1.

$$\begin{aligned}\int \frac{1}{x^2\sqrt{x^2-9}}\,dx &\xlongequal[dx=3\sec\theta\tan\theta\ d\theta]{x=3\sec\theta} \int \frac{1}{3^3\sec^2\theta\cdot 3\tan\theta}\cdot 3\sec\theta\tan\theta\ d\theta\\ &= \frac{1}{9}\int \cos\theta\ d\theta\\ &= \frac{1}{9}\sin\theta + C\end{aligned}$$

2.

$$\begin{aligned}\int x^3\sqrt{9-x^2}\,dx &\xlongequal[dx=3\cos\theta\ d\theta]{x=3\sin\theta} \int 3^3\sin^3\theta\cdot 3\cos\theta\cdot 3\cos\theta\ d\theta\\ &= 3^5\int \sin^3\theta\cos^2\theta\ d\theta\\ &= 3^5\int \sin\theta(\cos^2\theta-\cos^4\theta)\ d\theta\\ &\xlongequal[du=-\sin\theta\ d\theta]{u=\cos\theta} -3^5\int \cos^2\theta-\cos^4\theta\ d\theta\\ &= -3^5\int \frac{1}{2}(1+\cos(2\theta))-(1+2\cos(2\theta))^2\ d\theta\\ &= -3^3\int \frac{1}{2}+\cos(2\theta)-1-2\cos(2\theta)-\cos^2(2\theta)\ d\theta\\ &= -3^5\int -\frac{1}{2}-\cos(2\theta)-\frac{1}{2}-\frac{1}{2}\cos(4\theta)\ d\theta\\ &= 3^5\int 1+\cos(2\theta)+\frac{1}{2}\cos(4\theta)\ d\theta\\ &= 3^5\theta+\frac{3^5}{2}\sin(2\theta)+\frac{3^5}{8}\sin(4\theta)+C\end{aligned}$$

$\boxed{3.}$

$$\begin{aligned}\int \frac{x^3}{\sqrt{x^2+9}}\,dx &\xlongequal[dx=3\sec^2\theta\ d\theta]{x=3\tan\theta} \int \frac{3^3\tan^3\theta}{3\sec\theta}\cdot 3\sec^2\theta\ d\theta\\ &= 3^3\int \tan^3\theta\sec\theta\ d\theta\\ &= 3^3\int \sec\theta\tan\theta\sec^2\theta+\sec\theta\tan\theta\ d\theta\\ &\xlongequal[\sec\theta\tan\theta\ d\theta]{u=\sec\theta} 3^3\int u^2+1\ du\\ &= 9u^3+27u+C\end{aligned}$$

4.

$$\begin{aligned}\int_0^{2\sqrt{3}} \frac{x^3}{\sqrt{16-x^2}}\,dx &\xlongequal[dx=4\cos\theta\ d\theta]{x=4\sin\theta} \int_0^{\frac{\pi}{3}} \frac{4^3\sin^3\theta}{4\cos\theta}\cdot 4\cos\theta\ d\theta\\ &= 4^3\int_0^{\frac{\pi}{3}} \sin\theta(1-\cos^2\theta)\ d\theta\\ &= 4^3\int_0^{\frac{\pi}{3}} \sin\theta\ d\theta + 4^3\int_0^{\frac{\pi}{3}} -\sin\theta\cos^2\theta\ d\theta\\ &\xlongequal[du=-\sin\theta\ d\theta]{u=\cos\theta} -4^3[\cos\theta]_0^{\frac{\pi}{3}} + 4^3\int_1^{\frac{1}{2}} u^2\ du\\ &= 2^5+\frac{2^6}{3}u^3\Big|_1^{\frac{1}{2}}\\ &= \frac{40}{3}\end{aligned}$$

5.

$$\int_{\sqrt{2}}^{2} \frac{1}{t^3\sqrt{t^2-1}}\ dt \xlongequal[dt=\sec\theta\tan\theta\ d\theta]{t=\sec\theta} \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{\sec^3\theta\tan\theta}\cdot\sec\theta\tan\theta\ d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \cos^2\theta\ d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} 1+\cos(2\theta)\ d\theta$$

$$= \theta + \frac{1}{2}\sin(2\theta)\Big|_{\frac{\pi}{4}}^{\frac{\pi}{3}}$$

$$= [(\frac{\pi}{3}+\frac{1}{2}\cdot\frac{\sqrt{3}}{2}) - (\frac{\pi}{4}+\frac{1}{2}\cdot 1)]$$

$$= \frac{\pi+3\sqrt{3}-6}{12}$$

6.

