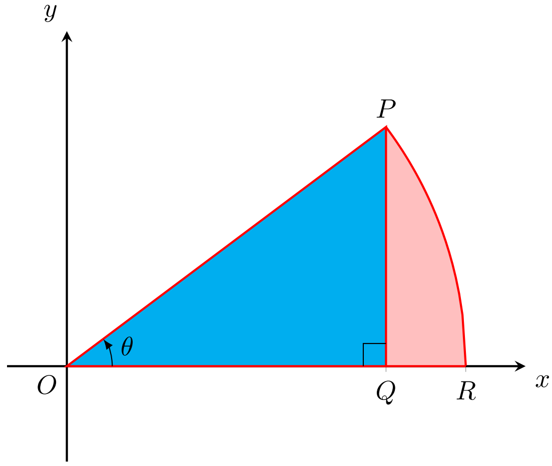
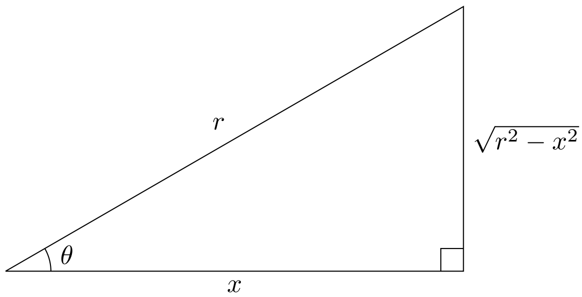
Prove the formula $a = \frac{1}{2}r^2\theta$ for the area of a sector of a circle with radius r and central angle θ. [*Hint:* Assume $0 < \theta < \pi/2$ and place the center of the circle at the origin so it has equation $x^2 + y^2 = r^2$. Then A is the sum of the area of the triangle POQ and the area of the region PQR in the figure.]



$$A = \frac{r^2}{2}\sin\theta\cos\theta + \int_{r\cos\theta}^{r} \sqrt{r^2 - x^2}\ dx$$

let $x = r\cos\theta,\ dx = -r\sin\theta\ d\theta$



Now we will solve the indefinite integral $\int \sqrt{r^2 - x^2}\ dx$.

$$\begin{aligned}
\int \sqrt{r^2 - x^2}\ dx &\xlongequal[dx=-r\sin\theta\ d\theta]{x=r\cos\theta} -r^2 \int \sin^2\theta\ d\theta \\
&= -\frac{r^2}{2}\int 1 - \cos 2\theta\ d\theta \\
&= -\frac{r^2}{2}[\theta - \frac{1}{2}\sin 2\theta] + C \\
&= -\frac{r^2}{2}[\theta - \sin\theta\cos\theta] + C \\
&= -\frac{r^2}{2}[\theta - \frac{\sqrt{r^2 - x^2}}{r}\cdot\frac{x}{r}] + C \\
&= -\frac{r^2}{2}\arccos\frac{x}{r} + \frac{x}{2}\sqrt{r^2 - x^2} + C
\end{aligned}$$

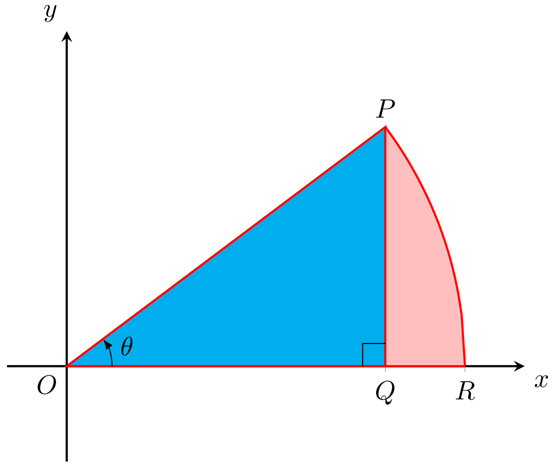
Now we will apply the original boundaries.

$$\begin{aligned}
\int_{r\cos\theta}^{r} \sqrt{r^2 - x^2}\ dx &= \left[-\frac{r^2}{2}\arccos\frac{x}{r} + \frac{x}{2}\sqrt{r^2 - x^2}\right]_{r\cos\theta}^{r} \\
&= \frac{r^2}{2}\theta - \frac{r^2}{2}\sin\theta\cos\theta
\end{aligned}$$

Solving, we get:

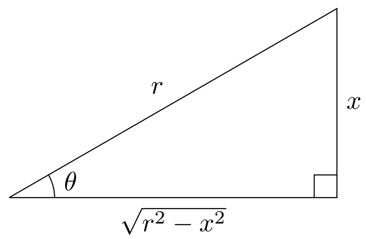
$$\frac{r^2}{2}\sin\theta\cos\theta + \int_{r\cos\theta}^{r} \sqrt{r^2 - x^2}\ dx = \frac{r^2}{2}\sin\theta\cos\theta + \frac{r^2}{2}\theta - \frac{r^2}{2}\sin\theta\cos\theta = \frac{r^2}{2}\theta$$

Prove the formula $a = \frac{1}{2}r^2\theta$ for the area of a sector of a circle with radius r and central angle θ. [*Hint:* Assume $0 < \theta < \pi/2$ and place the center of the circle at the origin so it has equation $x^2 + y^2 = r^2$. Then A is the sum of the area of the triangle POQ and the area of the region PQR in the figure.]



$$A = \frac{r^2}{2}\sin\theta\cos\theta + \int_{r\cos\theta}^{r}\sqrt{r^2 - x^2}\,dx$$

let $x = r\sin\theta,\ dx = r\cos\theta\ d\theta$



Now we will solve the indefinite integral $\int\sqrt{r^2 - x^2}\,dx$.

$$\begin{aligned}
\int\sqrt{r^2 - x^2} &\overset{x = r\sin\theta}{\underset{dx = r\cos\theta\ d\theta}{=}} r^2\int\cos^2\theta\ d\theta \\
&= \frac{r^2}{2}\int 1 + \cos 2\theta\ d\theta \\
&= \frac{r^2}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right] + C \\
&= \frac{r^2}{2}[\theta + \sin\theta\cos\theta] + C \\
&= \frac{r^2}{2}\left[\arcsin\frac{x}{r} + \frac{x}{r}\cdot\frac{\sqrt{r^2 - x^2}}{r}\right] + C
\end{aligned}$$

Now we will apply the original boundary conditions:

$$\begin{aligned}
\frac{r^2}{2}\left[\arcsin\frac{x}{r} + \frac{x}{r}\cdot\frac{\sqrt{r^2 - x^2}}{r}\right]_{r\cos\theta}^{r} &= \left[\frac{r^2}{2}\cdot\frac{\pi}{2} + 0\right] - \left[\frac{r^2}{2}\cdot\arcsin(\cos\theta) + \frac{r^2}{2}\cos\theta\sin\theta\right] \\
&= \left[\frac{r^2}{2}\cdot\frac{\pi}{2} + 0\right] - \left[\frac{r^2}{2}\left(\frac{\pi}{2} - \theta\right) + \frac{r^2}{2}\cos\theta\sin\theta\right] \\
&= \frac{r^2}{2}\theta - \frac{r^2}{2}\cos\theta\sin\theta
\end{aligned}$$

Therefore,

$$\frac{r^2}{2}\sin\theta\cos\theta + \int_{r\cos\theta}^{r}\sqrt{r^2 - x^2}\,dx = \frac{r^2}{2}\sin\theta\cos\theta + \frac{r^2}{2}\theta - \frac{r^2}{2}\cos\theta\sin\theta = \frac{r^2}{2}\theta$$