We are integrating functions of the form:
$R(x) =$ polynomial $+$ proper fraction $=$ polynomial $+\sum$ (partial fractions)

EXAMPLE Evaluate $\displaystyle\int \frac{3x^2+1}{x^3+1}\,dx$

First we use partial fraction decomposition to re-express the integrand as the sum of its partial fractions. Recall the formula $(a^3+b^3) = (a+b)(a^2-ab+b^2)$

$$\begin{aligned}\frac{3x^2+1}{(x+1)(x^2-x+1)} &= \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1} \\ &= \frac{A(x^2-x+1)+(Bx+C)(x+1)}{(x+1)(x^2-x+1)} \\ &= \frac{(A+B)x^2+(-A+B+C)x+(A+C)}{(x+1)(x^2-x+1)}\end{aligned}$$

Now we can obtain a system of linear equations. We could use more advanced techniques, such as Cramer's Rule or Gaussian Elimination, but let's do it the way Dr. Solomonovich showed us.

$$\begin{cases} A+B=3 \\ -A+B+C=0 \\ A+C=1 \end{cases}$$

$$\begin{aligned}&2A+B+C-(-A+B+C)=4 \\ &3A=4 \quad \boxed{A=\frac{4}{3}} \\ &B = 3-\frac{4}{3} = \frac{8}{3} \\ &C = 1-\frac{4}{3} = \frac{-1}{3}\end{aligned}$$

This lets us rewrite the integral;

$$\begin{aligned}&\Rightarrow \frac{4}{3}\int\frac{1}{x+1}\,dx + \frac{1}{3}\int\frac{5x-1}{x^2-x+1}\,dx \\ &= \begin{pmatrix} \text{completed square } = \left(x-\frac{1}{2}\right)^2+\frac{3}{4} \\ x-\frac{1}{2}=t \quad dx = \,dt \quad x=t+\frac{1}{2}\end{pmatrix} \\ &= \frac{4}{3}\ln|x+1| + \frac{1}{3}\int\frac{5t+\frac{5}{2}-1}{t^2+\frac{3}{4}}\,dt \\ &= \frac{4}{3}\ln|x+1| + \frac{1}{3}\left[\int\frac{t}{t^2+\frac{3}{4}}\,dt + \int\frac{3/2}{t^2+\frac{3}{4}}\,dt\right]\end{aligned}$$

The following formula is easy to derive using the difference of squares formula and partial fractions; that being said, Dr. Solomonovich thinks it is useful to remember. I choose to do it by derivation because I can't remember all these formulas, but it's up to you.

$$\boxed{\int \frac{1}{x^2 - a^2}\, dx = \frac{1}{2}\ln\left|\frac{x-a}{x+a}\right| + C}$$

We then went over the fourth (and final) partial fraction decomposition technique - for denominators which contain powers of irreducible quadratics.

$$\begin{aligned}\int \frac{Ax+B}{(x^2 - px + q)^k}\, dx &\overset{\text{complete}}{\underset{\text{square}}{=\!=\!=}} \int \frac{At+C}{(t^2+a^2)^k}\, dt \\ &= A\int \frac{t}{(t^2+a^2)^k}\, dt + C\int \frac{1}{(t^2+a^2)^k}\, dt\end{aligned}$$

Both terms are worthy of their own questions, tbh. So, we will treat them separately.

$$\begin{aligned}A\int \frac{t}{(t^2+a^2)^k}\, dt &= \begin{pmatrix} t\ dt = \frac{1}{2}\ d(t^2+a^2) \\ t^2 + a^2 = u \end{pmatrix} \\ &= \frac{1}{2}\int \frac{du}{u^k} = \frac{1}{2}\frac{u^{-k+1}}{-k+1} + C \\ &\overset{\text{back}}{\underset{\text{substitute}}{=\!=\!=}} \frac{1}{2}\frac{(t^2+a^2)^{-k+1}}{-k+1} + C\end{aligned}$$

Then we did the second one;

$$
\begin{aligned}
\int \frac{1}{(t^2+a^2)^k}\,dt &= \frac{1}{a^2}\int \frac{a^2}{(t^2+a^2)^k}\,dt \\
&= \frac{1}{a^2}\int \frac{a^2+t^2-t^2}{(t^2+a^2)^k}\,dt \\
&= \frac{1}{a^2}\left[\int (t^2+a^2)^{-k+1}\,dt - \int \frac{t^2}{t^2+a^2}\,dt\right] \\
&= \begin{pmatrix} t = u & du = \ dt \\ dV = \dfrac{t\ dt}{(t^2+a^2)^k} & V = \dfrac{1}{2}\displaystyle\int \dfrac{d\ (t^2+a^2)}{(t^2+a^2)^k} \\ & \quad = \dfrac{1}{2}(t^2+a^2)^{-k+1} \end{pmatrix} \\
&= \left[t\cdot\frac{1}{2}\frac{(t^2+a^2)^{-k+1}}{-k+1} - \frac{1}{2(-k+1)}\cdot\int \frac{dt}{(t^2+a^2)^{k-1}}\right] \\
&= \left[\frac{t}{2(1-k)}\cdot(t^2+a^2)^{1-k} - \frac{1}{2(-k+1)}\cdot\int (t^2+a^2)^{-k+1}\,dt\right] \\
&\text{So, } \int \frac{1}{(t^2+a^2)^k}\,dt = \frac{1}{a^2}\left[\int (t^2+a^2)^{-k+1}\,dt\right. \\
&\qquad \left. - \left[\frac{t}{2(1-k)}\cdot(t^2+a^2)^{1-k} - \frac{1}{2(-k+1)}\cdot\int (t^2+a^2)^{-k+1}\,dt\right]\right]
\end{aligned}
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And Dr. Solomonovich mentioned integration by trigonometric substitution
(Friendly heads up: if you haven't *mastered* the trig formulas yet, *now* is the time.)