Use Of LaTeX And Other Software For Illustrating Mathematical Concepts

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April 19, 2024

Outline

The Basic Problem That I Studied

How Mathematicians Make Math Diagrams

My work focused on using LaTeX to illustrate some mathematical concepts related to analysis in the extended complex plane, with special consideration being given to the stereographic projection, a tool for understanding complex transformations.

What Is LaTeX Anyway?

LaTeX Is Not Rubber

LaTeX is a document preparation system that enables mathematical communication without data loss.

Where Do Textbook Diagrams Come From?

There are numerous packages which extend the capabilities of LaTeX. Some important graphical packages include;

1. Ti*k*Z And Its Related Packages/Libraries
   1. pgfplots, tikz-3dplot & tkz-euclide packages
   2. spath3 library
2. Asymptote & SageTeX

Making Algorithms That Write LaTeX

You can design algorithms which iteratively make frames of paramaterized graphs that can be appended into a GIF.

The Geometry Of Complex Numbers

What Are Complex Numbers?

Complex numbers are binomials of real and imaginary components; they can be represented as 2-tuples, enabling them to be plotted as points in a Cartesian plane; this is called the *complex plane*, $\mathbb{C}$.

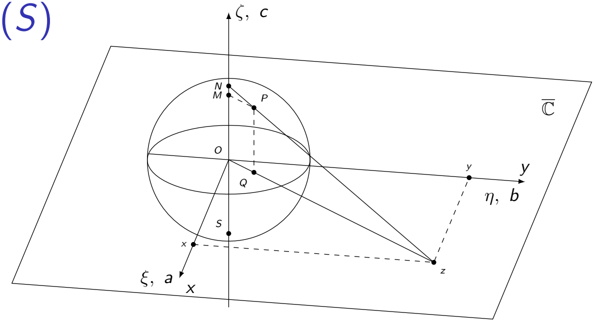
The Polar Form Of A Complex Number

Euler's Formula, $e^{i\theta} = \cos(\theta) + i\sin(\theta)$, lets us represent complex numbers in terms of an angle (the *argument*, $\arg(z)$) and a radius (the *modulus*, $|z|$). That is, $z = |z|e^{i\arg(z)}$.

Complex Arithmetic

To add complex numbers, we separately sum the real and imaginary parts. For multiplication, the polar representation is useful because complex numbers obey the fundamental exponential properties. For instance, $e^{i\theta}e^{i\mu} = e^{i(\theta+\mu)}$

The Riemann Sphere (Σ) And Its Stereographic Projection (S)



$$\boxed{\Sigma : \xi^2 + \eta^2 + \zeta^2 = 1}$$

$$\triangle NzO \sim \triangle NPM \Rightarrow \frac{\overline{ON}}{\overline{MN}} = \frac{\overline{Oz}}{\overline{MP}} = \frac{\overline{Oz}}{\overline{OQ}}$$

$$\Rightarrow \frac{\overline{Oz}}{\overline{OQ}} = \frac{\overline{ON}}{\overline{MN}} = \frac{1}{1-\zeta}$$

$$\Rightarrow S : (\xi, \eta, \zeta) \to (x, y, 0) = \frac{1}{1-\zeta}(\xi, \eta, 0) = \frac{\xi + i\eta}{1-\zeta} \qquad (1)$$

Defining The Extended Complex Plane ($\overline{\mathbb{C}}$)

The Stereographic Image Of N

All points on Σ excepting N have a stereographic image on $\mathbb{C}$.

Similar to how parallel lines when drawn in perspective meet at a vanishing point, as you increase your radial distance infinitely from the origin **in any direction** on $\mathbb{C}$, the inverse stereographic mapping converges at N on Σ. So we can suggest that the stereographic image of N meets $\mathbb{C}$ at ∞.

This lets us define the extended complex plane $\overline{\mathbb{C}}$, which is the union of $\mathbb{C}$ and the point-at-infinity, $\{\infty\}$.

