We discussed improper integrals, which are integrals with unbounded limits of integration.

$$\int_a^\infty f(x)\ dx = \lim_{b\to\infty}\int_a^b f(x)\ dx \tag{1}$$
$$\int_{-\infty}^a f(x)\ dx = \lim_{c\to-\infty}\int_c^a f(x)\ dx \tag{2}$$
$$\int_{-\infty}^\infty f(x)\ dx = \lim_{b\to\infty}\lim_{c\to-\infty}\int_c^b f(x)\ dx \tag{3}$$

If the limit exists, then the integral converges; if it does not, then the integral is divergent.

Integrals of the form

$$\int_1^\infty \frac{1}{x^P}\ dx$$

are convergent if $P > 1$, and divergent for $P \leq 1$. This integral is often used for comparison tests.

EXAMPLE $\displaystyle\int_1^\infty \frac{3+\sin 2x + e^{-x}}{x^2+1}\ dx$

$$3+\sin 2x + e^{-x} < 5 \Rightarrow \frac{3+\sin 2x+e^{-x}}{x^2+1} < \frac{5}{x^2+1} < \frac{5}{x^2}$$

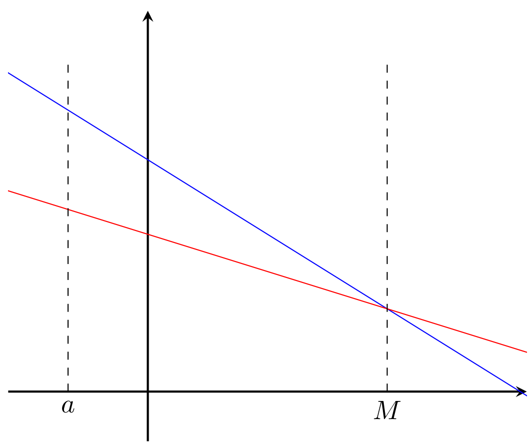
Compare to $\displaystyle\int_1^\infty \frac{5}{x^2}\ dx = -\frac{5}{x}\Big|_1^\infty \Rightarrow$ converges.

Comparison Test 1

$$I_1 = \int_a^\infty f(x)\ dx \qquad I_2 = \int_b^\infty g(x)\ dx$$

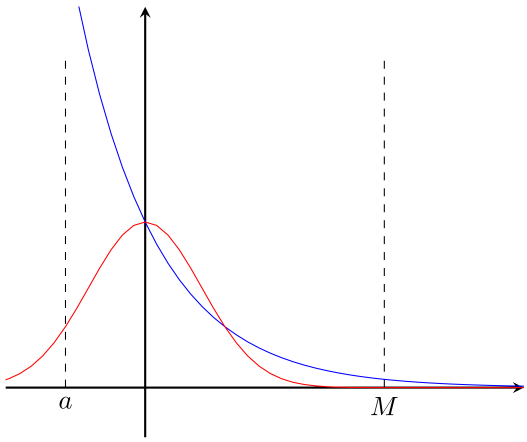
$$f(x) \leq g(x) \quad \text{for all} \quad x > M$$

If I_1 diverges, then I_2 diverges; if I_2 converges, then I_1 converges.



EXAMPLE $\int_0^\infty e^{-x^2}\ dx$

$$\text{for } x > 1,\ x^2 > x \Rightarrow e^{-x^2} < e^{-x}$$



So, $\int_0^\infty e^{-x^2}\ dx = \int_0^1 e^{-x^2}\ dx + \int_1^\infty e^{-x^2}\ dx = \mathit{finite\ number}\ +\ \mathit{comparison\ test\ integral}$

$$\int_1^\infty e^{-x}\ dx = e^{-1} \therefore\ \mathit{integral\ is\ convergent}$$

The following substitution is also useful,

$$\int_{-\infty}^{c} f(x)\ dx = (x = -t) = \int_{\infty}^{-c} f(-t)\ d(-t) = \int_{-c}^{\infty} f(-t)\ dt$$