28}$$

We can also plug in the Fourier mode expansion into the left hand side of this equation to get (ignoring the operators that annihilate the vacuum):

$$\langle 0|T\hat{\psi}^\dagger_\alpha(x)\hat{\psi}_\beta(y)|0\rangle \overset{y^0\geq x^0}{=} -\langle 0|\hat{\psi}_\beta(y)\hat{\psi}^\dagger_\alpha(x)|0\rangle = \sum_{\boldsymbol{pq}} \frac{-u_\beta(\boldsymbol{p})u^\dagger_\alpha(\boldsymbol{q})}{V\sqrt{|\boldsymbol{p}\ ||\boldsymbol{q}|}} e^{-i\boldsymbol{qx}+i\boldsymbol{py}} \langle 0|\hat{c}_{\boldsymbol{p}}(y^0)\hat{c}^\dagger_{\boldsymbol{q}}(x^0)|0\rangle. \tag{7.29}$$

It turns out that these two ways of computing the correlator will match if the field creation and annihilation operators evolve according to the free field Hamiltonian, i.e. $\hat{c}^\dagger_{\boldsymbol{p}}(t) = e^{i\hat{H}t}\hat{c}^\dagger_{\boldsymbol{p}}$.

Chapter 8

The LSZ Reduction Formula and The Feynman Rules

In typical QFT the connection between n-point functions, where the Feynman diagrams come from (Equation 6.66), and scattering amplitudes is given by the LSZ-formula [47]. Note that if we use a propagator in this chapter, we will use the simplified form (Equation 7.17) to simplify expressions. The downside to this is that we assume that the eigenvalues of M are all comparably close to each. Another assumption we will be making in this chapter is that τ, the time it takes for the effects of the overlaps to become significant for the propagation of a single particle, is very large.

8.1 The Lehmann–Symanzik–Zimmermann (LSZ) Reduction Formula

We have showed that up to some time scale τ the field behaves like a normal quantum field. We use this fact to define the asymptotic field, which is close to a solution of the Schrödinger equation with a free overlapped field Hamiltonian:

$$\psi_{\text{free}}(x) = \int \frac{d^3\boldsymbol{k}}{\sqrt{|\boldsymbol{k}|V}} u(\boldsymbol{k})\hat{c}_{\boldsymbol{k}} e^{i(k_s(\boldsymbol{k}))^\mu x_\mu} + u(\boldsymbol{k})\hat{d}^\dagger_{\boldsymbol{k}} e^{-i(k_s(\boldsymbol{k}))^\mu x_\mu}, \tag{8.1}$$

where we have defined $(k_s(\boldsymbol{k}))^\mu = (k_{s(\boldsymbol{k})}, \boldsymbol{k})^\mu$ and $k_{s(\boldsymbol{k})} = k_{s'}$ for $\boldsymbol{k} \in s'$. We can also take an inverse fourier transform of this to get:

$$\hat{c}^\dagger_{\boldsymbol{k}}(t) = \int \frac{d^3\boldsymbol{x}}{\sqrt{V|\boldsymbol{k}|}} e^{i(k_s(\boldsymbol{k}))^\mu x_\mu} \psi^\dagger_{\text{free}}(x) u(\boldsymbol{k}). \tag{8.2}$$

We can then take the limit where this approximation is assumed to hold:

$$\hat{c}^\dagger_{\boldsymbol{k}_{\text{in}}} = \lim_{t\to-\tau} \hat{c}^\dagger_{\boldsymbol{k}}(t) \qquad \hat{c}^\dagger_{\boldsymbol{k}_{\text{out}}} = \lim_{t\to\tau} \hat{c}^\dagger_{\boldsymbol{k}}(t). \tag{8.3}$$

We want to look at scattering matrix element $\langle f|i\rangle$ of n incoming particles and n' outgoing particles. We will assume that at the time scales probed here, $-\tau \ll t \ll \tau$, the direct effects of the overlaps can be neglected; i.e. $\{\hat{c}^\dagger_{\boldsymbol{p}_{\text{out}}}, \hat{c}_{\boldsymbol{p}'_{\text{out}}}\}$ are assumed to vanish. This is standard in scattering experiments, as it is assumed that there are no interactions when the particles are away from each other, like at the start or end of the scattering experiment (c.f. discussion on wave packets in sub-section 5.2.2). While we won't be interested in the direct effect of the overlaps, they may still have indirect influence when calculating S-Matrices by modifying other local interactions. As also discussed in sub-section 5.2.1, in general scattering experiments are ill designed to probe long-range interactions such as these overlaps.

$$\langle f|i\rangle = \langle \boldsymbol{p}'_1, \boldsymbol{p}'_2, \dots \boldsymbol{p}'_{n'}|\boldsymbol{p}, \boldsymbol{p}_1, \boldsymbol{p}_2, \dots \boldsymbol{p}_n\rangle \tag{8.4}$$

$$= \langle \boldsymbol{p}'_1, \boldsymbol{p}'_2, \dots \boldsymbol{p}'_{n'}|\hat{c}^\dagger_{\boldsymbol{p}_{\text{in}}} - \hat{c}^\dagger_{\boldsymbol{p}_{\text{out}}}|\boldsymbol{p}_1, \boldsymbol{p}_2, \dots \boldsymbol{p}_n\rangle \tag{8.5}$$

$$= \frac{1}{\sqrt{\mathcal{Z}}}\int \frac{d^3\boldsymbol{x}}{\sqrt{V|\boldsymbol{p}|}} e^{i(k_s(\boldsymbol{p}))^\mu x_\mu}\langle \alpha'|\left(\psi^\dagger_{\text{in}}(x) - \psi^\dagger_{\text{out}}(x)\right)u(\boldsymbol{p})|\alpha\rangle, \tag{8.6}$$

where we have defined $\langle \boldsymbol{p}'_1, \boldsymbol{p}'_2, \dots \boldsymbol{p}'_{n'}| = \langle \alpha'|$ and $|\boldsymbol{p}_1, \boldsymbol{p}_2, \dots \boldsymbol{p}_n\rangle = |\alpha\rangle$ for briefness. Because of our assumptions, we can anti-commute $\hat{c}^\dagger_{\boldsymbol{p}_{\text{out}}}$ with all the operators in $\langle \alpha'|$ until we reach and annihilate the vacuum. We have also assumed that:

$$\lim_{t\to\tau}\int \frac{d^3\boldsymbol{x}}{\sqrt{V|\boldsymbol{p}|}} e^{i(k_s(\boldsymbol{p}))^\mu x_\mu}\langle \alpha'|\psi^\dagger(x)u(\boldsymbol{p})|\alpha\rangle \tag{8.7}$$

$$\approx \sqrt{\mathcal{Z}}\int \frac{d^3\boldsymbol{x}}{\sqrt{V|\boldsymbol{p}|}} e^{i(k_s(\boldsymbol{p}))^\mu x_\mu}\langle \alpha'|\psi^\dagger_{\text{free}}(x)u(\boldsymbol{p})|\alpha\rangle, \tag{8.8}$$

and we have a similar thing for the "in" field. With these assumptions we have:

$$\langle f|i\rangle = \left(\lim_{t\to-\tau} - \lim_{t\to\tau}\right)\frac{1}{\sqrt{\mathcal{Z}}}\int \frac{d^3\boldsymbol{x}}{\sqrt{V|\boldsymbol{p}|}} e^{i(k_s(\boldsymbol{p}))^\mu x_\mu}\langle \alpha'|\psi^\dagger u(\boldsymbol{p})|\alpha\rangle \tag{8.9}$$

$$= -\int_{-\tau}^{\tau} dt \frac{1}{\sqrt{\mathcal{Z}}}\partial_0 \int \frac{d^3\boldsymbol{x}}{\sqrt{V|\boldsymbol{p}|}} e^{i(k_s(\boldsymbol{p}))^\mu x_\mu}\langle \alpha'|\psi^\dagger u(\boldsymbol{p})|\alpha\rangle \tag{8.10}$$

$$= -\int \frac{d^4x}{\sqrt{V\mathcal{Z}|\boldsymbol{p}|}} e^{i(k_s(\boldsymbol{p}))^\mu x_\mu}\langle \alpha'|\left(ik_{s(\boldsymbol{p})}\psi^\dagger + \partial_0\psi^\dagger\right)u(\boldsymbol{p})|\alpha\rangle \tag{8.11}$$

$$= \int \frac{d^4x}{\sqrt{\mathcal{Z}V|\boldsymbol{p}|}} e^{i(k_s(\boldsymbol{p}))^\mu x_\mu}\langle \alpha'|\left(-\psi^\dagger i\sigma^i \boldsymbol{p}_i \frac{k_{s(\boldsymbol{p})}}{|\boldsymbol{p}|} - \partial_0\psi^\dagger\right)u(\boldsymbol{p})|\alpha\rangle \tag{8.12}$$

$$= \int \frac{d^4x}{\sqrt{\mathcal{Z}V|\boldsymbol{p}|}} e^{i(k_s(\boldsymbol{p}))^\mu x_\mu}\langle \alpha'|\left(\sigma^i \frac{\overleftarrow{\partial_i}}{|\boldsymbol{p}|} k_{s(\boldsymbol{p})}\psi^\dagger - \partial_0\psi^\dagger\right)u(\boldsymbol{p})|\alpha\rangle \tag{8.13}$$

