5. Polynomials

Polynomials

Irreducibility

$f(x) \in F[x]$ is **reducible** over $F \iff f(x) = g(x)h(x)$ for some $g(x), h(x) \in F[x]$ of lower degree.
Otherwise f is **irreducible** over F

Irreducibility over $\mathbb{Q}$ and $\mathbb{Z}$
1.

- Let $f \in \mathbb{Z}[x]$ be irreducible $\Rightarrow f$ is irreducible over $\mathbb{Q}$
- Contrapositive: if $f \in \mathbb{Z}[x]$ factors over $\mathbb{Q}$ then it factors over $\mathbb{Z}$

2.

- Let p be a prime and suppose that $f(x)[Z[x]$ with $deg f(x) > 1$
- Let $\bar{f}(x)$ be the polynomial in $\mathbb{Z}_p[x]$ obtained from $f(x)$ by reducing all the coefficients of $f(x) \bmod p$.
- If $f(x)$ is irreducible over $\mathbb{Z}_p$ and $deg \bar{f}(x) = deg f(x) \Rightarrow f(x)$ is irreducible over $\mathbb{Q}$

Kronecker's Theorem

- Let F be a field
- Let $f(x)$ be a nonconstant polynomial in $F[x]$.
- Then there is an extension field E of F in which $f(x)$

Proof
$f(x)$ has an irreducible factor $p(x)$. It's enough to construct a field E where $p(x)$ has a 0

Let's try $F[x]/\langle p(x)\rangle$ with the one-to-one mapping $\phi : F \to E; \phi(a) = a + \langle p(x)\rangle$

Let $p(x) = a_n x^n + ... + a_0$
Then
$p(x + \langle p(x)\rangle) =$
$= a_n((x + \langle p(x)\rangle)^n + ... + a_0 =$
$= a_n(x^n + \langle p(x)\rangle + ... + a_0 =$
$= a_n x^n + a_{n-1}x^{n-1} + ... + a_0 + \langle p(x)\rangle =$
$= p(x) + \langle p(x)\rangle =$
$= 0 + \langle p(x)\rangle$