$$\int_{1}^{2} \frac{\sqrt{x^2-1}}{x}\ dx \xlongequal[dx=\sec\theta\tan\theta\ d\theta]{x=\sec\theta} \int_{0}^{\frac{\pi}{3}} \tan^2\theta\ d\theta$$

$$= \int_{0}^{\frac{\pi}{3}} \sec^2\theta - 1\ d\theta$$

$$= \tan\theta - \theta\Big|_{0}^{\frac{\pi}{3}}$$

$$= \sqrt{3} - \frac{\pi}{3}$$

$\boxed{7.}$

$$\int \frac{1}{x^2\sqrt{25-x^2}}\ dx \xlongequal[dx=5\cos\theta\ d\theta]{x=5\sin\theta} \int \frac{1}{25\sin\theta\cdot 5\cos\theta}\cdot 5\cos\theta\ d\theta$$

$$= \frac{1}{25}\int \sec\theta\ d\theta$$

$$= \frac{1}{25}\int \frac{\sec^2\theta+\sec\theta\tan\theta}{\sec\theta+\tan\theta}\ d\theta$$

$$\xlongequal[du=\sec\theta\tan\theta+\sec^2\theta\ d\theta]{u=\sec\theta+\tan\theta} \frac{1}{25}\int \frac{1}{u}\ du$$

$$= \frac{1}{25}\ln|\sec\theta+\tan\theta| + C$$

8.

$$\int \frac{x^3}{\sqrt{x^2+100}}\ dx \xlongequal[dx=10\sec^2\theta\ d\theta]{x=10\tan\theta} 100\int \cos\theta\ d\theta$$

$$= 100\sin\theta + C$$

9.

$$\int \frac{dx}{\sqrt{x^2+16}} \xlongequal[dx=4\sec^2\theta\ d\theta]{x=4\tan\theta} \frac{1}{4}\int \cos\theta\ d\theta$$

$$= \frac{1}{4}\sin\theta + C$$

10.

$$
\begin{aligned}
\int \frac{t^5}{\sqrt{t^2+2}}\ dt \xlongequal[dt=\sqrt{2}\sec^2\theta\ d\theta]{t=\sqrt{2}\tan\theta} &\int \frac{2^{\frac{5}{2}}\tan^5\theta}{\sqrt{2}\sec\theta}\cdot\sqrt{2}\sec^2\theta\ d\theta \\
&= 2^{\frac{5}{2}}\int \sin^5\theta\cos^4\theta\ d\theta \\
&= 2^{\frac{5}{2}}\int \sin\theta(1-\cos^2\theta)^2\cos^4\theta\ d\theta \\
&= 2^{\frac{5}{2}}\int \sin\theta(1-\cos^2\theta)(\cos^4\theta-\cos^6\theta)\ d\theta \\
&= -2^{\frac{5}{2}}\int -\sin\theta(\cos^4\theta-2\cos^6\theta+\cos^8\theta)\ d\theta \\
&\xlongequal[du=-\sin\theta\ d\theta]{u=\cos\theta} -2^{\frac{5}{2}}\int u^4-2u^6+u^8\ du \\
&= -2^{\frac{5}{2}}[\frac{u^5}{5}-2\frac{u^7}{7}+\frac{u^9}{9}]+C
\end{aligned}
$$

11.

$$
\begin{aligned}
\int \sqrt{1-4x^2}\ dx \xlongequal[du=2\ dx]{u=2x} &\frac{1}{2}\int\sqrt{1-u^2}\ du \\
&\xlongequal[du=\cos\theta\ d\theta]{u=\sin\theta} \frac{1}{2}\int \cos^2\theta\ d\theta \\
&= \frac{1}{4}\int 1+\cos(2\theta)\ d\theta \\
&= \frac{1}{4}[\theta+\frac{1}{2}\sin(2\theta)]+C
\end{aligned}
$$

12.

$$
\begin{aligned}
\int_0^1 x\sqrt{x^2+4}\ dx \xlongequal[dx=2\sec^2\theta\ d\theta]{x=2\tan\theta} &8\int_0^{\arctan\frac{1}{2}} \tan\theta\sec^3\theta\ d\theta \\
&\xlongequal[du=\sec\theta\tan\theta\ d\theta]{u=\sec\theta} 8\int_1^{\sec(\arctan\frac{1}{2})} u^2\ du \\
&= \frac{8}{3}u^3\Big]_1^{\sec(\arctan\frac{1}{2})}\Big|_{u=\sec\theta} \\
&= \frac{8}{3}\sec^3\theta\Big]_0^{\arctan\frac{1}{2}} \\
&= \frac{8}{3}(1+\tan^2\theta)^{\frac{3}{2}}\Big]_0^{\arctan\frac{1}{2}}\Big|_{\theta=\arctan\frac{x}{2}} \\
&= \frac{8}{3}(1+\tan^2(\arctan\frac{x}{2}))^{\frac{3}{2}}\Big]_0^1 \\
&= \frac{8}{3}(1+\frac{x^2}{4})^{\frac{3}{2}}\Big]_0^1 \\
&= \frac{5\sqrt{5}-8}{3}
\end{aligned}
$$

