Exercise 6.5.

1) When any of Alice's or Bob's spin operators acts on a product state, the result is a product state

2) Show that in a product state, the expectation of any component of $\vec{\sigma}$ or $\vec{\tau}$ is the same as it would be in the individual spin states.

Solution.

1) Let $|AB\rangle$ be a product state and let $\sigma_w = \sigma_x$ or σ_y or σ_z. Then $|A'\} \equiv \sigma_w |A\}$ is another of Alice's states. Thus

$$\sigma_w |AB\rangle = (\sigma_w \otimes \boldsymbol{I})\left(|A\} \otimes |B\rangle\right) = \sigma_w |A\} \otimes \boldsymbol{I}|B\rangle = A'\} \otimes |B\rangle = |A'B\rangle$$

is a product state, and similarly for Bob.

2) In part 1, if $|A\}$ is normalized (i.e., a unit vector) then so is $|A'\}$ since $\det \sigma_w = -1$ (whether $\sigma_w = \sigma_x$, σ_y, or σ_z). If, in addition, $|B\rangle$ is normalized, then so is $|AB\rangle$:

First we have to define what we mean for $|AB\rangle$ to be a unit vector. We would want it to satisfy $\left(\langle (AB)^* |\right)\left(|AB\rangle\right) = 1$. Since A and B cannot be co-mingled, we must define $\langle (AB)^*| = \langle A^* B^*|$. Since $|A\}$ and $|B\rangle$ are normalized, $\langle A^*|A\rangle = 1 = \langle B^*|B\rangle$. Hence $\langle A^*B^*|AB\rangle = \langle A^*|A\rangle\langle B^*|B\rangle = 1$ if we define $\langle CD|AB\rangle \equiv \langle C|D\rangle\langle A|B\rangle$ for any $\langle CD|$.

Since both $|A\}$ and $|AB\rangle$ are normalized, then equation (4.14) in the book is valid:

$$\langle \sigma_w \rangle \overset{(4.14)}{=} \langle AB|\sigma_w|AB\rangle = \left(\{A|\sigma_w|A\}\right)\left(\langle B|B\rangle\right) \overset{(4.14)}{=} \langle \sigma_w \rangle \langle \boldsymbol{I} \rangle = \langle \sigma_w \rangle$$

where the first σ_w is in the product state and the final σ_w is in Alice's spin state.

Of course, the proof for τ is similar.