2022Fall Probability & Mathematical Statistics 2023/12/09

Homework 9

Professor: Ziyu Shao & Dingzhu Wen *Due: 2023/12/17 10:59pm*

1. Let X and Y be i.i.d. Expo(λ), and transform them to $T = X + Y$, $W = X/Y$. Find the marginal PDFs of T and W, and the joint PDF of T and W.

2. Let X, Y, Z be i.i.d. Unif$(0, 1)$, and $W = X + Y + Z$. Find the PDF of W.

3. Let (X, Y) be Bivariate Normal with $X \sim \mathcal{N}(0, \sigma_1^2)$ and $Y \sim \mathcal{N}(0, \sigma_2^2)$ marginally and with $\text{Corr}(X, Y) = \rho$. Find a constant c such that $Y - cX$ is independent of X.

4. (a) Let $U_1, \ldots, U_n$ be i.i.d. Unif$(0, 1)$. Let $U_{(j)}$ be the jth order statistic, $U_{(k)}$ be the kth order statistic, where $1 \le j < k \le n$. Find the joint PDF of $U_{(j)}$ and $U_{(k)}$.

   (b) Let $X \sim \text{Bin}(n, p)$ and $B \sim \text{Beta}(j, n - j + 1)$, where n is a positive integer and j is a positive integer with $j \le n$. Show using a story about order statistics that
   $$P(X \ge j) = P(B \le p).$$
   This shows that the CDF of the continuous r.v. B is closely related to the CDF of the discrete r.v. X, and is another connection between the Beta and Binomial.

   (c) Show that
   $$\int_0^x \frac{n!}{(j-1)!(n-j)!} t^{j-1}(1-t)^{n-j} dt = \sum_{k=j}^{n} \binom{n}{k} x^k (1-x)^{n-k},$$
   without using calculus, for all $x \in [0, 1]$ and j, n positive integers with $j \le n$.

5. (a) Let $p \sim \text{Beta}(a, b)$, where a and b are positive real numbers. Find $E(p^2(1-p)^2)$, fully simplified (Γ should not appear in your final answer).

   Two teams, A and B, have an upcoming match. They will play five games and the winner will be declared to be the team that wins the majority of games. Given p, the outcomes of games are independent, with probability p of team A winning and $(1 - p)$ of team B winning. But you don't know p, so you decide to model it as an r.v., with $p \sim \text{Unif}(0, 1)$ a priori (before you have observed any data).

   To learn more about p, you look through the historical records of previous games between these two teams, and find that the previous outcomes were, in chronological order, $AAABBAABAB$. (Assume that the true value of p has not been changing over time and will be the same for the match, though your *beliefs* about p may change over time.)

$G > \ell$

$T \sim \text{Gamma}(k+1, \ell)$

$(T_{k+1} > 1)$

Gamma (k +1 , λ) > t

Pois (λt) ≤ k

(b) Does your posterior distribution for p, given the historical record of games between A and B, depend on the specific order of outcomes or only on the fact that A won exactly 6 of the 10 games on record? Explain.

(c) Find the posterior distribution for p, given the historical data.

$\sum \text{Expo}(1) =$

The posterior distribution for p from (c) becomes your new prior distribution, and the match is about to begin!

$\text{Expo}(\lambda) = \text{Gamma}(1, \lambda)$

(d) Conditional on p, is the indicator of A winning the first game of the match positively correlated with, uncorrelated with, or negatively correlated with the indicator of A winning the second game of the match? What about if we only condition on the historical data?

$T_k \geq 1$

(e) Given the historical data, what is the expected value for the probability that the match is not yet decided when going into the fifth game (viewing this probability as an r.v. rather than a number, to reflect our uncertainty about it)?

$N_1 \leq k$

6. (Optional Challenging Problem) If $X \sim \text{Pois}(\lambda)$, $Z \sim \text{Gamma}(k+1, 1)$, where k is a nonnegative integer. Use **two different methods** to show the Poisson-Gamma Duality holds:

a, λ

$$P(X \leq k) = P(Z > \lambda).$$

$k+1$ Expo(1)

$$\frac{1}{\Gamma(a)} (\lambda y)^a e^{-\lambda y} \frac{1}{y}$$

$$\int_{\lambda}^{+\infty} \frac{1}{k!} y^k e^{-y} dy$$

$=$

$$e^{-\lambda} \sum_{t=0}^{k} \frac{\lambda^t}{t!}$$

$$= \frac{1}{\Gamma(k+1)} \int_{\lambda}^{+\infty} y^{k+1} e^{-y} \frac{1}{y} dy$$

$$\frac{1}{k!}$$

$$\int_{\lambda}^{+\infty} y^k e^{-y} dy$$

$$= \frac{1}{k+1} ($$