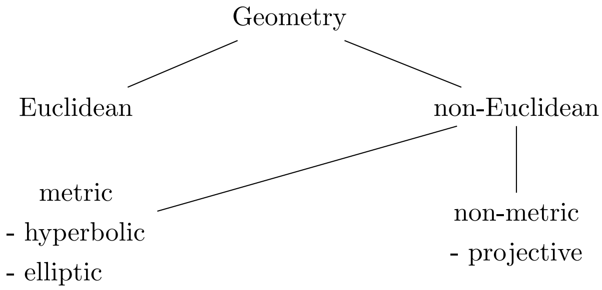
PREFACE

WHAT IS MODERN GEOMETRY?



ANALYTIC VS. SYNTHETIC GEOMETRY

Analytic Geometry → our approach	vs.	Synthetic Geometry → axiomic
• firmy rooted in Descartes • principal concept is congruence (or geometric transformation) • Erlanger Problem		• obsolete • so, we use analytic

INTRODUCTION

In non-Euclidean universes, some Euclidean conceptions are misconceptions, and many Euclidean Theorems are false. Our universe may be non-Euclidean.

In spherical geometry, straight lines (geodesics) are arcs of great circles. Higher dimensional analogues are called elliptic geometries.

Most important arehyperbolic geometries. They are similar to spherical, except they curve the other way.

In our treatment, we emphasize the transformational perspective in Klein's Erlanger Problem.

Part I

BACKGROUND

There are two ways to present Geometry:

Synthetic

$$\text{axioms} \xrightarrow{\text{pure reasoning}} \text{deductive results}$$

Analytic

$$\begin{array}{c} \text{geometric} \\ \text{model} \end{array} \xrightarrow[\text{mathematical realm}]{\text{computation within}} \begin{array}{c} \text{deductive} \\ \text{results} \end{array}$$

- plane
- sphere
- etc.

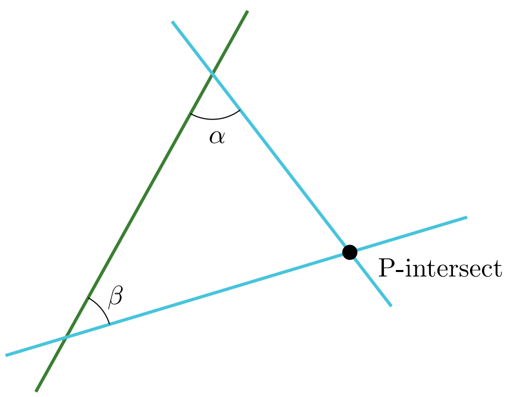
Chapter 1

Some History

Euclide's postulates

1. Two points determine a unique straight line.
2. A finite line can be made any length. Though this does not necessarily imply that a non-finite line is infinite...
3. A point and a radius determine a unique circle
4. All right angles are "congruent"
5. If two non-parallel straight lines intersect a line, they will meet on the side where the interior angles are acute; this is the "parallel postulate." Equivalently, there exists exactly one parallel line through a point not on a given line; this is "Playfair's postulate."

   That is, $\alpha + \beta < 180° \iff \exists$ P-intercept



DISCOVERY OF NON-EUCLIDEAN GEOMETRY

Assuming that postulates 1 through 4 are true, the following are equivalent definitions of postulate 5;

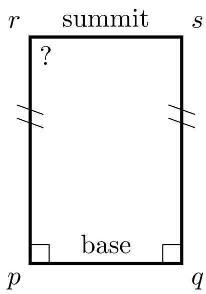
(a) Two parallels are equidistant.

(b) If a line intersects one of two paralllels, then it intersects the other.

(c) Given a triangle, a similar triangle can be constructed of any size.

(d) The angular sum of a triangle is 180°.

Saccheri's attempt to prove the parallel postulate let to many theorems - which he rejected for being repugnant to the nature of a straight line - which were actually true in hyperbolic geometry!



He was able to prove that $\angle r = \angle s$. Euclid's postulate is equivalent to $\angle r = 90°$, so Saccheri set about treating separately angles of r which were obtuse and acute. Obtuse angles of r led to a contradiction, but treatment of r as though it were acute gave birth to the aforementioned theorems.

