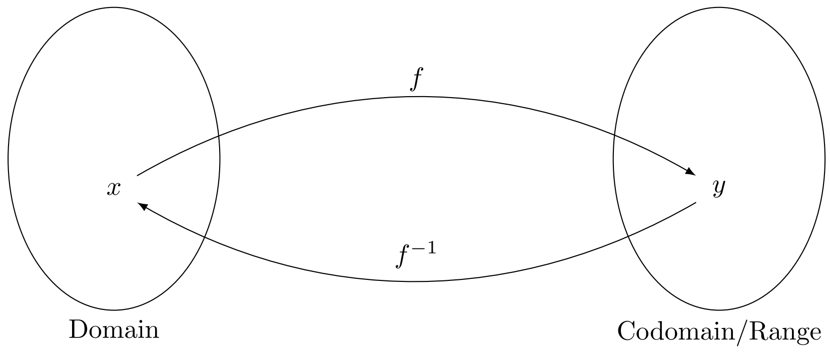

For every x in the domain of f, there exists a $y = f(x)$ such that $x \in \mathbb{R}$ and $y \in \mathbb{R}$.

Terminology

x - independent variable	"y is the image of x"
y - the value of the function	"x is a pre-image of y"

Take $y = x^2$, for example. For every $y \geq 0$, there are two pre-images: $x_1 = \sqrt{y}$ and $x_2 = -\sqrt{y}$.

A function f is called a **one-to-one function** if it never takes on the same value twice; that is,

$$f(x_1) \neq f(x_2) \qquad \text{whenever } x_1 \neq x_2$$

Equivalently, a function f is $1 \to 1$ if every image of f has exactly one pre-image.

Also equivalently, f is $1 \to 1$ if it is monotonic. That is, $x_1 < x_2$ implies either $y_1 < y_2$ or $y_1 > y_2$ for all x in the domain of f.

Now to the point; if f is a $1 \to 1$ function, then there exists an **inverse function** denoted by f^{-1}. If a function maps x into y, then its inverse function maps y back into x. That is,

$$\boxed{f^{-1}(f(x)) = x} \longrightarrow$$ Must be satisfied for whole domain of inverse.

The textbook gives this definition:

Let f be a one-to-one function with domain A and range B. Then its **inverse function** f^{-1} has domain B and range A and is defined by

$$f^{-1}(y) = x \iff f(x) = y$$

for any y in B.

EXAMPLE Given $f(x) = x^2 - 3x + 1$, define f^{-1}. Restrict the domain if necessary.

First we solve for x in terms of y.

$$\text{Let } y = f(x) \quad x = f^{-1}(y)$$

$$\begin{aligned}
y &= x^2 - 3x + 1 \\
0 &= x^2 - 3x + 1 - y \\
\text{Generally, } x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
\text{But here, } x &= \frac{-b + \sqrt{b^2 - 4ac}}{2a} = \frac{3 + \sqrt{4y + 5}}{2}
\end{aligned}$$

We write $+$ instead of $\pm$, restricting the range of f^{-1} so that it abides by the definition of a function (ie. an input cannot have more than one output). Note that the choice of $+$ over $-$ was an arbitrary decision.

Finally, we verify that $f^{-1}(f(x)) = x$.

$$\begin{aligned}
f^{-1}(f(x)) &= \frac{3}{2} + \sqrt{y + \frac{5}{4}} \\
&= \frac{3}{2} + \sqrt{\frac{5}{4} + x^2 - 3x + 1} \\
&= \frac{3}{2} + \sqrt{\left(x - \frac{3}{2}\right)^2} \\
&= \frac{3}{2} + x - \frac{3}{2} = x
\end{aligned}$$

EXAMPLE Sketch $\sqrt[3]{1 - x^3} \quad -\infty < x < \infty$

Our Math Professor recommmends analyzing the asymptotic behaviour first when sketching curves. A rigorous step-by-step prodecure for curve sketching (which includes asymptotic analysis, just not first) can be found in Stewart.

Asymptotic Analysis Strategy 1:
$x \to \infty \qquad x^3 >> 1$
Since $x^3 >> 1$, we can neglect the 1 and say $y \sim \sqrt[3]{-x^3} = -x$ where $y > -x$

Asymptotic Analysis Strategy 2:
$y = \sqrt[3]{1 - x^3} = \sqrt[3]{(-x)^3(1 - 1/x^3)} = -x\sqrt[3]{1 - \frac{1}{x^3}}$
and since $\sqrt[3]{1 - \frac{1}{x^3}}$ tends to 1, $y \sim -x$

In this example, after the asymptotic analysis, Our Math Professor checked the x and y-intercepts.
When $x = 0$, $y = 1$. And when $y = 0$, $x = 1$.

In this example, these two forms of curve analysis have given us enough information to draw the curve. I feel that it's important to note that different functions have different idiosyncracies, and may require different forms of analysis to sketch. But anyways, here's the finished product!