Data science for networked data

Po-Ling Loh

University of Wisconsin-Madison
Department of Statistics

Big Data Meetup
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Joint work with:

Justin Khim (UPenn), Varun Jog (UW-Madison), Ashley Hou (UW-Madison), Wen Yan (Southeast University), and Muni Pydi (UW-Madison)

Key problems in network modeling

1. Given data from a network, how do we estimate the network?
2. How do we model dynamic processes over a network?
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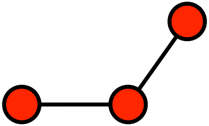

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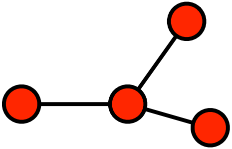

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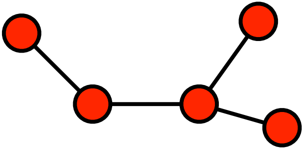

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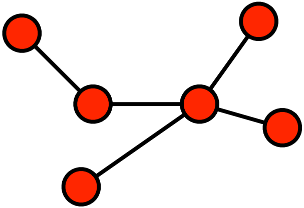

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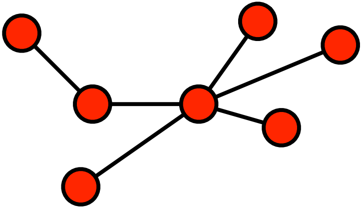

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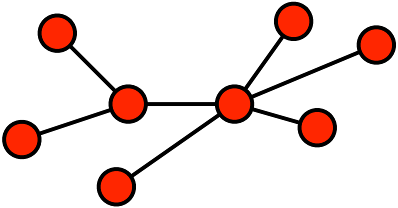

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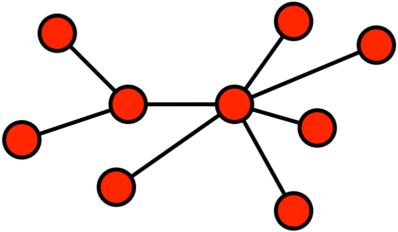

Prelude: Network estimation

Graphical models

- Method for constructing connectivity network from matrix of data

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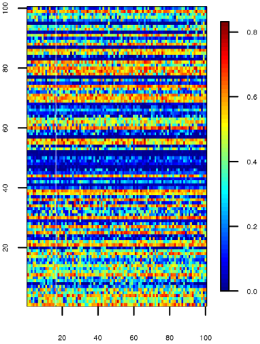
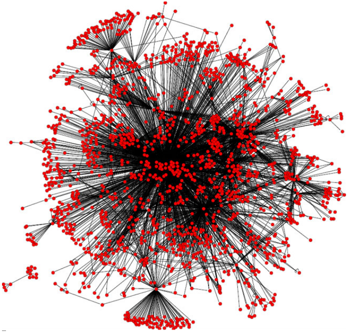


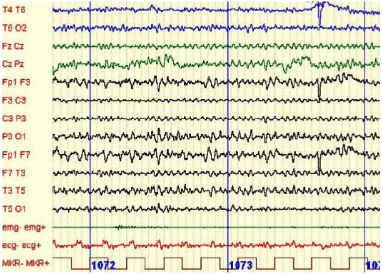
gene expression (mRNA) data



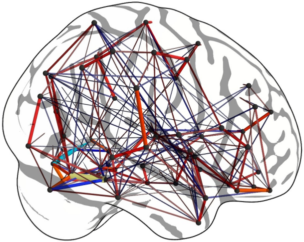
E. coli network

Graphical models

- Method for constructing connectivity network from matrix of data



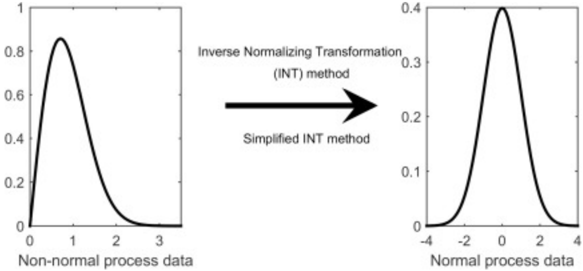
fMRI/EEG readings



“functional connectivity” network

Graphical models

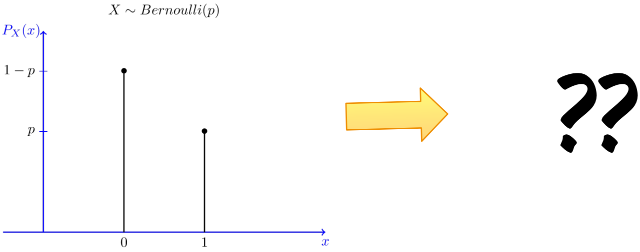
- Mathematical analysis derived for Gaussian data



- In practice, transform data to Gaussian before applying algorithm

Graphical models

- But not all data are transformable!