In hyperbolic geometry;

(a) There are an **infinite** number of parallel lines through a given point not on a line.

(b) The lines mentioned in the parallel postulate do **not** have to meet.

(c) Two coplanar lines are **never** equidistant.

(d) A line may intersect one of two parallel lines, not **not** necessarily the other.

(e) Two similar triangles are always congruent.

(f) The sum of the angles of a triangle is **less** than 180°.

Further Developments

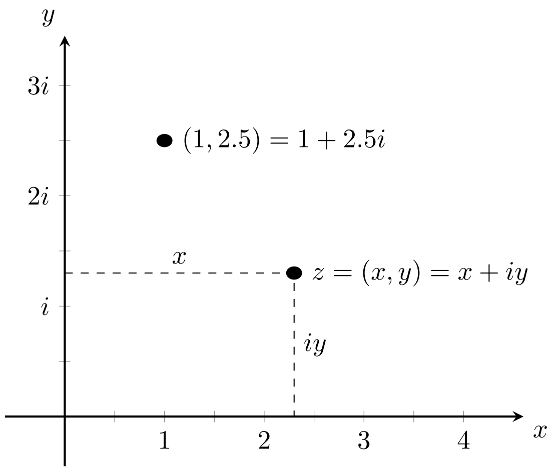
- → projective geometry
  - no parallel lines
- → Affine geometry
  - the geometry of linear algebra
- → multidimensional geometries
- → Differential geometry
  - geometry of curved surfaces (and lines)

There are no parallel lines in spherical (A.K.A elliptic) geometry, the angularsum of a trianlge exceeds $180°$. As in hyperbolic geometry, similar triangles are congruent. Postulates 1. and 2. needed modification, as lines here are not infinite and two points can determine more than one line.

Chapter 2

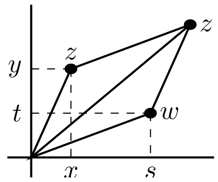
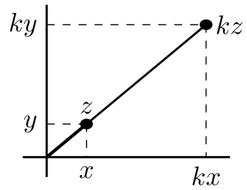
COMPLEX NUMBERS

A complex number is a point $z = (x, y)$ in the Cartesian plane in which the x-axis is measured in ordinary (real) units and the y-axis is measured in a different (imaginary) unit i, where $i^2 = -1$. $x = \mathrm{Re}(z)$, and $y = \mathrm{Im}(z)$. NOTE that the real and imaginary components of z are REAL numbers (they are coefficients).



OPERATIONS ON COMPLEX NUMBERS

Complex numbers inherit properties of vector addition (i.e. the parallelogram law) and scalar multiplication. Scalar multiplication of a complex number by a real number simply multiplies both components by the scalar coefficient.

	Algebra	**Geometry**
vector addition:	$z = (x, y) \quad w = (s, t)$ $z + w = (x + s, y + t)$	
scalar multiplication:	$z = (x, y) \quad (k \text{ real})$ $kz = (kx, ky)$	

Using vector addition and scalar multiplication, we can write the complex number $z = (x, y) = x(1, 0) + y(0, 1) = x + iy$, where $z = x + iy$ is its Cartesian representation.

Questions:

A- find Cartesian form

(a) $(2 + 3i) + (1 - 6i) = 3 - 3i$

(b) $4(-2 + 5i) - (-7 - 2i) = -1 + 22i$

B- explain the commutative property of complex addition algebraically and geometrically

(a) $(a + bi) + (c + di) = (c + di) + (a + bi)$
$(a + c) + (b + d)i = (c + a) + (d + b)i$

We have reduced complex addition to the addition of the real number components, treated as scalar coefficients. Therefore, by the commutative property of real numbers, complex addition too is commutative.

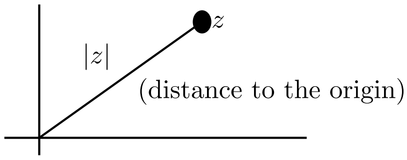
(b) Per the parallelogram law, complex addition is commutative.

