Exercise 3.4 Let $\hat{n} = \begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}$ be a direction (i.e., unit 3-vector) in spherical coordinates. Compute the eigenvalues and eigenvectors for

$$\sigma_n = \begin{pmatrix} n_z & n_x - i n_y \\ n_x + i n_y & -n_z \end{pmatrix}.$$

Solution.

$$\sigma_n = \begin{pmatrix} \cos\theta & \sin\theta(\cos\phi - i\sin\phi) \\ \sin\theta(\cos\phi + i\sin\phi) & -\cos\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix}.$$

To find the eigenvalues we solve the characteristic equation:

$$0 = \det(\sigma_n - \lambda I) = \begin{vmatrix} \cos\theta - \lambda & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta - \lambda \end{vmatrix} = -\cos^2\theta + \lambda^2 - \sin^2\theta$$

$$\Rightarrow \boxed{\lambda_1 = 1} \text{ and } \boxed{\lambda_2 = -1}.$$

To find the eigenvectors we assume $|\lambda\rangle = \begin{pmatrix} \cos\omega \\ \sin\omega e^{i\phi} \end{pmatrix}$ and then solve the eigenvector equation $\sigma_n|\lambda\rangle = \lambda|\lambda\rangle$:

$$\begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix}\begin{pmatrix} \cos\omega \\ \sin\omega e^{i\phi} \end{pmatrix} = \begin{pmatrix} \lambda\cos\omega \\ \lambda\sin\omega e^{i\phi} \end{pmatrix}$$

$$\begin{pmatrix} \cos\theta\cos\omega + \sin\theta\sin\omega \\ e^{i\phi}[\sin\theta\cos\omega - \cos\theta\sin\omega] \end{pmatrix} = \begin{pmatrix} \lambda\cos\omega \\ e^{i\phi}[\lambda\sin\omega] \end{pmatrix}$$

$$\begin{pmatrix} \cos(\theta - \omega) \\ \sin(\theta - \omega) \end{pmatrix} = \begin{pmatrix} \lambda\cos\omega \\ \lambda\sin\omega \end{pmatrix}.$$

For λ = 1:

$$\begin{cases} \cos(\theta - \omega) = \cos\omega \\ \sin(\theta - \omega) = \sin\omega \end{cases}$$

$$\Rightarrow\ \theta-\omega=\omega\ \Rightarrow\ \omega=\frac{\theta}{2}\ \Rightarrow\ \boxed{\left|\lambda_1\right\rangle=\begin{pmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}e^{i\phi}\end{pmatrix}}$$

For λ = -1:

$$\begin{cases}\cos(\theta-\omega)=-\cos\omega\\ \sin(\theta-\omega)=-\sin\omega\end{cases}\Leftrightarrow\begin{cases}\cos(\theta-\omega)=\cos(\omega+\pi)\\ \sin(\theta-\omega)=\sin(\omega+\pi)\end{cases}$$

$$\Rightarrow\ \theta-\omega=\omega+\pi\ \Rightarrow\ \omega=\frac{\theta}{2}-\frac{\pi}{2}$$

$$\Rightarrow\ \left|\lambda_2\right\rangle=\begin{pmatrix}\cos\omega\\ \sin\omega\, e^{i\phi}\end{pmatrix}=\begin{pmatrix}\cos\left(\frac{\theta}{2}-\frac{\pi}{2}\right)\\ \sin\left(\frac{\theta}{2}-\frac{\pi}{2}\right)e^{i\phi}\end{pmatrix}=\begin{pmatrix}\sin\frac{\theta}{2}\\ -\cos\frac{\theta}{2}e^{i\phi}\end{pmatrix}$$

$$\boxed{\left|\lambda_2\right\rangle=\begin{pmatrix}\sin\frac{\theta}{2}\\ -\cos\frac{\theta}{2}e^{i\phi}\end{pmatrix}}.$$ ■

Note 1: Multiplying either $\left|\lambda_1\right\rangle$ or $\left|\lambda_2\right\rangle$ by a phase factor of $e^{-i\phi}$ yields an equivalent solution.

Note 2: Had we assumed $\left|\lambda\right\rangle=\begin{pmatrix}\cos\omega\\ \sin\omega\end{pmatrix}$ as we did in Exercise 3.3, without the phase factor, that would have been equivalent to setting ϕ = 0. That would have meant that $\hat{n}=\begin{pmatrix}n_x\\ n_y\\ n_z\end{pmatrix}=\begin{pmatrix}\sin\theta\\ 0\\ \cos\theta\end{pmatrix}$ and so we could not have obtained a solution for general $\hat{n}$.