Exercise 3.4 Let $\hat{n}=\begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix}=\begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}$ be a direction (i.e., unit 3-vector) in spherical coordinates. Compute the eigenvalues and eigenvectors for

$$\sigma_n=\begin{pmatrix} n_z & n_x-in_y \\ n_x+in_y & -n_z \end{pmatrix}.$$

Solution.

$$\sigma_n=\begin{pmatrix} \cos\theta & \sin\theta(\cos\phi-i\sin\phi) \\ \sin\theta(\cos\phi+i\sin\phi) & -\cos\theta \end{pmatrix}=\begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix}.$$

To find the eigenvalues we solve the characteristic equation:

$$0=\det(\sigma_n-\lambda I)=\begin{vmatrix} \cos\theta-\lambda & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta-\lambda \end{vmatrix}=-\cos^2\theta+\lambda^2-\sin^2\theta$$

$$\Rightarrow \boxed{\lambda_1=1} \text{ and } \boxed{\lambda_2=-1}.$$

To find the eigenvectors we assume $|\lambda\rangle=\begin{pmatrix} \cos\omega \\ \sin\omega e^{i\phi} \end{pmatrix}$ and then solve the eigenvector equation $\sigma_n|\lambda\rangle=\lambda|\lambda\rangle$:

$$\begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix}\begin{pmatrix} \cos\omega \\ \sin\omega e^{i\phi} \end{pmatrix}=\begin{pmatrix} \lambda\cos\omega \\ \lambda\sin\omega e^{i\phi} \end{pmatrix}$$

$$\begin{pmatrix} \cos\theta\cos\omega+\sin\theta\sin\omega \\ e^{i\phi}[\sin\theta\cos\omega-\cos\theta\sin\omega] \end{pmatrix}=\begin{pmatrix} \lambda\cos\omega \\ e^{i\phi}[\lambda\sin\omega] \end{pmatrix}$$

$$\begin{pmatrix} \cos(\theta-\omega) \\ \sin(\theta-\omega) \end{pmatrix}=\begin{pmatrix} \lambda\cos\omega \\ \lambda\sin\omega \end{pmatrix}.$$

For λ = 1:

$$\begin{cases} \cos(\theta-\omega)=\cos\omega \\ \sin(\theta-\omega)=\sin\omega \end{cases}$$

$$\Rightarrow \quad \theta - \omega = \omega \quad \Rightarrow \quad \omega = \frac{\theta}{2} \quad \Rightarrow \quad \boxed{\left|\lambda_1\right\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} e^{i\phi} \end{pmatrix}}$$

For λ = -1:

$$\begin{cases} \cos(\theta-\omega) = -\cos\omega \\ \sin(\theta-\omega) = -\sin\omega \end{cases} \quad \Leftrightarrow \quad \begin{cases} \cos(\theta-\omega) = \cos(\omega+\pi) \\ \sin(\theta-\omega) = \sin(\omega+\pi) \end{cases}$$

$$\Rightarrow \quad \theta - \omega = \omega + \pi \quad \Rightarrow \quad \omega = \frac{\theta}{2} - \frac{\pi}{2}$$

$$\Rightarrow \quad \left|\lambda_2\right\rangle = \begin{pmatrix} \cos\omega \\ \sin\omega e^{i\phi} \end{pmatrix} = \begin{pmatrix} \cos\left(\frac{\theta}{2} - \frac{\pi}{2}\right) \\ \sin\left(\frac{\theta}{2} - \frac{\pi}{2}\right) e^{i\phi} \end{pmatrix} = \begin{pmatrix} \sin\frac{\theta}{2} \\ -\cos\frac{\theta}{2} e^{i\phi} \end{pmatrix}$$

$$\boxed{\left|\lambda_2\right\rangle = \begin{pmatrix} \sin\frac{\theta}{2} \\ -\cos\frac{\theta}{2} e^{i\phi} \end{pmatrix}}. \quad \blacksquare$$

Note 1: Multiplying either $\left|\lambda_1\right\rangle$ or $\left|\lambda_1\right\rangle$ by a phase factor of $e^{-i\phi}$ yields an equivalent solution.

Note 2: Had we assumed $\left|\lambda\right\rangle = \begin{pmatrix} \cos\omega \\ \sin\omega \end{pmatrix}$ as we did in Exercise 3.3, without the phase factor, that would have been equivalent to setting ϕ = 0. That would have meant that $\hat{n} = \begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix} = \begin{pmatrix} \sin\theta \\ 0 \\ \cos\theta \end{pmatrix}$ and so we could not have obtained a solution for general $\hat{n}$.