Exercise 4.2. If M and L are Hermitian, then i [M, L] is Hermitian but not necessarily [M, L].

Proof. Given $M^{\dagger}=M$ and $L^{\dagger}=L$, we wish to show that $\left\{i\left[M,L\right]\right\}^{\dagger}=i\left[M,L\right]$. Let $M=\left[m_{ij}\right]$. Then $iM=\left[i\,m_{ij}\right]$ and

$$\left(i\mathbf{M}\right)^{\dagger} = \left[\left(i\,\mathbf{m}_{_{\!\!\mathit{j}\,i}}\right)^{\!\star}\right] = \left[i^{\star}\,\mathbf{m}_{_{\!\!\mathit{j}\,i}}\right]^{\!\star} = \left[-i\,\mathbf{m}_{_{\!\!\mathit{j}\,i}}\right]^{\!\star} = -i\left[\mathbf{m}_{_{\!\!\mathit{j}\,i}}\right]^{\!\star} = -i\mathbf{M}^{\dagger} \; .$$

So

$$\begin{aligned} \left\{i\big[M,L\big]\right\}^{\dagger} &= \left\{i\big(ML - LM\big)\right\}^{\dagger} = \left\{(IM\big)L - L\big(IM\big)\right\}^{\dagger} = L^{\dagger}\big(iM\big)^{\dagger} - \big(iM\big)^{\dagger}L^{\dagger} \\ &= -iL^{\dagger}M^{\dagger} + iM^{\dagger}L^{\dagger} = i\big(M^{\dagger}L^{\dagger} - L^{\dagger}M^{\dagger}\big) = i\big[M^{\dagger},L^{\dagger}\big] = i\big[M,L\big] \end{aligned}$$

However,

$$\begin{bmatrix} \mathbf{M}, \mathbf{L} \end{bmatrix}^{\dagger} = \begin{pmatrix} \mathbf{M} \mathbf{L} - \mathbf{L} \mathbf{M} \end{pmatrix}^{\dagger} = \mathbf{L}^{\dagger} \mathbf{M}^{\dagger} - \mathbf{M}^{\dagger} \mathbf{L}^{\dagger} = \mathbf{L} \mathbf{M} - \mathbf{M} \mathbf{L} = \begin{bmatrix} \mathbf{L}, \mathbf{M} \end{bmatrix} = - \begin{bmatrix} \mathbf{M}, \mathbf{L} \end{bmatrix},$$

not equal to [M,L] unless M and L commute; i.e., unless [M,L] = 0.