$\boxed{4.5}$ Let $\vec{n}$ be any unit 3-vector and $H = \frac{\hbar\omega}{2}\vec{\sigma}\cdot\vec{n}$. Find the energy eigenvalues and eigenvectors: $E_1, E_2, |E_1\rangle$, and $|E_2\rangle$.

Solution. In spherical coordinates, $\vec{n} = \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{bmatrix}$

By problem 3.4, $\lambda = \pm 1$ and $|\lambda_1\rangle = \begin{bmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}e^{i\phi} \end{bmatrix}$ and $|\lambda_2\rangle = \begin{bmatrix} \sin\frac{\theta}{2} \\ -\cos\frac{\theta}{2}e^{i\phi} \end{bmatrix}$.

$\therefore \boxed{E_1 = \frac{\hbar\omega}{2}\lambda_1 = \frac{\hbar\omega}{2}}$, $\boxed{E_2 = \frac{\hbar\omega}{2}\lambda_2 = -\frac{\hbar\omega}{2}}$,

$\boxed{|E_1\rangle = \frac{\hbar\omega}{2}|\lambda_1\rangle = \frac{\hbar\omega}{2}\begin{bmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}e^{i\phi} \end{bmatrix}}$ and $\boxed{|E_2\rangle = \frac{\hbar\omega}{2}\begin{bmatrix} \sin\frac{\theta}{2} \\ -\cos\frac{\theta}{2}e^{i\phi} \end{bmatrix}}$ ∎