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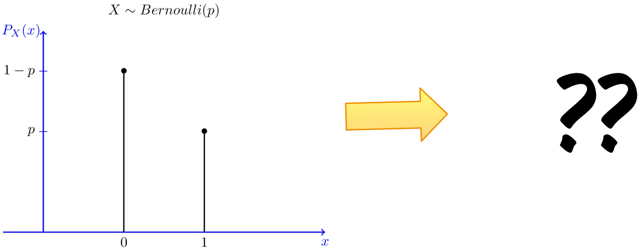

- We have developed new methods for estimating graphical models for discrete (count) data

Graphical models

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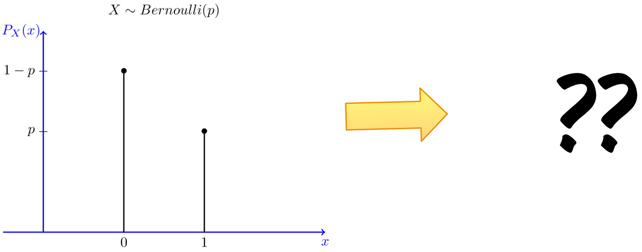


- We have developed new methods for estimating graphical models for discrete (count) data
- **However, life is more than network estimation...**

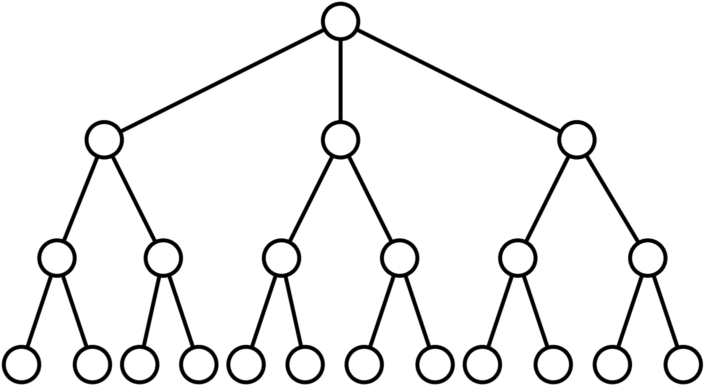
Outline

1. Statistical inference
   - Confidence sets for source estimation
   - Graph hypothesis testing
2. Resource allocation
   - Influence maximization
   - Budget allocation
   - Network immunization
3. Local algorithms

Statistical inference

Justin Khim
(UPenn)

Source estimation



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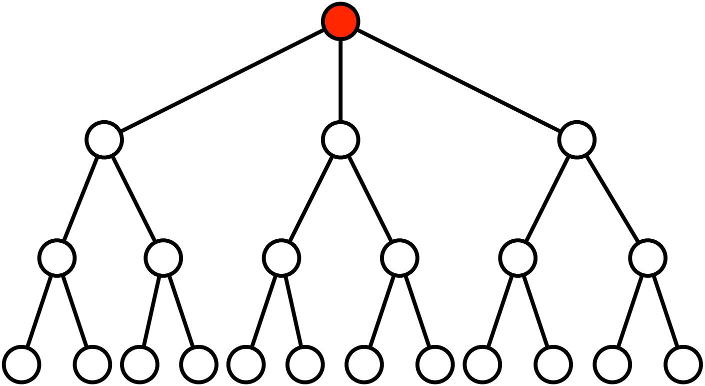

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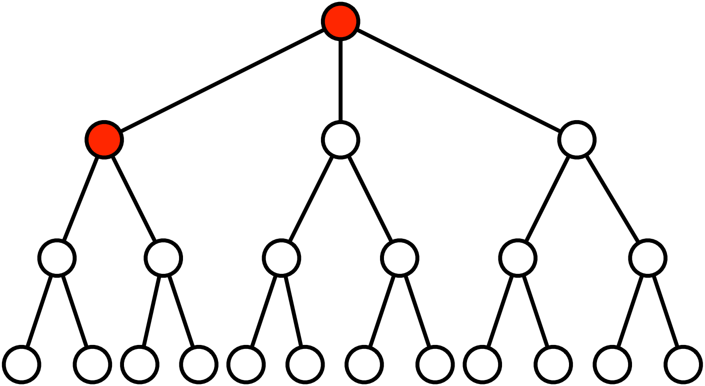

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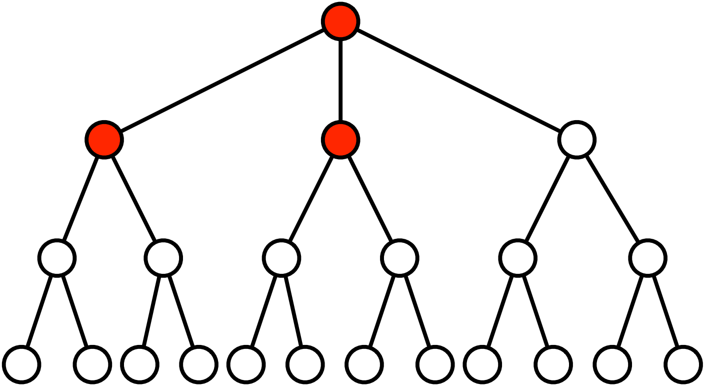

