

3.4 Let $\vec{n} = \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} = \begin{bmatrix} \sin\theta \cos\phi \\ \sin\theta \sin\phi \\ \cos\theta \end{bmatrix}$ be a direction in \mathbb{R}^3 in spherical coordinates

Compute the eigenvectors and eigenvalues for $\sigma_n = \begin{bmatrix} n_z & n_x - i n_y \\ n_x + i n_y & -n_z \end{bmatrix}$.

Solution: $\sigma_n = \begin{bmatrix} \cos\theta & \sin\theta(\cos\phi - i\sin\phi) \\ \sin\theta(\cos\phi + i\sin\phi) & -\cos\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{bmatrix}$

$$0 = \det(\sigma_n - \lambda I) = \begin{vmatrix} \cos\theta - \lambda & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta - \lambda \end{vmatrix} = -\cos^2\theta + \lambda^2 - \sin^2\theta \Rightarrow \boxed{\lambda = \pm 1}$$

Caution: If we let $|\lambda\rangle = \begin{bmatrix} \cos\omega \\ \sin\omega \end{bmatrix}$ as in 3.3 and solve $\sigma_n |\lambda\rangle = \lambda |\lambda\rangle$, we can only achieve solutions for $\vec{n} = \begin{bmatrix} n_x \\ 0 \\ n_z \end{bmatrix}$, directions in xz -plane. To allow a y -component we must introduce $e^{i\phi}$ into $|\lambda\rangle$.

Let $|\lambda\rangle = \begin{bmatrix} \cos\omega \\ \sin\omega e^{i\phi} \end{bmatrix}$. Note $\| |\lambda\rangle \| = 1$. So

$$\begin{bmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{bmatrix} \begin{bmatrix} \cos\omega \\ \sin\omega e^{i\phi} \end{bmatrix} = \sigma_n |\lambda\rangle = \lambda |\lambda\rangle = \begin{bmatrix} \lambda \cos\omega \\ \lambda \sin\omega e^{i\phi} \end{bmatrix}$$

$$\Rightarrow \begin{cases} \cos\theta \cos\omega + \sin\theta \sin\omega = \lambda \cos\omega \\ e^{i\phi} [\sin\theta \cos\omega - \cos\theta \sin\omega] = e^{i\phi} [\lambda \sin\omega] \end{cases} \Leftrightarrow \begin{cases} \cos(\theta - \omega) = \lambda \cos\omega \\ \sin(\theta - \omega) = \lambda \sin\omega \end{cases}$$

$$\lambda_1 = 1: \begin{cases} \cos(\theta - \omega) = \cos\omega \\ \sin(\theta - \omega) = \sin\omega \end{cases} \Rightarrow \theta - \omega = \omega \text{ or } \omega = \frac{\theta}{2} \Rightarrow \boxed{|\lambda_1\rangle = \begin{bmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} e^{i\phi} \end{bmatrix}}$$

$$\lambda_2 = -1: \begin{cases} \cos(\theta - \omega) = -\cos\omega \\ \sin(\theta - \omega) = -\sin\omega \end{cases} \Rightarrow \theta - \omega = \omega + \pi \text{ or } \omega = \frac{\theta}{2} - \frac{\pi}{2}$$

$$\cos\omega = \cos\left(\frac{\theta}{2} - \frac{\pi}{2}\right) = \cos\frac{\theta}{2} \cos\left(\frac{\pi}{2}\right) + \sin\frac{\theta}{2} \sin\left(\frac{\pi}{2}\right) = \sin\frac{\theta}{2}$$

$$\sin\omega = \sin\left(\frac{\theta}{2} - \frac{\pi}{2}\right) = \sin\frac{\theta}{2} \cos\left(\frac{\pi}{2}\right) - \cos\frac{\theta}{2} \sin\left(\frac{\pi}{2}\right) = -\cos\frac{\theta}{2}$$

So $\boxed{|\lambda_2\rangle = \begin{bmatrix} \sin\frac{\theta}{2} \\ -\cos\frac{\theta}{2} e^{i\phi} \end{bmatrix}}$

Extra: $P(1) = |\langle u | \lambda_1 \rangle|^2 = \left| \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} e^{i\phi} \end{bmatrix} \right|^2 = \cos^2\frac{\theta}{2} \checkmark$

$P(-1) = \sin^2\frac{\theta}{2}$ since $|e^{i\phi}| = 1 \checkmark$

Note: Multiplying $|\lambda_1\rangle$ or $|\lambda_2\rangle$ by $e^{i\phi}$ yields equivalent solutions