While not all algebraic properties are preserved for the point at infinity, the following four prove to be useful;

$$\frac{a}{0} = \infty; \;\; a + \infty = \infty; \;\; a \cdot \infty = \infty; \;\; \frac{a}{\infty} = 0$$

Deriving $S^{-1} : \overline{\mathbb{C}} \to \Sigma$ I

The Inverse Of Stereographic Projection

$$\text{Recall (1):} \quad S : P = \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix} \to z = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = \frac{\xi + i\eta}{1 - \zeta}$$

$$\Rightarrow x = \frac{\xi}{1-\zeta}; \;\; y = \frac{\eta}{1-\zeta} \tag{2}$$

$$\text{From (2):} \quad |z|^2 = z\bar{z} = (x+iy)(x-iy) = \frac{\xi^2 + \eta^2}{(1-\zeta)^2}$$

$$\text{Notice:} \quad \Sigma : \xi^2 + \eta^2 + \zeta^2 = 1 \Rightarrow \xi^2 + \eta^2 = 1 - \zeta^2 \tag{3}$$

Deriving $S^{-1} : \overline{\mathbb{C}} \to \Sigma$ II

We use difference of squares to obtain

$$\frac{\xi^2+\eta^2}{(1-\zeta)^2} = \frac{1-\zeta^2}{(1-\zeta)^2} = \frac{1+\zeta}{1-\zeta}$$

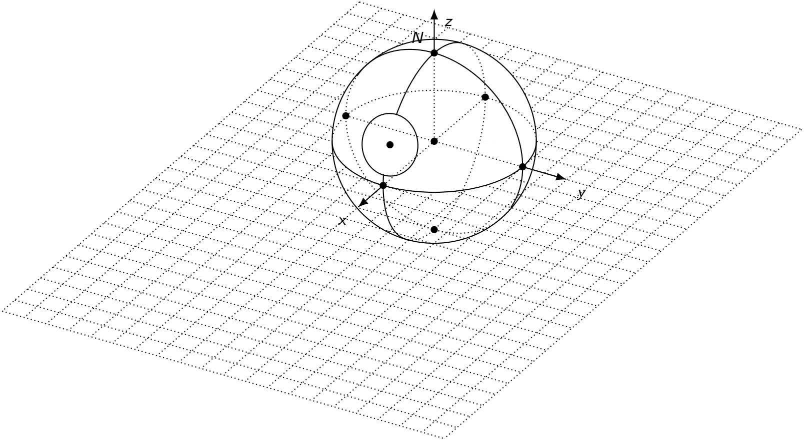
Allowing us to define ζ in terms of z;

$$\Rightarrow 1+\zeta = |z|^2 - |z|^2\zeta \Rightarrow \zeta(|z|^2+1) = |z|^2 = 1 \Rightarrow \boxed{\zeta = \frac{|z|^2-1}{|z|^2+1}}$$

Which lets us extract ξ and η from (1):

$$\Rightarrow 1-\zeta = 1 - \frac{|z|^2-1}{|z|^2+1} = \frac{2}{|z|^2+1} \Rightarrow \left.\begin{aligned} \xi &= \frac{2x}{1+|z|^2} \\ \eta &= \frac{2y}{1+|z|^2} \\ \zeta &= \frac{|z|^2-1}{|z|^2+1} \end{aligned}\right\} \tag{4}$$

Intersecting Σ With Planes I



Intersecting Σ With Planes (II)

Circles Are Obtained By Cutting Σ By Planes

$$\text{Plane} : A\xi + B\eta + C\zeta + D = 0 \tag{5}$$

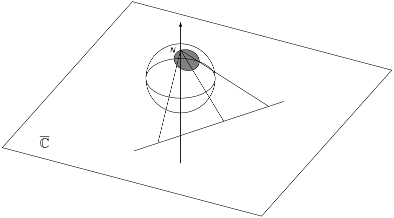
To see the maps of the circles on $\overline{\mathbb{C}}$, we substitute (4) into (5);

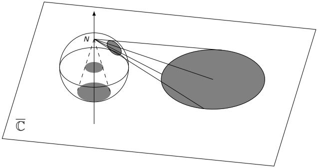
$$\frac{2Ax}{1+|z|^2} + \frac{2By}{1+|z|^2} + \frac{C(|z|^2-1)}{|z|^2+1} + D = 0 \tag{6}$$

