We proceeded with the lecture material: Integration by Parts. This follows from the product rule of differentiation - essentially, if we are integrating a product, it lets us simplify one of the functions that is being multiplied (through differentiation). You choose which one to differentiate based on whether the resulting integral becomes easier or not. Recall that the $\int$ sign is basically an operation which recovers something based on its differential.

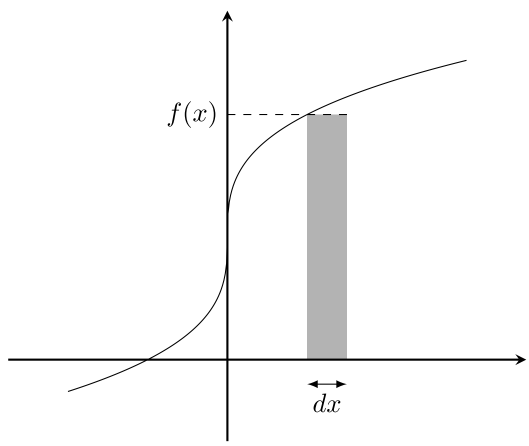
$$\begin{aligned}\frac{d}{dx}[uv] &= u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx} \\ d(uv) &= u\ dv + v\ du \\ u\ dv &= \ d(uv) - v\ du \\ \int u\ dv &= \int\ d(uv) - \int v\ du\end{aligned}$$

$$\int u\ dv = uv - \int v\ du \tag{1}$$

EXAMPLE Solve $\displaystyle\int x3^x\ dx$

Our known methods of substitution don't work here - try them if you want, they only make the integral more complicated. But for this nail, we now have a hammer - Integration by Parts!

$$\begin{aligned}\int x3^x\ dx &= \begin{pmatrix} u = x & du = \ dx \\ dv = 3^x\ dx & v = 3^x / \ln 3\end{pmatrix} \\ &= x\frac{3^x}{\ln 3} - \int \frac{3^x}{\ln 3}\ dx \\ &= \frac{x3^x}{\ln 3} - \frac{3^x}{(\ln 3)^2} + C\end{aligned}$$



This neat little graphic shows how we can derive the area differential from the x differential. That is, $dA = f(x)\ dx$. And we just use the symbol which recovers something from its differential to find the area: $\int\ dA = A = \int f(x)\ dx$.

EXAMPLE Evaluate $\displaystyle\int (x^2+3x)\cos 2x\ dx$

When doing Integration by parts, we choose as u the term which becomes simpler upon differentiation. In this case, the polynomial (x^2+3x) fits this bill.

$$\begin{aligned}
\int (x^2+3x)\cos 2x\ dx &= \begin{pmatrix} u = x^2+3x & du = (2x+3)\ dx \\ dV = \cos 2x\ dx & V = \dfrac{1}{2}\sin 2x\ dx \end{pmatrix} \\
&= (x^2+3x)\cdot\frac{1}{2}\sin 2x - \int \frac{1}{2}\sin(2x)\cdot(2x+3)\ dx \\
&= \frac{1}{2}(x^2+3x)\sin 2x - \frac{1}{2}\int (2x+3)\cdot\sin 2x\ dx \\
&= \begin{pmatrix} 2x+3 = u & 2x\ dx = \ du \\ \sin 2x\ dx = \ dV & \dfrac{1}{2}\cos 2x = V \end{pmatrix} \\
&= \frac{1}{2}(x^2+3x)\sin 2x - \frac{1}{2}\left(-\frac{1}{2}(2x+3)\cos(2x) - \int\left(-\frac{1}{2}\cos(2x)\right)\cdot 2\ dx\right) \\
&= \frac{1}{2}(x^2+3x)\sin(2x) + \frac{1}{4}(2x+3)\cos(2x) - \frac{1}{2}\cdot\frac{1}{2}\sin(2x) + C \\
&= \frac{1}{2}(\sin 2x)(x^2+3x-\frac{1}{2}) + \frac{1}{4}(2x+3)\cos 2x + C
\end{aligned}$$

An alternative to Integration by Parts is Integration by Undetermined Coefficients, but it isn't taught in the course.

Recall:

$$\begin{aligned}
\int \cos(ax+b)\ dx &= \cos(ax)\ d(ax)\cdot\frac{1}{a} \\
&= \frac{1}{a}\int \cos t\ dt \\
&= \frac{1}{a}\sin t + C = \frac{1}{a}\sin(ax) + C
\end{aligned}$$

Likewise,

$$\begin{aligned}
\int \sin(ax+b) &= \cdots \\
\cdots &= -\frac{1}{a}\cos(ax+b) + C
\end{aligned}$$

EXAMPLE Evaluate $\int \arctan x\ dx$

$$\int \arctan x\ dx = \begin{pmatrix} \arctan x = u & du = \dfrac{1}{1+x^2}\ dx \\ dx = \ dv & V = x \end{pmatrix}$$

$$= x \arctan x - \int \frac{x}{1+x^2}\ dx$$

$$= \begin{pmatrix} x^2+1 = t \\ 2x\ dx = \ dt \\ x\ dx = \dfrac{1}{2}\ dt \end{pmatrix}$$

$$= x \arctan x - \frac{1}{2}\int \frac{dt}{t} = x \arctan x - \frac{1}{2}\ln|t| + C$$

$$= x \arctan x - \frac{1}{2}\int \frac{dt}{t} = x \arctan x - \frac{1}{2}\ln|x^2+1| + C$$

EXAMPLE Evaluate $\int 5^{x^2+4} \cdot x\ dx$

$$\int 5^{x^2+4} \cdot x\ dx = \begin{pmatrix} x^2+4 = t \\ 2x\ dx = \ dt \end{pmatrix}$$

$$= \frac{1}{2} \cdot \frac{5^{x^2+4}}{\ln 5} + C$$