$\boxed{13.}$

$$
\begin{aligned}
\int \frac{\sqrt{x^2-9}}{x^3}\ dx \xlongequal[dx=3\sec\theta\tan\theta\ d\theta]{x=3\sec\theta} &\frac{1}{3}\int \sin^2\theta \\
&= \frac{1}{3}\int 1-\cos(2\theta)\ d\theta \\
&= \frac{\theta}{3}-\frac{1}{6}\sin(2\theta)+C
\end{aligned}
$$

14.

$$
\begin{aligned}
\int \frac{du}{u\sqrt{5-u^2}} \xlongequal[du=\sqrt{5}\cos\theta\ d\theta]{u=\sqrt{5}\sin\theta} &\int \frac{d\theta}{\sqrt{5}\sin\theta} \\
&= -\frac{1}{\sqrt{5}}\csc\theta\cot\theta+C
\end{aligned}
$$

15.

$$\begin{aligned}
\int_0^a x^2\sqrt{a^2-x^2}\ dx &\xlongequal[dx=a\cos\theta\ d\theta]{x=a\sin\theta} a^4\int_0^a \sin^2\theta\cos^2\theta\ d\theta \\
&= a^4\int_0^a (1-\cos(2\theta))(1+\cos(2\theta))\ d\theta \\
&= a^4\int_0^a 1-\cos^2(2\theta)\ d\theta \\
&= -a^4\int_0^a \cos(4\theta)\ d\theta \\
&= -\frac{a^4}{4}\sin(4\theta)\Big|_0^a \\
&= -\frac{a^4}{4}\sin(4a)
\end{aligned}$$

16.

$$\begin{aligned}
\int_{\sqrt{2}/3}^{2/3}\frac{dx}{x^5\sqrt{9x^2-1}} &\xlongequal[du=3\ dx]{u=3x} 3^4\int_{\sqrt{2}}^{2}\frac{1}{u^5\sqrt{u^2-1}}\ du \\
&\xlongequal[du=\sec\theta\tan\theta\ d\theta]{u=\sec\theta} 3^4\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{\sec\theta\tan\theta}{\sec^5\theta\tan\theta}\ d\theta \\
&= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\cos^4\theta\ d\theta \\
&= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\left(\frac{1+\cos(2\theta)}{2}\right)^2\ d\theta \\
&= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{1}{4}+\frac{1}{2}\cos(2\theta)+\frac{1}{4}\cos^2(2\theta)\ d\theta \\
&= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{3}{8}+\frac{1}{2}\cos(2\theta)+\frac{1}{8}\cos(4\theta)\ d\theta \\
&= \frac{3}{8}\theta+\frac{1}{4}\sin(2\theta)+\frac{1}{32}\sin(4\theta)\Big|_{\frac{\pi}{4}}^{\frac{\pi}{3}} \\
&= \left[\frac{\pi}{8}+\frac{\sqrt{3}}{8}-\frac{\sqrt{3}}{64}\right]-\left[\frac{3\pi}{32}\right]
\end{aligned}$$

$\boxed{17.}$

$$\begin{aligned}
\int\frac{x}{\sqrt{x^2-7}}\ dx &\xlongequal[dx=\sqrt{7}\sec\theta\tan\theta\ d\theta]{x=\sqrt{7}\sec\theta} \int\frac{\sqrt{7}\sec\theta}{\sqrt{7}\tan\theta}\cdot\sqrt{7}\sec\theta\tan\theta\ d\theta \\
&= \sqrt{7}\int\sec^2\theta\ d\theta \\
&= \sqrt{7}\tan\theta+C
\end{aligned}$$

18.

$$\begin{aligned}
\int\frac{dx}{[(ax)^2-b^2]^{3/2}} &= \int\frac{dx}{\left(\sqrt{a^2(x^2-b^2/a^2)}\right)^3} \\
&= \int\frac{dx}{\left(a\sqrt{(x^2-b^2/a^2)}\right)^3} \\
&\xlongequal[dx=\frac{b}{a}\sec\theta\tan\theta\ d\theta]{x=\frac{b}{a}\sec\theta} \int\frac{\frac{b}{a}\sec\theta\tan\theta\ d\theta}{a^3\cdot\frac{b^3}{a^3}\tan^3\theta} \\
&= \frac{1}{ab^2}\int\sec\theta\cot^2\theta\ d\theta \\
&= \frac{1}{ab^2}\int\frac{\cos\theta}{\sin^2\theta}\ d\theta \\
&\xlongequal[du=\cos\theta\ d\theta]{u=\sin\theta}
\end{aligned}$$

