$\boxed{4.5}$ Let $\vec{n}$ be any unit 3-vector and $H = \frac{\hbar\omega}{2}\vec{\sigma}\cdot\vec{n}$. Find the energy eigenvalues and eigenvectors: $E_1, E_2, |E_1\rangle$, & $|E_2\rangle$.

Solution.

In spherical coordinates, $\vec{n} = \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{bmatrix}$.

By exercise 3.4, $\lambda = \pm 1$, $|\lambda_1\rangle = \begin{bmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}e^{i\phi} \end{bmatrix}$, and $|\lambda_2\rangle = \begin{bmatrix} \sin\frac{\theta}{2} \\ -\cos\frac{\theta}{2}e^{i\phi} \end{bmatrix}$.

$\therefore$ $\boxed{E_1 = \frac{\hbar\omega}{2}\lambda_1 = \frac{\hbar\omega}{2} \text{ and } E_2 = \frac{\hbar\omega}{2}\lambda_2 = -\frac{\hbar\omega}{2}}$.

Let $|E_1\rangle = \begin{bmatrix} x \\ y \end{bmatrix}$. $H|E_1\rangle = \frac{\hbar\omega}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \frac{\hbar\omega}{2}\begin{bmatrix} x \\ -y \end{bmatrix}$ and

$E_1|E_1\rangle = \frac{\hbar\omega}{2}\begin{bmatrix} x \\ y \end{bmatrix}$ $\Rightarrow y = 0$. Since $x^2+y^2=1$, $x = \pm 1$. We can choose $+1$.

So $\boxed{|E_1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}}$

Let $|E_2\rangle = \begin{bmatrix} x \\ y \end{bmatrix}$. $H|E_2\rangle = \frac{\hbar\omega}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \frac{\hbar\omega}{2}\begin{bmatrix} x \\ -y \end{bmatrix}$

$E_2|E_2\rangle = -\frac{\hbar\omega}{2}\begin{bmatrix} x \\ y \end{bmatrix}$ $\Rightarrow x = 0 \Rightarrow y = \pm 1$. Choose $+1$.

$\therefore$ $\boxed{|E_2\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}}$