$$= \frac{i}{\sqrt{\mathcal{Z}}}\int \frac{d^4x}{\sqrt{V|\boldsymbol{p}|}} e^{i(k_s(\boldsymbol{p}))^\mu x_\mu}(i\sigma^\mu \partial^{\boldsymbol{p}}_\mu \langle \alpha'|\psi^\dagger(x)|\alpha\rangle)u(\boldsymbol{p}), \tag{8.14}$$

where the fundamental theorem of calculus was used in the second step, and then we used the product rule on the third step. On the fourth step we moved the minus sign inside the integral and used the eigenvalue equation for $u(\boldsymbol{p})$ (Equation 3.9). On the last step we used partial integration and ignored boundary terms, we also defined $\partial_\mu^{\boldsymbol{p}} = (\partial_0, -(k_{s(\boldsymbol{p})}/|\boldsymbol{p}|)\partial_i)^T$. We can repeat this process, now defining $\langle\alpha''|\hat{c}_{\boldsymbol{p}_{\text{out}}} = \langle\alpha'|$ with a $\hat{c}_{\boldsymbol{p}_{\text{out}}}$ in $\langle\alpha'|$ to get:

$$\begin{aligned}
\langle\alpha'|\psi^\dagger(x)|\alpha\rangle &= \langle\alpha''|\hat{c}_{\boldsymbol{p_1}_{\text{out}}}\psi^\dagger(x) - \psi^\dagger(x)\hat{c}_{\boldsymbol{p_1}_{\text{in}}}|\alpha\rangle && (8.15)\\
&= \left(\lim_{y^0\to-\tau} - \lim_{y^0\to\tau}\right)\frac{1}{\sqrt{\mathcal{Z}}}\int\frac{d^3\boldsymbol{y}}{\sqrt{V|\boldsymbol{p}_1|}}e^{-i(k_s(\boldsymbol{p}_1))^\mu y_\mu}\times && (8.16)\\
&\quad \times u^\dagger(\boldsymbol{p}_1)\langle\alpha''|T\hat{\psi}(y)\hat{\psi}^\dagger(x)|\alpha\rangle, && (8.17)
\end{aligned}$$

and continue as before. The final result is:

$$\langle f|i\rangle = \int\left[\prod_{i=1}\frac{i\sigma^\mu\partial_{y_{i_\mu}}^{\boldsymbol{p}_i'}u(\boldsymbol{p}_i')e^{i(k_{s(\boldsymbol{p}_i)})^\mu y_{i_\mu}}d^4y_i}{\sqrt{\mathcal{Z}V|\boldsymbol{p}_i'|}}\right]^\dagger\mathcal{T}(x_1,\dots)\left[\prod_{i=1}\frac{i\sigma^\mu\overleftarrow{\partial_{x_{i_\mu}}^{\boldsymbol{p}_i}}u(\boldsymbol{p}_i)e^{i(k_{s(\boldsymbol{p}_i)})^\mu x_{i_\mu}}d^4x_i}{\sqrt{\mathcal{Z}V|\boldsymbol{p}_i|}}\right], \tag{8.18}$$

with:

$$\mathcal{T}(x_1,x_2\dots x_n,y_1,y_2\dots y_{n'}) = \langle 0|T\hat{\psi}(x_1)\dots\hat{\psi}(x_n)\hat{\psi}^\dagger(y_1)\dots\hat{\psi}^\dagger(y_{n'})|0\rangle. \tag{8.19}$$

Noting that the † changes the ordering of the operators, such that the derivative on the left act only on the $\mathcal{T}$ function, like the one in the right. The above is very similar to the usual LSZ formula for fermions as given in [67], with the only necessary replacement being:

$$\partial_\mu \to \partial_\mu^{\boldsymbol{p}} = \left(\partial_0, -\frac{k_{s(\boldsymbol{p})}}{|\boldsymbol{p}|}\partial_i\right)^T. \tag{8.20}$$

8.2 The Feynman Rules

8.2.1 Preliminaries

Recall that the n-point correlation functions in the LSZ formula, $\langle 0|T\psi(x_1)\dots\psi^\dagger(y_1)\dots|0\rangle$, can be given as a sum of propagators, all multiplying each other (Equation 6.66). So want to know what happens when we multiply the propagators by the factors given in the LSZ formula:

$$\int d^4y\ \Delta_\psi(x,y)\frac{i\sigma^\mu\overleftarrow{\partial_\mu^{\boldsymbol{p}'}}u(\boldsymbol{p}')e^{-(ik_s(\boldsymbol{p}'))^\mu y_\mu}}{\sqrt{V|\boldsymbol{p}'|}} =$$

$$\overset{(7.15)}{=}\int\frac{-d\omega d^4y}{2\pi V^{3/2}\sqrt{|\boldsymbol{p}'|}}\sum_s\sum_{\boldsymbol{p},\boldsymbol{q}\in s}\frac{\Sigma(\boldsymbol{q},\boldsymbol{p})i(\omega+\sigma^i k_s\boldsymbol{q}_i/|\boldsymbol{p}'|)u(\boldsymbol{p}')}{\omega^2-k_s^2+i\epsilon}\times$$

$$\times M_{\boldsymbol{pq}}e^{i\omega x^0-i(\omega+k_s)y^0-i(\boldsymbol{q}-\boldsymbol{p}')\boldsymbol{y}+i\boldsymbol{px}} \tag{8.21}$$

$$=\int\frac{-d\omega}{\sqrt{V|\boldsymbol{p}'|}}\sum_s\sum_{\boldsymbol{q},\boldsymbol{p}\in s}\frac{\Sigma(\boldsymbol{q},\boldsymbol{p})(\omega+\sigma^i k_s\boldsymbol{q}_i/|\boldsymbol{p}'|)u(\boldsymbol{p}')}{\omega^2-k_s^2+i\epsilon}M_{\boldsymbol{pq}}e^{i\omega x^0+i\boldsymbol{px}}\delta(\omega+k_s)\delta_{-\boldsymbol{qp}'} \tag{8.22}$$

$$=-\int\frac{d\omega}{\sqrt{V|\boldsymbol{p}'|}}\sum_s\sum_{\boldsymbol{p}\in s}\frac{u(\boldsymbol{p})u^\dagger(\boldsymbol{p}')}{\sqrt{|\boldsymbol{p}||\boldsymbol{p}'|}}\frac{i(\omega+k_s)u(\boldsymbol{p}')}{\omega+k_s+i\epsilon}M_{\boldsymbol{pp}'}e^{i\omega x^0+i\boldsymbol{px}}\delta(\omega+k_s) \tag{8.23}$$

$$=-\frac{1}{\sqrt{V}}\sum_{\boldsymbol{p}\in s}\frac{u(\boldsymbol{p})}{\sqrt{|\boldsymbol{p}|}}M_{\boldsymbol{pp}'}e^{i\omega x^0+i\boldsymbol{px}}, \tag{8.24}$$

where in the third step we had:

$$\frac{\delta(\omega+k_s)}{\omega^2-k_s^2}\Sigma(\boldsymbol{q},\boldsymbol{p})=\frac{\delta(\omega+k_s)}{\omega^2-k_s^2}(k_s-\omega)\frac{u(\boldsymbol{p})u^\dagger(\boldsymbol{p}')}{\sqrt{|\boldsymbol{p}||\boldsymbol{p}'|}}-\frac{\delta(\omega+k_s)}{\omega^2-k_s^2}(k_s+\omega)\frac{u(-\boldsymbol{p})u^\dagger(-\boldsymbol{p}')}{\sqrt{|\boldsymbol{p}||\boldsymbol{p}'|}}, \tag{8.25}$$

of which the second term reduces to 0. Then, the terms on the right hand side of the LSZ formula have the interpretation of fixing the incoming particles's momenta and energy to the specified values. They have to be "on shell", meaning that if they have momentum $\boldsymbol{p}\in s$ then they have energy k_s of the corresponding shell. Given Equation 7.18 and the fact that the factors on the LSZ formula are the adjoint of those of the right, we can surmise that those factors will fix the momenta and energy of the outgoing particles in a similar manner. Next let's look at two propagators multiplied together:

$$\Delta(x,y)\Delta(y,z)=\int\frac{d\omega' d\omega}{(2\pi V)^2}\sum_{s,s'}\sum_{\substack{\boldsymbol{p},\boldsymbol{q}\in s\\ \boldsymbol{p}',\boldsymbol{q}'\in s}}\frac{\Sigma(\boldsymbol{q},\boldsymbol{p},k_s)}{\omega^2-k_s^2+i\epsilon}M_{\boldsymbol{pq}}e^{i\omega x^0+i(\omega'-\omega)y^0+i(\boldsymbol{p}'-\boldsymbol{q})\boldsymbol{y}+i\boldsymbol{px}}\times \tag{8.26}$$

$$\times\frac{\Sigma(\boldsymbol{q}',\boldsymbol{p}',k_{s'})}{\omega'^2-k_{s'}^2+i\epsilon}M_{\boldsymbol{p}'\boldsymbol{q}'}e^{-i\omega' z^0-i\boldsymbol{q}'\boldsymbol{z}}. \tag{8.27}$$

