Exercise 4.6. Carry out the ket recipe with $H = \frac{\hbar\omega}{2}\sigma_z$, final observable $\sigma_x$, initial state $|u\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and observable $\sigma_y$ measured at time *t*. Find the possible outcomes and their probabilities of occurrence.

Solution. This exercise is the culmination of a problem started in Exercise 3.4 and continued in 4.5. There is an unfortunate choice of wording in the 4.6 problem statement. We should not confuse "initial state" in the problem wording with "initial state" in step 2 of the ket recipe. They are quite different.

Where the problem wording says we are given initial state $|u\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, it simply means that as developed in chapter 2, formula (2.13), we should use the following additional formulas:

$$|d\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix},\ |i\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix},\ |o\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \end{pmatrix},\ |r\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix},\ |l\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

$$|\sigma_z\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ \text{and}\ |\sigma_y\rangle = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

Step 1. $H = \frac{\hbar\omega}{2}\sigma_z$

Step 2. Where the problem statement says the final observable is $\sigma_x$, it means that the state vector of the system as defined on page 70 and illustrated on page 37 is

$$|\Psi(0)\rangle = |r\rangle.$$

That is, what is called the "initial state" in Step 2 is what was previously called "state vector".

Step 3. From Exercise 4.5 we learned that

$$E_1 = \frac{\hbar w}{2},\ E_2 = -\frac{\hbar w}{2},\ |E_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ \text{and}\ |E_2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Step 4. $\alpha_1(0) = \langle E_1|r\rangle = (1\ \ 0)\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}$ and

$$\alpha_2(0) = \langle E_2|l\rangle = (1\ \ 0)\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}}.$$

Step 5. $\left|\Psi(0)\right\rangle = \sum_{j=1}^{2}\alpha_j(0)\left|E_j\right\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}.$

Step 6. $\left|\Psi(t)\right\rangle = \sum_{j=1}^{2}\alpha_j(t)\left|E_j\right\rangle = \alpha_1(t)\begin{pmatrix}1\\0\end{pmatrix} + \alpha_2(t)\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}\alpha_1(t)\\\alpha_2(t)\end{pmatrix}.$

Step 7. $\Psi(t) = \begin{pmatrix}\alpha_1(0)e^{-\frac{i}{\hbar}E_1 t}\\\alpha_2(0)e^{-\frac{i}{\hbar}E_2 t}\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{2}}e^{-\frac{i\hbar\omega}{2\hbar}E_1 t}\\\frac{1}{\sqrt{2}}e^{\frac{i\hbar\omega}{2\hbar}E_2 t}\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}e^{-\frac{i\omega t}{2}}\\e^{\frac{i\omega t}{2}}\end{pmatrix}.$

Step 8. From Exercise 4.5, $\sigma_y$ has eigenvalues ±1 and eigenvectors $\left|i\right\rangle$ and $\left|o\right\rangle$. Thus

$$P_1(t) = \left\langle i\middle|\Psi(t)\right\rangle\left\langle\Psi(t)\middle|i\right\rangle = \left\{\frac{1}{\sqrt{2}}(1\ \ \text{-}i)\frac{1}{\sqrt{2}}\begin{pmatrix}e^{-\frac{1}{2}i\omega t}\\e^{\frac{1}{2}i\omega t}\end{pmatrix}\right\}\left\{\frac{1}{\sqrt{2}}\left(e^{\frac{1}{2}i\omega t}\ \ e^{-\frac{1}{2}i\omega t}\right)\frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}\right\}$$

$$= \frac{1}{4}\left(e^{-\frac{1}{2}i\omega t} - ie^{-\frac{1}{2}i\omega t}\right)\left(e^{\frac{1}{2}i\omega t} + ie^{-\frac{1}{2}i\omega t}\right) = \frac{1}{2}\left[1 + i\left(\frac{e^{-i\omega t}}{2} - \frac{e^{i\omega t}}{2}\right)\right]$$

$$= \frac{1}{2}(1 + \sin\omega t) \quad \text{by De Moivre's Theorem}$$

$$P_{-1}(t) = \left\langle o\middle|\Psi(t)\right\rangle\left\langle\Psi(t)\middle|o\right\rangle = \left\{\frac{1}{\sqrt{2}}(1\ \ i)\frac{1}{\sqrt{2}}\begin{pmatrix}e^{-\frac{1}{2}i\omega t}\\e^{\frac{1}{2}i\omega t}\end{pmatrix}\right\}\left\{\frac{1}{\sqrt{2}}\left(e^{\frac{1}{2}i\omega t}\ \ e^{-\frac{1}{2}i\omega t}\right)\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}\right\}$$

$$= \frac{1}{4}\left(e^{-\frac{1}{2}i\omega t} + ie^{\frac{1}{2}i\omega t}\right)\left(e^{\frac{1}{2}i\omega t} - ie^{-\frac{1}{2}i\omega t}\right) = \frac{1}{2}\left[1 - i\left(\frac{e^{-i\omega t}}{2} - \frac{e^{i\omega t}}{2}\right)\right]$$

$$= \frac{1}{2}(1 - \sin\omega t)$$

Check: $P_1(t) + P_{-1}(t) = 1$ ✔

Another check: Both probabilities vary between 0 and 1. ✔

The conclusion is that $\sigma_y$ varies as a sinusoidal wave over time with a mean value of $1/2$.