Exercise 4.6. Carry out the ket recipe with $H=\frac{\hbar\omega}{2}\sigma_z$, final observable σ_x, initial state $|u\rangle=\begin{pmatrix}1\\0\end{pmatrix}$, and observable σ_y measured at time *t*. Find the possible outcomes and their probabilities of occurrence.

Solution. I believe the initial state and final observables were inadvertently reversed in the problem statement because reversing them makes this exercise the natural culmination of a problem started in Exercise 3.4 (represent $\hat{n}$ in spherical coordinates), and continued in 4.5 (for $H=\frac{\hbar\omega}{2}\vec{\sigma}\bullet\hat{n}$, find the energy eigenvectors and eigenvalues $|E_1\rangle, |E_2\rangle, E_1$, and E_2 for direction $\hat{n}$). It also becomes a means to confirm the results of Exercise 4.4 that the 3-vector operator $\vec{\sigma}$ precesses clockwise around the direction of the magnetic field. The problem as stated, as others have shown, results in σ_y unchanging over time, a not very interesting result. I provide both results, doing the reversed problem 1st.

I thus assume an initial state $|r\rangle$ (corresponds to observable σ_x) and a final observable σ_z (corresponds to state $|u\rangle$). Moreover, the following spin formulas become valid because they were developed under the assumption that system is prepared in some state $|A\rangle$ and then measured in the up direction:

$$|u\rangle=\begin{pmatrix}0\end{pmatrix},\ |d\rangle=\begin{pmatrix}0\\1\end{pmatrix},\ |i\rangle=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix},\ |o\rangle=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix},\ |r\rangle=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix},$$
$$|l\rangle=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix},\ \sigma_x=\begin{pmatrix}0&1\\1&0\end{pmatrix},\ \sigma_y=\begin{pmatrix}0&-i\\i&0\end{pmatrix},\ \text{and}\ \sigma_z=\begin{pmatrix}1&0\\0&-1\end{pmatrix}.$$

Step 1. Find *H*. $H=\frac{\hbar\omega}{2}\sigma_z$

Step 2. Prepare a state vector. The state vector that corresponds to the observable σ_x is

$$|\Psi(0)\rangle=|r\rangle\ .$$

Step 3. Find the energy eigenvalues and eigenvectors of *H*. From Exercise 4.5 we learned that for a generic direction $\hat{n}=\begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix}=\begin{pmatrix} \sin\phi\cos\theta \\ \sin\phi\sin\theta \\ \cos\phi \end{pmatrix}$ that $H=\frac{\hbar\omega}{2}\sigma_n$ has the following energy eigenvalues and corresponding eigenvectors:

$$E_1=\frac{\hbar\omega}{2},\ E_2=-\frac{\hbar\omega}{2},\ \left|E_1\right\rangle=\begin{pmatrix} \cos\frac{\phi}{2} \\ \sin\frac{\phi}{2}e^{i\theta} \end{pmatrix},\ \text{and}\ \left|E_2\right\rangle=\begin{pmatrix} \sin\frac{\phi}{2} \\ -\cos\frac{\phi}{2}e^{i\theta} \end{pmatrix}.$$

$H=\frac{\hbar\omega}{2}\sigma_z$, and for $\sigma_n=\sigma_z$ we get that $\hat{n}=\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}=\begin{pmatrix} \sin\phi\cos\theta \\ \sin\phi\sin\theta \\ \cos\phi \end{pmatrix}\Rightarrow\phi=0$.

Hence

$$\left|E_1\right\rangle=\begin{pmatrix} \cos\frac{\phi}{2} \\ \sin\frac{\phi}{2}e^{i\theta} \end{pmatrix}=\begin{pmatrix} 1 \\ 0 \end{pmatrix}=\left|u\right\rangle \ ✔$$

$$\left|E_2\right\rangle=\begin{pmatrix} \sin\frac{\phi}{2} \\ -\cos\frac{\phi}{2}e^{i\theta} \end{pmatrix}=\begin{pmatrix} 0 \\ -e^{i\theta} \end{pmatrix}.$$

$\left|E_2\right\rangle$ is a unit vector for any value of θ. However, in Exercise 4.5 it was shown that $\left|E_1\right\rangle=\left|\lambda_1\right\rangle$ and $\left|E_2\right\rangle=\left|\lambda_2\right\rangle$ where $\left|\lambda_1\right\rangle$ and $\left|\lambda_2\right\rangle$ are the eigenvectors of $\sigma_n=\sigma_z$. Since $\left|\lambda_1\right\rangle=\left|u\right\rangle$ and $\left|\lambda_2\right\rangle=\left|d\right\rangle$ we have that

$$\left|E_2\right\rangle=\left|d\right\rangle=\begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ (and consequently } \theta=\pi\text{).} \ ✔$$

Step 4. Calculate $\alpha_j(0)=\left\langle E_j\middle|\Psi(0)\right\rangle$.