Then we multiply (6) by $|z|^2 + 1$ and simplify, getting;

$$2Ax + 2By + (C+D)|z|^2 - C + D = 0 \tag{7}$$

The Intersection Set On $\overline{\mathbb{C}}$





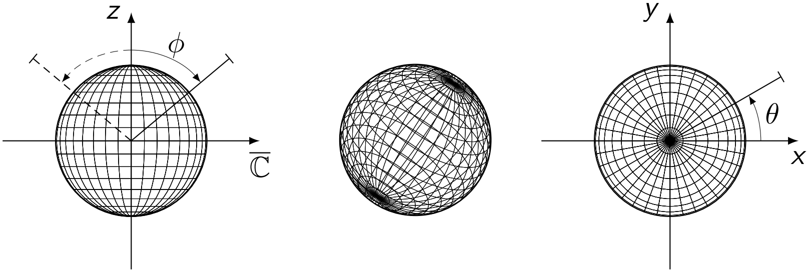
Planes Through N Make Lines

(*i*) If $C + D = 0$, then (7) reduces to $Ax + By - C = 0$ - a straight line in $\overline{\mathbb{C}}$. Geometrically, this means the plane intersects N.

(*ii*) If $C + D \neq 0$, then we can divide (7) by $(C + D)$ and complete the square, giving the equation of a circle;

$$\left(x + \frac{A}{C+D}\right)^2 + \left(y + \frac{B}{C+D}\right)^2 = \frac{A^2 + B^2}{(C+D)^2} + D - C$$

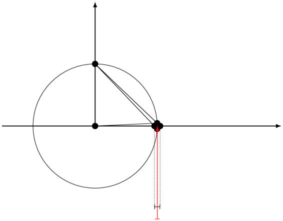
Spherical Coordinates On Σ and $\overline{\mathbb{C}}$ I

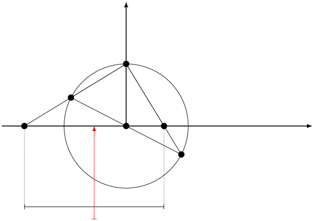


Latitudinal And Longitudinal (Spherical) Coordinates

$$0 \leq \phi \leq 180 \qquad 0 \leq \theta \leq 360$$

Spherical Coordinates On Σ and $\overline{\mathbb{C}}$ II



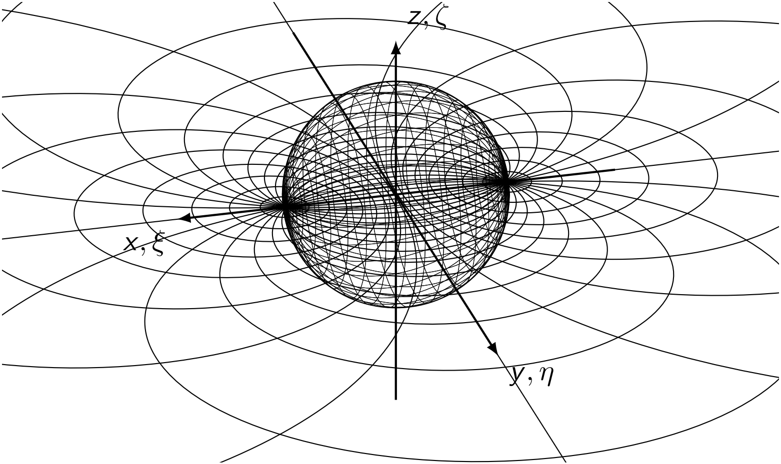


Latitudinal And Longitudinal Lines Of A Rotated Σ on $\overline{\mathbb{C}}$

$$C_{lat} = \frac{1}{\cos\theta} \qquad R_{lat} = \sqrt{\frac{1}{\cos^2\theta} - \cos\theta}$$

$$C_{lon} = -\frac{\sin\theta}{\cos\theta} \qquad R_{lon} = \frac{1}{\cos\theta}$$

A Rotation Of Σ On $\overline{\mathbb{C}}$ In Spherical Coordinates



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Thank You!

Questions And Comments

I want to hear what you think; if you're interested in making your own math diagrams, feel free to approach me and I can connect you to resources.