Integrating over y, then, will produce a delta function:

$$\int d^4y\Delta(x,y)\Delta(y,z)=\int\frac{d\omega}{2\pi V}\sum_s\sum_{\boldsymbol{p},\boldsymbol{q}\in s}\frac{\Sigma(\boldsymbol{q},\boldsymbol{p},k_s)}{\omega^2-k_s^2+i\epsilon}M_{\boldsymbol{pq}}e^{i\omega x^0+i\boldsymbol{px}}\times \tag{8.28}$$

$$\times\sum_{q'\in s}\frac{\Sigma(\boldsymbol{q}',\boldsymbol{q},k_{s'})}{\omega^2-k_s^2+i\epsilon}M_{\boldsymbol{qq}'}e^{-i\omega z^0-i\boldsymbol{q}'\boldsymbol{z}} \tag{8.29}$$

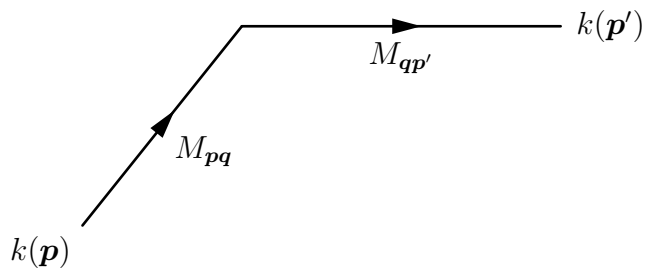
If there are more propagators leading to the same point y we can see that the *4-momentum will be conserved at the vertex*, even if the **3**-momentum a particle has at one end of the propagator need not be the same as at the other end, although note that the frequency, ω, will be. Putting everything together we may calculate:

$$\frac{1}{\mathcal{Z}}\int d^4x d^4y d^4z \left[\frac{i\sigma^\mu \partial^{\boldsymbol{p}}_{x_\mu} u(\boldsymbol{p}) e^{(ik_s(\boldsymbol{p}))^\mu x_\mu}}{\sqrt{V|\boldsymbol{p}|}}\right]^\dagger \Delta_\psi(x,y)\Delta_\psi(y,z)\frac{i\sigma^\mu \overleftarrow{\partial}^{\boldsymbol{p}}_{z_\mu} u(\boldsymbol{p}') e^{-(ik_s(\boldsymbol{p}'))^\mu z_\mu}}{\sqrt{V|\boldsymbol{p}'|}} \tag{8.30}$$

$$\overset{(8.21)}{=}\frac{1}{\mathcal{Z}}\int \frac{d^4y}{V}\left[\sum_{\boldsymbol{q}\in s}\frac{u(\boldsymbol{q})}{\sqrt{|\boldsymbol{q}|}}M_{\boldsymbol{pq}}e^{i\omega y^0+i\boldsymbol{qy}}\right]^\dagger \sum_{\boldsymbol{q}'\in s}\frac{u(\boldsymbol{q}')}{\sqrt{|\boldsymbol{q}'|}}M_{\boldsymbol{q'p'}}e^{i\omega y^0+i\boldsymbol{q'y}} \tag{8.31}$$

$$=\frac{2\pi}{\mathcal{Z}}\sum_{q\in s}M_{\boldsymbol{pq}}M_{\boldsymbol{qp'}}, \tag{8.32}$$

which we may organize in a diagram as:



Note, however, that this not a diagram that would appear in the series given by the LSZ formula. As stated then, deflection caused by the overlaps are assumed to be too small at the timescales of $-\tau < t < \tau$. In terms of the action, we would need an interaction in order to have a vertex that joins two propagators. Still we can take this as an example to start to recognize the Feynman rules for this new theory.

8.2.2 Overlapped Feynman Rules

The calculations of this chapter suggest only 2 changes to the Feynman rules caused by the overlapping of degrees of freedom:

- New propagator for the overlapped fields, Equation 7.15, in which the starting **3**-momentum may be different from the ending **3**-momentum, but energy is conserved.
- External legs of diagrams come with frequency $\omega \approx k_s$ of the respective momentum shell (instead of $E = |\boldsymbol{p}|$); so that the "*4* momentum" of that leg reads $k(\boldsymbol{p}) = (k_s, \boldsymbol{p})$ for $\boldsymbol{p} \in s$. Note that this is not the real *4* momentum of the particle, as it's not an Eigenstate of the actual *4* momentum.

Chapter 9

Conclusions and Outlook

Standard quantum field theory can be split up into two main parts; quantum systems, that is systems that evolve according to quantum mechanics, and their spatial distribution. This is the view that a quantum field is nothing more than a collection of coupled quantum harmonic oscillators [81], each existing in a point in space, and each coupling to their nearest neighbors. We have seen how we can change this second component only, the spatial distribution of these oscillators, how many there are, and how they couple to each other, in order to vastly downsize the amount of them needed and bring this amount to agreement with the holographic principle. From this perspective, the overlaps introduced in [32] are the natural choice to get a quantum field to exhibit holography as described by the holographic principle.

This partial tuning of QFT, has meant that a surprising amount of tools and constructions of QFT can still be applied. We have seen how one can define particle- and multi-particle-states in a very similar way to what is standard practice in QFT. Even more surprising is the fact that these particle and multi-particle states behave in a comparable way to their un-overlapped counterparts. To add to this we have demonstrated that one can carry over this construction to the path integral formalism, and in this formalism, we have been able to solve the theory using the same techniques and procedures as in normal QFT. The reward for this effort is the option to quickly bring perturbative QFTs with interactions to this overlapped framework via the changes to the Feynman rules as laid out at the end of the chapter 8

At the same time, however, we should recognize that the original motivation for exploring quantum fields was to get a Lorentz invariant quantum theory, and Lorentz invariance is broken in several ways by the overlapping procedure. Although, even before introducing the overlaps, we require a UV cutoff or regularization in order to count the amount of degrees of freedom in the quantum field in the first place. This, too, breaks Lorentz invariance. To add to this, Lorentz symmetry is used extensively in normal QFT. For instance, it is used in the typical derivation of the LSZ formula, and it is also used to justify the eigenstates of the standard QFT being particle states, as these have all the right symmetries. This is something the particle states presented here fail to do.

Given all of the examples laid out in this work, one could jump to the conclusion that to even formulate the holographic principle we need to brake Lorentz invariance, putting into question the hope that we can use both of these principles to figure out which degrees we should quantize with 1.1. To definitively conclude that these principles are incompatible, however, is beyond the scope of this work. One could argue that this tension was to be expected; the holographic principle was discovered exploring space-times that are very different from minkowski space. Taking this and the —already mentioned— strong relation between typical, un-overlapped, QFT and minkowski symmetry into account; it is right to question the validity of normal, un-overlapped QFT when exploring more general, perhaps quantum-dynamic, space-times.

To the end described above, The overlaps presented in this work can represent a vast class of models, if we take the overlapping matrix, F, to be a general matrix. Among these models we can even get back standard QFT for $F = \mathbf{1}$. In essence, each of these models corresponds to a different vector space $\mathcal{H}_V$, as described in chapters 3 and 4 with different couplings between neighbors as defined by the Hamiltonian or Lagrangian. Intriguingly, this also has the interpretation of different classical "field" theories that are later quantized by 1.1, or its anti-commutator equivalent, this was first recognized in [32]. Regardless of how one comes to this vast space of models, there is a lot of directions to next take this line of research once here.

9.1 Outlook

As a first step, we can try to improve upon the choice made by [32] of having the column vectors of F come from a random sample from the unit n-sphere. For instance, we can find the matrix that best reduces the overlap between different fourier modes, while keeping F rank deficient. This will likely mean that the resulting theory behaves even more like standard QFT, and where the non-localities would be even less pronounced. Another potential area of improvement is some dependence on the momenta themselves; like having $F_{\boldsymbol{pq}}$ be a function of $\boldsymbol{p}\cdot\boldsymbol{q}$, potentially so that the overlap is large only if the angle between $\boldsymbol{p}$ and $\boldsymbol{q}$ is small. In general, we can be guided to better choices of F by following reasonable symmetries. Some form of locality would be ideal and so would the isotropy of space-time. This approach of letting symmetries guide what degrees of freedom are overlapped would be closely linked to the research program of [53].

If we decide to explore F matrices that give us theories with more symmetry, it makes sense to attempt to get close to Lorentz symmetry. In that task, we should move beyond the non-relativistic version of the holographic principle presented in the introduction to the covariant entropy bound as given in [12]. Perhaps this would involve overlapping not only different fourier modes, **3**-momenta, but also $\not{4}$-momenta, at least for a massive fields. However, it's not at all clear at this stage if this is even possible in anything resembling quantum mechanics with unitary time evolution. Furthermore, we have seen how the present theory doesn't conserve momentum, and it superficially seems that a theory of overlapping $\not{4}$-momenta won't conserve energy either.