$$\alpha_1(0)=\left\langle E_1\middle|\Psi(0)\right\rangle=\left\langle u\middle|r\right\rangle=\left(\left\langle u\right|\right)\left(\left|r\right\rangle\right)=\begin{pmatrix} 1 & 0 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}=\frac{1}{\sqrt{2}}$$

$$\alpha_2(0)=\left\langle E_2\middle|\Psi(0)\right\rangle=\left\langle d\middle|r\right\rangle=\begin{pmatrix} 0 & 1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}=\frac{1}{\sqrt{2}}.$$

Step 5. $\left|\Psi(0)\right\rangle = \sum_{j=1}^{2} \alpha_j(0)\left|E_j\right\rangle$.

$$\left|\Psi(0)\right\rangle = \sum_{j=1}^{2} \alpha_j(0)\left|E_j\right\rangle = \frac{1}{\sqrt{2}}\left(\left|u\right\rangle + \left|d\right\rangle\right) = \frac{1}{\sqrt{2}}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right] = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Sanity check: $\left|\Psi(0)\right\rangle = \left|r\right\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\left|\Psi(0)\right\rangle$ in Step 2 is same as in Step 5 ✔

Step 6. Expand $\left|\Psi(t)\right\rangle = \sum_{j=1}^{2} \alpha_j(t)\left|E_j\right\rangle$ in terms of $\left\{\alpha_j(t)\right\}$

$$\left|\Psi(t)\right\rangle = \sum_{j=1}^{2} \alpha_j(t)\left|E_j\right\rangle = \alpha_1(t)\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \alpha_2(t)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha_1(t) \\ \alpha_2(t) \end{pmatrix}.$$

Step 7. Replace $\alpha_j(t)$ in (6) with $\alpha_j(0)e^{-\frac{i}{\hbar}E_j t}$.

$$\left|\Psi(t)\right\rangle = \begin{pmatrix} \alpha_1(0)e^{-\frac{i}{\hbar}E_1 t} \\ \alpha_2(0)e^{-\frac{i}{\hbar}E_2 t} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}}e^{-\frac{i}{\hbar}\frac{\hbar\omega}{2}t} \\ \frac{1}{\sqrt{2}}e^{-\frac{i}{\hbar}\left(-\frac{\hbar\omega}{2}\right)t} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{-\frac{i\omega t}{2}} \\ e^{\frac{i\omega t}{2}} \end{pmatrix}.$$

Step 8. Specify a new observable at time *t*, compute its eigenvalues $\{\lambda\}$ and eigenvectors $\{\left|\lambda\right\rangle\}$, and calculate the probabilities of the outcomes.

According to the problem statement, the observable σ_y is measured at time *t*.

Thus $\hat{n} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \sin\phi\cos\theta \\ \sin\phi\sin\theta \\ \cos\phi \end{pmatrix} \Rightarrow \phi = \frac{\pi}{2} = \theta$. From Exercise 3.4,

$$\lambda_1 = +1,\ \left|\lambda_1\right\rangle = \begin{pmatrix} \cos\frac{\phi}{2} \\ \sin\frac{\phi}{2}e^{i\theta} \end{pmatrix} = \begin{pmatrix} \cos\frac{\pi}{4} \\ \sin\frac{\pi}{4}e^{\frac{\pi}{2}i} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} = \left|i\right\rangle,$$

$$\lambda_2 = -1 \text{ , and } \left|\lambda_2\right\rangle = \begin{pmatrix} \sin\frac{\phi}{2} \\ -\cos\frac{\phi}{2}\, e^{i\theta} \end{pmatrix} = \begin{pmatrix} \sin\frac{\pi}{4} \\ -\cos\frac{\pi}{4} e^{\frac{\pi}{2}i} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix} = \left|o\right\rangle.$$

Thus

$$\begin{aligned}
\mathrm{P}_1(t) &= \left\langle \lambda_1 \middle| \Psi(t) \right\rangle \left\langle \Psi(t) \middle| \lambda_1 \right\rangle = \left\langle i \middle| \Psi(t) \right\rangle \left\langle \Psi(t) \middle| i \right\rangle \\
&= \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-\frac{1}{2}i\omega t} \\ e^{\frac{1}{2}i\omega t} \end{pmatrix} \right\} \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} e^{\frac{1}{2}i\omega t} & e^{-\frac{1}{2}i\omega t} \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} \right\} \\
&= \frac{1}{4}\left(e^{-\frac{1}{2}i\omega t} - i e^{\frac{1}{2}i\omega t}\right)\left(e^{\frac{1}{2}i\omega t} + i e^{-\frac{1}{2}i\omega t}\right) = \frac{1}{4}\left(1 + i e^{-i\omega t} - i e^{i\omega t} + 1\right) \\
&= \frac{1}{2} + \frac{i}{4}\left[\left(\cos\omega t - i\sin\omega t\right) - \left(\cos\omega t + i\sin\omega t\right)\right] \quad \text{(de Moivre's Theorem)} \\
&= \frac{1}{2}\left(1 + \sin\omega t\right)
\end{aligned}$$