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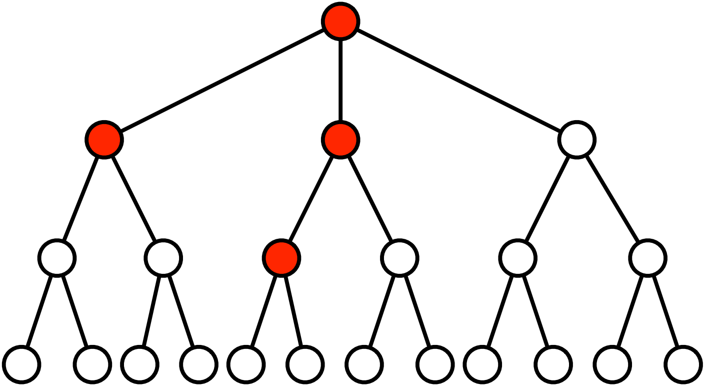

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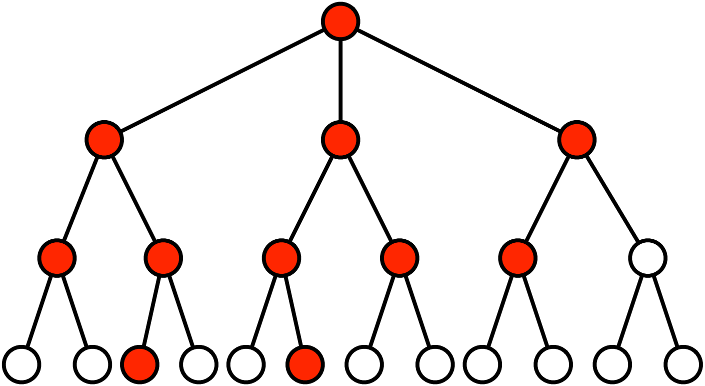

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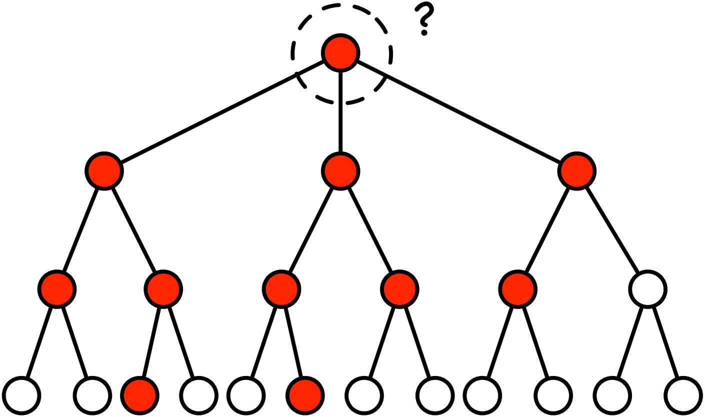
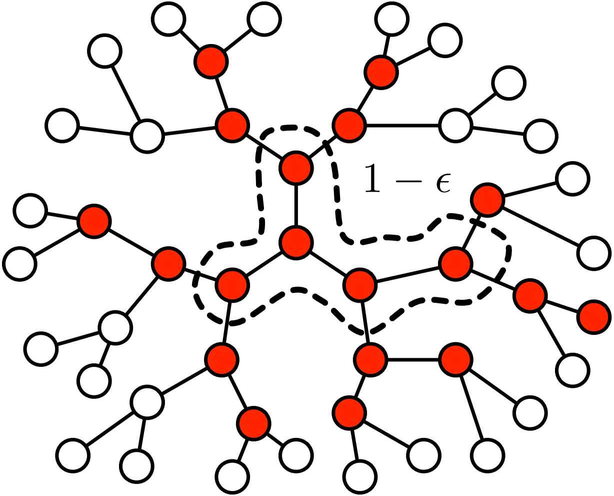


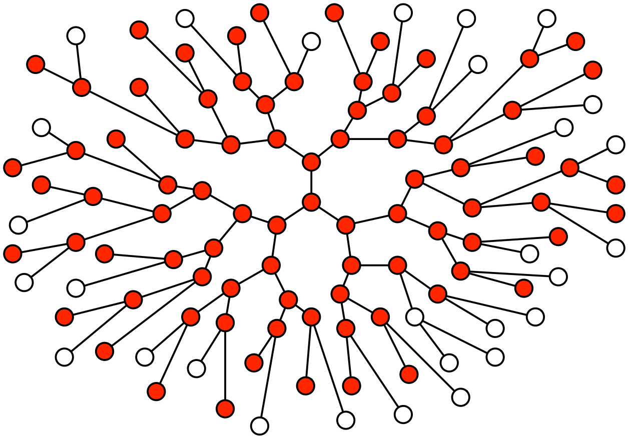
Confidence sets

- **Instead:** Find a *confidence set* that includes root node with probability at least $1 - \epsilon$



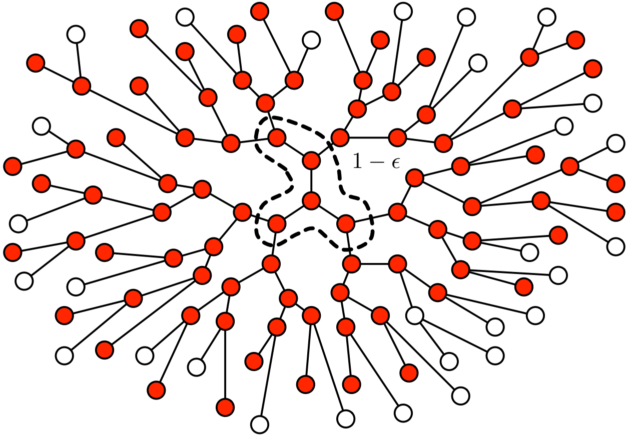
Confidence sets

- **Question:** How does size of confidence set grow with number of infected nodes n?



