Theorem. Let V be an n-dimensional complex vector space and L a matrix. L has at least one eigenvalue, and each distinct eigenvalue has a corresponding (non-zero) eigenvector. In particular, L has at least one eigenvector. If L is Hermitian then V has an orthonormal basis consisting of eigenvectors of L.

Proof. Let $p_L(\lambda) = \det(L - \lambda \boldsymbol{I})$ be the characteristic polynomial of L. $p_L(\lambda)$ is a polynomial in λ. By the Fundamental Theorem of Algebra, $p_L(\lambda) = (\lambda - \lambda_1)^{p_1} \cdots (\lambda - \lambda_r)^{p_r}$ where r is a positive integer, $p_1 + \cdots + p_r = n$, and $\lambda_1, \cdots, \lambda_n$ are roots of $p_L(\lambda)$, some possibly with multiplicity p_i greater than 1.So, for all i, $p_L(\lambda_i) = 0$. By footnote (*) there is a non-zero vector $|\lambda_i\rangle$ such that $L|\lambda_i\rangle = \lambda_i|\lambda_i\rangle$. So λ_i is an eigenvalue of L with eigenvector $|\lambda_i\rangle$. This proves that every matrix L has at least one eigenvalue and that each distinct eigenvalue has a corresponding (non-zero) eigenvector.

Now suppose that L is Hermitian. Then all eigenvalues λ are real. WLOG we can assume $|\lambda_1\rangle$ is a unit vector, $\langle\lambda_1|\lambda_1\rangle = 1$. Define the null space $N = \{|v\rangle : \langle v|\lambda_1\rangle = 0\}$. It is easy to see that N is a vector subspace of L. Since dim $\{\alpha|\lambda_1\rangle : \alpha \in \mathbb{C}\}$ is a 1-dimensional subspace, the orthogonal subspace N has dimension $n - 1$. Claim $LN \subseteq N$:

Let $|v\rangle \in N$. We need to show that $L|v\rangle \in N$. Since L is Hermitian, $L|v\rangle \leftrightarrow \langle v|L$. So we need to show that $\langle v|L|\lambda_1\rangle = 0$:

$$\langle v|L|\lambda_1\rangle = \langle v|\lambda_1|\lambda_1\rangle = \lambda_1\langle v|\lambda_1\rangle = 0 \quad ✔$$

Let $L_2 = L$ restricted to N. Repeating our logic above, $p_L(L_2)$ has a real root λ_2 that is an eigenvalue of L_2 with corresponding unit eigenvector $|\lambda_2\rangle$. Since $|\lambda_2\rangle \in N$, $\langle\lambda_1|\lambda_2\rangle = 0 \Rightarrow |\lambda_1\rangle \perp |\lambda_2\rangle$.

Using the (n – 2)-dimensional null space of L_2 as above we generate $|\lambda_3\rangle \perp |\lambda_2\rangle$, and since $|\lambda_3\rangle \in N$, $|\lambda_3\rangle \perp |\lambda_1\rangle$ also.

Continuing this process we eventually obtain the orthonormal basis $\{|\lambda_i\rangle : i = 1, \cdots, n\}$. ■

(*) Suppose we have *n* equations in *n* unknowns:

$$\begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = 0 \\ \vdots \\ a_{n1}x_1 + \cdots + a_{nn}x_n = 0 \end{cases} \Leftrightarrow \quad A|x\rangle = |0\rangle$$

If det *A* = 0, then there is an $|x\rangle \neq 0$ such that $A|x\rangle = |0\rangle$.

Let $A = L - \lambda_i \boldsymbol{I}$. So $\det A = \det\left(L - \lambda_i \boldsymbol{I}\right) = p_L\left(\lambda_i\right) = 0 \quad \Rightarrow \quad \exists$ non-zero $|x\rangle = |\lambda_i\rangle$ such that $\left(L - \lambda_i \boldsymbol{I}\right)|\lambda_i\rangle = \mathrm{A}|x\rangle = 0$. That is, $L|\lambda_i\rangle = \lambda_i|\lambda_i\rangle$ and $|\lambda_i\rangle \neq 0$.