RELATED GEOMETRIC NOTIONS

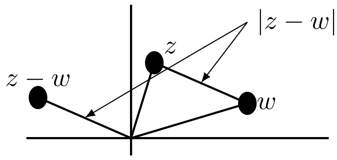
Worthy of special note is the modulus.

Algebra | **Geometry**

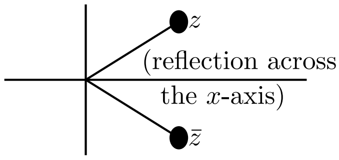
modulus: $|z| = \sqrt{x^2 + y^2}$



distance between points: $|z - w| = \sqrt{(x - s)^2 + (y - t)^2}$



conjugate: $\bar{z} = x - iy$



Themodulus has two important properties:

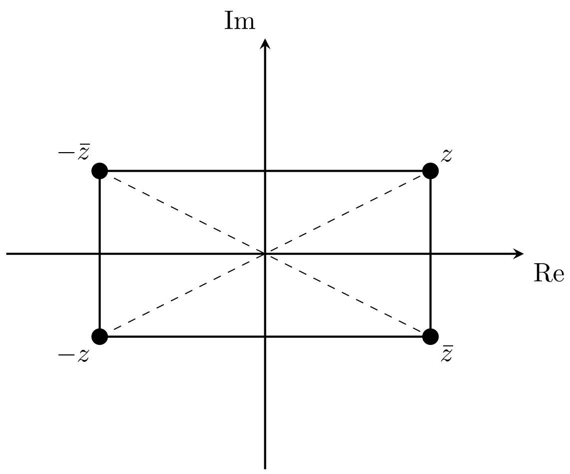
(a) Homogeneity: $|kz| = |k||z|$ (where k is real)

(b) Triangular Inequality: $|z + w| \leq |z| + |w|$

According to Euclidean geometry, one side of a triangle is always less than or equal to the sum of the other sides. This proves the triangle inequality.

Questions:

C- Let $z = 2 + i$. Plot z, $-z$, $\bar{z}$, and $-\bar{z}$ on the same pair of axes. What shape is outlines? Note its relation to the coordinate axes.



The shape outlined by the positive and negative complex point-conjugate pairs is a rectangle that is centered at the origin. Due to this, it is symmetric about both coordinate axes.

D- What is $|2 + i|$?
$z = 2 + i = 2(1, 0) + 1(0, 1) = (2, 1)$, so $|z| = \sqrt{2^2 + 1^2} = \sqrt{5}$.

COMPLEX MULTIPLICATION

Let $z = x + iy$ and $w = s + it$. Then, $zw = (x + iy)(s + it) = (xs - yt) + i(ys + xt)$. Questions:

E- Evaluate products.

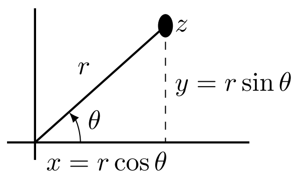
(a) $i^2 = -1$ by definition. More interesting is the polar transformation.
(b) $(1 + i)^2 = 1 + 2i + i^2 = 2i$
(c) $(4 + 5i)(1 - i) = 4 - 4i + 5i - 5i^2 = 9 + i$

POLAR FORM

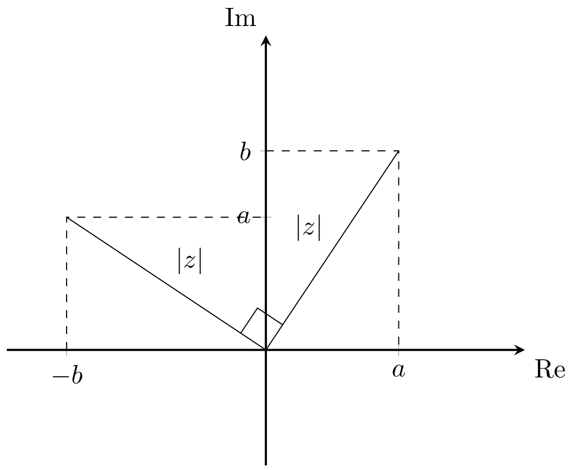
Every nonzero complex number has two square roots. This can be proven algebraically, but is also a simple consequence of geometric complex multiplication - which depends on understanding the polar form of complex numbers, as derived by Euler's formula.