As previously mentioned, the results of chapter 8, provide us with the opportunity to explore theories with overlaps *and* interactions. It is, then, a logical next step to use the outlined changes to the Feynman rules to effectively overlap well proven theories like quantum electrodynamics and quantum chromo-dynamics and see if or how the overlapping procedure changes observable quantities. For instance, one can calculate quantum corrections to the magnetic moments of particles. As already discussed, these experiments constitute some of the most precise measurements in the history of physics, and that may result in strong constraints on the overlapping matrices.

We can already get a qualitative idea of how these overlaps will change the value of scattering matrices. The changes to the Feynman rules indicate that every time two propagators meet at a vertex, we need to sum up all the momenta in the shell. This can be interpreted to be the result of the non-local nature of the overlapped fields. In terms of scattering amplitudes, it means that the vertex will get a contribution depending on the amount of fourier modes in the shell. This would, in turn, easily be absorbed as a contribution to the bare coupling constant. Thus, if there are different amount of momenta in the different shells, the coupling constant will change in accordance to the shell involved. This is very intriguing, because *re-normalization predicts exactly this kind of behavior*.

In general the relationship between the re-normalization group and the overlaps is worth looking into, as re-normalization always plays a big role in perturbative QFT, and the S-matrices calculated with it. In this context of perturbative QFT, the overlapping of degrees of freedom may be effectively the same thing as the coarse-graining of degrees of freedom of re-normalization. Just as we have overlapped fourier modes within each momentum shell in QFT, one can also coarse-grain momenta in QFT or condensed matter systems[80]. The details, of course will depend in the exact nature of the overlapping matrix, F, but at present, it is very likely that for some choices of F, the overlapping may be the same as the action of a member of the re-normalization group.

The idea that the renormalization group is a particular case of the overlapping procedure is an interesting direction to take this research into, particularly since the clearest challenge with perturbative canonical quantum gravity lies with the inability to re-normalize it. This naturally leads one to ask whether a version of the overlapping, some choice of F, motivated by the holographic principle could achieve what simple coarse-graining couldn't, and produce a UV-complete theory of quantum gravity. However, to explore this hypothesis we require a better understanding of the connection between the overlaps and the renormalization group.

There is also the possibility that renormalization is a completely different phenomenon, and it cannot be described via overlaps, but if that is the case another direction for future research open up: Can the contributions from overlaps account for the difference between theoretical values of the parameters of the standard model and the observed parameters? Some work in this vein was done when [32] showed that the overlaps brought the vacuum energy value closer to the observed value. With the tools developed in chapter 8 we can do the same for other parameters, such as the Higgs mass, or the coupling constants. This is assuming that the bare quantities have their origin in a UV-theory, like string theory or a grand unified theory.

Appendix A

Expectation Values for Minors of Determinants

To simplify things we assume that $\{\boldsymbol{p}_i\}_{i=1}^{i=m} \subset s$. The case where multiple shells are involved can be dealt with as in Equation 4.19. First we start from the definition of the determinant of a matrix, as a sum over all possible permutations of pairs of indices weighted by the sign of the permutation, and then we commute the sum with the expectation value:

$$\mathbb{E}\left[\det\left(M_{\mathbb{C}(\{\boldsymbol{p}_i\}_{i=1}^{i=m})^2}\right)\right] = \sum_{\pi\in\mathcal{S}(\{\boldsymbol{p}_i\}_{i=1}^{i=m})} \mathbb{E}\left[\operatorname{sign}(\pi)\prod_{j=1}^{m} M_{\boldsymbol{p}_j\boldsymbol{p}_{\pi(j)}}\right] \tag{A.1}$$

We will use the fact that any permutation lets itself be expressed as a product of disjoint cycles. We need to show that if permutation π has the disjoint cycle decomposition $\pi = \sigma_1\sigma_2\ldots\sigma_l$, if σ_i is a h_i-cycle, then we have:

$$\mathbb{E}\left[\operatorname{sign}(\pi)\prod_{j=1}^{m} M_{\boldsymbol{p}_j\boldsymbol{p}_{\pi(j)}}\right] \propto \frac{1}{n_s^{h_1-1}}\frac{1}{n_s^{h_2-1}}\cdots\frac{1}{n_s^{h_l-1}}. \tag{A.2}$$

Since the cycles are disjoint (meaning $\operatorname{Im}(\sigma_j)\cap\operatorname{Im}(\sigma_i) = \{\}$ for $i \neq j$), we can write the expectation value of the whole product as the product of the expectation value for each cycle:

$$\mathbb{E}\left[\prod_{j=1}^{m} M_{\boldsymbol{p}_j\boldsymbol{p}_{\pi(j)}}\right] = \mathbb{E}\left[\prod_{j\in\operatorname{Im}(\sigma_1)} M_{\boldsymbol{p}_j\boldsymbol{p}_{\sigma_1(j)}}\right]\mathbb{E}\left[\prod_{j\in\operatorname{Im}(\sigma_2)} M_{\boldsymbol{p}_j\boldsymbol{p}_{\sigma_2(j)}}\right]\ldots \tag{A.3}$$

Note that we have $h_i := |\operatorname{Im}(\sigma_i)|$. To calculate each factor, we write the matrix component $M_{\boldsymbol{p}_j\boldsymbol{p}_{\pi(j)}}$ as the dot product of vectors v_k and v_{k+1} with k going from 0 to h_i, each v_k having

n_s components, v_k^n. We further choose labeling $v_0 = v_h$, and order the product such that:

$$\prod_{j\in\mathrm{Im}(\sigma_i)} M_{\boldsymbol{p}_j\boldsymbol{p}_{\sigma_i(j)}} = \prod_{k=0}^{h_i-1}\left(\sum_{n=1}^{n_s} v_k^n v_{k+1}^n\right) \tag{A.4}$$

$$= \prod_{k=0}^{h_i-1}\left(\sum_{n_k=1}^{n_s} v_k^{n_k} v_{k+1}^{n_k}\right) \tag{A.5}$$

$$= \sum_{n_1=1}^{n_s}\sum_{n_2=1}^{n_s}\cdots\sum_{n_{h_i}=1}^{n_s}\prod_{k=0}^{h_i-1} v_k^{n_k} v_{k+1}^{n_k} \tag{A.6}$$

$$= \sum_{n_1=1}^{n_s}\sum_{n_2=1}^{n_s}\cdots\sum_{n_{h_i}=1}^{n_s}\prod_{k=1}^{h_i} v_k^{n_k} v_k^{n_{k-1}}. \tag{A.7}$$

$$\tag{A.8}$$

Noting that the expectation value of different v_ks is independent and thus the expectation value of the product is the product of the expectation values, calculating the expectation value of the above is straightforward using the results from appendix C:

$$\mathbb{E}\left[\prod_{j\in\mathrm{Im}(\sigma_i)} M_{\boldsymbol{p}_j\boldsymbol{p}_{\sigma_i(j)}}\right] = \sum_{n_1=1}^{n_s}\sum_{n_2=1}^{n_s}\cdots\sum_{n_{h_i}=1}^{n_s}\prod_{k=1}^{h_i}\mathbb{E}\left[v_k^{n_k} v_k^{n_{k-1}}\right] \tag{A.9}$$

$$= \sum_{n_1=1}^{n_s}\sum_{n_2=1}^{n_s}\cdots\sum_{n_{h_i}=1}^{n_s}\prod_{k=1}^{h_i}\frac{\delta_{n_k n_{k-1}}}{n_s} \tag{A.10}$$

$$= \left(\frac{1}{n_s}\right)^{h_i}\cdot n_s = \frac{1}{n_s^{h_i-1}}. \tag{A.11}$$

The above equation along with Equation A.3 gives Equation A.2. This means we may now express the sum in Equation A.1 as a power series over $1/n_s$. The leading order of said power series, must be given by the unique permutation that decomposes as only 1-cycles, the identity. The next order is given by the permutations that let themselves be written as a single transposition; There are "m choose 2" of said transpositions possible, and they all have negative parity, thus:

$$\sum_{\pi\in\mathcal{S}(\{\boldsymbol{p}_i\}_{i=1}^{i=m})}\mathbb{E}\left[\mathrm{sign}(\pi)\prod_{j=1}^{m} M_{\boldsymbol{p}_j\boldsymbol{p}_{\pi(j)}}\right] = 1 - \binom{m}{2}\frac{1}{n_s} + \mathcal{O}\left(\frac{1}{n_s^2}\right). \tag{A.12}$$

We proceed to test this equation with some numerical experiments, plotted in the following figures:

