Calculate the eigenvectors and eigenvalues of σ_n if $\vec{n} = \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix}$

$\boxed{3.3}$ Solution: $\sigma_n = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$. Let $|\lambda\rangle = \begin{bmatrix} \cos\phi \\ \sin\phi \end{bmatrix}$ $\quad \sigma_n|\lambda\rangle = \lambda \begin{bmatrix} \cos\phi \\ \sin\phi \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} \begin{bmatrix} \cos\phi \\ \sin\phi \end{bmatrix} = \begin{bmatrix} \lambda\cos\phi \\ \lambda\sin\phi \end{bmatrix}$$

$$\Rightarrow \begin{cases} \cos(\theta-\phi) = \cos\theta\cos\phi + \sin\theta\sin\phi = \lambda\cos\phi \\ \sin(\theta-\phi) = \sin\theta\cos\phi - \cos\theta\sin\phi = \lambda\sin\phi \end{cases}$$

$$\Rightarrow \tan(\theta-\phi) = \tan\phi \Rightarrow \overset{(1)}{\phi = \theta-\phi} \text{ or } \overset{(2)}{\phi - \pi = \theta - \phi}$$

(1): $\lambda_1 = \dfrac{\cos(\theta-\phi)}{\cos\phi} = \dfrac{\cos\phi}{\cos\phi} = 1$ and $|\lambda_1\rangle = \begin{bmatrix} \cos\phi \\ \sin\phi \end{bmatrix} = \begin{bmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{bmatrix}$ ✓

(2): $\lambda_2 = \dfrac{\sin(\theta-\phi)}{\sin\phi} = \dfrac{\sin(\phi-\pi)}{\sin\phi} = \dfrac{-\sin\phi}{\sin\phi} = -1$ ✓

Now $\cos\phi = \cos\left(\frac{\theta}{2}+\frac{\pi}{2}\right) = -\sin\frac{\theta}{2}$ and $\sin\phi = \sin\left(\frac{\theta}{2}+\frac{\pi}{2}\right) = \cos\frac{\theta}{2}$

$$\Rightarrow |\lambda_2\rangle = \begin{bmatrix} \cos\phi \\ \sin\phi \end{bmatrix} = \begin{bmatrix} -\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{bmatrix}$$ ✓

Note: We could have 1st solved for λ_1 & λ_2:

$$0 = \det(\sigma_n - \lambda I) = \begin{vmatrix} \cos\theta-\lambda & \sin\theta \\ \sin\theta & -\cos\theta-\lambda \end{vmatrix} = -\cos^2\theta + \lambda^2 - \sin^2\theta \Rightarrow \lambda = \pm 1$$ ✓

Then

$\lambda_1 = 1$: $\begin{cases} \cos(\theta-\phi) = \cos\phi \\ \sin(\theta-\phi) = \sin\phi \end{cases} \Rightarrow \theta-\phi = \phi \Rightarrow \phi = \frac{\theta}{2} \Rightarrow |\lambda_1\rangle = \begin{bmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{bmatrix}$ ✓

$\lambda_2 = -1$: $\begin{cases} \cos(\theta-\phi) = -\cos\phi \\ \sin(\theta-\phi) = -\sin\phi \end{cases} \Rightarrow \theta-\phi = \phi-\pi \Rightarrow \phi = \frac{\theta}{2} - \frac{\pi}{2} \Rightarrow |\lambda_2\rangle = \begin{bmatrix} \sin\frac{\theta}{2} \\ -\cos\frac{\theta}{2} \end{bmatrix}$ ✓

Note 2: Another way to solve for λ_1 & λ_2:

$$\begin{cases} \lambda_1\lambda_2 = \det(\sigma_n) = -\cos^2\theta - \sin^2\theta = -1 \\ \lambda_1 + \lambda_2 = \mathrm{Tr}(\sigma_n) = 0 \end{cases} \Rightarrow \lambda_2 = -\lambda_1 \Rightarrow \lambda_1(-\lambda_1) = -1 \Rightarrow \lambda_1 = 1, \lambda_2 = -1$$ ✓