Confidence sets

- **It doesn't!**



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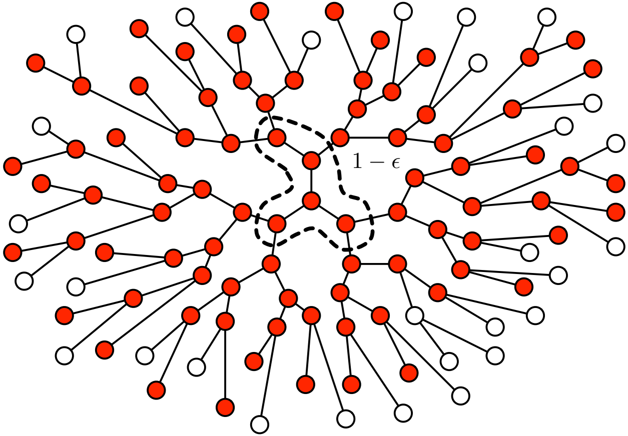
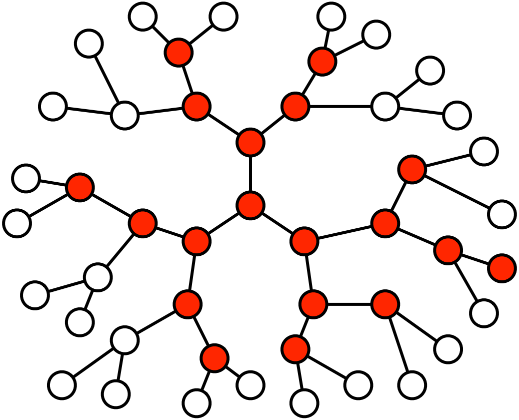


- **Rough interpretation:** No "information loss" about source as disease spreads

Inference algorithm

- Select nodes that are most "central" to network of infected individuals



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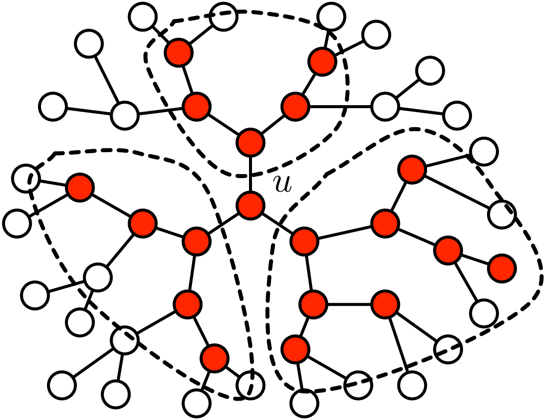

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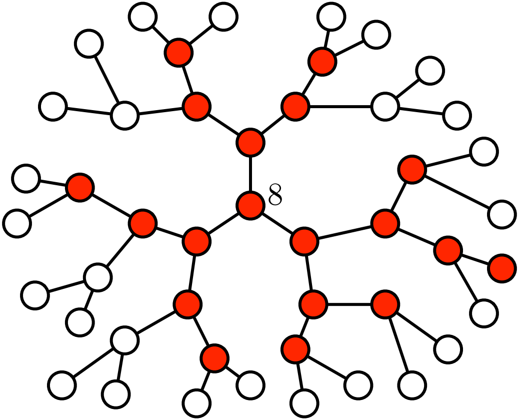

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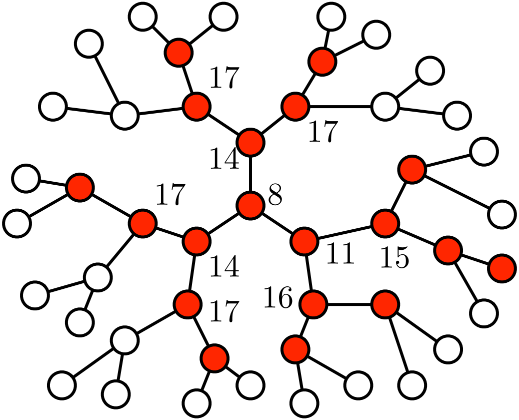
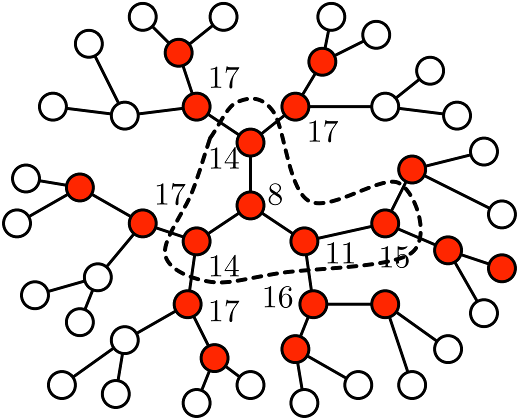


Inference algorithm

- Select $K(\epsilon)$ nodes with smallest values



Theory for confidence sets

Theorem

Suppose $d \geq 3$. Then the min-max subtree estimator with $K_\psi(\epsilon) = \frac{C(d)}{\epsilon}$ yields a $1 - \epsilon$ confidence set for the root.

