

Exercise 4.6. Carry out the ket recipe with $H = \frac{\hbar\omega}{2}\sigma_z$, final observable σ_x , initial state $|u\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and observable σ_y measured at time t . Find the possible outcomes and their probabilities of occurrence.

Solution. I believe the initial state and final observables were inadvertently reversed in the problem statement because reversing them makes this exercise the natural culmination of a problem started in Exercise 3.4 (represent \hat{n} in spherical coordinates), and continued in 4.5 (for $H = \frac{\hbar\omega}{2}\vec{\sigma} \cdot \hat{n}$, find the energy eigenvectors and eigenvalues $|E_1\rangle, |E_2\rangle, E_1$, and E_2 for direction \hat{n}). It also becomes a means to confirm the results of Exercise 4.4 that the 3-vector operator $\vec{\sigma}$ precesses clockwise around the direction of the magnetic field. The problem as stated, as others have shown, results in σ_y unchanging over time, a not very interesting result. I provide both results, doing the reversed problem 1st.

I thus assume an initial state $|r\rangle$ (corresponds to observable σ_x) and a final observable σ_z (corresponds to state $|u\rangle$). Moreover, the following spin formulas become valid because they were developed under the assumption that system is prepared in some state $|A\rangle$ and then measured in the up direction:

$$|u\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |d\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, |i\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}, |o\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}, |r\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

$$|l\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \text{ and } \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Step 1. Find H . $H = \frac{\hbar\omega}{2}\sigma_z$

Step 2. Prepare a state vector. The state vector that corresponds to the observable σ_x is

$$|\Psi(0)\rangle = |r\rangle.$$

Step 3. Find the energy eigenvalues and eigenvectors of H . From Exercise 4.5

we learned that for a generic direction $\hat{n} = \begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix} = \begin{pmatrix} \sin\phi \cos\theta \\ \sin\phi \sin\theta \\ \cos\phi \end{pmatrix}$ that $H = \frac{\hbar\omega}{2} \sigma_n$

has the following energy eigenvalues and corresponding eigenvectors:

$$E_1 = \frac{\hbar\omega}{2}, \quad E_2 = -\frac{\hbar\omega}{2}, \quad |E_1\rangle = \begin{pmatrix} \cos\frac{\phi}{2} \\ \sin\frac{\phi}{2} e^{i\theta} \end{pmatrix}, \quad \text{and} \quad |E_2\rangle = \begin{pmatrix} \sin\frac{\phi}{2} \\ -\cos\frac{\phi}{2} e^{i\theta} \end{pmatrix}.$$

$$H = \frac{\hbar\omega}{2} \sigma_z, \quad \text{and for } \sigma_n = \sigma_z \text{ we get that } \hat{n} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \sin\phi \cos\theta \\ \sin\phi \sin\theta \\ \cos\phi \end{pmatrix} \Rightarrow \phi = 0.$$

Hence

$$|E_1\rangle = \begin{pmatrix} \cos\frac{\phi}{2} \\ \sin\frac{\phi}{2} e^{i\theta} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |u\rangle \quad \checkmark$$

$$|E_2\rangle = \begin{pmatrix} \sin\frac{\phi}{2} \\ -\cos\frac{\phi}{2} e^{i\theta} \end{pmatrix} = \begin{pmatrix} 0 \\ -e^{i\theta} \end{pmatrix}.$$

$|E_2\rangle$ is a unit vector for any value of θ . However, in Exercise 4.5 it was shown

that $|E_1\rangle = |\lambda_1\rangle$ and $|E_2\rangle = |\lambda_2\rangle$ where $|\lambda_1\rangle$ and $|\lambda_2\rangle$ are the eigenvectors of

$\sigma_n = \sigma_z$. Since $|\lambda_1\rangle = |u\rangle$ and $|\lambda_2\rangle = |d\rangle$ we have that

$$|E_2\rangle = |d\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (\text{and consequently } \theta = \pi). \quad \checkmark$$

Step 4. Calculate $\alpha_j(0) = \langle E_j | \Psi(0) \rangle$.

$$\alpha_1(0) = \langle E_1 | \Psi(0) \rangle = \langle u | r \rangle = \left(\langle u | \right) \left(| r \rangle \right) = (1 \ 0) \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}$$

$$\alpha_2(0) = \langle E_2 | \Psi(0) \rangle = \langle d | r \rangle = (0 \ 1) \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}.$$

Step 5. $|\Psi(0)\rangle = \sum_{j=1}^2 \alpha_j(0) |E_j\rangle$.

$$|\Psi(0)\rangle = \sum_{j=1}^2 \alpha_j(0) |E_j\rangle = \frac{1}{\sqrt{2}}(|u\rangle + |d\rangle) = \frac{1}{\sqrt{2}} \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right] = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

$$\text{Sanity check: } |\Psi(0)\rangle = |r\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$|\Psi(0)\rangle$ in Step 2 is same as in Step 5 ✓