We can use the modulus and basic trig definitions to re-express z in a form which can be converted into a polar exponential using Euler's formula. The polar angle is in standard position.

$$\begin{aligned} z &= x + iy \\ &= r\cos(\theta) + ir\sin(\theta) \\ &= r(\cos(\theta) + i\sin(\theta)) \\ \textbf{Polar form:} \quad &= |z|e^{i\theta} \end{aligned}$$



Euler's identity can be derived without the use of Taylor's Series by differentiating $f(\theta) = e^{-i\theta}(\cos\theta + i\sin\theta)$ to prove it is constant, finding that constant (1), and finally multiplying both sides of the equality by $e^{i\theta}$. Algebraically, it makes sense that multiplication by i yields a 90 degree rotation of the point about the origin, because "the magnitude of the real and imaginary parts swap, and the real part of the new number is negative." I'll omit the trigonometricproof, as it is trivial.



Very importantly, the fundamental properties of exponentiation hold for complex exponents.

$$\begin{aligned}
e^z e^w &= e^{z+w} \\
e^{i\theta} e^{i\mu} &= (\cos\theta + i\sin\theta)(\cos\mu + i\sin\mu) \\
&= (\cos\theta\cos\mu - \sin\theta\sin\mu) + i(\sin\theta\cos\mu + \cos\theta\sin\mu) \\
&= \cos(\theta+\mu) + i\sin(\theta+\mu) \\
&= e^{i(\theta+\mu)}
\end{aligned}$$

It is useful to define an operation which finds the angle of a complex number in standard position, this operation is called the **argument** of z.

$$\arg(z) = \theta \pmod{2\pi}$$

Questions:

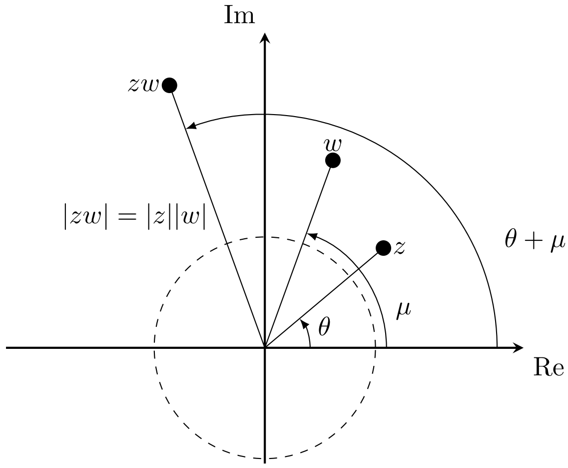
F- find the polar form

(a) $1 + i$
$\theta = \arctan(1) = \frac{\pi}{4}$ and $|z| = \sqrt{1^2 + 1^2} = \sqrt{2}$ so, $1 + i = \sqrt{2}e^{\frac{\pi}{4}\cdot i}$

(b) $1 - i$
$\theta = \arctan(-1/1) = \arctan(-1) = -\frac{\pi}{4}$ and $|z| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$ so, $1 - i = \sqrt{2}e^{-\frac{\pi}{4}\cdot i}$

THE GEOMETRY OF COMPLEX MULTIPLICATION

Since multiplying complex numbers in their exponential form just multiplies their moduli and adds their angles, we can visualize the geometry of this quite easily. That is, if $z = |z|e^{i\theta}$and $w = |w|e^{i\mu}$, then $zw = |z||w|e^{i(\theta+\mu)}$.



