$\boxed{3.5}$ Prepare a system in direction $\hat{m}$ and make a measurement in direction $\hat{n}$. Find $P(1)$.

Solution. In spherical coordinates, $\vec{m} = \begin{bmatrix} m_x \\ m_y \\ m_z \end{bmatrix} = \begin{bmatrix} \sin\theta_1 \cos\phi_1 \\ \sin\theta_1 \sin\phi_1 \\ \cos\theta_1 \end{bmatrix}$ and

$\vec{n} = \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} = \begin{bmatrix} \sin\theta_2 \cos\phi_2 \\ \sin\theta_2 \sin\phi_2 \\ \cos\theta_2 \end{bmatrix}$. From problem 3.4, the eigenvalues and eigenvectors

for σ_m are $\mu_1 = 1$, $\mu_2 = -1$, $|\mu_1\rangle = \begin{bmatrix} \cos\frac{\theta_1}{2} \\ \sin\frac{\theta_1}{2} e^{i\phi_1} \end{bmatrix}$, and $|\mu_2\rangle = \begin{bmatrix} -\sin\frac{\theta_1}{2} \\ \cos\frac{\theta_1}{2} e^{i\phi_1} \end{bmatrix}$.

For σ_n they are $\lambda_1 = 1$, $\lambda_2 = -1$, $|\lambda_1\rangle = \begin{bmatrix} \cos\frac{\theta_2}{2} \\ \sin\frac{\theta_2}{2} e^{i\phi_2} \end{bmatrix}$, and $|\lambda_2\rangle = \begin{bmatrix} -\sin\frac{\theta_2}{2} \\ \cos\frac{\theta_2}{2} e^{i\phi_2} \end{bmatrix}$.

Let w be the angle between $\vec{m}$ and $\vec{n}$. Since they are unit vectors,

$\vec{m} \cdot \vec{n} = |\vec{m}||\vec{n}| \cos w = \cos w$. That is

$\cos w = \sin\theta_1 \cos\phi_1 \sin\theta_2 \cos\phi_2 + \sin\theta_1 \sin\phi_1 \sin\theta_2 \sin\phi_2 + \cos\theta_1 \cos\theta_2$

We wish to show $P(1) = \cos^2\frac{w}{2} = \frac{1+\cos w}{2}$ (half-angle formula). Now,

$\frac{1+\cos w}{2} = \frac{1}{2}\left[1 + \sin\theta_1 \cos\phi_1 \sin\theta_2 \cos\phi_2 + \sin\theta_1 \sin\phi_1 \sin\theta_2 \sin\phi_2 + \cos\theta_1 \cos\theta_2\right]$

The system is prepared in the direction $\hat{m}$, so $|\mu_1\rangle$ is the state vector of the system. We measure in direction $\hat{n}$, so σ_n is the observable (Hermitian operator). Thus

$P(1) \overset{3.11}{=} \langle\mu_1|\lambda_1\rangle\langle\lambda_1|\mu_1\rangle$. $\langle\lambda_1| = \begin{bmatrix} \cos\frac{\theta_2}{2} & \sin\frac{\theta_2}{2} e^{-i\phi_2} \end{bmatrix}$, so

(i) $\langle\lambda_1|\mu_1\rangle = \cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2} + \sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2} e^{-i(\phi_2-\phi_1)}$. $\langle\mu_1| = \begin{bmatrix} \cos\frac{\theta_1}{2} & \sin\frac{\theta_1}{2} e^{-\phi_1} \end{bmatrix}$, so

(ii) $\langle\mu_1|\lambda_1\rangle = \cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2} + \sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2} e^{i(\phi_2-\phi_1)}$. Thus, since $\cos\phi = \frac{e^{-i\phi}}{2} + \frac{e^{i\phi}}{2}$,

$$P(1) \overset{(i,ii)}{=} \cos^2\frac{\theta_1}{2}\cos^2\frac{\theta_2}{2} + \sin\frac{\theta_1}{2}\cos\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\cos\frac{\theta_2}{2}\left[e^{i(\phi_2-\phi_1)} + e^{-i(\phi_2-\phi_1)}\right] + \sin^2\frac{\theta_1}{2}\sin^2\frac{\theta_2}{2}$$

$$= \frac{1+\cos\theta_1}{2}\,\frac{1+\cos\theta_2}{2} + 2\sin\frac{\theta_1}{2}\cos\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\cos\frac{\theta_2}{2}\cos(\phi_2-\phi) + \frac{1-\cos\theta_1}{2}\,\frac{1-\cos\theta_2}{2}$$ (by Half angle & sum formulas)

$$= \frac{1}{4}(1+\cancel{\cos\theta_1}+\cancel{\cos\theta_2}+\cos\theta_1\cos\theta_2) + \frac{1}{2}\sin\theta_1\sin\theta_2\cos(\phi_2-\phi_1) + \frac{1}{4}(1-\cancel{\cos\theta_1}-\cancel{\cos\theta_2}+\cos\theta_1\cos\theta_2)$$

$$= \frac{1}{2}\left[1 + \cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2(\cos\phi_2\cos\phi_1 + \sin\phi_2\sin\phi_1)\right]$$

$$= \frac{1}{2}\left[1 + \sin\theta_1\cos\phi_1\sin\theta_2\cos\phi_2 + \sin\theta_1\sin\phi_1\sin\theta_2\sin\phi_2 + \cos\theta_1\cos\theta_2\right]$$

$$= \frac{1+\cos w}{2} = \cos^2\frac{w}{2} \checkmark$$