Step 6. Expand $|\Psi(t)\rangle = \sum_{j=1}^2 \alpha_j(t) |E_j\rangle$ in terms of $\{\alpha_j(t)\}$

$$|\Psi(t)\rangle = \sum_{j=1}^2 \alpha_j(t) |E_j\rangle = \alpha_1(t) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \alpha_2(t) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha_1(t) \\ \alpha_2(t) \end{pmatrix}.$$

Step 7. Replace $\alpha_j(t)$ in (6) with $\alpha_j(0) e^{-\frac{i}{\hbar} E_j t}$.

$$|\Psi(t)\rangle = \begin{pmatrix} \alpha_1(0) e^{-\frac{i}{\hbar} E_1 t} \\ \alpha_2(0) e^{-\frac{i}{\hbar} E_2 t} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} e^{-\frac{i}{\hbar} \frac{\hbar\omega}{2} t} \\ \frac{1}{\sqrt{2}} e^{-\frac{i}{\hbar} \left(-\frac{\hbar\omega}{2}\right) t} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-\frac{i\omega t}{2}} \\ e^{\frac{i\omega t}{2}} \end{pmatrix}.$$

Step 8. Specify a new observable at time t , compute its eigenvalues $\{\lambda\}$ and eigenvectors $\{|\lambda\rangle\}$, and calculate the probabilities of the outcomes.

According to the problem statement, the observable σ_y is measured at time t .

$$\text{Thus } \hat{n} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \sin\phi \cos\theta \\ \sin\phi \sin\theta \\ \cos\phi \end{pmatrix} \Rightarrow \phi = \frac{\pi}{2} = \theta. \text{ From Exercise 3.4,}$$

$$\lambda_1 = +1, \quad |\lambda_1\rangle = \begin{pmatrix} \cos\frac{\phi}{2} \\ \sin\frac{\phi}{2} e^{i\theta} \end{pmatrix} = \begin{pmatrix} \cos\frac{\pi}{4} \\ \sin\frac{\pi}{4} e^{i\frac{\pi}{2}} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} = |i\rangle,$$

$$\lambda_2 = -1, \text{ and } |\lambda_2\rangle = \begin{pmatrix} \sin \frac{\phi}{2} \\ -\cos \frac{\phi}{2} e^{i\theta} \end{pmatrix} = \begin{pmatrix} \sin \frac{\pi}{4} \\ -\cos \frac{\pi}{4} e^{\frac{\pi}{2}i} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix} = |o\rangle.$$

Thus

$$\begin{aligned} P_1(t) &= \langle \lambda_1 | \Psi(t) \rangle \langle \Psi(t) | \lambda_1 \rangle = \langle i | \Psi(t) \rangle \langle \Psi(t) | i \rangle \\ &= \left\{ \frac{1}{\sqrt{2}} (1 \ -i) \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-\frac{1}{2}i\omega t} \\ e^{\frac{1}{2}i\omega t} \end{pmatrix} \right\} \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} e^{\frac{1}{2}i\omega t} & e^{-\frac{1}{2}i\omega t} \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} \right\} \\ &= \frac{1}{4} \left(e^{-\frac{1}{2}i\omega t} - i e^{\frac{1}{2}i\omega t} \right) \left(e^{\frac{1}{2}i\omega t} + i e^{-\frac{1}{2}i\omega t} \right) = \frac{1}{4} (1 + i e^{-i\omega t} - i e^{i\omega t} + 1) \\ &= \frac{1}{2} + \frac{i}{4} [(\cos \omega t - i \sin \omega t) - (\cos \omega t + i \sin \omega t)] \quad (\text{de Moivre's Theorem}) \\ &= \frac{1}{2} (1 + \sin \omega t) \end{aligned}$$

$$\begin{aligned} P_{-1}(t) &= \langle \lambda_2 | \Psi(t) \rangle \langle \Psi(t) | \lambda_2 \rangle = \langle o | \Psi(t) \rangle \langle \Psi(t) | o \rangle \\ &= \left\{ \frac{1}{\sqrt{2}} (1 \ i) \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-\frac{1}{2}i\omega t} \\ e^{\frac{1}{2}i\omega t} \end{pmatrix} \right\} \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} e^{\frac{1}{2}i\omega t} & e^{-\frac{1}{2}i\omega t} \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix} \right\} \\ &= \frac{1}{4} \left(e^{-\frac{1}{2}i\omega t} + i e^{\frac{1}{2}i\omega t} \right) \left(e^{\frac{1}{2}i\omega t} - i e^{-\frac{1}{2}i\omega t} \right) = \frac{1}{4} (1 - i e^{-i\omega t} + i e^{i\omega t} + 1) \\ &= \frac{1}{2} + \frac{i}{4} [(\cos \omega t + i \sin \omega t) - (\cos \omega t - i \sin \omega t)] \\ &= \frac{1}{2} + \frac{i}{4} (2i \sin \omega t) = \frac{1}{2} (1 - \sin \omega t) \end{aligned}$$

Check: $P_1(t) + P_{-1}(t) = 1$ ✓

Another check: Both probabilities vary between 0 and 1. ✓

The conclusion is that σ_y varies as a sinusoidal wave over time between 0 and 1 with a mean value of $\frac{1}{2}$.