A basic formula that is still worth remembering is

$$\arg(zw) = \arg(z) + \arg(w) \pmod{2\pi}$$

Complex Arithmetic

1) Find the Cartesian form of these complex numbers:

   (a) $4(1 + 2i) - 2(5 - i) = 4 + 8i - 10 + 2i = -6 + 10i$
   (b) $(1 + i\sqrt{2}) + \sqrt{2}(\pi + i) = 1 + \pi\sqrt{2} + i(\sqrt{2} + 1)$
   (c) $2(4 + i) - (1 + 6i) = 7 - 4i$
   (d) $(1 - 2(i + 3(1 - 4i))) = -5 + 22i$

2) Find the Cartesian form of these complex numbers:

   (a) $(1 + i)(1 - i) = 2$
   (b) $(5 + 10i)(-2 + 3i) = -40 - 5i$
   (c) $(1 + i)^3 = -2 + 2i$
   (d) $(1 + 2i)(3 - 4i)(5 + 6i) = (11 + 2i)(5 + 6i) = 43 + 76i$

3) Prove these properties of the conjugate and the modulus:

(a) $|kz| = |k||z|$ (k real)

$$\begin{aligned}
|kz| &= |k\mathrm{Re}(z) + ik\mathrm{Im}(z)| \\
&= \sqrt{(k\mathrm{Re}(z))^2 + (k\mathrm{Im}(z))^2} \\
&= \sqrt{k^2(\mathrm{Re}(z))^2 + k^2(\mathrm{Im}(z))^2} \\
&= \sqrt{k^2((\mathrm{Re}(z))^2 + (\mathrm{Im}(z))^2)} \\
&= \sqrt{k^2}\sqrt{(\mathrm{Re}(z))^2 + (\mathrm{Im}(z))^2)} \\
&= |k||z|
\end{aligned}$$

And this is consistent with all of the geometric notions.

i) This is the vector addition of parallel position vectors, k times. This is equivalent to scalar multipication.

ii) This is consistent with multiplying the components in scalar multiplication to recieve a ”scaled” vector.

iii) This is also consistent with polar multiplication, as we multiply the magnitudes, and add the angles, one of which is zero.

(b) $|z|^2 = z\bar{z}$

$$\begin{aligned}
|z|^2 &= (\sqrt{(\mathrm{Re}(z))^2 + (\mathrm{Im}(z))^2})^2 \\
&= (\mathrm{Re}(z))^2 + (\mathrm{Im}(z))^2 \\
&= (\mathrm{Re}(z))^2 - (i\mathrm{Im}(z))^2 \\
&= (\mathrm{Re}(z) + i\mathrm{Im}(z))(\mathrm{Re}(z) - i\mathrm{Im}(z)) \\
&= z\bar{z}
\end{aligned}$$

This also makes geometric sense when multiplying in their polar forms, considering that the angles are additive inverses and the moduli being equal are squared.

(c) $|wz| = |w||z|$

proof 1:

$$\text{Let } a = \mathrm{Re}(w); \quad b = \mathrm{Im}(w); \quad c = \mathrm{Re}(z); \quad d = \mathrm{Im}(z)$$

$$\begin{aligned}
|wz| &= \sqrt{(ac - bd)^2 + (ad + cb)^2} \\
&= \sqrt{a^2c^2 - 2acbd + b^2d^2 + a^2d^2 + adcb + c^2b^2} \\
&= \sqrt{a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2} \\
&= \sqrt{(a^2 + b^2)(c^2 + d^2)} \\
&= \sqrt{a^2 + b^2}\sqrt{c^2 + d^2} \\
&= |w||z|
\end{aligned}$$

Geometrically, this can be explained in the light of polar form multiplication. In polar multiplication, we multiply the moduli and add the angles; this results in a complex number whos modulus is equal to the product of the moduli of its factors.

proof 2:

$$\text{Let } z = x + iy = |z|e^{i\theta}; \quad w = u + iv = |w|e^{i\phi}$$

$$\begin{aligned} |zw| &= ||z|e^{i\theta}|w|e^{i\phi}| = |z||w||e^{i\theta}||e^{i\phi}| \\ &= |z||w| \cdot 1 \cdot 1 = |z||w| \end{aligned}$$

*only valid if 5(a) is proved

(d) $\overline{zw} = \bar{z}\bar{w}$

$$\text{Let } a = \text{Re}(z); \quad b = \text{Im}(z); \quad c = \text{Re}(w); \quad d = \text{Im}(w)$$

$$\begin{aligned} \overline{zw} &= (ac - bd) - i(ad + cb) \\ &= ac + bdi^2 - adi - cbi \\ &= (a - bi)(c - di) \\ &= \bar{z}\bar{w} \end{aligned}$$

This makes complete geometric sense. The product of two complex numbers is at an angle with is the sum of each of their arguments. The product of their conjugates has an angle which is the sum of their negated arguments. That is, they are symmetric about the Re axis. And since they are of equal magnitude, we can deduce that they are indeed conjugates of each other.