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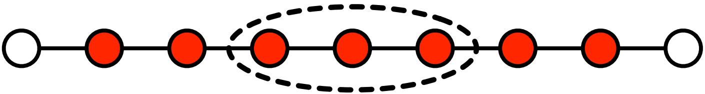
- **Note:** Cannot construct finite confidence set for $d = 2$; need set of size $K = \Theta(\sqrt{n})$



Extensions and open directions

- Similar result holds for broader class of “regular” trees
- **Robustness:** Confidence set eventually settles down after finitely many steps

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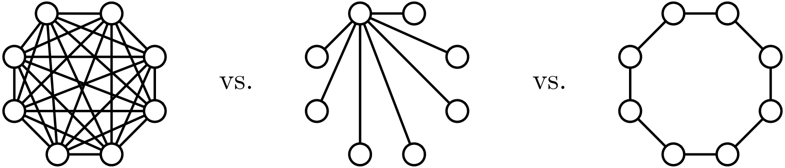
Open directions:

- What if underlying graph is not a tree?
- What if network is asymmetric?
- What if nodes can heal?

Graph testing



- **Question:** Can we use epidemic data to infer network structure?

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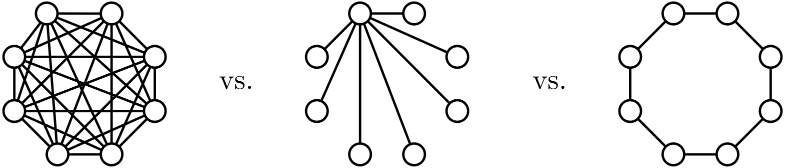

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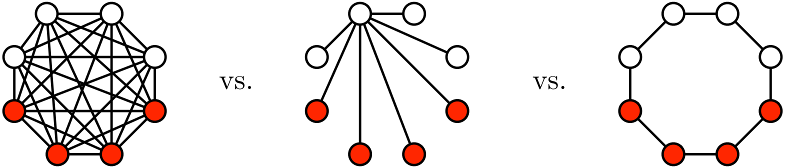
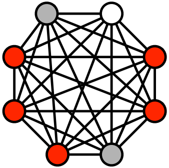


Graph testing

- **Observations:** Infection status of n nodes in graph
  - k infected nodes (1)
  - c censored (nonreporting) nodes ($\star$)
  - $n - k - c$ uninfected nodes (0)



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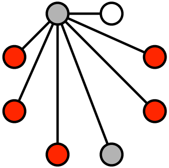

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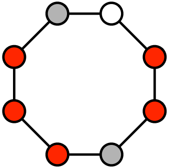

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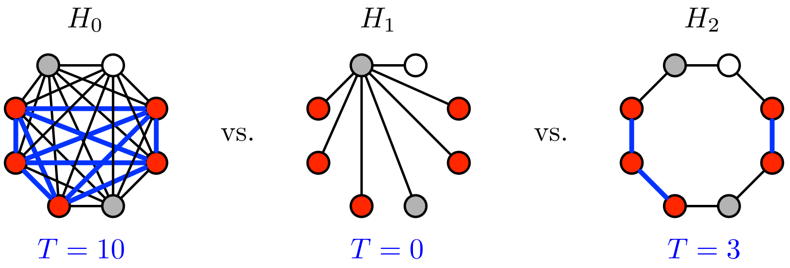
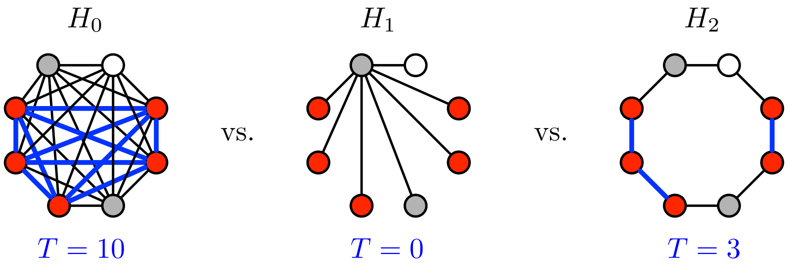


- Compute test statistic

$$T = \# \text{ edges between infected nodes}$$

Graph testing



- Compute test statistic

$$T = \# \text{ edges between infected nodes}$$

- Need to construct proper rejection rule based on T, derive validity of hypothesis test

