Exercise 3.1. Let V be an n-dimensional vector space and L a matrix. L has at least one (non-zero) eigenvector. If L is Hermitian then V has an orthonormal basis consisting of eigenvectors of L.

Proof. Let $p_L(\lambda) = \det(L - \lambda \boldsymbol{I})$ be the characteristic polynomial of L. $p_L(\lambda)$ is a polynomial in λ. By the Fundamental Theorem of Algebra, $p_L(\lambda) = K(\lambda - \lambda_1)^{p_1} \cdots (\lambda - \lambda_r)^{p_r}$ where K is a complex constant, r is a positive integer, $p_1 + \cdots + p_r = n$, and $\lambda_1, \cdots, \lambda_n$ are roots of $p_L(\lambda)$, some possibly with multiplicity p_i greater than 1. In particular, $p_L(\lambda_1) = 0$. By footnote (*) there is a non-zero vector $|\lambda_1\rangle$ such that $L|\lambda_1\rangle = \lambda_1|\lambda_1\rangle$. So λ_1 is an eigenvalue of L with eigenvector $|\lambda_1\rangle$. This proves that any matrix L has at least one eigenvector.

Now suppose that L is Hermitian. Then λ is real. WLOG we can assume $|\lambda_1\rangle$ is a unit vector, $\langle\lambda_1|\lambda_1\rangle = 1$. Define the null space $N = \{|v\rangle : \langle v|\lambda_1\rangle = 0\}$. It is easy to see that N is a vector subspace of L. Since dim $\{\alpha|\lambda_1\rangle : \alpha \in \mathbb{C}\}$ is a 1-dimensional subspace, the orthogonal subspace N has dimension n – 1. Claim $LN \subseteq N$:

Let $|v\rangle \in N$. We need to show that $L|v\rangle \in N$. Since L is Hermitian, $L|v\rangle \leftrightarrow \langle v|L$. So we need to show that $\langle v|L|\lambda_1\rangle = 0$.:

$$\langle v|L|\lambda_1\rangle = \langle v|\lambda_1|\lambda_1\rangle = \lambda_1\langle v|\lambda_1\rangle = 0 \quad ✔$$

Let $L_2 = L$ restricted to N. Repeating our logic above, $p_L(L_2)$ has a real root λ_2 that is an eigenvalue of L_2 with corresponding unit eigenvector $|\lambda_2\rangle$. Since $|\lambda_2\rangle \in N$, $\langle\lambda_1|\lambda_2\rangle = 0 \Rightarrow |\lambda_1\rangle \perp |\lambda_2\rangle$.

Using the (n – 2)-dimensional null space of L_2 as above we generate $|\lambda_3\rangle \perp |\lambda_2\rangle$, and since $|\lambda_3\rangle \in N$, $|\lambda_3\rangle \perp |\lambda_1\rangle$ also.

Continuing this process we eventually obtain the orthonormal basis $\{|\lambda_i\rangle : i = 1, \cdots, n\}$. ■

(*) Suppose we have *n* equations in *n* unknowns:

$$\begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = 0 \\ \vdots \\ a_{n1}x_1 + \cdots + a_{nn}x_n = 0 \end{cases} \quad \Leftrightarrow \quad A|x\rangle = |0\rangle$$

If det $A \neq 0$, then A^{-1} exists. Left multiplying by A^{-1} yields $|x\rangle = A^{-1}|0\rangle = |0\rangle$ as the unique solution for the system. That is, $x_1 = \cdots = x_n = 0$ is the unique solution.

If det A = 0, then there is not a unique solution. Since $|0\rangle$ is still a solution, there must be another (non-zero) solution $|x\rangle$; that is, $A|x\rangle = |0\rangle$ with $|x\rangle \neq 0$.

Returning our attention to *L*, we have λ_1 and need to find $|\lambda_1\rangle \neq 0$ such that $L|\lambda_1\rangle = \lambda_1|\lambda_1\rangle$. Let $A = L - \lambda_1 \boldsymbol{I}$. Consider the system of *n* equations in *n* unknowns $A|x\rangle = 0$. So $\det A = \det\left(L - \lambda_1 \boldsymbol{I}\right) = p_L\left(\lambda_1\right) = 0 \quad \Rightarrow \quad \exists$ non-zero $|x\rangle = |\lambda_1\rangle$ such that $\left(L - \lambda_1 \boldsymbol{I}\right)|\lambda_1\rangle = \mathrm{A}|x\rangle = 0$. That is, $L|\lambda_1\rangle = \lambda|\lambda_1\rangle$ and $|\lambda_1\rangle \neq 0$.