[3.5] Prepare a system in direction $\hat{m}$ and make a measurement in direction $\hat{n}$. Show that $\mathrm{P}(1)=\cos^2\frac{\omega}{2}$ where ω is the angle between $\hat{m}$ and $\hat{n}$.

Solution. In spherical coordinates

$$\hat{m}=\begin{pmatrix} m_x \\ m_y \\ m_z \end{pmatrix}=\begin{pmatrix} \sin\theta_1\ \cos\phi_1 \\ \sin\theta_1\ \sin\phi_1 \\ \cos\theta_1 \end{pmatrix} \text{ and } \hat{n}=\begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix}=\begin{pmatrix} \sin\theta_2\ \cos\phi_2 \\ \sin\theta_2\ \sin\phi_2 \\ \cos\theta_2 \end{pmatrix}. \tag{1}$$

From Problem 3.4 the eigenvalues and eigenvectors for σ_m are

$$\mu_1=1,\ \ \mu_2=-1,\ \left|\mu_1\right\rangle=\begin{pmatrix} \cos\frac{\theta_1}{2} \\ \sin\frac{\theta_1}{2}e^{i\phi_1} \end{pmatrix}, \text{ and } \left|\mu_2\right\rangle=\begin{pmatrix} \sin\frac{\theta_1}{2} \\ -\cos\frac{\theta_1}{2}e^{i\phi_1} \end{pmatrix}.$$

For σ_n they are

$$\lambda_1=1,\ \ \lambda_2=-1,\ \left|\lambda_1\right\rangle=\begin{pmatrix} \cos\frac{\theta_2}{2} \\ \sin\frac{\theta_2}{2}e^{i\phi_2} \end{pmatrix}, \text{ and } \left|\lambda_2\right\rangle=\begin{pmatrix} \sin\frac{\theta_2}{2} \\ -\cos\frac{\theta_1}{2}e^{i\phi_2} \end{pmatrix}.$$

Since $\hat{m}\bullet\hat{n}=|\hat{m}||\hat{n}|\cos\omega=\cos\omega$ we have

$$\cos\omega=\sin\theta_1\cos\phi_1\sin\theta_2\cos\phi_2+\sin\theta_1\sin\phi_1\sin\theta_2\sin\phi_2+\cos\theta_1\cos\theta_2 \tag{2}$$

Recall the Half-Angle Formula for cosine:

$$\cos^2\frac{\theta}{2}=\frac{1+\cos\theta}{2} \tag{3}$$

So

$$\begin{aligned}\cos^2\frac{\omega}{2}&\overset{(3)}{=}\frac{1+\cos\omega}{2}\\ &\overset{(2)}{=}\frac{1}{2}\left(1+\sin\theta_1\cos\phi_1\sin\theta_2\cos\phi_2+\sin\theta_1\sin\phi_1\sin\theta_2\sin\phi_2+\cos\theta_1\cos\theta_2\right)\end{aligned} \tag{4}$$

The system is prepared in the direction $\hat{m}$ so $\left|\mu_1\right\rangle$ is the state vector. We measure in direction $\hat{n}$ so σ_n is the observable (Hermitian operator). Thus

$$\mathrm{P}(1)=\left\langle\mu_1\middle|\lambda_1\right\rangle\left\langle\lambda_1\middle|\mu_1\right\rangle \tag{5}$$

Since $\left\langle\lambda_1\right|=\begin{pmatrix}\cos\frac{\theta_2}{2} & \sin\frac{\theta_2}{2}e^{-i\phi_2}\end{pmatrix}$,

$$\left\langle\lambda_1\middle|\mu_1\right\rangle=\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}+\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}e^{-i(\phi_2-\phi_1)} \tag{6}$$

Since $\left\langle \mu_1 \right| = \begin{pmatrix} \cos\frac{\theta_1}{2} & \sin\frac{\theta_1}{2} e^{-i\phi_1} \end{pmatrix}$,

$$\left\langle \mu_1 \middle| \lambda_1 \right\rangle = \cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2} + \sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2} e^{i(\phi_2-\phi_1)} \tag{7}$$

To simplify the multiplication of (6) and (7) we require several trig identities. DeMoivre's Theorem ($e^{i\theta} = \cos\theta + i\sin\theta$) can be used to easily show

$$\cos\phi = \frac{e^{-i\phi}}{2} + \frac{e^{i\phi}}{2} \tag{8}$$

The Half-Angle Formula for sine is

$$\sin^2\frac{\theta}{2} = \frac{1-\cos\theta}{2} \tag{9}$$

The Double-Angle Formula for sine is

$$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} \tag{10}$$

The cosine formula for Difference of Two Angles is

$$\cos(\phi_2-\phi_1) = \cos\phi_2\cos\phi_1 + \sin\phi_2\sin\phi_1 \tag{11}$$

Finally

$$\begin{aligned}
\mathrm{P}(1) &\overset{(5)}{=} \left\langle \mu_1 \middle| \lambda_1 \right\rangle \left\langle \lambda_1 \middle| \mu_1 \right\rangle \\
&\overset{(6,\,7)}{=} \cos^2\frac{\theta_1}{2}\cos^2\frac{\theta_2}{2} + \sin\frac{\theta_1}{2}\cos\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\cos\frac{\theta_2}{2}\left(e^{i(\phi_2-\phi_1)} + e^{-i(\phi_2-\phi_1)}\right) \\
&\quad + \sin^2\frac{\theta_1}{2}\sin^2\frac{\theta_2}{2} \\
&\overset{(3,\,8,\,9)}{=} \frac{1+\cos\theta_1}{2}\frac{1+\cos\theta_2}{2} + 2\sin\frac{\theta_1}{2}\cos\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\cos\frac{\theta_2}{2}\cos(\phi_2-\phi_1) \\
&\quad + \frac{1-\cos\theta_1}{2}\frac{1-\cos\theta_2}{2} \\
&\overset{(10)}{=} \frac{1}{4}\left(1 + \cancel{\cos\theta_1} + \cancel{\cos\theta_2} + \cos\theta_1\cos\theta_2\right) + \frac{1}{2}\sin\theta_1\sin\theta_2\cos(\phi_2-\phi_1) \\
&\quad + \frac{1}{4}\left(1 - \cancel{\cos\theta_1} - \cancel{\cos\theta_2} + \cos\theta_1\cos\theta_2\right) \\
&\overset{(11)}{=} \frac{1}{2}\left[1 + \cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\left(\cos\phi_2\cos\phi_1 + \sin\phi_2\sin\phi_1\right)\right] \\
&= \frac{1}{2}\left[1 + \sin\theta_1\cos\phi_1\sin\theta_2\cos\phi_2 + \sin\theta_1\sin\phi_1\sin\theta_2\sin\phi_2 + \cos\theta_1\cos\theta_2\right] \\
&\overset{(4)}{=} \cos^2\frac{\omega}{2}
\end{aligned}$$