Exercise 1.2.6.
Find the standard equation of the circle passing through $(-2, 1)$ and tangent to the line $3x - 2y = 6$ at the point $(4, 3)$. Sketch.

1. We will solve $3x - 2y = 6$ for y and graph it.

$$\begin{aligned} 3x - 2y &= 6 \\ 3x - 6 &= 2y \\ \frac{3}{2}x - 3 &= y \end{aligned}$$

2. We will plot the points $(-2, 1)$ and $(4, 3)$.

3. We will graph the line perpendicular to $y = \frac{3}{2}x - 3$ that passes through the point $(4, 3)$.

   We know the slope is the negative reciprocal of $\frac{3}{2}$ and we have a point so we can solve for the y intercept by plugging these values into the $y = mx + b$ form.

$$\begin{aligned} y &= mx + b \\ 3 &= -\frac{2}{3}(4) + b \\ \frac{9}{3} &= -\frac{8}{3} + b \\ b &= \frac{17}{3} \\ y &= -\frac{2}{3}x + \frac{17}{3} \end{aligned}$$

4. the distance from the center of the circle to the point $(4, 3)$ is the same as the distance to the point $(-2, 1)$. So we will set the distance formulas equal to each other and solve for y. Then we will graph this and find its intercept with $y = \frac{3}{2}x - 3$ which will be the center of the circle.

$$\begin{aligned} \sqrt{(x - (-2))^2 + (y - 1)^2} &= \sqrt{(x - 4)^2 + (y - 3)^2} \\ (x - (-2))^2 + (y - 1)^2 &= (x - 4)^2 + (y - 3)^2 \\ (x + 2)^2 + (y - 1)^2 &= (x - 4)^2 + (y - 3)^2 \\ x^2 + 4x + 4 + y^2 - 2y + 1 &= x^2 - 8x + 16 + y^2 - 6y + 9 \\ 4x - 2y + 5 &= -8x - 6y + 25 \\ 4y &= -12x + 20 \\ y &= -3x + 5 \end{aligned}$$

5. now we will solve for the intercept of $y = -3x + 5$ and $y = -\frac{2}{3}x + \frac{17}{3}$, and plot it.

$$\begin{aligned}
-3x + 5 &= -\frac{2}{3}x + \frac{17}{3} \\
-\frac{9}{3}x + \frac{15}{3} &= -\frac{2}{3}x + \frac{17}{3} \\
\frac{7}{3} &= -\frac{2}{3} \\
x &= -\frac{2}{7} \\
y &= -3\left(-\frac{2}{7}\right) + \frac{35}{7} \\
y &= \frac{41}{7}
\end{aligned}$$

6. Now we will solve for the radius of the circle.

$$\begin{aligned}
\sqrt{(\Delta x)^2 + (\Delta y)^2} &= \sqrt{\left(4 - \left(-\frac{2}{7}\right)\right)^2 + \left(3 - \frac{41}{7}\right)^2} \\
&= \sqrt{\left(\frac{30}{7}\right)^2 + \left(\frac{400}{49}\right)^2} \\
&= \sqrt{\frac{1300}{49}} \\
&= \frac{10\sqrt{13}}{7}
\end{aligned}$$

7. now we express the formula for the circle, and draw it.

$$\left(x + \frac{2}{7}\right)^2 + \left(y - \frac{41}{7}\right)^2 = \frac{1300}{49}$$