[1.1] Given

(0) $\langle A|B\rangle \in \mathbb{C}$ (since it is defined to be an inner product)

(1) $\langle C|\{|A\rangle + |B\rangle\} = \langle C|A\rangle = \langle C|B\rangle$

(2) $\langle B|A\rangle = \langle A|B\rangle *$

Prove

(a) $\{\langle A| + \langle B|\}|C\rangle = \langle A|C\rangle + \langle B|C\rangle$

(b) $\langle A|A\rangle \in \mathbb{R}$

Solution.

(a) From (0), let $\langle C|A\rangle = a + bi$ and $\langle C|B\rangle = c + di$. Then

(3) Claim $(\langle C|A\rangle + \langle C|B\rangle)* = \langle C|A\rangle * + \langle C|B\rangle *$:

$$[(a+bi)+(c+di)]* = [(a+c)+(b+d)i]* = (a+c)-(b+d)i$$
$$= (a+bi)+(c+di) = (a+bi)*+(c+di)*$$

So,

$$\{\langle A| + \langle B|\}|C\rangle \overset{(2)}{=} \langle C|\{|A\rangle + |B\rangle\}\rangle * \overset{(1)}{=} (\langle C|A\rangle + \langle C|B\rangle)*$$
$$\overset{(3)}{=} \langle C|A\rangle * + \langle C|B\rangle * \overset{(2)}{=} \langle A|C\rangle + \langle B|C\rangle$$

(b) From (0), let $\langle A|A\rangle = x + yi$. By (2),

$$x - yi = \langle A|A\rangle * \overset{(2)}{=} \langle A|A\rangle = x + yi \quad \Rightarrow \quad y = 0$$
$$\Rightarrow \quad \langle A|A\rangle = x \in \mathbb{R}$$