3.1 Let V be an n-dimensional vector space over $\mathbb{C}$ and T a Hermitian operator with matrix A in some basis. Then there are n orthonormal eigenvectors of A that span V.

Lemma 3. Let A be Hermitian. Then $Ax = \lambda x$ iff $\bar{x}^T A = \lambda \bar{x}^T$.

Proof: Let $A = [a_{ij}] = [b_{ij} + c_{ij} i]$ and $x = [x_j] = [u_j + v_j i]$ (col vector)

$$Ax = [b_{ij} + c_{ij} i][u_j + v_j i] = \left[\sum_j (b_{ij}u_j - c_{ij}v_j) + \left\{\sum_j (b_{ij}v_j + c_{ij}u_j)\right\} i\right]$$

$\lambda x = [\lambda u_j] + [\lambda v_j] i$. Thus

$$Ax = \lambda x \iff \left\{ \begin{array}{l} \vdots \\ \sum_j (b_{ij}u_j - c_{ij}v_j) = \lambda u_j \text{ and } \sum_j (b_{ij}v_j + c_{ij}u_j) = \lambda v_j \\ \vdots \end{array} \right.$$

A is Hermitian $\iff a_{ij} = \bar{a}_{ji}$. $\bar{x}^T = [\bar{x}_i] = [u_i - v_j i]$ (row vector)

$\bar{A}^T = [\bar{a}_{ji}] = [a_{ij}] = A$. $\bar{a}_{ji} = b_{ji} - c_{ji} i = b_{ij} + c_{ij} i \overset{(1)}{\Rightarrow} b_{ji} = b_{ij}$ and $c_{ji} = -c_{ij}$.

$$\bar{x}^T A = [u_i - v_j i][b_{ij} + c_{ij} i] = \left[\sum_i (b_{ij}u_i + c_{ij}v_i) + \left\{\sum_i (c_{ij}u_i - b_{ij}v_i)\right\} i\right]$$

$\lambda \bar{x}^T = [\lambda u_i] + [\lambda v_i] i$. Thus

$$\bar{x}^T A = \lambda \bar{x}^T \iff \left\{ \begin{array}{l} \vdots \\ \sum_i (b_{ij}u_i + c_{ij}v_i) = \lambda u_i \text{ and } \sum_i (c_{ij}u_i - b_{ij}v_i) = -\lambda v_i \\ \vdots \end{array} \right.$$

$$\iff \left\{ \begin{array}{l} \vdots \\ \sum_j (b_{ji}u_j + c_{ji}v_j) = \lambda u_j \text{ and } \sum_j (c_{ji}u_j - b_{ji}v_j) = -\lambda v_j \\ \vdots \end{array} \right.$$

$$\overset{(1)}{\iff} \left\{ \begin{array}{l} \vdots \\ \sum_j (b_{ij}u_j - c_{ij}v_j) = \lambda u_j \text{ and } \sum_j (c_{ij}u_j + b_{ij}v_j) = \lambda v_j \quad \checkmark \\ \vdots \end{array} \right.$$

Bra-ket Version of Lemma 3. Let A be Hermitian. Then $A|x\rangle = \lambda|x\rangle \iff \langle x|A = \langle x|\lambda$

Proof Recall: $|x\rangle = [x_i] \iff \langle x| = [x_i^*]$ ($x^* = \bar{x}$), $\lambda|x\rangle \leftrightarrow \langle x|\lambda^*$, and $A|x\rangle \iff \langle x|A^\dagger$. So $A|x\rangle = \lambda|x\rangle \iff \langle x|A^\dagger = \langle x|\lambda^*$. But A is Hermitian $\Rightarrow A^\dagger = A$ and $\lambda^* = \lambda$. So $A|x\rangle = \lambda|x\rangle \iff \langle x|A = \langle x|\lambda$.

Lemma 4. Let A be Hermitian and λ_i a root of the char poly $p_A(\lambda)$ having multiplicity $m \geq 2$. Then $\exists$ m orthonormal eigenvectors corresp to λ_i

Proof: λ_i has at least one eigenvector which can be taken to be a unit vector. That is $\exists\, x_i \neq 0$ s.t. (1) $\bar{x}_i^T x_i = 1$ and (2) $A x_i = \lambda_i x_i$. By lemma, (3) $\bar{x}_i^T A = \lambda_i \bar{x}_i$

Let $Y = [y_2 \cdots y_n]$ where y_i are orthogonal unit column vectors and set $B = [x_i\ Y]$. Since B has orthonormal column vectors, B is orthogonal, i.e. (4) $B^{-1} = B^{+}$. $B^{+} = \begin{bmatrix} \bar{x}_i^T \\ Y^{+} \end{bmatrix}$, a matrix with row vectors $\bar{x}_i^T$ & $\{\bar{y}_k^T\}$. $AB = A[x_i\ Y] = [Ax_i\ AY] = [\lambda_i x_i\ AY]$.