19.

$$
\begin{aligned}
\int \frac{\sqrt{1+x^2}}{x}\, dx &\xlongequal[dx=\sec^2\theta\ d\theta]{x=\tan\theta} \int \frac{\sec\theta}{\tan\theta}\cdot\sec^2\theta\ d\theta \\
&= \int \csc\theta\sec^2\theta\ d\theta \\
&= \int \csc\theta + \tan^2\theta\csc\theta \\
&= \int \csc\theta + \tan\theta\sec\theta \\
&= \int \frac{\csc^2\theta+\csc\theta\cot\theta}{\csc\theta+\cot\theta} + \sec\theta\tan\theta\ d\theta \\
&\xlongequal[du=-\csc\theta\cot\theta-\csc^2\theta\ d\theta]{u=\csc\theta+\cot\theta} -\int\frac{du}{u} + \int \sec\theta\tan\theta\ d\theta \\
&= \ln|\csc\theta+\cot\theta| + \sec\theta + C
\end{aligned}
$$

20.

$$
\begin{aligned}
\int \frac{t}{\sqrt{25-t^2}}\ dt &\xlongequal[dt=5\cos\theta\ d\theta]{t=5\sin\theta} \int \frac{5\sin\theta}{5\cos\theta}\cdot 5\cos\theta\ d\theta \\
&= -5\cos\theta + C
\end{aligned}
$$

21.

$$
\begin{aligned}
\int_0^{0.6} \frac{x^2}{\sqrt{9-25x^2}}\ dx &= \int_0^{\frac{3}{5}} \frac{x^2}{5\sqrt{\frac{9}{25}-x^2}}\ dx \\
&\xlongequal[dx=\frac{3}{5}\cos t\ dt]{x=\frac{3}{5}\sin t} \int_0^{\frac{\pi}{2}} \frac{\frac{3^2}{5^2}\sin^2 t}{5\cdot\frac{3}{5}\cos t}\cdot\frac{3}{5}\cos t\ dt \\
&= \frac{3^2}{5^3}\int_0^{\frac{\pi}{2}} \sin^2 t\ dt \\
&= \frac{3^2}{2\cdot 5^3}\int_0^{\frac{\pi}{2}} 1-\cos(2t)\ dt \\
&= \frac{3^2}{2\cdot 5^3}[t-\frac{1}{2}\sin(2t)]_0^{\frac{\pi}{2}} \\
&= \frac{3^2}{2^2\cdot 5^3}\pi
\end{aligned}
$$

22.

23.

$$
\begin{aligned}
\int \sqrt{5+4x-x^2}\ dx &= \int \sqrt{9-(x-2)^2}\ dx \\
&\xlongequal[du=\ dx]{u=x-2} \int \sqrt{9-u^2}\ du \\
&\xlongequal[du=3\cos t\ dt]{u=3\sin t} 3\int \cos u\ du \\
&= 3\sin u + C
\end{aligned}
$$

24.

$$
\begin{aligned}
\int \frac{dt}{\sqrt{t^2-6t+13}} &= \int \frac{dt}{\sqrt{(t-3)^2+4}} \\
&\xlongequal[du=\ dt]{u=t-3} \int \frac{du}{\sqrt{u^2+4}} \\
&\xlongequal[du=2\sec^2 t\ dt]{u=2\tan t} \int \sec t\ dt \\
&= \int \frac{\sec^2 t+\sec t\tan t}{\sec t+\tan t}\ dt \\
&\xlongequal[dw=\sec t\tan t+\sec^2 t\ dt]{w=\sec t+\tan t} \ln|\sec t+\tan t| + C
\end{aligned}
$$

25.

$$\int \frac{x}{\sqrt{x^2+x+1}}\ dx = \int \frac{x}{\sqrt{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}}\ dx$$

$$\xlongequal[dx=\frac{\sqrt{3}}{2}\sec^2\theta\ \ d\theta]{x=\frac{\sqrt{3}}{2}\tan\theta-\frac{1}{2}}$$

26.

$$\begin{aligned}\int \frac{x^2}{(3+4x-4x^2)^{3/2}}\ dx &= \int \frac{x^2}{\left(\sqrt{-4[x^2-4x]+3}\right)^3} \\ &= \int \frac{x^2}{\left(\sqrt{19-4(x-2)^2}\right)^3} \\ &= \frac{1}{8}\int \frac{x^2}{\left(\sqrt{\frac{19}{4}-(x-2)^2}\right)^3}\end{aligned}$$

$$\xlongequal[dx=\frac{\sqrt{19}}{2}\cos\theta+\frac{1}{2}\theta\ \ d\theta]{x=\frac{\sqrt{19}}{2}\sin\theta+2}$$