Infection model

- Parameters λ, η
  - For each node v, generate $T_v \sim Exp(\lambda)$
  - For each edge (u, v), generate $T_{uv} \sim Exp(\eta)$
- Infection time of any vertex v is $t_v = \min_{u \in N(v)} \{T_u + T_{uv}\} \wedge T_v$

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- Infection time of any vertex v is $t_v = \min_{u \in N(v)} \{T_u + T_{uv}\} \wedge T_v$
- Observation vector corresponds to infection states at a certain time
- Subset of censored nodes chosen uniformly at random

Permutation test

- **Goal:** For $\alpha \in (0,1)$, construct rejection rule such that

$$P(\text{reject} \mid H_0 \text{ is true}) \leq \alpha$$

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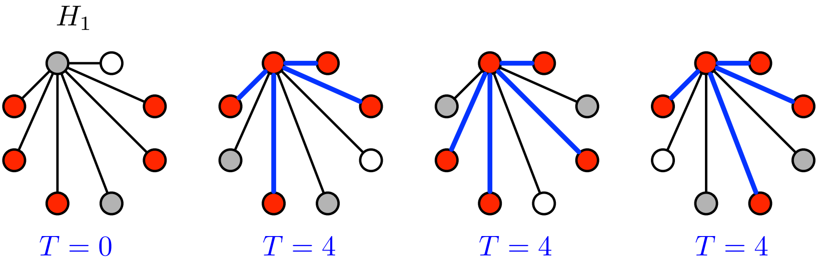
- Use *permutation test* that computes T for $\binom{n}{k,c,n-k-c}$ reassignments of infected/nonreporting/uninfected nodes



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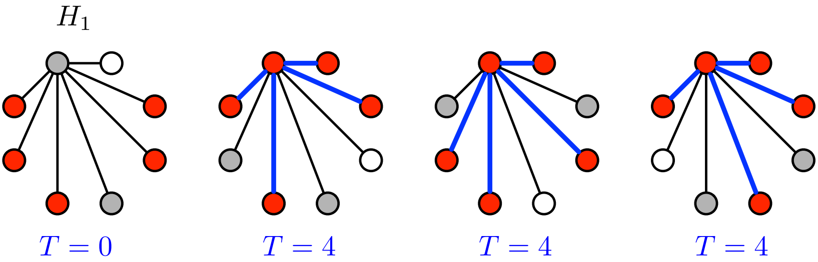
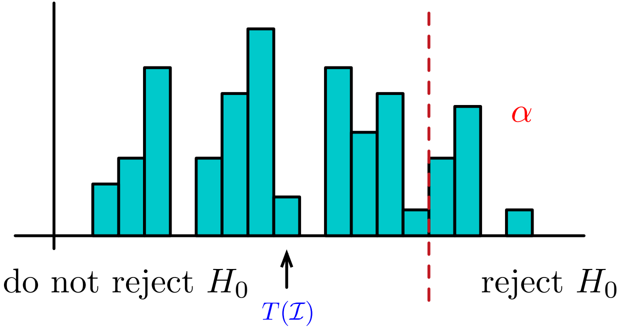


- Based on (randomly chosen) permutations, compute p-value/rejection region and reject H_0 if (p-value of T) $\le \alpha$

Permutation test



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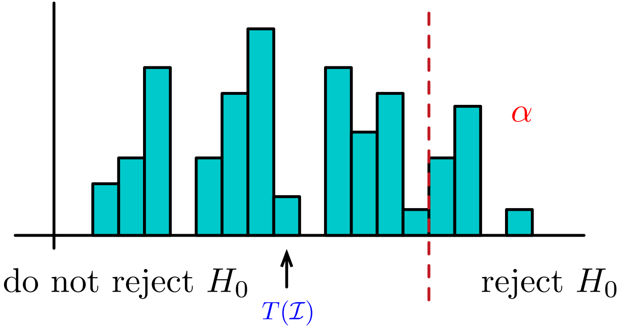

- In practice, sufficient to compute empirical distribution from large number of random permutations

Theory for permutation test

- Success depends on *symmetries* of underlying networks rather than parameters λ, η
- Consider $\Pi_0 = \text{Aut}(G_0)$ and $\Pi_1 = \text{Aut}(G_1)$, subsets of S_n

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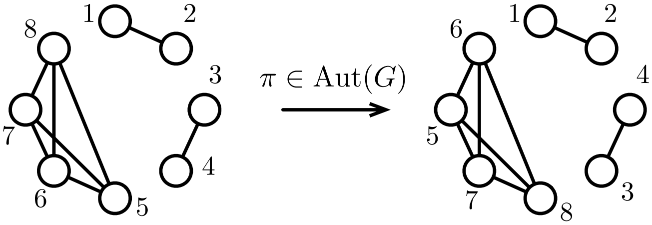

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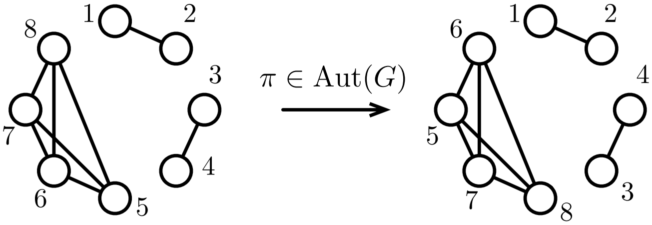

Theorem

Let π be drawn uniformly from S_n. If $\Pi_1 \Pi_0 = S_n$, the permutation test controls Type I error at level α.

