5.2. Let L and M be observables and $|\Psi\rangle$ a normalized simultaneous eigenvector of L and M. Set (i) $\bar{L} = L - \langle L \rangle \boldsymbol{I}$ and $\bar{M} = M - \langle M \rangle \boldsymbol{I}$.

1) Show that $\Delta L^2 = \langle \bar{L}^2 \rangle$ and $\Delta M^2 = \langle \bar{M}^2 \rangle$

2) Show that $[\bar{L}, \bar{M}] = [L, M]$

3) Show that (5.13) $\Delta L \, \Delta M \geq \frac{1}{2} \left| \langle \Psi | [L, M] | \Psi \rangle \right|$

Solution. Recall that if $\{\lambda\}_{\lambda \in \Lambda}$ is the set of eigenvalues of an observable L then

(ii) $\langle L \rangle \overset{(4.11)}{=} \sum_{\lambda} \lambda \mathrm{P}_L(\lambda)$

where $\mathrm{P}_L(\lambda)$ is the probability that λ is the outcome of measurement L.

Notation: In the book, equation (4.11) used $\mathrm{P}(\lambda)$ because L was obvious. However, in this problem we have L, L^2, and $\bar{L}$. For the sake of clarity, we write $\mathrm{P}_L(\lambda)$ rather than $\mathrm{P}(\lambda)$.

It is straight-forward to check that the eigenvalues and eigenvectors of L^2 are $\{\lambda^2\}$ and $\{|\lambda\rangle\}$, respectively. Since $\mathrm{P}_{L^2}(\lambda^2) = \mathrm{P}_L(\lambda)$,

(iii) $\langle L^2 \rangle \overset{(ii)}{=} \sum_{\lambda} \lambda^2 \mathrm{P}_L(\lambda)$.

1) Let $\{\lambda\}$ and $\{\mu\}$ be the sets of eigenvalues of L and M, respectively. Define

(iv) $\bar{\lambda} = \lambda - \langle L \rangle$ and $\bar{\mu} = \mu - \langle M \rangle$.

Claim $\{\bar{\lambda}\}$, $\{|\lambda\rangle\}$, $\{\bar{\mu}\}$, and $\{|\mu\rangle\}$ are the eigenvalues and eigenvectors of $\bar{L}$ and $\bar{M}$:

$$\bar{L}|\lambda\rangle \overset{(i)}{=} \left(L - \langle L \rangle \boldsymbol{I}\right)|\lambda\rangle = L|\lambda\rangle - \langle L \rangle \boldsymbol{I}|\lambda\rangle = \lambda|\lambda\rangle - \langle L \rangle|\lambda\rangle = \left(\lambda - \langle L \rangle\right)|\lambda\rangle \overset{(iv)}{=} \bar{\lambda}|\lambda\rangle$$

and similarly for M.

By (i), the distribution of $\bar{L}$ is simply the distribution of L shifted by $\langle L \rangle$ to have a mean of zero. Therefore, since $\{\bar{\lambda}\}$ are the eigenvalues of $\bar{L}$,

(v) $\mathrm{P}_{\bar{L}}(\bar{\lambda}) = \mathrm{P}_L(\lambda)$,

(vi) $\Delta L =$ Standard deviation of L = Standard deviation of $\bar{L} = \Delta \bar{L}$

and similarly

(vii) $\Delta M = \Delta\bar{M}$.

Hence

$$\left(\Delta L\right)^2 = \text{Variance of L} \overset{(5.4)}{=} \sum_{\lambda}\left(\lambda - \left\langle L\right\rangle\right)^2 \mathrm{P}_L\left(\lambda\right) \overset{(\text{iv, } v)}{=} \sum_{\lambda}\bar{\lambda}^{\,2}\, \mathrm{P}_{\bar{L}}\left(\bar{\lambda}\right) \overset{(\text{iii})}{=} \left\langle \bar{L}^2\right\rangle \quad ✔$$

and similarly for *M*. ✔

2)

$$\bar{L}\bar{M} \overset{(\text{i})}{=} \left(L - \left\langle L\right\rangle \boldsymbol{I}\right)\left(M - \left\langle M\right\rangle \boldsymbol{I}\right) = LM + \left\langle L\right\rangle\left\langle M\right\rangle \boldsymbol{I} - \left\langle L\right\rangle M - \left\langle M\right\rangle L$$

$$\bar{M}\bar{L} \overset{(\text{i})}{=} \left(M - \left\langle M\right\rangle \boldsymbol{I}\right)\left(L - \left\langle L\right\rangle \boldsymbol{I}\right) = ML + \left\langle M\right\rangle\left\langle L\right\rangle \boldsymbol{I} - \left\langle L\right\rangle M - \left\langle M\right\rangle L$$

Subtracting, we get

(viii) $\left[\bar{L},\bar{M}\right] = \bar{L}\bar{M} - \bar{M}\bar{L} = LM - ML = \left[L,M\right]$ ✔

3) $$\Delta L\,\Delta M \overset{(\text{vi, vii})}{=} \Delta\bar{L}\,\Delta\bar{M} \overset{(*)}{\geq} \frac{1}{2}\left\langle \Psi\left|\left[\bar{L},\bar{M}\right]\right|\Psi\right\rangle \overset{(\text{viii})}{=} \left\langle \Psi\left|\left[L,M\right]\right|\Psi\right\rangle \quad ✔$$

(*) The book showed that this inequality, 5.13, holds because $\left\langle \bar{L}\right\rangle$ = 0 = $\left\langle \bar{M}\right\rangle$.

Note. I did not use step 1) in the proof of steps 2) and 3). Hence I also did not use (ii) – (v).