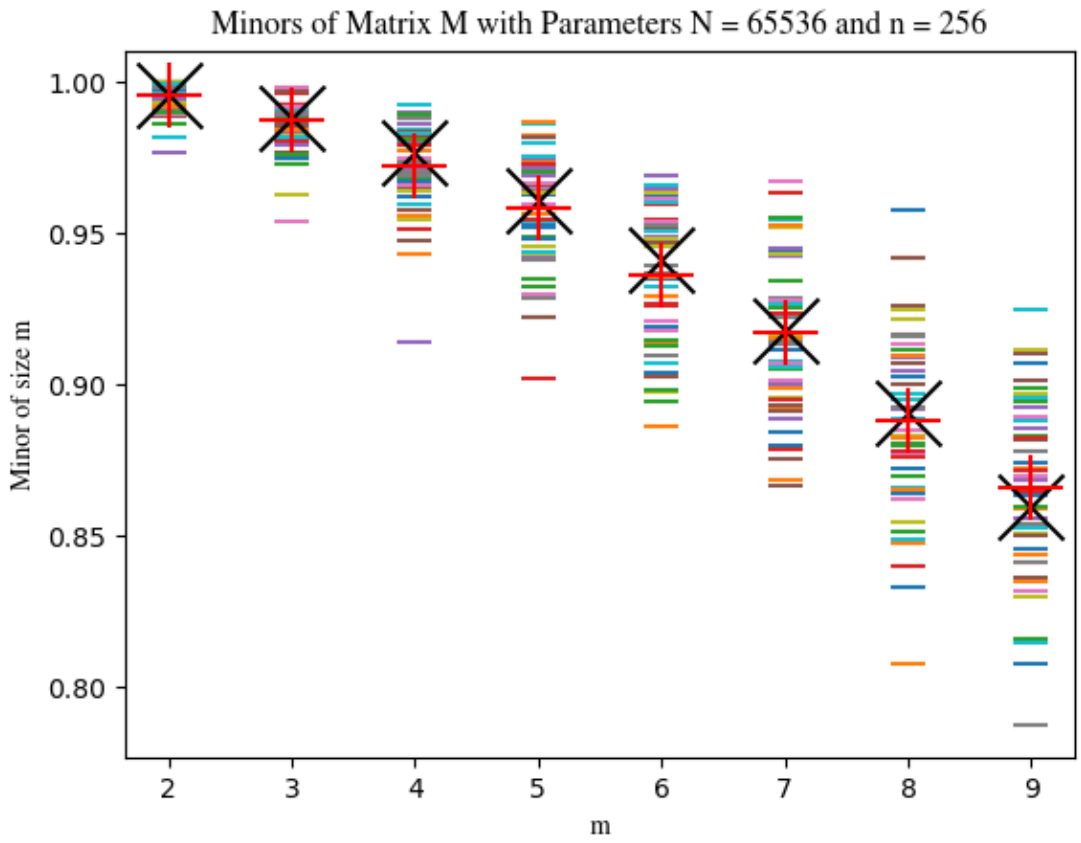


Figure A.1: Numerical simulation of the determinants of square sub-matrices of M (also called minors of M) of different sizes, m. The parameters of M are $n_s = 256$ and $N_s = 256^2$. Lines of different colors indicate different sub-matrices of the same size, here 100 are plotted. The red "+"s indicate the average of all determinants, or in other words, the numerically observed left hand side of Equation A.12. The black "X"s are the first two terms on the right hand side of Equation A.12.

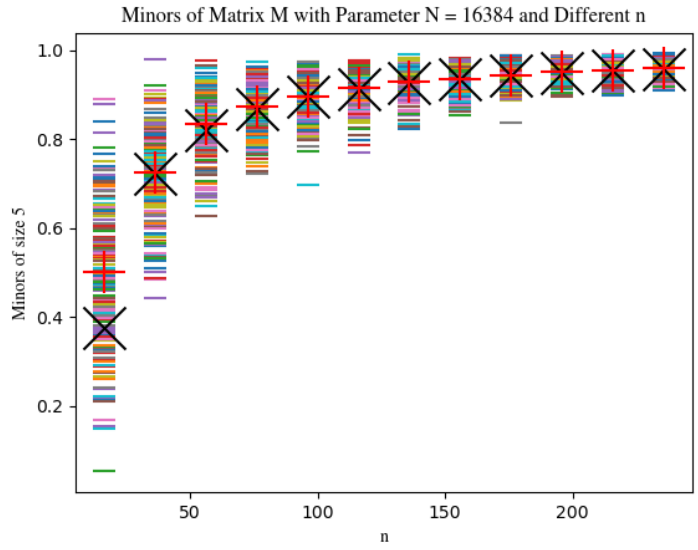


Figure A.2: Numerical simulation of the determinants of square sub-matrices of M (also called minors of M) of the same size, 5. The Parameter N_s is fixed at $N_s = 128^2$, but varying n_s. The markers are the same as figure A with the other difference being that here 500 different determinants are plotted

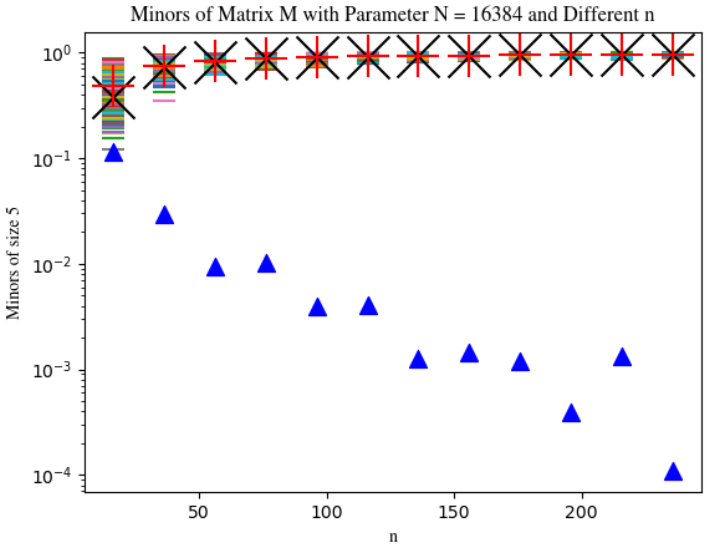


Figure A.3: The same determinants as A, now on a log scale. The blue triangles represent the difference between the first two terms on the right hand side Equation A.12 and the numerically observed left hand side. According to the same equation we expect this difference to be dominated by a term $\propto 1/n_s^2$, which looks like a line on a log scale.

Appendix B

Approximations for the Evolution Operator

To show that Equation 5.27 holds and to see how well it holds, we first will show that for $0 < n < N_s$, $n \in \mathbb{N}$ the following holds:

$$\mathbb{E}\left[(M^{n+1})_{\boldsymbol{pq}}\right] = \delta_{\boldsymbol{pq}} \left(\frac{N_s}{n_s}\right)^n + \frac{n(n+1)}{2} \sum_{k=0}^{n-1} \mathcal{O}\left(\left(\frac{N_s}{n_s}\right)^k n^{2(n-k-1)}\right). \tag{B.1}$$

However, this doesn't immediately imply Equation 5.27. The problem with the above expression is that n grows with the terms in the power series of the exponential of the evolution operator, and thus in a power series the second term may produce a large and growing divergence from the standard result. To make sure that this is not an issue we will show that truncating the power series for large, but not too large $n < N_s/n_s$ (given appropriately small time scales and low energies) does not result in a large deviation from the actual result. This means we need to calculate upper bounds for 2 deviations from the given expression; a deviation caused by truncating the power series and a deviation caused by the second term in the above equation, assuming said truncation.

B.1 Asymptotics for Moments of the Anti-commutators

We show Equation B.1 inductively: For $n = 1$:

$$\mathbb{E}\left[(M^2)_{\boldsymbol{pq}}\right] = \left(\frac{N_s - 1}{n_s} + 1\right) \delta_{\boldsymbol{pq}} = \delta_{\boldsymbol{pq}} \left(\frac{N_s}{n_s}\right)^1 + \frac{1(1+1)}{2} \underbrace{\delta_{\boldsymbol{pq}} \left(1 - 1/n_s\right)}_{\mathcal{O}(1)}. \tag{B.2}$$

Next we assume the asymptotic holds for n. For $n \to n+1$ we get:

$$\mathbb{E}\left[(M^{n+2})_{\boldsymbol{q}_0\boldsymbol{q}_{n+2}}\right] = \mathbb{E}\left[\sum_{\boldsymbol{q}_0\ \boldsymbol{q}_1\cdots\boldsymbol{q}_{n+2}} M_{\boldsymbol{q}_0\boldsymbol{q}_1} M_{\boldsymbol{q}_1\boldsymbol{q}_2}\dots M_{\boldsymbol{q}_{n+1}\boldsymbol{q}_{n+2}}\right] \tag{B.3}$$

$$= \mathbb{E}\left[\sum_{v_1}\sum_{v_2}\cdots\sum_{v_{n+1}}\prod_{j=0}^{n+1} v_j \cdot v_{j+1}\right], \tag{B.4}$$

where the v_i are some of the N_s normalized, n_s-dimensional vectors. We can use the fact that the expectation value is linear to move it inside the sums, we can also split the sum into a term where all vectors are different (which we can do as $n \ll N_s$) and one where at least 2 vectors are the same:

$$\mathbb{E}\left[(M^{n+2})_{\boldsymbol{q}_0\boldsymbol{q}_{n+2}}\right] = \sum_{v_1\neq v_2\neq\dots v_{n+1}} \mathbb{E}\left[\prod_{j=0}^{n+1} v_j \cdot v_{j+1}\right] + \sum_{i<k}\sum_{v_1}\sum_{v_2}\cdots\sum_{v_{n+1}} \delta_{v_i v_k}\mathbb{E}\left[\prod_{j=0}^{n+1} v_j \cdot v_{j+1}\right]. \tag{B.5}$$