Extensions and open directions

- Characterization of condition $\Pi_1 \Pi_0 = S_n$ for various graph families
- Bounds on Type II error for specific graphs
- Conditioning on identity of censored nodes

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Open directions:

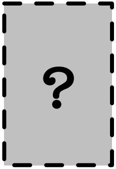
- How to identify which graphs to use as null/alternative hypotheses?
- Inhomogeneous λ and η?
- Confidence sets for underlying network?

Resource allocation

Justin Khim
(UPenn)

Varun Jog
(UW-Madison)

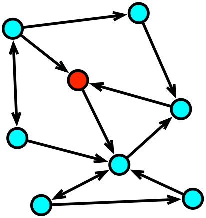
Ashley Hou
(UW-Madison)



Wen Yan
(Southeast University)

Influence maximization (with Justin Khim and Varun Jog)

- **New goal:** Seed a network to "infect" as many nodes as possible
- Useful for information dissemination, marketing, etc.



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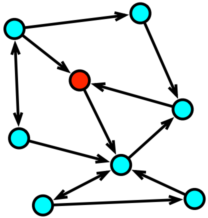

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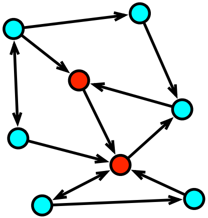

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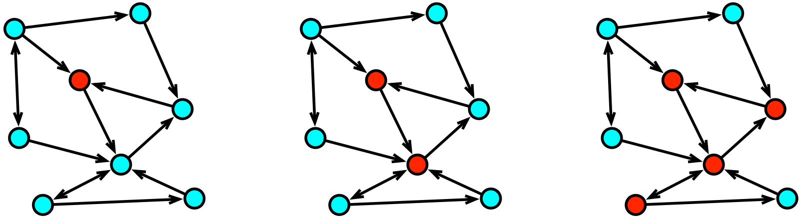

t = 0 t = 1 t = 2

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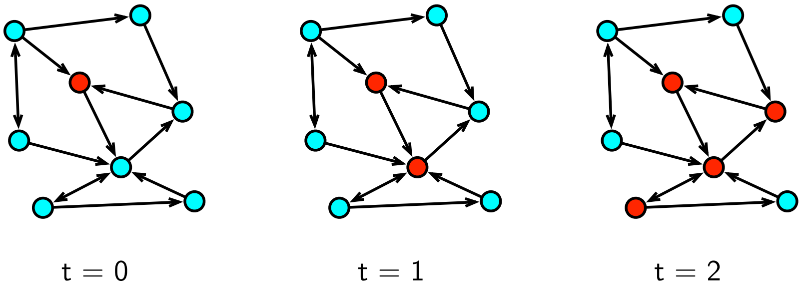

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Influence maximization (with Justin Khim and Varun Jog)

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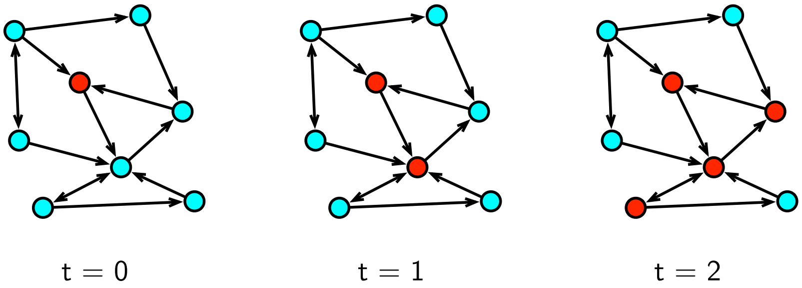

Questions

1. If k nodes may be infected initially, which nodes should be selected to maximize infection spread?
2. How to determine maximal set efficiently?

Model: Linear threshold model (broadly, triggering models)

- Edges have weights (b_{ij}), satisfying $\sum_j b_{ji} \leq 1$
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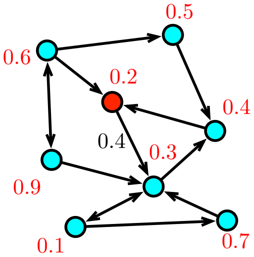

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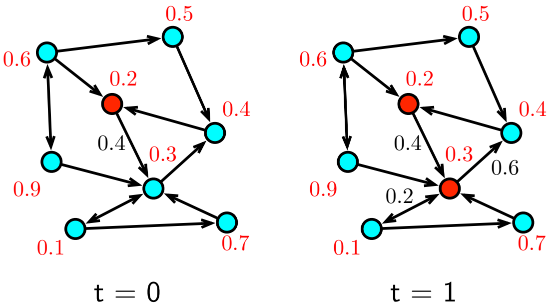

