$\boxed{6.8}$ ~~Charlie prepares spin in state $|T_1\rangle = \frac{1}{\sqrt{2}}(|ud\rangle + |du\rangle)$.~~ What are $\langle \sigma_z \tau_z \rangle$, $\langle \sigma_x, \tau_x \rangle$, and $\langle \sigma_y, \tau_y \rangle$ when either $|T_2\rangle = \frac{1}{\sqrt{2}}(|uu\rangle + |dd\rangle)$ or $|T_3\rangle = \frac{1}{\sqrt{2}}(|uu\rangle - |dd\rangle)$ are prepared

Solution. $\langle T_2 | T_2 \rangle = \frac{1}{2}(\overset{1}{\langle uu|uu\rangle} + \overset{0}{\langle uu|dd\rangle} + \overset{0}{\langle dd|uu\rangle} + \overset{1}{\langle dd|dd\rangle}) = 1$

and $\langle T_3 | T_3 \rangle = \frac{1}{2}(\overset{1}{\langle uu|uu\rangle} - \overset{0}{\langle uu|dd\rangle} - \overset{0}{\langle dd|uu\rangle} + \overset{1}{\langle dd|dd\rangle}) = 1$

$\sigma_z \tau_z |T_2\rangle = \sigma_z \frac{1}{\sqrt{2}}(|uu\rangle - |dd\rangle) = \frac{1}{\sqrt{2}}(|uu\rangle + |dd\rangle) = |T_2\rangle$

Since $|T_2\rangle$ is an eigenvector of $\sigma_z \tau_z$ with eigenvalue $+1$, $\langle \sigma_z \tau_z \rangle = +1$ ✓

Another way: $\langle \sigma_z \tau_z \rangle = \langle T_2 | \sigma_z \tau_z | T_2 \rangle = \langle T_2 | T_2 \rangle = +1$ ✓

$\sigma_x \tau_x |T_2\rangle = \sigma_x \frac{1}{\sqrt{2}}(|ud\rangle + |du\rangle) = \frac{1}{\sqrt{2}}(|dd\rangle + |uu\rangle) = |T_2\rangle$

as with $\sigma_z \tau_z$, $\langle \sigma_x \tau_x \rangle = +1$ ✓

$\sigma_y \tau_y |T_2\rangle = \sigma_y \frac{1}{\sqrt{2}}(i|ud\rangle - i|du\rangle) = \frac{i}{\sqrt{2}}(i|dd\rangle + i|uu\rangle) = -\frac{1}{\sqrt{2}}(|uu\rangle + \langle dd\rangle$
$= -|T_2\rangle$

Since -1 is the eigenvalue, $\langle \sigma_y \tau_y \rangle = -1$ ✓

Note 1. all three pairs are completly corrrelated

Note 2. $\boxed{\vec{\sigma} \cdot \vec{\tau} |T_2\rangle = [\sigma_x \tau_x + \sigma_y \tau_y + \sigma_z \tau_z] |T_2\rangle = |T_2\rangle \text{ and } \vec{\sigma} \cdot \vec{\tau} |T_3\rangle = |T_3\rangle \Rightarrow |T_2\rangle \text{ (as well as } |T_1\rangle) \text{ is an eigenvector of } \vec{\sigma} \cdot \vec{\tau} \text{ with eigenvalue } +1}$

$\sigma_z \tau_z |T_3\rangle = \sigma_z \frac{1}{\sqrt{2}}(|uu\rangle + |dd\rangle) = \frac{1}{\sqrt{2}}(|uu\rangle - |dd\rangle) = |T_3\rangle$

$\Rightarrow \langle \sigma_z \tau_z \rangle = 1$

$\sigma_x \tau_x |T_3\rangle = \sigma_x \frac{1}{\sqrt{2}}(|ud\rangle - |du\rangle) = \frac{1}{\sqrt{2}}(|dd\rangle - |uu\rangle) = -|T_3\rangle$

$\Rightarrow \langle \sigma_x \tau_x \rangle = -1$

$\sigma_y \tau_y |T_3\rangle = \sigma_y \frac{1}{\sqrt{2}}(-i|ud\rangle + i|du\rangle) = \frac{i}{\sqrt{2}}(i|dd\rangle - i|uu\rangle) = \frac{1}{\sqrt{2}}(-|dd\rangle + |uu\rangle)$
$= |T_3\rangle$

$\Rightarrow \langle \sigma_y \tau_y \rangle = +1$

Note 1 $\boxed{|T_1\rangle, |T_2\rangle, \& |T_3\rangle \text{ are eigenvectors of } \vec{\sigma} \cdot \vec{\tau} \text{ w/ triply degenerate eigenvalue } +1}$

Note 2: all three pairs are again completly correlated