

We are integrating functions of the form:

$$R(x) = \text{polynomial} + \text{proper fraction} = \text{polynomial} + \sum (\text{partial fractions})$$

**EXAMPLE** Evaluate  $\int \frac{3x^2 + 1}{x^3 + 1} dx$

First we use partial fraction decomposition to re-express the integrand as the sum of its partial fractions. Recall the formula  $(a^3 + b^3) = (a + b)(a^2 - ab + b^2)$

$$\begin{aligned} \frac{3x^2 + 1}{(x + 1)(x^2 - x + 1)} &= \frac{A}{x + 1} + \frac{Bx + C}{x^2 - x + 1} \\ &= \frac{A(x^2 - x + 1) + (Bx + C)(x + 1)}{(x + 1)(x^2 - x + 1)} \\ &= \frac{(A + B)x^2 + (-A + B + C)x + (A + C)}{(x + 1)(x^2 - x + 1)} \end{aligned}$$

Now we can obtain a system of linear equations. We could use more advanced techniques, such as Cramer's Rule or Gaussian Elimination, but let's do it the way Our Math Professor showed us.

$$\begin{cases} A + B = 3 \\ -A + B + C = 0 \\ A + C = 1 \end{cases}$$

$$2A + B + C - (-A + B + C) = 4$$

$$3A = 4 \quad \boxed{A = \frac{4}{3}}$$

$$B = 3 - \frac{4}{3} = \frac{8}{3}$$

$$C = 1 - \frac{4}{3} = \frac{-1}{3}$$

This lets us rewrite the integral;

$$\begin{aligned} &\Rightarrow \frac{4}{3} \int \frac{1}{x + 1} dx + \frac{1}{3} \int \frac{5x - 1}{x^2 - x + 1} dx \\ &= \left( \begin{array}{l} \text{completed square} = \left(x - \frac{1}{2}\right)^2 + \frac{3}{4} \\ x - \frac{1}{2} = t \quad dx = dt \quad x = t + \frac{1}{2} \end{array} \right) \\ &= \frac{4}{3} \ln|x + 1| + \frac{1}{3} \int \frac{5t + \frac{5}{2} - 1}{t^2 + \frac{3}{4}} dt \\ &= \frac{4}{3} \ln|x + 1| + \frac{1}{3} \left[ \int \frac{t}{t^2 + \frac{3}{4}} dt + \int \frac{3/2}{t^2 + \frac{3}{4}} dt \right] \end{aligned}$$

The following formula is easy to derive using the difference of squares formula and partial fractions; that being said, Our Math Professor thinks it is useful to remember. I choose to do it by derivation because I can't remember all these formulas, but it's up to you.

$$\boxed{\int \frac{1}{x^2 - a^2} dx = \frac{1}{2} \ln \left| \frac{x - a}{x + a} \right| + C}$$

We then went over the fourth (and final) partial fraction decomposition technique - for denominators which contain powers of irreducible quadratics.

$$\begin{aligned} \int \frac{Ax + B}{(x^2 - px + q)^k} dx &\stackrel{\text{complete}}{\text{square}} \int \frac{At + C}{(t^2 + a^2)^k} dt \\ &= A \int \frac{t}{(t^2 + a^2)^k} dt + C \int \frac{1}{(t^2 + a^2)^k} dt \end{aligned}$$

Both terms are worthy of their own questions, tbh. So, we will treat them separately.

$$\begin{aligned} A \int \frac{t}{(t^2 + a^2)^k} dt &= \left( \begin{array}{l} t \, dt = \frac{1}{2} d(t^2 + a^2) \\ t^2 + a^2 = u \end{array} \right) \\ &= \frac{1}{2} \int \frac{du}{u^k} = \frac{1}{2} \frac{u^{-k+1}}{-k+1} + C \\ &\stackrel{\text{back}}{\text{substitute}} \frac{1}{2} \frac{(t^2 + a^2)^{-k+1}}{-k+1} + C \end{aligned}$$

Then we did the second one;

$$\begin{aligned}
\int \frac{1}{(t^2 + a^2)^k} dt &= \frac{1}{a^2} \int \frac{a^2}{(t^2 + a^2)^k} dt \\
&= \frac{1}{a^2} \int \frac{a^2 + t^2 - t^2}{(t^2 + a^2)^k} dt \\
&= \frac{1}{a^2} \left[ \int (t^2 + a^2)^{-k+1} dt - \int \frac{t^2}{t^2 + a^2} dt \right] \\
&= \left( \begin{array}{l} t = u \quad du = dt \\ dV = \frac{t dt}{(t^2 + a^2)^k} \quad V = \frac{1}{2} \int \frac{d(t^2 + a^2)}{(t^2 + a^2)^k} \\ \qquad \qquad \qquad = \frac{1}{2} (t^2 + a^2)^{-k+1} \end{array} \right) \\
&= \left[ t \cdot \frac{1}{2} \frac{(t^2 + a^2)^{-k+1}}{-k+1} - \frac{1}{2(-k+1)} \cdot \int \frac{dt}{(t^2 + a^2)^{k-1}} \right] \\
&= \left[ \frac{t}{2(1-k)} \cdot (t^2 + a^2)^{1-k} - \frac{1}{2(-k+1)} \cdot \int (t^2 + a^2)^{-k+1} dt \right] \\
\text{So, } \int \frac{1}{(t^2 + a^2)^k} dt &= \frac{1}{a^2} \left[ \int (t^2 + a^2)^{-k+1} dt \right. \\
&\quad \left. - \left[ \frac{t}{2(1-k)} \cdot (t^2 + a^2)^{1-k} - \frac{1}{2(-k+1)} \cdot \int (t^2 + a^2)^{-k+1} dt \right] \right]
\end{aligned}$$

And Our Math Professor mentioned integration by trigonometric substitution  
(Friendly heads up: if you haven't *mastered* the trig formulas yet, *now* is the time.)