Exercise 4.5 Let $\hat{n}$ be any direction (i.e., unit 3-vector) and $H = \frac{\hbar\omega}{2}\vec{\sigma} \bullet \vec{n}$. Find the energy eigenvalues and eigenvectors E_1, E_2, $\left|E_1\right\rangle$, and $\left|E_2\right\rangle$.

Solution.

In spherical coordinates $\hat{n} = \begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}$.

From Exercise 3.4 the eigenvalues and eigenvectors of σ_n are respectively

$$\lambda_1 = +1, \quad \left|\lambda_1\right\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}\, e^{i\phi} \end{pmatrix}$$

and

$$\lambda_2 = -1, \quad \left|\lambda_2\right\rangle = \begin{pmatrix} \sin\frac{\theta}{2} \\ -\cos\frac{\theta}{2}\, e^{i\phi} \end{pmatrix}.$$

Since $H = \frac{\hbar\omega}{2}\hat{\sigma}_n$, we have that

$$H\left|\lambda_1\right\rangle = \frac{\hbar\omega}{2}\hat{\sigma}_n\left|\lambda_1\right\rangle = \frac{\hbar\omega}{2}\lambda_1\left|\lambda_1\right\rangle = \frac{\hbar\omega}{2}\left|\lambda_1\right\rangle$$

and

$$H\left|\lambda_2\right\rangle = \frac{\hbar\omega}{2}\hat{\sigma}_n\left|\lambda_2\right\rangle = \frac{\hbar\omega}{2}\lambda_2\left|\lambda_2\right\rangle = -\frac{\hbar\omega}{2}\left|\lambda_2\right\rangle.$$

Claim $E_1 = \frac{\hbar\omega}{2}$, $\left|E_1\right\rangle = \left|\lambda_1\right\rangle$, $E_2 = -\frac{\hbar\omega}{2}$, and $\left|E_2\right\rangle = \left|\lambda_2\right\rangle$ are the respective eigenvalues and eigenvectors of H:

$$H\left|E_1\right\rangle = \frac{\hbar\omega}{2}\hat{\sigma}_n\left|\lambda_1\right\rangle = \frac{\hbar\omega}{2}\lambda_1\left|\lambda_1\right\rangle = \frac{\hbar\omega}{2}\left|\lambda_1\right\rangle = E_1\left|E_1\right\rangle \quad ✔$$

$$H\left|E_2\right\rangle = \frac{\hbar\omega}{2}\hat{\sigma}_n\left|\lambda_2\right\rangle = \frac{\hbar\omega}{2}\lambda_2\left|\lambda_2\right\rangle = -\frac{\hbar\omega}{2}\left|\lambda_2\right\rangle = E_2\left|E_2\right\rangle \quad ✔$$

Thus

$$\boxed{E_1 = \frac{\hbar\omega}{2}}, \quad \boxed{\left|E_1\right\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}\, e^{i\phi} \end{pmatrix}}$$

and

$$\boxed{E_2 = -\frac{\hbar\omega}{2}}, \quad \boxed{\left|E_1\right\rangle = \begin{pmatrix} \sin\frac{\theta}{2} \\ -\cos\frac{\theta}{2}\, e^{i\phi} \end{pmatrix}}.$$