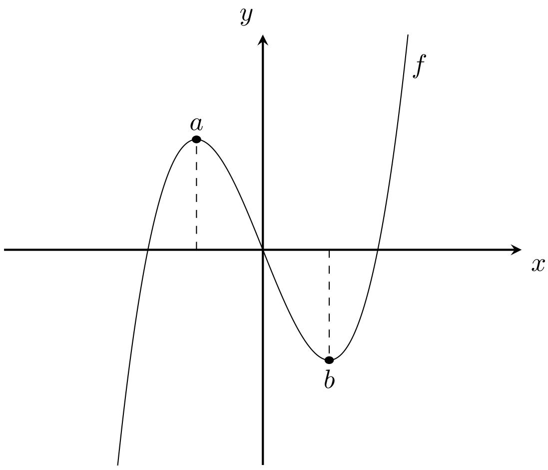
At the beginning of class, one of our classmates asked about defining an inverse for a non one-to-one function. The way to do this is to define an inverse function for each interval where the original function is one-to-one.



$$D = (-\infty, a] \cup (a, b] \cup (b, \infty)$$

This particular function is one-to-one on three intervals, so we would define an inverse for each and it would be understood from the context of future questions which one should be used.

After answering the question, we moved on to some differentiation and integration of exponential and logarithmic functions.

$(e^x)' = e^x$	So, $\int e^x \; dx = e^x + C$
$(a^x)' = a^x \ln a$	So, $\int a^x \; dx = \frac{a^x}{\ln a} + C$

At this point, we differentiated the natural logarithmic function, making use of the fact that the derivatives of a function is the same as the reciprocal of the derivative of its inverse.

$$\boxed{\frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)}}$$

So,

$$\begin{aligned} \text{Let } y &= \ln x \\ x &= e^y \\ \frac{dx}{dy} &= e^y = x \\ \therefore (\ln x)' &= \frac{1}{x} \end{aligned}$$

Reversing this logic, we can now obtain the indefinite integral of the reciprocal function for all $x > 0$.

$$\int \frac{1}{x}\, dx \text{ where } x > 0 = \ln x + C$$

And by using a substitution, we can obtain the indefinite integral of the reciprocal function for all $x < 0$.

$$\text{Let } t = -x,\ dt = -dx \Rightarrow \int \frac{1}{t}\, dt = \ln t + C$$

Therefore,

$$\int \frac{1}{x}\, dx \text{ where } x < 0 = \ln x + C$$

We then recalled the piecewise definition of the absolute value function.

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

And applied it to our definition of the indefinite integral of the reciprocal function.

$$\boxed{\begin{aligned} \int \frac{1}{x}\, dx &= \begin{cases} \ln x + C & \text{if } x > 0 \\ \ln(-x) + C & \text{if } x < 0 \end{cases} \\ &= \ln |x| + C \end{aligned}}$$

When it comes to examinations, use this formula because we were taught it in class. That being said, the formula is actually incorrect, as I pointed out in class. Since it is beyond the scope of the course, I will not demonstrate this fact here, but if you're interested in discovering the truth of the matter, A Mathematician from the Univerity of Victoria, Dr. Trefor Bazett, published a very nice Youtube video explaining it. The video is called "Your calculus prof lied to you (probably)".

Anyways, back to the course material. Using the change of base formula in addition to what we have just discussed, we can easily arrive at the derivative of a logarithm of arbitrary positive base.

$$\boxed{\begin{aligned} \log_a x &= \tfrac{\ln x}{\ln a} \\ (\log_a x)' &= \tfrac{1}{\ln a} \cdot \tfrac{1}{x} \end{aligned}}$$

After this, we spent some time in class solving exercises.

EXAMPLE Evaluate $\lim\limits_{x\to\infty} e^{-x^2}$

$$\lim_{x\to\infty} e^{-x^2} = \begin{pmatrix} -x^2 & = t \\ t & \to -\infty \end{pmatrix} = \lim_{t\to-\infty} e^t = 0+$$

EXAMPLE Evaluate $\lim\limits_{x\to 2^-} e^{\frac{3}{2-x}}$

$$\text{Let } 2-x=t \Rightarrow x=2-t$$

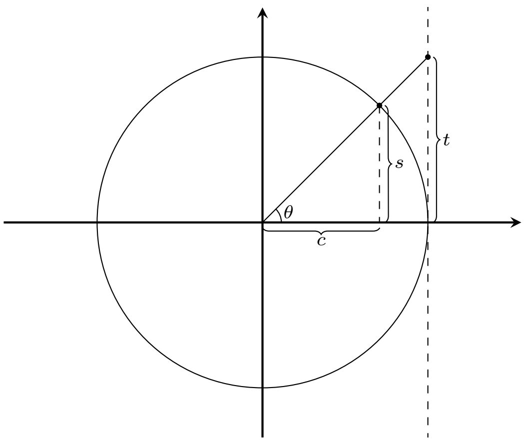
So,

$$\begin{aligned}\lim_{x\to 2^-} e^{\frac{3}{2-x}} &= \begin{pmatrix} x & \to 2^- \\ t & \to -0^+ \end{pmatrix} \\ &= \lim_{t\to 0^+} e^{\frac{3}{t}} \\ &= +\infty\end{aligned}$$

EXAMPLE Evaluate $\lim\limits_{x\to \frac{\pi}{2}^+} e^{\tan x}$

To answer this question, it may be useful to recall the geometric (unit circle) definitions of the trigonometric functions.

Let $s=\sin\theta,\ c=\cos\theta,\ t=\tan\theta$



So,

$$\lim_{x\to \frac{\pi}{2}^+} e^{\tan x} = \begin{pmatrix} \tan x & = u \\ u & \to -\infty \end{pmatrix} = \lim_{u\to -\infty} e^u = 0$$

EXAMPLE Differentiate $y=e^{-2t}\cos 4t$

This is a product and chain rule exercise.

$$y' = -2e^{-2t}\cos 4t - 4e^{-2t}\sin 4t$$

EXAMPLE Differentiate $y=\ln(\cos 3x^2)$

This is a chain rule exercise.

$$y' = \frac{1}{\cos 3x^2} \cdot (-\sin 3x^2) \cdot 6x$$

With the conclusion of the exercised, we moved onto our final theoretical point: differentiating functions where the variable is in both the exponent and the base.

Solution 1:

$$\begin{aligned}
y &= x^x \\
\ln y &= x \ln x \\
\frac{1}{y} y' &= 1 + \ln x \\
\therefore y' &= x^x(\ln x + 1)
\end{aligned}$$

Solution 2:

$$\begin{aligned}
y &= x^x \\
&= e^{\ln(x^x)} \\
&= e^{x \ln x} \\
\therefore y' &= e^{x \ln x}(\ln x + 1) \\
&= x^x(1 + \ln x)
\end{aligned}$$

And finally, we differentiated one function exponentiated to another.

$$\begin{aligned}
y &= f(x)^{g(x)} \\
\ln y &= g(x) \ln f(x) \\
\frac{1}{y} y' &= g'(x) \ln f(x) + g(x) \cdot \frac{1}{f(x)} \cdot f'(x) \\
\therefore y' &= x^x[g'(x) \ln f(x) + g(x) \cdot \frac{1}{f(x)} cdot f'(x)]
\end{aligned}$$