Note Ax_i is a column vector and AY is an $(n-1)\times n$ array. Now

$$B^{-1}AB \overset{(4)}{=} B^{+}AB = \begin{bmatrix} \bar{x}_i^T \\ Y^{+} \end{bmatrix} [\lambda_i x_i\ AY] = \begin{bmatrix} \lambda_i \bar{x}_i^T x_i & \bar{x}_i^T AY \\ \lambda_i Y^{+} x_i & Y^{+}AY \end{bmatrix} = \begin{bmatrix} \lambda_i & 0 \\ 0 & Y^{+}AY \end{bmatrix}$$ since:

(i) $\bar{x}_i^T x_i \overset{(1)}{=} 1$, (ii) $\bar{x}_i^T AY \overset{(3)}{=} \lambda_i \bar{x}_i Y = 0$ because $x_i \perp y_k\ \forall k$, and

(iii) $\lambda_i Y^{+} x_i = 0$, again because $x_i \perp y_k\ \forall k$.

$B^{-1}AB$ is similar to A hence it has same eigenvalues. The characteristic polynomial is

$$p_A(\lambda) = p_{B^{-1}AB}(\lambda) = \det[B^{-1}AB - \lambda I_n] = \det \begin{bmatrix} \lambda_i - \lambda & 0 \\ 0 & Y^{+}AY - \lambda I_{n-1} \end{bmatrix}$$

$= (\lambda_i - \lambda)\det(Y^{+}AY - \lambda I_{n-1})$ (expansion by minors)

$= (\lambda_i - \lambda)^m g(\lambda)$ (because λ has multiplicity m)

$\therefore \det(Y^{+}AY - \lambda I_{n-1}) = (\lambda_i - \lambda)^{m-1} g(\lambda) \Rightarrow \mathrm{Det}(Y^{+}AY - \lambda_i I_{n-1}) = 0$.

The null space of a matrix C is $N(C) = \{x : Cx = 0\}$ and has dimension $n - \mathrm{rank}(C)$.

Set $C = BAB^{-1} - \lambda_i I_n$. Then $\dim[N(C)] = n - \mathrm{rank}(C) = n - (n-m) = m$.

We now proceed by induction. Let $m = 2$. $\mathrm{Dim}[N(C)] = 2 \Rightarrow \exists$ pair of orthonormal vectors x_1 and x_2 that span $N(C)$. So, for $k = 1, 2$, $C x_{ik} = 0$ or $BAB^{-1} x_{ik} = \lambda_i x_{ik}$.

Thus x_1 and x_2 are eigenvectors of A corresp to λ_i

Let $m = 3$. Starting with x_1 and x_2 we can find another $(n-2)$ vectors $y_3, \cdots, y_n$ s.t. $\{x_1, x_2, y_3, \cdots, y_n\}$ forms an orthonormal basis for V. Repeating the process we generate 3 orthonormal vectors that span C and are thus eigenvectors of A corresp to λ_i.

Continuing by induction we generate m orthonormal eigenvectors of A that corresp to λ_i

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We now prove 3.1.

$\lambda \in \mathbb{C}$ is an eigenvalue of A

$\Leftrightarrow$ $A\lambda = \lambda v$ for some $0 \neq v \in V$

$\Leftrightarrow$ $A - \lambda I$ is singular (i.e., $\dim A(V) < n$)

$\Leftrightarrow$ $\det(A - \lambda I) = 0$

$\Leftrightarrow$ $p_A(\lambda) = 0$ where $p_A(\lambda) = \det(A - \lambda I)$ is the characteristic polynomial of A.

$p_A(\lambda)$ is an nth order polynomial in the variable λ. By the Fundamental Theorem of Algebra, $p_A(\lambda) = K(\lambda - \lambda_1)^{m_1}(\lambda - \lambda_2)^{m_2} \cdots (\lambda - \lambda_k)^{m_I}$ for some constant K.

That is, the n roots of $p_A(\lambda)$ are all eigenvalues of A.

By Lemma 4 there are m_k orthonormal eigenvectors corresponding to λ_k, each of which is orthogonal to any eigenvector corresponding to some other λ by Lemma 1 (in Notes).

Thus the entire collection consists of n orthonormal vectors which constitutes a basis for V. ∎