Integration by partial fractions is based on one basic concept. If you have a quotient of two polynomials where the numerator has a degree that is less than the denominator, then you can break it apart into a sum of fractions - one for each factor of the denominator. Basically, we do the following formula backwards.

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + cb}{bd}$$

If the numerator's degree is equal to or greater than the denominator's, then we first do polynomial long division:

$$\begin{array}{r l}
 & x^2 - x - 6 \\ \cline{2-2}
x - 2 \big) & x^3 - 3x^2 - 4x + 12 \\
 & \underline{x^3 - 2x^2} \\
 & -x^2 - 4x \\
 & \underline{-x^2 + 2x} \\
 & -6x + 12 \\
 & \underline{-6x + 12} \\
 & 0
\end{array}$$

If this is hard, I cannot more highly recommend Stewart's Review of Algebra (free download).

There are four cases that we will look at. And I need to mention that they are not exclusive of eachother - one integrand may be a combination of a few. The first one is where the denominator is comprised of linear coefficients (of degree 1!), and we split into a sum of fractions where the numerators are constants. The second is for linear factors of degrees higher than one; essentially, we add one fraction for each. Each of our partial fractions will have that linear raised to an exponent - from 1, to the power of the of the original linear expression. For example;

$$\frac{7x + 3}{(5x + 6)^5} = \frac{C_1}{(5x + 6)^1} + \cdots + \frac{C_5}{(5x + 6)^5}$$

The third and fourth cases are analygous, except their denominators are either repeated or non-repeated irreducible quadratics, and the numerator becomes a linear instead of a constant. For case 3, when we get our partial fractions, we complete the square in the denominator and use u-substitution and the derivative of arctan to get our answer. For case 4, we can substitute $u =$irreducible quadratic, and make the numerator du

CASE: 1 linear factors of degree 1

$$\frac{c_1 x^n + \cdots + c_n x^0}{(a_1 x + b_1) \times \cdots \times (a_{m \geq n} x + b_{m \geq n})} = \frac{A_1}{a_1 x + b_1} + \cdots + \frac{A_{m \geq n}}{(a_{m \geq n} x + b_{m \geq n})}$$

*solve with $\int \frac{a}{bx+c}\, dx = a \ln |bx + c| + C$

CASE: 2 linear factors of degree greater than 1

$$\frac{c_1 x^n + \cdots + c_n x^0}{(a_1 x + b_1)^{m \geq n}} = \frac{A_1}{a_1 x + b_1} + \cdots + \frac{A_m}{(a_1 x + b_1)^m}$$

*solve with $\int \frac{a}{bx+c}\, dx = a \ln |bx + c| + C$ and u-substitution to make integrals of the same form

CASE: 3 irreducible quadratic factors of degree 1

$$\frac{a_1x^0+\cdots+a_{2n}x^{2n-1}}{(\text{quadratic}_1)\times\cdots\times(\text{quadratic}_n)}=\frac{A_1x+B_1}{\text{quadratic}_1}+\cdots+\frac{A_nx+B_n}{\text{quadratic}_n}$$

*solve using u-substitution, and/or completing the square followed by making use of the derivative of arctan

CASE: 4 irreducible quadratic factors of degree greater than 1

$$\frac{a_1x^0+\cdots a_{2n-1}x^{2n-1}}{(\text{quadratic})^n}=\frac{A_1x+B_1}{\text{quadratic}}+\cdots+\frac{A_nx+B_n}{(\text{quadratic})^n}$$

Selected Questions:

39. $\displaystyle\int \frac{1}{(x\sqrt{x+1})}\,dx$

$$\begin{aligned}\int \frac{1}{(x\sqrt{x+1})}\,dx &= 2\int \frac{1}{x}\cdot\frac{a}{2\sqrt{x+1}}\,dx\\ &= (u=\sqrt{x+1})\\ &= 2\int \frac{1}{u^2-1}\,du\\ &= \text{ trivial}\end{aligned}$$

40. $\displaystyle\int \frac{dx}{2\sqrt{x+3}+x}$

$$\begin{aligned}\int \frac{dx}{2\sqrt{x+3}+x} &= (u=\sqrt{x+3})\\ &= \int \frac{2u\ du}{2u+u^2-3}\\ &= \text{ trivial}\end{aligned}$$

41. $\displaystyle\int \frac{\sqrt{x}}{x-4}\,dx$

$$\begin{aligned}\int \frac{\sqrt{x}}{x-4}\,dx &= (u=\sqrt{x})\\ &= \int \frac{2u^2\ du}{u^2-4}\\ &= \text{ trivial}\end{aligned}$$

42. $\displaystyle\int \frac{1}{1+\sqrt[3]{x}}\,dx$

$$\begin{aligned}\int \frac{1}{1+\sqrt[3]{x}}\,dx &= (u=\sqrt[3]{x})\\ &= \int \frac{3u^2\ du}{1+u}\\ &= \text{ trivial}\end{aligned}$$

43. $\int \frac{x^3}{\sqrt[3]{x^2+1}}\ dx$

$$\begin{aligned}
\int \frac{x^3}{\sqrt[3]{x^2+1}}\ dx &= \int \frac{x^2}{\sqrt[3]{x^2+1}}\ xdx \\
&= (u = \sqrt[3]{x^2+1};\ du = 2x\ dx) \\
&= \frac{1}{2}\int \frac{u^3-1}{u}\ du \\
&= \text{ trivial}
\end{aligned}$$

46. $\int \frac{\sqrt{1+\sqrt{x}}}{x}\ dx$

$$\begin{aligned}
\int \frac{\sqrt{1+\sqrt{x}}}{x}\ dx &= (u = \sqrt{x};\ u^2 = x;\ dx = 2u\ du) \\
&= 2\int \frac{\sqrt{1+u}}{u^2}\ du \\
&= \text{ trivial after onemore rationalizing substitution}
\end{aligned}$$