4) Verify algebraically that complex multiplication satisfies the distributive law: $z(w + u) = zw + zu$.

This is the product of a binomial and the sum of two other binomials.

$$\begin{aligned} z(w+u) &= (\text{Re}(z) + i\text{Im}(z))((\text{Re}(w) + i\text{Im}(w)) + (\text{Re}(u) + i\text{Im}(u))) \\ &= (((\text{Re}(z) + i\text{Im}(z))(\text{Re}(w) + i\text{Im}(w))) + ((\text{Re}(z) + i\text{Im}(z))(\text{Re}(u) + i\text{Im}(u)))) \\ &= zw + zu \end{aligned}$$

And we could do it in polar form too by exponentiating to the arguments.

Complex Arithmetic

5) Evaluate these exponentials (that is, express them in Cartesian and/or polar form):

(a) $e^{1+i\pi} = \boxed{e \cdot e^{i\pi}} = \boxed{-e}$

(b) $\boxed{e^{1\pi/2}} = \boxed{i}$

(c) $e^{e^{\ln \pi + i\pi/2}} = \boxed{e^{i\pi}} = \boxed{-1}$

6) Verify that the law of exponents (i.e. $e^z e^w = e^{z+w}$) holds with base e and arbitrary complex exponents.

My proposed solution supposes this by saying $e^z = e^x e^{iy}$; what is a more fundamental way of saying this?

$$\text{Let } z = x + iy = |z|e^{i\theta}; \quad w = u + iv = |w|e^{i\mu}$$

$$\begin{aligned} e^z e^w &= e^x e^{iy} e^u e^{iv} \\ e^{i\theta} e^{i\mu} &= (\cos\theta + i\sin\theta)(\cos\mu + i\sin\mu) \\ &= (\cos\theta\cos\mu - \sin\theta\sin\mu) + i(\sin\theta\cos\mu + \cos\theta\sin\mu) \\ &= \cos(\theta+\mu) + i\sin(\theta+\mu) \\ &= e^{i(\theta+\mu)} \end{aligned}$$

7) Revisit when have learned series

8) cosine(0) plus isin(0) = 1. also, exp(0)=1.

Polar Form

9) convert to polar form

(a) $-1 = e^{i\pi}$

(b) $i\sqrt{3} = \sqrt{3}e^{\frac{\pi}{2}i}$

(c) $4i = 4e^{\frac{\pi}{2}i}$

(d) $5 - 5i\sqrt{3} = \sqrt{5^2 + (5\sqrt{3})^2}e^{i\arctan(-5\sqrt{3}/5)} = 10e^{i\arctan(-\sqrt{3})} = 10e^{i\frac{\pi}{3}}$

10) express in cartesian/polar form

(a) $\frac{1}{i} = \frac{e^{0i}}{e^{\frac{\pi}{2}i}} = e^{0i - \frac{\pi}{4}i} = e^{-\frac{\pi}{2}i}$

(b) $\frac{1}{1+i} = e^{0i}/e^{\frac{\pi}{4}i} = e^{-\frac{\pi}{4}i}$

(c) $e^{\frac{\pi}{4}i}/e^{i\frac{\pi}{2}} = e^{-\frac{\pi}{4}i}$

(d)

(e)

11) use polar form to show that every complex number has a multiplicative inverse 1/z.

$$|z|e^{i\arg(z)} \cdot \frac{1}{|z|}e^{-i\arg(z)} = 1$$

12. Use polar form to show that every complex number $z \neq 0$ has two square roots.

$$\forall z = re^{i\theta} \in \mathbb{C} : z \neq 0 \exists w_1 = \sqrt{r}e^{i\frac{\theta}{2}};\ w_2 = \sqrt{r}e^{i\frac{\theta}{2}+\pi} : w_1^2 = w_2^2 = z$$