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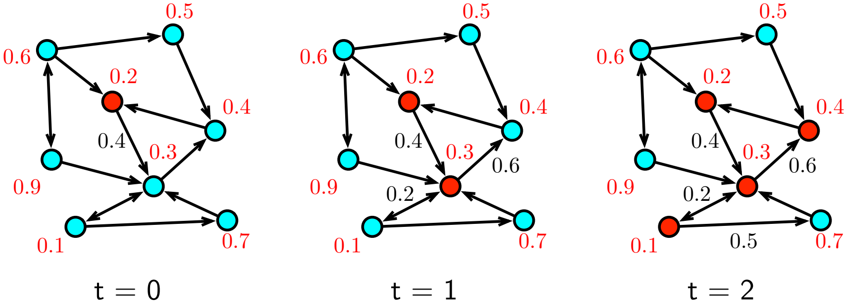

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- However, method involves approximating $\mathcal{I}$ at each iteration of greedy algorithm via simulations

Key contributions

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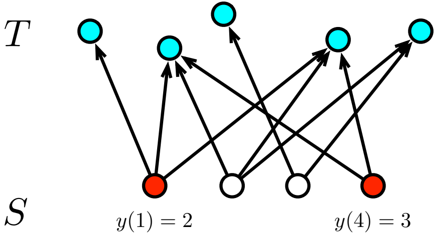
- Leads to significant speed-ups:

	LB_1	LB_2	UB	Simulation
Erdös-Renyi	**1.00**	2.36	27.43	710.58
Preferential attachment	**1.00**	2.56	28.49	759.83
$2D$-grid	**1.00**	2.43	47.08	1301.73

Budget allocation (with Ashley Hou)

- **Problem:** Given fixed budget to distribute amongst influencers, how to optimally allocate resources?



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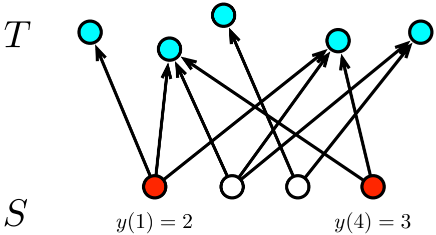

- **Mathematical formulation:** If resources $\{y(s)\}_{s \in S}$ are allocated among source nodes S, probability of influencing customer t is

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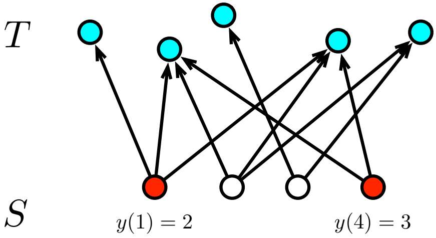

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so we solve max $\sum_{t \in T} I_t(y)$ s.t. $\sum_{s \in S} y(s) \leq B$

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- **Goal:** Develop efficient algorithms for robust budget allocation with provable approximation guarantees
- Ingredients: Maximization of min of submodular functions, extensions to integer lattices and budget constraints

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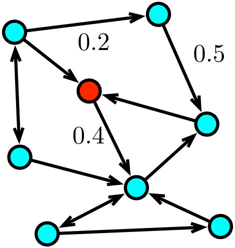
- **Goal:** Given a budget of interventions at nodes/edges of a graph, how to optimally distribute resources to retard an epidemic?
- Interested in *fractional immunization*, which only decreases infectiveness of nodes/edges



Network immunization

- Formulation as influence maximization problem:

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- **Challenges:**
  1. Bilevel optimization problem involving discrete and continuous variables
  2. No computable closed-form expression for $\mathcal{I}$ or $\nabla \mathcal{I}$

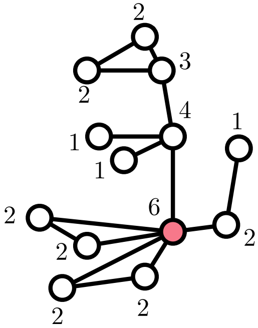
Local algorithms

Muni Pydi
(UW-Madison)

Varun Jog
(UW-Madison)

Maximizing graph functions

- Given function f defined on nodes of a graph
- **Examples:** Degree, age of node, power/population level, etc.



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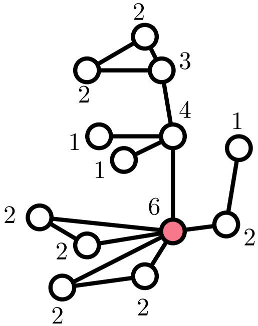

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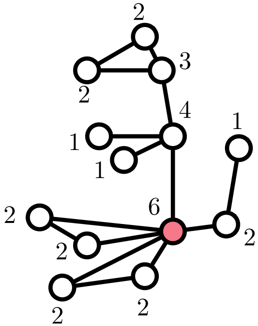

- **Goal:** Maximize f by "walking" along edges and querying values
- Could use "vanilla random walk" with transition probabilities $P_{ij} = \frac{w_{ij}}{d_i}$, but can we leverage smoothness/structure of graph function?

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- **Idea:** Build a density p_f maximized wherever f is maximized, hope that MH algorithm finds maximizers quickly

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- **Theoretical results:** Rates of convergence in TV distance, hitting time bounds for both algorithms in terms of graph/function characteristics

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