Similarly, σ_x varies as a sine wave between 0 and 1, and $\sigma_z = 0 \ \forall t$.

For the record, working the problem as stated yields:

$$1. H = \frac{\hbar \omega}{2} \sigma_z$$

$$2. |\Psi(0)\rangle = |u\rangle$$

$$3. |E_1\rangle = |u\rangle, |E_2\rangle = |d\rangle, E_1 = \frac{\hbar \omega}{2}, E_2 = -\frac{\hbar \omega}{2}$$

$$4. \alpha_1(0) = \langle E_1 | \Psi(0) \rangle = \langle u | u \rangle = 1$$

$$\alpha_2(0) = \langle E_2 | \Psi(0) \rangle = \langle d | u \rangle = 0$$

$$5. |\Psi(0)\rangle = \alpha_1(0) |E_1\rangle + \alpha_2(0) |E_2\rangle = |u\rangle + |0\rangle = |u\rangle$$

$$6. |\Psi(t)\rangle = \alpha_1(t) |E_1\rangle + \alpha_2(t) |E_2\rangle = \alpha_1(t) |u\rangle + \alpha_2(t) |d\rangle = \begin{pmatrix} \alpha_1(t) \\ \alpha_2(t) \end{pmatrix}$$

$$7. |\Psi(t)\rangle = \begin{pmatrix} \alpha_1(0) e^{-\frac{i}{\hbar} E_1 t} \\ \alpha_2(0) e^{-\frac{i}{\hbar} E_2 t} \end{pmatrix} = \begin{pmatrix} e^{-\frac{i}{\hbar} \frac{\hbar \omega}{2} t} \\ 0 \end{pmatrix} = \begin{pmatrix} e^{-\frac{1}{2} i \omega t} \\ 0 \end{pmatrix}$$

8. Compute the eigenvectors and eigenvalues of σ_y at time t :

$$\hat{n} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \sin \phi \cos \theta \\ \sin \phi \sin \theta \\ \cos \phi \end{pmatrix} \Rightarrow \phi = \frac{\pi}{2} = \theta.$$

From Exercise 3.4,

$$\lambda_1 = 1, |\lambda_1\rangle = \begin{pmatrix} \cos \frac{\phi}{2} \\ \sin \frac{\phi}{2} e^{i\theta} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} i \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} = |i\rangle$$

and

$$\lambda_2 = -1, |\lambda_2\rangle = \begin{pmatrix} \sin \frac{\phi}{2} \\ -\cos \frac{\phi}{2} e^{i\theta} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} i \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix} = |o\rangle.$$

At time t :

$$\begin{aligned} P(1) &= \langle \lambda_1 | \Psi(t) \rangle \langle \Psi(t) | \lambda_1 \rangle = \langle i | \Psi(t) \rangle \langle \Psi(t) | i \rangle \\ &= \frac{1}{2} \left[\begin{pmatrix} 1 & -i \end{pmatrix} \begin{pmatrix} e^{-\frac{1}{2}i\omega t} \\ 0 \end{pmatrix} \right] \left[\begin{pmatrix} e^{\frac{1}{2}i\omega t} & 0 \end{pmatrix} \begin{pmatrix} 1 \\ i \end{pmatrix} \right] = \frac{1}{2} e^{-\frac{1}{2}i\omega t} e^{\frac{1}{2}i\omega t} = \frac{1}{2} \end{aligned}$$

and

$$\begin{aligned} P(-1) &= \langle \lambda_2 | \Psi(t) \rangle \langle \Psi(t) | \lambda_2 \rangle = \langle o | \Psi(t) \rangle \langle \Psi(t) | o \rangle \\ &= \frac{1}{2} \left[\begin{pmatrix} 1 & i \end{pmatrix} \begin{pmatrix} e^{-\frac{1}{2}i\omega t} \\ 0 \end{pmatrix} \right] \left[\begin{pmatrix} e^{\frac{1}{2}i\omega t} & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -i \end{pmatrix} \right] = \frac{1}{2} e^{-\frac{1}{2}i\omega t} e^{\frac{1}{2}i\omega t} = \frac{1}{2} \end{aligned}$$

$\Rightarrow \sigma_y$ is unchanging over time (i.e., $P(1)$ and $P(2)$ are not expressed in terms of time).