$\boxed{6.7}$ Charlie prepares spin in state $|T_1\rangle = \frac{1}{\sqrt{2}}(|ud\rangle + |du\rangle)$.

Find $\langle \sigma_z \tau_z \rangle$, $\langle \sigma_x, \tau_x \rangle$, and $\langle \sigma_y, \tau_y \rangle$.

Solution: $\langle T_1 | T_1 \rangle = \frac{1}{2}\left(\overset{1}{\langle ud|ud\rangle} + \overset{0}{\langle ud|du\rangle} + \overset{0}{\langle du|ud\rangle} + \overset{1}{\langle du|du\rangle}\right) = 1$

$\sigma_z \tau_z |T_1\rangle = \sigma_z \frac{1}{\sqrt{2}}(-|ud\rangle + |du\rangle) = \frac{1}{\sqrt{2}}(-|ud\rangle - |du\rangle) = -|T_1\rangle$

$\therefore \langle \sigma_z \tau_z \rangle = \langle T_1 | \sigma_z \tau_z | T_1 \rangle = -\langle T_1 | T_1 \rangle = -1$ ✓ Note $\mathrm{Corr}(\sigma_z, \tau_z) = -1$

$\sigma_x \tau_x |T_1\rangle = \sigma_x \frac{1}{\sqrt{2}}(|uu\rangle + |dd\rangle) = \frac{1}{\sqrt{2}}(|du\rangle + |ud\rangle) = |T_1\rangle$

$\therefore \langle \sigma_x \tau_x \rangle = \langle T_1 | T_1 \rangle = 1$ ✓ $\mathrm{Corr}(\sigma_x, \tau_x) = 1$

$\sigma_y \tau_y |T_1\rangle = \sigma_y \frac{1}{\sqrt{2}}(-i|uu\rangle + i|dd\rangle) = \frac{i}{\sqrt{2}}(-i|du\rangle - i|ud\rangle) = \frac{1}{\sqrt{2}}(|du\rangle + |ud\rangle)$

$= |T_1\rangle$

$\therefore \langle \sigma_y \tau_y \rangle = \langle T_1 | T_1 \rangle = 1$ ✓ and $\mathrm{Corr}(\sigma_y, \tau_y) = 1$

$\boxed{\text{All three are completely correlated}}$

Corollary: If system is prepared in state $|T_1\rangle$ then

$\boxed{\vec{\sigma} \cdot \vec{\tau} |T_1\rangle = (\sigma_x \tau_x + \sigma_y \tau_y + \sigma_z \tau_z)|T_1\rangle = 1|T_1\rangle \Rightarrow |T_1\rangle \text{ is eigenvector of } \vec{\sigma} \cdot \vec{\tau} \text{ w/ eigenvalue } +1}$

Note: In all 3 cases, $|T_1\rangle$ is an eigenvector. So when the eigenvalue $= -1$, $\langle \circ \rangle = -1$ and when the eigenvalue $= +1$, $\langle \circ \rangle = +1$.