The first term is easy to calculate, it was done in Equation A.11. It is equal to:

$$\sum_{v_1\neq v_2\neq\dots q_{n+1}} \mathbb{E}\left[\prod_{j=0}^{n+1} v_j \cdot v_{j+1}\right] = \delta_{v_0 v_{n+1}}\left(\frac{N_s}{n_s}\right)^{n+1}. \tag{B.6}$$

Before we proceed, note that the expectation value of products like the one above are always non-negative, such products may be written as a polynomial over vector components, said components are distributed with mean 0. This means that the expectation value of odd powers of the components in the polynomial are 0 and all even powers are positive:

$$\mathbb{E}\left[\prod_{j=0}^{n+1} v_j \cdot v_{j+1}\right] = \mathbb{E}[\cdots + (v_0^1)^{n_1}(v_0^1)^{n_2}(v_1^2)^{n_3}\dots] = \cdots + \mathbb{E}[(v_0^1)^{n_1}]\mathbb{E}[(v_0^1)^{n_2}]\mathbb{E}[(v_1^2)^{n_3}]\cdots \geq 0. \tag{B.7}$$

This also means that (for $i \in \{1 \dots n\}$):

$$\begin{aligned}
\mathbb{E}\left[\prod_{j=0}^{n+1} v_j \cdot v_{j+1}\right] &= \mathbb{E}\left[\frac{(v_{i-1}\cdot v_i)(v_i \cdot v_{i+1})}{(v_{i-1}\cdot v_i)(v_i \cdot v_{i+1})}\prod_{j=0}^{n+1} v_j \cdot v_{j+1}\right] && \text{(B.8)}\\
&= \mathbb{E}\left[\frac{(v_{i-1}^{v_i} v_{i+1}^{v_i})}{(v_{i-1}\cdot v_i)(v_i \cdot v_{i+1})}\prod_{j=0}^{n+1} v_j \cdot v_{j+1}\right] && \text{(B.9)}\\
&\leq \mathbb{E}\left[\frac{(v_{i-1}^{v_i} v_{i+1}^{v_i})}{(v_{i-1}\cdot v_i)(v_i \cdot v_{i+1})}\prod_{j=0}^{n+1} v_j \cdot v_{j+1}\right] && \text{(B.10)}\\
&\quad + \mathbb{E}\left[\frac{\sum_{i\neq v_i}(v_{i-1}^{i} v_{i+1}^{i})}{(v_{i-1}\cdot v_i)(v_i \cdot v_{i+1})}\prod_{j=0}^{n+1} v_j \cdot v_{j+1}\right]\\
&\quad = \mathbb{E}\left[\frac{(v_{i-1}\cdot v_{i+1})}{(v_{i-1}\cdot v_i)(v_i \cdot v_{i+1})}\prod_{j=0}^{n+1} v_j \cdot v_{j+1}\right], && \text{(B.11)}
\end{aligned}$$

where, to evaluate the dot products, we picked v_i as a basis vector, and then an orthonormal basis around it. With this inequality in hand we can preform the induction step. To show the induction we need only for the second term of Equation B.5 to be bound by the appropriate asymptotic:

$$\begin{aligned}
&\sum_{i<k}\sum_{v_1}\sum_{v_2}\cdots\sum_{v_{n+1}} \delta_{v_i v_k}\mathbb{E}\left[\prod_{j=0}^{n+1} v_j \cdot v_{j+1}\right] && \text{(B.12)}\\
&\leq \sum_{i<k}\sum_{v_1}\sum_{v_2}\cdots\sum_{v_{n+1}} \delta_{v_i v_k}\mathbb{E}\left[\frac{(v_{i-1}\cdot v_{i+1})}{(v_{i-1}\cdot v_i)(v_i \cdot v_{i+1})}\prod_{j=0}^{n+1} v_j \cdot v_{j+1}\right] && \text{(B.13)}\\
&= \sum_{i<k}\mathbb{E}\left[(M^{n+1})_{v_0 v_{n+2}}\right] && \text{(B.14)}\\
&= \frac{(n+1)(n+2)}{2}\left(\delta_{\boldsymbol{p}v}\left(\frac{N_s}{n_s}\right)^n + \sum_{k=0}^{n-1}\mathcal{O}\left(\left(\frac{N_s}{n_s}\right)^k n^{2(n-k)}\right)\right) && \text{(B.15)}\\
&= \frac{(n+1)(n+2)}{2}\sum_{k=0}^{n}\mathcal{O}\left(\left(\frac{N_s}{n_s}\right)^k n^{2(n-k)}\right). && \text{(B.16)}
\end{aligned}$$

We used the induction assumption in the third step, as the delta function only gets rid of the sum over v_i and nothing else, as $(v_{i-1}\cdot v_i)(v_i\cdot v_{i+1})$ is already removed from the product. Thus Equation B.1 holds for all $0 < n < N_s$, $n \in \mathbb{N}$. Numerical tests of said equation are plotted in figures B.1 - B.5

B.2 Remainder After Truncation of The Exponential Series

Equation B.1 suggests that there will be a deviation (the second term in said equation) from the standard result when exponentiating the matrix M. We want to get an idea of how large the deviation will be if we truncate the series on some term K:

$$\mathbb{E}\left[\left(Me^{\frac{itk_s n_s}{N_s}M}\right)_{\boldsymbol{pq}}\right] - M_{\boldsymbol{pq}} \overset{!}{\approx} \sum_{n=1}^{K} \frac{(itk_s)^n}{n!}\left(\frac{n_s}{N_s}\right)^n \left[\delta_{\boldsymbol{pq}}\left(\frac{N_s}{n_s}\right)^n \right. \tag{B.17}$$

$$\left. + \frac{n(n+1)}{2}\sum_{k=0}^{n-1}\mathcal{O}\left(\left(\frac{N_s}{n_s}\right)^k n^{2(n-k-1)}\right)\right] \tag{B.18}$$

$$\approx \sum_{n=1}^{K}\frac{(itk_s)^n}{n!}\left(\delta_{\boldsymbol{pq}} + \sum_{k=0}^{n-1}\mathcal{O}\left(\left(\frac{n_s n^2}{N_s}\right)^{n-k}\right)\right). \tag{B.19}$$

Focusing on the deviation (the second term), we can get rid of the second sum, since it is clear what terms might be trouble for large n. Using Stirling's approximation for large n, $n! \approx \sqrt{2\pi n}(n/e)^n$ we get:

$$\sum_{n=1}^{K}\frac{(itk_s)^n}{n!}\mathcal{O}\left(\left(\frac{n_s n^2}{N_s}\right)^n\right) \approx \sum_{n=0}^{K}\frac{(ie)^n}{\sqrt{2\pi n}}\mathcal{O}\left(\left(tk_s\frac{n_s}{N_s}n\right)^n\right) \tag{B.20}$$

$$\overset{\text{large } n}{=} \mathcal{O}\left(\left(tk_s\frac{n_s}{N_s}K\right)^K\right). \tag{B.21}$$

To explore the cases where the numbers become very large and comparable with N_s/n_s, it is instructive to define $tk_sK = \kappa(N_s/n_s)^\varepsilon$, for $\kappa, \varepsilon = \mathcal{O}(1)$. To get:

$$\mathcal{O}\left(\left(tk_s\frac{n_s}{N_s}K\right)^K\right) = \mathcal{O}\left(\left(\frac{n_s}{N_s}\right)^{(1-\varepsilon)\left(\frac{N_s}{n_s}\right)^\varepsilon\frac{\kappa}{tk_s}}\right). \tag{B.22}$$

Plugging in, for instance, $\varepsilon = 1/2$ and also $tk_s \propto \sqrt{N_s/n_s}$ (but $tk_s < \sqrt{N_s/n_s}$), gives:

$$\mathcal{O}\left(\left(\frac{n_s}{N_s}\right)^{(1-\varepsilon)\left(\frac{N_s}{n_s}\right)^\varepsilon\frac{\kappa}{tk_s}}\right) = \mathcal{O}\left(\left(\frac{n_s}{N_s}\right)^{\kappa/2}\right). \tag{B.23}$$

