Learning Outcome: determine if $\int$ is convergent and evaluate it if it is.

If a function $f(x)$ is less than or equal to another function $g(x)$ for all $x \in \mathbb{R}$ such that $x \geq M$, then the following is true;

(a) $\displaystyle\int_M^\infty f(x)\ dx$ convg. $\implies \displaystyle\int_M^\infty g(x)\ dx$ convg.

(b) $\displaystyle\int_M^\infty g(x)\ dx$ diverg. $\implies \displaystyle\int_M^\infty g(x)\ dx$ diverg.

EXAMPLE $\displaystyle\int_{10}^\infty \frac{1}{x\ln^2 x}\ dx$

$$\begin{aligned}\int_{10}^\infty \frac{1}{x\ln^2 x}\ dx &= (\ln x = u,\ du = 1/x\ dx)\\ &= \int_{\ln 10}^\infty \frac{1}{u^2}\ du \text{ convg. by p-test}\end{aligned}$$

EXAMPLE $\displaystyle\int_{10}^\infty \frac{1+\sin^2 3x + \arctan x}{x\ln^2 x}\ dx$

$$\text{numerator } < 1 + 1 + \pi/2 < 4$$

So,

$$\int_{10}^\infty \frac{1+\sin^2 3x + \arctan x}{x\ln^2 x}\ dx < \int_{10}^\infty \frac{4}{x\ln^2 x}\ dx \text{ convg. by p-test}$$

Therefore, our original integral is convergent by comparison.

O, o - notation:

$$f(x) = O(g(x)) \iff \exists c : f(x) \leq cg(x)$$

A function is a big O of another function if it is strictly less than or equal to a scalar multiple of another function.

$$f(x) = O(g(x)) \wedge \int_a^\infty g(x)\ dx \text{ convg.} \implies \int_a^\infty f(x)\ dx \text{ convg.}$$

If a function is big O of another function and that function is convergent on the domain, then our original function is also. Similarly;

$$f(x) = O(g(x)) \wedge \int_a^\infty f(x)\ dx \text{ diverg.} \implies \int_a^\infty g(x)\ dx \text{ diverg.}$$

$f(x) \sim g(x)$ when $x \to \infty$ if $f(x) = O(g(x)$ and $g(x) = O(f(x))$

EXAMPLE $5g(x) < f(x) < 9g(x)$

If $f(x) \sim g(x)$, then they converg or diverge simultaneously. This implies

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = c$$

and

$$\left| \frac{f(x)}{g(x)} - c \right| < \varepsilon \implies (c - \varepsilon)g(x) < f(x) < (c + \varepsilon)g(x)$$

or, "$f(x) \sim g(x)$ when $x \to \infty$."

EXAMPLE $I_1 = \int_1^\infty \frac{2+e^{-x}}{x}\ dx > I_2 = \int_1^\infty \frac{2}{x}\ dx$ diverg. $\therefore I_1$ diverg. too

EXAMPLE $I_1 = \int_1^\infty \frac{2-e^{-x}}{x}\ dx$

$$\lim_{x \to]infty} \left(\frac{2 - e^{-x}}{x} \div \frac{2}{x} \right) = \lim_{x \to \infty} \frac{2 - e^{-x}}{2} = 1$$

so,

$$\int_1^\infty \frac{2 - e^{-x}}{x}\ dx \text{ and } \int_1^\infty \frac{2}{x}\ dx \text{ are similar, i.e. converg or diverge simultaneously}$$

EXAMPLE $\int_0^\infty \frac{\arctan x}{2+e^x}\ dx$

$$\frac{-\pi/2}{2 + e^x} < \frac{\arctan x}{e^x + 2} < \frac{\pi/2}{2 + e^x} < \frac{\pi}{2 + e^x}$$

$$\implies \int_0^\infty \frac{\pi}{2} \frac{1}{e^x}\ dx = \lim_{b \to \infty} \frac{\pi}{2}(-e^{-x})\Big|_0^b = \frac{\pi}{2} \therefore \text{ converg.}$$

MODIFICATION $\int_0^\infty \frac{\arctan x}{e^x - x^2 - 3}\ dx$

While it is not generally true that

$$\frac{\arctan x}{e^x - x^2 - 3} < \frac{\pi}{2e^x}$$

but it is the truth that

$$\lim_{x \to \infty} \frac{\arctan x}{e^x - x^2 - 3} \div \frac{1}{e^x} = \frac{\pi}{2}$$

since the terms are equivalent, or "similar"

EXAMPLE $\int_1^\infty \frac{x\sqrt{x}-2x+1}{x^3+2x^2+x+5}\ dx$

$$\text{inegrand } < \frac{x\sqrt{x}}{x^3} = x^{-3/2}$$

$$\int x^{-3/2}\ dx \text{ convg.} \therefore \int \text{ convg.}$$

MODIFICATION $\int_{10}^\infty \frac{\arctan x}{x^3-2x^2-5x-6}\ dx$

$$\lim_{x\to\infty} \text{ integrand} \div \text{ dominating terms} = 1$$

Inproper $\int$ with a singularity (discontinuity);

$$f(x) : x \in [a,b) \cup (b,c] \to \mathbb{R} \implies \int_a^c f(x)\ dx = \lim_{n\to b^-} \int_a^n f(x) + \lim_{n\to b^+} \int_n^c f(x)\ dx$$