$\boxed{3.4}$ Let $\vec{n} = \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{bmatrix}$ be a direction in $\mathbb{R}^3$ in spherical coordinates

Compute the eigenvectors and eigenvalues for $\sigma_n = \begin{bmatrix} n_z & n_x - i n_y \\ n_x + i n_y & -n_z \end{bmatrix}$.

Solution: $\sigma_n = \begin{bmatrix} \cos\theta & \sin\theta(\cos\phi - i\sin\phi) \\ \sin\theta(\cos\phi + i\sin\phi) & -\cos\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{bmatrix}$

$$0 = \det(\sigma_n - \lambda I) = \begin{vmatrix} \cos\theta - \lambda & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta - \lambda \end{vmatrix} = -\cos^2\theta + \lambda^2 - \sin^2\theta \Rightarrow \boxed{\lambda = \pm 1}$$

Caution: If we let $|\lambda\rangle = \begin{bmatrix} \cos\omega \\ \sin\omega \end{bmatrix}$ as in 3.3 and solve $\sigma_n|\lambda\rangle = \lambda|\lambda\rangle$, we can only achieve solutions for $\vec{n} = \begin{bmatrix} n_x \\ 0 \\ n_z \end{bmatrix}$, directions in xz-plane. To allow a y-component we must introduce $e^{i\phi}$ into $|\lambda\rangle$.

Let $|\lambda\rangle = \begin{bmatrix} \cos\omega \\ \sin\omega e^{i\phi} \end{bmatrix}$. Note $\||\lambda\rangle\| = 1$. So

$$\begin{bmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{bmatrix} \begin{bmatrix} \cos\omega \\ \sin\omega e^{i\phi} \end{bmatrix} = \sigma_n|\lambda\rangle = \lambda|\lambda\rangle = \begin{bmatrix} \lambda\cos\omega \\ \lambda\sin\omega e^{i\phi} \end{bmatrix}$$

$$\Rightarrow \left\{ \begin{array}{l} \cos\theta\cos\omega + \sin\theta\sin\omega = \lambda\cos\omega \\ e^{i\phi}[\sin\theta\cos\omega - \cos\theta\sin\omega] = e^{i\phi}[\lambda\sin\omega] \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} \cos(\theta - \omega) = \lambda\cos\omega \\ \sin(\theta - \omega) = \lambda\sin\omega \end{array} \right\}$$

$$\lambda_1 = 1: \left\{ \begin{array}{l} \cos(\theta-\omega) = \cos\omega \\ \sin(\theta-\omega) = \sin\omega \end{array} \right\} \Rightarrow \theta - \omega = \omega \text{ or } \omega = \frac{\theta}{2} \Rightarrow \boxed{|\lambda_1\rangle = \begin{bmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} e^{i\phi} \end{bmatrix}}$$

$$\lambda_2 = -1: \left\{ \begin{array}{l} \cos(\theta-\omega) = -\cos\omega \\ \sin(\theta-\omega) = -\sin\omega \end{array} \right\} \Rightarrow \theta - \omega = \omega + \pi \text{ or } \omega = \frac{\theta}{2} - \frac{\pi}{2}$$

$$\cos\omega = \cos\left(\frac{\theta}{2} - \frac{\pi}{2}\right) = \cos\frac{\theta}{2}\cancelto{0}{\cos\left(\frac{\pi}{2}\right)} - \sin\frac{\theta}{2}\cancelto{-1}{\sin\left(-\frac{\pi}{2}\right)} = \sin\frac{\theta}{2}$$

$$\sin\omega = \sin\left(\frac{\theta}{2} - \frac{\pi}{2}\right) = \sin\frac{\theta}{2}\cancelto{0}{\cos\left(\frac{\pi}{2}\right)} + \cancelto{-1}{\sin\left(-\frac{\pi}{2}\right)}\cos\frac{\theta}{2} = -\cos\frac{\theta}{2}$$

So $\boxed{|\lambda_2\rangle = \begin{bmatrix} \sin\frac{\theta}{2} \\ -\cos\frac{\theta}{2} e^{i\phi} \end{bmatrix}}$ ∎

Extra: $P(1) = |\langle u|\lambda_1\rangle|^2 = \left| [1\ 0] \begin{bmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} e^{i\phi} \end{bmatrix} \right|^2 = \cos^2\frac{\theta}{2}$ ✓

$P(-1) = \sin^2\frac{\theta}{2}$ since $|e^{i\phi}| = 1$ ✓

Note: Multiplying $|\lambda_1\rangle$ or $|\lambda_2\rangle$ by $e^{-i\phi}$ yields equivalent solutions