$$\begin{aligned}
\mathrm{P}_{-1}(t) &= \left\langle \lambda_2 \middle| \Psi(t) \right\rangle \left\langle \Psi(t) \middle| \lambda_2 \right\rangle = \left\langle o \middle| \Psi(t) \right\rangle \left\langle \Psi(t) \middle| o \right\rangle \\
&= \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-\frac{1}{2}i\omega t} \\ e^{\frac{1}{2}i\omega t} \end{pmatrix} \right\} \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} e^{\frac{1}{2}i\omega t} & e^{-\frac{1}{2}i\omega t} \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix} \right\} \\
&= \frac{1}{4}\left(e^{-\frac{1}{2}i\omega t} + i e^{\frac{1}{2}i\omega t}\right)\left(e^{\frac{1}{2}i\omega t} - i e^{-\frac{1}{2}i\omega t}\right) = \frac{1}{4}\left(1 - i e^{-i\omega t} + i e^{i\omega t} + 1\right) \\
&= \frac{1}{2} + \frac{i}{4}\left[\left(\cos\omega t + i\sin\omega t\right) - \left(\cos\omega t - i\sin\omega t\right)\right] \\
&= \frac{1}{2} + \frac{i}{4}\left(2i\sin\omega t\right) = \frac{1}{2}\left(1 - \sin\omega t\right)
\end{aligned}$$

Check: $\mathrm{P}_1(t) + \mathrm{P}_{-1}(t) = 1$ ✔

Another check: Both probabilities vary between 0 and 1. ✔

The conclusion is that σ_y varies as a sinusoidal wave over time between 0 and 1 with a mean value of $\frac{1}{2}$.

Similarly, σ_x varies as a sine wave between 0 and 1, and $\sigma_z = 0 \ \forall t$.

For the record, working the problem as stated yields:

1. $H = \dfrac{\hbar\omega}{2}\sigma_z$
2. $\left|\Psi(0)\right\rangle = \left|u\right\rangle$
3. $\left|E_1\right\rangle = \left|u\right\rangle,\ \left|E_2\right\rangle = \left|d\right\rangle,\ E_1 = \dfrac{\hbar\omega}{2},\ E_1 = -\dfrac{\hbar\omega}{2}$
4. $\alpha_1(0) = \left\langle E_1 \middle| \Psi(0)\right\rangle = \left\langle u \middle| u\right\rangle = 1$

   $\alpha_2(0) = \left\langle E_2 \middle| \Psi(0)\right\rangle = \left\langle d \middle| u\right\rangle = 0$
5. $\left|\Psi(0)\right\rangle = \alpha_1(0)\left|E_1\right\rangle + \alpha_2(0)\left|E_2\right\rangle = \left|u\right\rangle + \left|0\right\rangle = \left|u\right\rangle$
6. $\left|\Psi(t)\right\rangle = \alpha_1(t)\left|E_1\right\rangle + \alpha_2(t)\left|E_2\right\rangle = \alpha_1(t)\left|u\right\rangle + \alpha_2(t)\left|d\right\rangle = \begin{pmatrix} \alpha_1(t) \\ \alpha_2(t) \end{pmatrix}$
7. $\left|\Psi(t)\right\rangle = \begin{pmatrix} \alpha_1(0)e^{-\frac{i}{\hbar}E_1 t} \\ \alpha_2(0)e^{-\frac{i}{\hbar}E_2 t} \end{pmatrix} = \begin{pmatrix} e^{-\frac{i}{\hbar}\frac{\hbar\omega}{2}t} \\ 0 \end{pmatrix} = \begin{pmatrix} e^{-\frac{1}{2}i\omega t} \\ 0 \end{pmatrix}$
8. Compute the eigenvectors and eigenvalues of σ_y at time t:

   $$\hat{n} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \sin\phi\cos\theta \\ \sin\phi\sin\theta \\ \cos\phi \end{pmatrix} \Rightarrow \phi = \frac{\pi}{2} = \theta.$$

   From Exercise 3.4,

   $$\lambda_1 = 1,\ \lambda_2 = -1,\ \left|\lambda_1\right\rangle = \begin{pmatrix} \cos\frac{\phi}{2} \\ \sin\frac{\phi}{2}e^{i\theta} \end{pmatrix} = \cdots = \left|i\right\rangle,\ \left|\lambda_2\right\rangle = \cdots = 0.$$

   At time *t:*

   $$\mathrm{P}(1) = \left\langle \lambda_1 \middle| \Psi(t)\right\rangle \left\langle \Psi(t) \middle| \lambda_1\right\rangle = \left\langle i \middle| \Psi(t)\right\rangle \left\langle \Psi(t) \middle| i\right\rangle$$

   $$= \frac{1}{2}\begin{pmatrix} 1 & -i \end{pmatrix}\begin{pmatrix} e^{-\frac{1}{2}i\omega t} \\ 0 \end{pmatrix}\begin{pmatrix} e^{\frac{1}{2}i\omega t} & 0 \end{pmatrix}\begin{pmatrix} 1 \\ i \end{pmatrix} = e^{-\frac{1}{2}i\omega t}e^{\frac{1}{2}i\omega t} = 1$$

   $$\mathrm{P}(-1) = 0$$

   $\Rightarrow \quad \sigma_y$ is unchanging over time.