Homework 6

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1. The *Beta distribution* with parameters $a = 3, b = 2$ has PDF

$$f(x) = 12x^2(1-x), \text{ for } 0 < x < 1.$$

Let X have this distribution. $4x^3 - 3x^4$

(a) Find the CDF of X. $\frac{1}{2} - \frac{3}{16}$

(b) Find $P(0 < X < 1/2)$.

(c) Find the mean and variance of X (without quoting results about the Beta distribution).

2. Let $U_1, \ldots, U_n$ be i.i.d. Unif$(0, 1)$, and $X = \max(U_1, \ldots, U_n)$. $1 - e^{-x} \cdot e^{-\ln n}$

(a) What is the PDF of X? $e^{-x-\log n}$

(b) What is $E[X]$?

3. the *Laplace distribution* has PDF $SX \le x$

$$f(x) = \frac{1}{2}e^{-|x|}$$

$P(X \le x, S = 1)$ $+P(X$

for all real x. The Laplace distribution is also called a *symmetrized Exponential* distribution. Explain this in the following two ways.

(a) Plot the PDFs and explain how they relate.

(b) Let $X \sim$ Expo(1) and S be a random sign (1 or -1, with equal probabilities), with S and X independent. Find the PDF of SX (by first finding the CDF), and compare the PDF of SX and the Laplace PDF. $P(X \le x) = 1 - e^{-x}$

4. The *Gumbel distribution* is the distribution of $-\log X$ with $X \sim$ Expo(1).

(a) Find the CDF of the Gumbel distribution. e^{-x}

(b) Let $X_1, X_2, \ldots$ be i.i.d. Expo(1) and let $M_n = \max(X_1, \ldots, X_n)$. Show that $M_n - \log n$ converges in distribution to the Gumbel distribution, i.e., as $n \to \infty$ the CDF of $M_n - \log n$ converges to the Gumbel CDF.

$P(-\log X \le x)$

$= P(X \le e^{x})$

$\left(\left(1 - \frac{l}{n}\right)^{\frac{n}{l}}\right)^{l}$

$P(Z \leq c)$

$\left(\frac{n-l}{n}\right)^n$

5. Let $Z \sim \mathcal{N}(0,1)$, and c be a nonnegative constant. Find $E[\max(Z-c,0)]$, in terms of the standard Normal CDF Φ and PDF φ.

6. (Optional Challenging Problem) Suppose $X \sim \mathcal{N}(m, \sigma^2)$, where m is an integer and σ is a real number. Let $Y = \lfloor X \rfloor$ be the integer part of X.

   (a) Find the PMF of Y

   (b) Find $E(Y)$

   (c) Find $\text{Var}(Y)$

$X = \sigma Z + m$

$P(Y \leq k) = P(\lfloor \quad \rfloor \leq k)$

$P(Y \leq k-1)$

$P(\sigma Z + m < k+1)$

$P\left(Z < \frac{k-m+1}{\sigma}\right)$