So far we have looked at the contribution of the deviation from the standard result, but truncating the series will result in itself in another deviation, it is given by [51]:

$$\left|\left|e^A - \sum_{n=0}^{K}\frac{A^n}{n!}\right|\right| \leq \max_{x\in[0,1]}\frac{||A^{K+1}e^{Ax}||}{(K+1)!} = R(K,A)\ , \tag{B.24}$$

where $||\cdot||$ is some matrix norm, which we will chose to be the Frobenius norm. In our case:

$$R\left(K, \frac{itk_s n_s}{N_s}M\right) = \max_{x\in[0,1]} \frac{\left|\left|\left(\frac{itk_s n_s}{N_s}M\right)^{K+1} e^{\frac{itk_s n_s}{N_s}Mx}\right|\right|}{(K+1)!} \tag{B.25}$$

$$\leq \max_{x\in[0,1]} \frac{\left|\left|\left(\frac{itk_s n_s}{N_s}M\right)\right|\right|^{K+1}\left|\left|e^{\frac{itk_s n_s}{N_s}Mx}\right|\right|}{(K+1)!} \tag{B.26}$$

$$\leq \frac{1}{(K+1)!}\left(\left|\frac{itk_s n_s}{N_s}\right|\sqrt{N+N(N-1)|\epsilon|^2}\right)^{K+1}, \tag{B.27}$$

where we used the fact that the Frobenius norm is sub multiplicative in the first step and then we used the JL-Theorem [45] from [32] to evaluate an upper bound on the norm explicitly. We can now use the lower bound $n! \geq (n/e)^n e$ once again to write:

$$R\left(K-1, \frac{itk_s n_s}{N_s}M\right) \leq e\left(\left|\frac{tk_s n_s e}{N_s K}\right|\sqrt{N+N(N-1)|\epsilon|^2}\right)^{K} \tag{B.28}$$

$$\leq e\left(\left|\frac{(tk_s)^2 e}{K}\frac{N_s^{1+\varepsilon}}{n_s^{1+\varepsilon}}\right|\sqrt{N+N(N-1)|\epsilon|^2}\right)^{K} \tag{B.29}$$

$$\leq e\left(\left|\frac{(tk_s)^2}{\kappa e}\right|\sqrt{\frac{n_s^{2+2\varepsilon}}{N_s^{1+2\varepsilon}}+\frac{n_s^{2+2\varepsilon}}{N_s^{2\varepsilon}}|\epsilon|^2}\right)^{K}. \tag{B.30}$$

Finally we can plug in the value for ϵ calculated in [32]:

$$R\left(K-1, \frac{itk_s n_s}{N_s}M\right) \leq e\left(\left|\frac{(tk_s)^2}{\kappa e}\right|\sqrt{\frac{n_s^{2+2\varepsilon}}{N_s^{1+2\varepsilon}}+8\frac{n_s^{1+2\varepsilon}}{N_s^{2\varepsilon}}\ln\left(\frac{N_s}{\sqrt{\delta}}\right)}\right)^{K}. \tag{B.31}$$

Plugging in, for instance, $\varepsilon = 1/2$ and also $tk_s \propto \sqrt{N_s/n_s}$ gives:

$$R\left(K-1, \frac{itk_s n_s}{N_s}M\right) \overset{\sim}{\leq} \left(\frac{1}{\kappa}\sqrt{\frac{n_s^5}{N_s^4}+8\frac{n_s^5}{N_s^3}\ln\left(\frac{N_s}{\sqrt{\delta}}\right)}\right)^{K}. \tag{B.32}$$

So we see that a truncation of the exponential series will result in vanishing errors even if $t \to \sqrt{N_s/n_s}$, Thus the approximations given are accurate. for very large N_s, n_s.

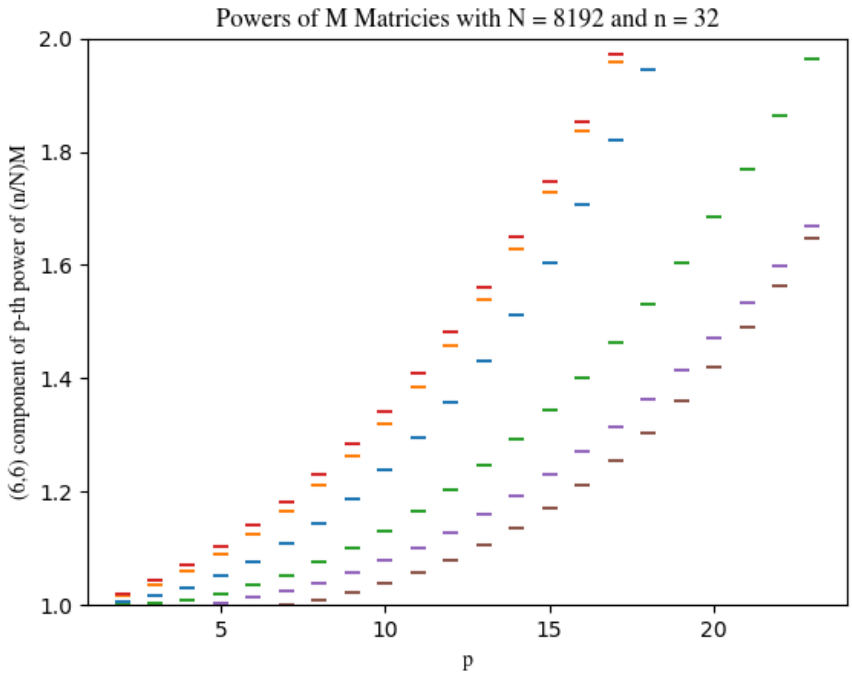


Figure B.1: 6 different realizations in different colors of the number $(n/N)^{p-1}(M^p)_{66}$. The matrix M has $N_s = 8192$ and $n_s = 32$. Notice that although the diagonal components of M blow up, they can be scaled down to something approaching 1 by scaling with n_s/N_s. However, even then the values do still grow with the power.

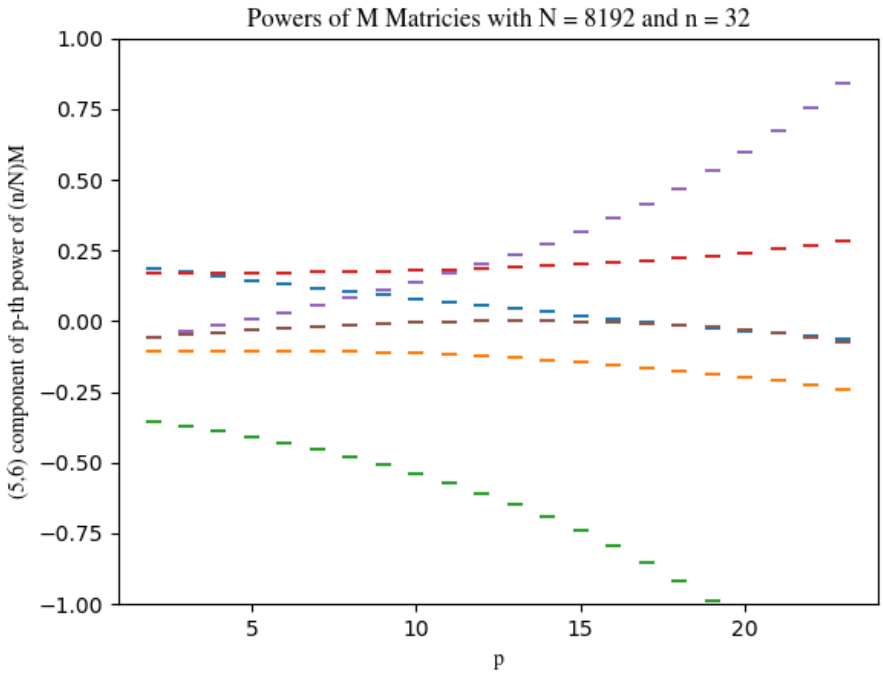


Figure B.2: 6 different realizations in different colors of the number $(n_s/N_s)^{p-1}(M^p)_{56}$. The matrix M has $N_s = 8192$ and $n_s = 32$. As expected the average is close to 0.

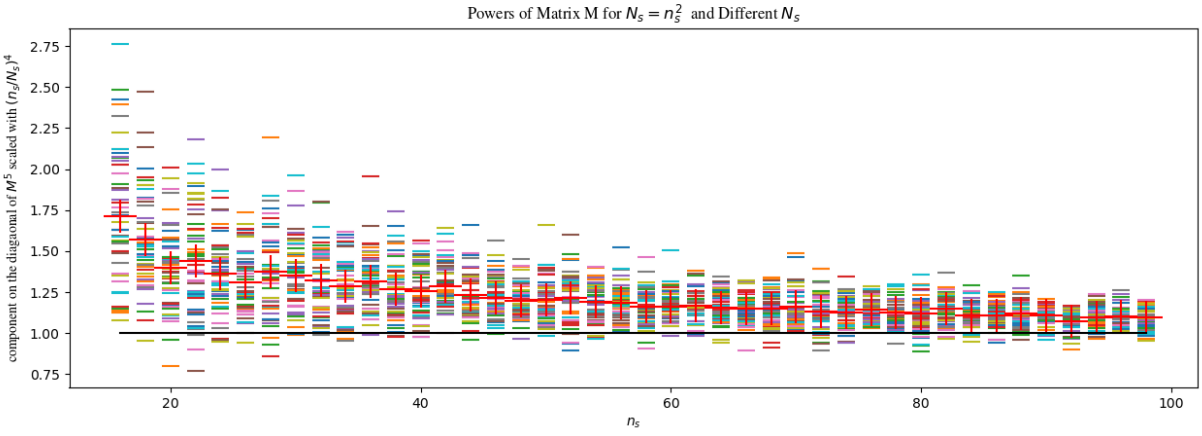


Figure B.3: 50 different diagonal components in different colors of the matrix $(n_s/N_s)^4 M^5$. For varying n_s and N_s, but fixed relative to each other as $N_s = n_s^2$. The red "+" represents the average over all the components, one can see that they very slowly approach 1 (the black line).

