Exercise 3.1. Let V be an *n*-dimensional complex vector space and *L* a matrix. *L* has at least one eigenvalue, and each distinct eigenvalue has a corresponding (non-zero) eigenvector. In particular, *L* has at least one eigenvector. If *L* is Hermitian then V has an orthonormal basis consisting of eigenvectors of *L*.

Proof. Let $p_L(\lambda) = \det(L - \lambda \boldsymbol{I})$. $p_L(\lambda)$ is clearly a polynomial in λ, and it is known as the characteristic polynomial of *L*. By the Fundamental Theorem of Algebra, there are complex roots λ_i of $p_L(\lambda)$ of multiplicity p_i such that

$p_L(\lambda) = (\lambda - \lambda_1)^{p_1} \cdots (\lambda - \lambda_r)^{p_r}$ where *r* is a positive integer and $p_1 + \cdots + p_r = n$. In particular, there is at least one λ_i, and for all *i*, $p_L(\lambda_i) = 0$. For each *i*, by footnote (*), there is a non-zero vector $|\lambda_i\rangle$ such that $L|\lambda_i\rangle = \lambda_i|\lambda_i\rangle$. That is, λ_i is an eigenvalue of *L* with eigenvector $|\lambda_i\rangle$. This proves that every matrix *L* has at least one non-zero eigenvector and that each distinct eigenvalue has a corresponding (non-zero) eigenvector.

Now suppose that *L* is Hermitian. Then all eigenvalues λ are real. WLOG we can assume $|\lambda_1\rangle$ is a unit vector, $\langle\lambda_1|\lambda_1\rangle = 1$. Define the null space

$N = \{|v\rangle : \langle v|\lambda_1\rangle = 0\}$. It is easy to see that *N* is a vector subspace of *L*. Since dim $\{\alpha|\lambda_1\rangle : \alpha \in \mathbb{C}\}$ is a 1-dimensional subspace, the orthogonal subspace *N* has dimension *n* – 1. Claim $LN \subseteq N$:

Let $|v\rangle \in N$. We need to show that $L|v\rangle \in N$. Since *L* is Hermitian, $L|v\rangle \leftrightarrow \langle v|L$. So, we need to show that $\langle v|L|\lambda_1\rangle = 0$:

$$\langle v|L|\lambda_1\rangle = \langle v|\lambda_1|\lambda_1\rangle = \lambda_1\langle v|\lambda_1\rangle = 0 \quad ✓$$

Let $L_2 = L$ restricted to *N*. Repeating our logic above, $p_L(L_2)$ has a real root λ_2 that is an eigenvalue of L_2 with corresponding unit eigenvector $|\lambda_2\rangle$. Since $|\lambda_2\rangle \in N$, $\langle\lambda_1|\lambda_2\rangle = 0 \Rightarrow |\lambda_1\rangle \perp |\lambda_2\rangle$.

Using the (*n* – 2)-dimensional null space of L_2 as above we generate $|\lambda_3\rangle \perp |\lambda_2\rangle$, and since $|\lambda_3\rangle \in N$, $|\lambda_3\rangle \perp |\lambda_1\rangle$ also.

Continuing this process, we eventually obtain the orthonormal basis $\{\,|\lambda_i\rangle : i = 1, \cdots, n\,\}$. ■

Footnote (*)

Suppose we have *n* equations in *n* unknowns:

$$\begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = 0 \\ \vdots \\ a_{n1}x_1 + \cdots + a_{nn}x_n = 0 \end{cases} \Leftrightarrow \quad A\left|x\right\rangle = \left|0\right\rangle$$

By Cramer's Rule, det $A \neq 0 \Leftrightarrow$ there exists a unique vector $\left|x\right\rangle$ such that $A\left|x\right\rangle = 0$. Since $A\left|0\right\rangle = 0$, $\left|x\right\rangle = \left|0\right\rangle$ is the unique solution. That is, det A ≠ 0 if and only if $\left|x\right\rangle = \left|0\right\rangle$ is the unique solution to $A\left|x\right\rangle = \left|0\right\rangle$.

So, if det A = 0, then there is an $\left|x\right\rangle \neq 0$ such that $A\left|x\right\rangle = \left|0\right\rangle$.

Fix *i* and let $A = L - \lambda_i \boldsymbol{I}$. Then

$\det A = \det\left(L - \lambda_i \boldsymbol{I}\right) = p_L\left(\lambda_i\right) = 0$.

$\Rightarrow \exists$ non-zero $\left|x\right\rangle$ such that $A\left|x\right\rangle = \left|0\right\rangle$.

Define the vector $\left|\lambda_i\right\rangle = \left|x\right\rangle \neq \left|0\right\rangle$. Then

$\left(L - \lambda_i \boldsymbol{I}\right)\left|\lambda_i\right\rangle = \mathrm{A}\left|x\right\rangle = 0$.

That is, for each λ_i there is a vector $\left|\lambda_i\right\rangle \neq 0$ such that $L\left|\lambda_i\right\rangle = \lambda_i\left|\lambda_i\right\rangle$.