Exercise 6.5.

1) When any of Alice's or Bob's spin operators acts on a product state, the result is a product state

2) Show that in a product state, the expectation of any component of $\vec{\sigma}$ or $\vec{\tau}$ is the same as it would be in the individual spin states.

Solution.

1) Let $|AB\rangle$ be a product state and let $\sigma_w = \sigma_x$ or $\sigma_y$ or $\sigma_z$. Then $|A'\} \equiv \sigma_w |A\}$ is another of Alice's states. Thus
$$\sigma_w |AB\rangle = (\sigma_w \otimes \boldsymbol{I})\big(|A\} \otimes |B\rangle\big) = \sigma_w |A\rangle \otimes \boldsymbol{I}|B\rangle = A'\} \otimes |B\rangle = |A'B\rangle$$
is a product state, and similarly for Bob.

2) In part 1, if $|A\}$ is normalized (i.e., a unit vector) then so is $|A'\}$ since $\det \sigma_w = -1$ (whether $\sigma_w = \sigma_x$, $\sigma_y$, or $\sigma_z$). If, in addition, $|B\rangle$ is normalized, then so is $|AB\rangle$:

First we have to define what we mean for $|AB\rangle$ to be a unit vector. For any $|CD\rangle$ we define $\langle CD|AB\rangle \equiv \langle C|A\rangle\langle D|B\rangle$. Now, since $|A\}$ and $|B\rangle$ are normalized, $\langle A|A\rangle = 1 = \langle B|B\rangle$. Hence $\langle AB|AB\rangle = \langle A|A\rangle\langle B|B\rangle = 1$.

Since both $|A\}$ and $|AB\rangle$ are normalized, then equation (4.14) in the book is valid:
$$\langle \sigma_w \rangle \overset{(4.14)}{=} \langle AB|\sigma_w|AB\rangle = \big(\{A|\sigma_w|A\}\big)\big(\langle B|B\rangle\big) \overset{(4.14)}{=} \langle \sigma_w \rangle \langle \boldsymbol{I} \rangle = \langle \sigma_w \rangle$$