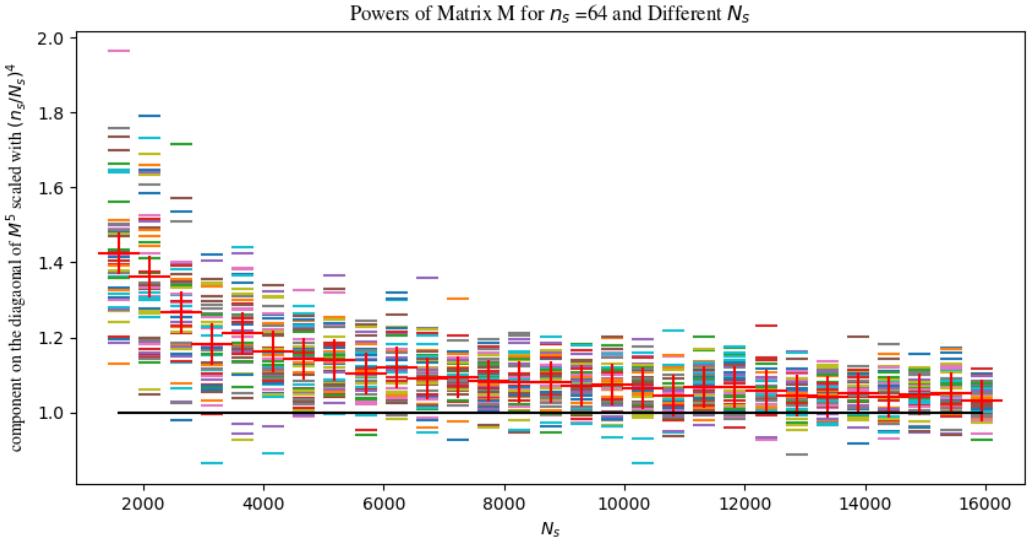


Figure B.4: Similar to B.2, now with a fixed $n_s = 64$ and variable N_s.

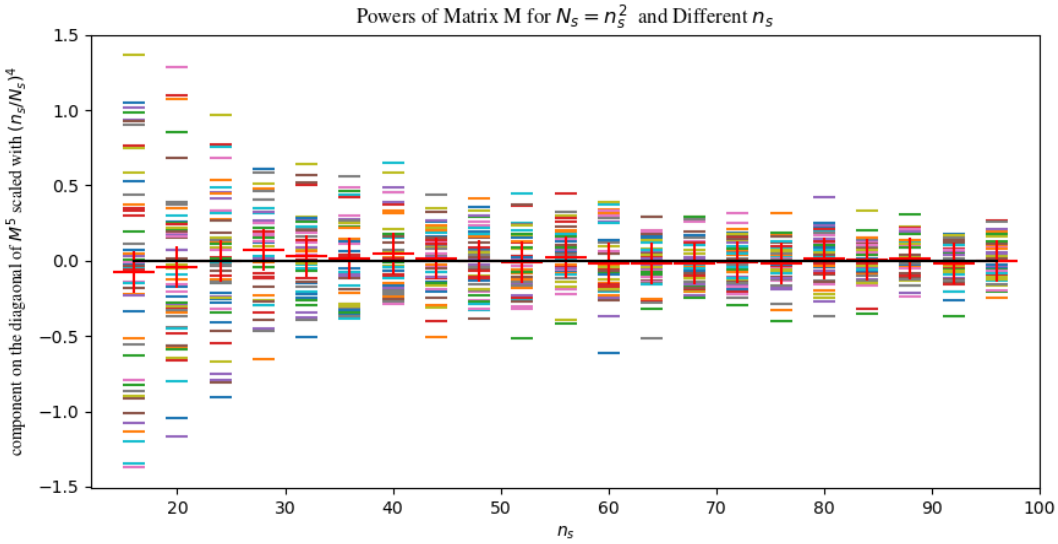


Figure B.5: Similar to B.2, but now we look at off diagonal components, these approach 0 (the black line) with larger n_s, N_s

Appendix C

Miscellaneous Properties of vectors Randomly Distributed in the unit n-Sphere

C.1 Moments of the Dot Product of Random Vectors

The Matrix elements $M_{\boldsymbol{qp}}$ are given by the dot product of two random vectors on an n-sphere. We call these two random vectors $v_{\boldsymbol{q}}$ and $v_{\boldsymbol{p}}$. Let us further pick n dimensional hyper-spherical coordinated such that $v_{\boldsymbol{q}}$ aligns with one of the axis. Thus, following [32], we see that:

$$\mathbb{E}\left(M_{\boldsymbol{qp}}^2\right) = \mathbb{E}\left((v_{\boldsymbol{q}} \cdot v_{\boldsymbol{p}})^2\right) \tag{C.1}$$

$$= \int_0^\pi P(\theta)(\cos(\theta))^2 \tag{C.2}$$

$$= \frac{\int_0^\pi d\theta\ (\sin(\theta))^{n-2}\left(1-(\sin(\theta))^2\right)}{\int_0^\pi d\theta\ (\sin(\theta))^{n-2}} \tag{C.3}$$

$$= 1 - \frac{\frac{2\pi}{2^{n+1}}\binom{n}{n/2}}{\frac{2\pi}{2^{n-1}}\binom{n-1}{n/2-1}} = 1 - \frac{\frac{n!}{((n/2)!)^2}}{4\frac{(n-2)!}{((n/2-1)!)^2}} = \frac{1}{n}. \tag{C.4}$$

We have also used:

$$\mathbb{E}\left(M_{\boldsymbol{qp}}^4\right) = \mathbb{E}\left((v_{\boldsymbol{q}} \cdot v_{\boldsymbol{p}})^4\right) \tag{C.5}$$

$$= \frac{\int_0^\pi d\theta\ (\sin(\theta))^{n-2}(\cos(\theta))^4}{\int_0^\pi d\theta\ (\sin(\theta))^{n-2}} \tag{C.6}$$

$$= \frac{\int_0^\pi d\theta\ ((\sin(\theta))^{n-2} - (\sin(\theta))^n)(\cos(\theta))^2}{\int_0^\pi d\theta\ (\sin(\theta))^{n-2}} \tag{C.7}$$

$$= \frac{1}{n} - \frac{\frac{2\pi}{2^{n+1}}\binom{n}{n/2} - \frac{2\pi}{2^{n+3}}\binom{n+2}{n/2+1}}{\frac{2\pi}{2^{n-1}}\binom{n-1}{n/2-1}} = \frac{3}{n(n+2)}. \tag{C.8}$$

The Integral over powers of the sine function can be calculated with the binomial theorem:

$$\int_0^\pi d\theta\ (\sin(\theta))^n = \frac{1}{2}\int_0^{2\pi} d\theta\ (\sin(\theta/2))^n \tag{C.9}$$

$$= \frac{1}{2}\int_0^{2\pi} d\theta\ \sum_{k=0}^{n}\binom{n}{k}\left(\frac{e^{i\theta/2}}{2i}\right)^{n-k}\left(\frac{-e^{-i\theta/2}}{2i}\right)^{k} \tag{C.10}$$

$$= \frac{1}{2}\frac{1}{(2i)^n}\sum_{k=0}^{n}\binom{n}{k}(-1)^k\int_0^{2\pi} d\theta\ e^{i\theta(n-2k)/2} \tag{C.11}$$

$$= \frac{1}{2}\frac{2\pi}{(2i)^n}\sum_{k=0}^{n}\binom{n}{k}(-1)^k\delta(n/2-k) \tag{C.12}$$

$$= \frac{2\pi}{2^{n+1}}\binom{n}{n/2}, \tag{C.13}$$

$$\tag{C.14}$$

or $= 0$ if n is an odd number.

C.2 Spectra of the Matrix $F^\dagger F$

The matrix $F^\dagger F$ is quite similar to a Wischart matrix. The matrix $F^\dagger F$ would follow a Wischart distribution, if we dropped the restriction that the sampled vectors are of unit length. When taking the limit $n, N \to \infty$, but $N > n$ the largest eigenvalues of such matrices tend to (proposition 5.1 of [26]):

$$\lambda_{\max} \overset{n,N\to\infty}{\longrightarrow} N\left(1+\sqrt{n/N}\right)^2, \tag{C.15}$$

while the smallest eigenvalue converges to (proposition 5.1 of [26]):

$$\lambda_{\min} \overset{n,N\to\infty}{\longrightarrow} N\left(1-\sqrt{n/N}\right)^2. \tag{C.16}$$

Thus we see that if $n/N \to 0$, the eigenvalues of the matrix $F^\dagger F$, λ_i will likely tend to:

$$\lambda_i \to N/n, \tag{C.17}$$

where we added a factor of $1/n$ to account for the different normalization of the matrix $F^\dagger F$.

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