$$\text{Recall: } \begin{cases} y = a^x, \quad a > 0 \\ e: \lim_{h \to 0} \dfrac{e^h - 1}{h} = 1 \\ (a^x)' = a^x \lim_{h \to 0} \dfrac{a^h - 1}{h} \end{cases}$$

Now we will define $(a^x)^{-1}$.

$$y = a^x \iff x = \log_a y$$

For x sufficiently large,
$a > 1 \qquad a^x > a^{\text{any}}$

$\log_a x < x^{\text{any positive}}$

Properties of Logarithms

1. $\log_a(xy) = \log_a(x) + \log_a(y)$

   Proof:

$$\begin{gathered} \text{Let } x = a^\alpha \quad y = a^\beta \\ (xy) = a^\alpha a^\beta = a^{\alpha+\beta} \\ \log_a(xy) = \alpha + \beta = \log_a x + \log_a y \end{gathered}$$

2. $\log_a x^k = k \log_a x$

   Proof:

$$\begin{gathered} \text{Let } x = a^\alpha \quad \alpha = \log_a x \\ x^k = (a^\alpha)^k = a^{k\alpha} \\ \log_a x^k = k\alpha = k \log_a x \end{gathered}$$

3. $\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$

   Proof omitted due to similarity to (1)

4. $\log_a x = \frac{\log_b x}{\log_b a}$

   Proof:

$$\begin{gathered} \text{Let } y = \log_a x \\ a^y = x \\ y \log_b(a) = \log_b(x) \\ \therefore \log_a x = \frac{\log_b x}{\log_b a} \end{gathered}$$

5. $\log_a 1 = 0$

   Proof:

$$a^0 = 1 \therefore 0 = \log_a 1$$

6. $\log_e(x) = \ln(x)$ and is called the "natural logarithm."

   Note that by e, Euler's ("Oilers") Constant is meant.

7. $\log_{10} x = \lg(x)$ and is called the "decimal logarithm"

Now, with all of these logarithm rules in our toolkit, we will once again take a crack at differentiating the exponential function - this time with much more success. In fact, due to the fact that the exponential derivative is a scalar multiple of the exponential function, we can with very much ease discover the antiderivative of the exponential function.

$$\begin{aligned}
(a^x)' &= (e^{x \ln a})' \\
&= e^{x \ln a} \cdot \ln a \\
&= a^x \cdot \ln a \\
\Rightarrow \int a^x \ dx &= \frac{1}{\ln a} \int a^x \cdot \ln a \ dx \\
&= \frac{a^x}{\ln a} + C
\end{aligned}$$

And finally, we will use our previously conceived definition of Euler's Constant to derive two more definitions.

Recall: e is the number s.t. $\lim\limits_{h \to 0} \frac{e^h - 1}{h} = 1$

$$\text{Let } e^h - 1 = t \to \left\{ \begin{pmatrix} h \to 0^+ & t \to 0^+ \\ h \to 0^- & t \to 0^- \end{pmatrix} \right.$$

$$\boxed{\lim_{t \to 0} \frac{\ln(1+t)}{t} = 1}$$

$$\text{If } \lim_{x \to \text{ any}} f(x) = 1, \text{ then } \lim_{x \to \text{ any}} \frac{1}{f(x)} = 1$$

$$\text{So, } \lim_{t \to 0} \frac{t}{\ln(1+t)} = 1$$

$$\boxed{\lim_{t \to 0} \frac{1}{t} \ln(1+t) = 1}$$

$$\lim_{t \to 0} \ln(1+t)^{\frac{1}{t}} = 1 = \ln(\lim_{t \to 0} (1+t)^{\frac{1}{t}})$$

$$\boxed{\lim_{t \to 0} (1+t)^{\frac{1}{t}} = e}$$

Let $t = 1/x$

$$\boxed{\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e}$$