$\boxed{6.10}$ Let $H = \frac{\hbar\omega}{2}\vec{\sigma}\cdot\vec{\tau}$.

(1) What are the possible energies of the system and what are the eigenvectors of H?

(2) If the system starts in state $|uu\rangle$ what is the state at time t?

(3) " " $|ud\rangle, |du\rangle$, or $|dd\rangle$, " ?

Solution (1) From 6.9, $\boxed{\text{the eigenvectors of } H \text{ are } |sing\rangle, |T_1\rangle, |T_2\rangle, \text{ and } |T_3\rangle.}$

The table shows the eigenvalues of $\vec{\sigma}\cdot\vec{\tau}$. $\boxed{\text{The possible energies}}$ are the eigenvalues of H, namely $\frac{\hbar\omega}{2}$ times the eigenvalues of $\vec{\sigma}\cdot\vec{\tau}$:

$\boxed{-3\frac{\hbar\omega}{2} \text{ and } \frac{\hbar\omega}{2} \text{ w/ degeneracy } 3}$

Eigenvalues of $\vec{\sigma}\cdot\vec{\tau}$ and its components

	$\|sing\rangle$	$\|T_1\rangle$	$\|T_2\rangle$	$\|T_3\rangle$
$\sigma_x\tau_x$	-1	1	1	-1
$\sigma_y\tau_y$	-1	1	-1	1
$\sigma_z\tau_z$	-1	-1	1	1
$\vec{\sigma}\cdot\vec{\tau}$	-3	1	1	1

(2) We use the cookbook Recipe to compute $|\Psi(t)\rangle$.

1) $H = \frac{\hbar\omega}{2}\vec{\sigma}\cdot\vec{\tau} = \frac{\hbar\omega}{2}[\sigma_x\tau_x + \sigma_y\tau_y + \sigma_z\tau_z]$

2) Prepare a state vector: $|\Psi(0)\rangle = |uu\rangle$

3) The energy eigenvalues and eigenvectors are as listed in (1)

4) $\alpha_{sing}(0) = \langle sing|\Psi(0)\rangle = \frac{1}{\sqrt{2}}(\langle ud| - \langle du|)|uu\rangle = 0$

$\alpha_{T_1}(0) = \langle T_1|\Psi(0)\rangle = \frac{1}{\sqrt{2}}(\langle ud| + \langle du|)|uu\rangle = 0$

$\alpha_{T_2}(0) = \langle T_2|\Psi(0)\rangle = \frac{1}{\sqrt{2}}(\langle uu| + \langle dd|)|uu\rangle = \frac{1}{\sqrt{2}}$

$\alpha_{T_3}(0) = \langle T_3|\Psi(0)\rangle = \frac{1}{\sqrt{2}}(\langle uu| - \langle dd|)|uu\rangle = \frac{1}{\sqrt{2}}$

5) Rewrite $|\Psi(0)\rangle = \alpha_{sing}(0)|sing\rangle + \alpha_{T_1}(0)|T_1\rangle + \alpha_{T_2}(0)|T_2\rangle + \alpha_{T_3}(0)|T_3\rangle$

$= \frac{1}{\sqrt{2}}(|T_2\rangle + |T_3\rangle)$

Sanity check: $|uu\rangle = |\Psi(0)\rangle = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}(|uu\rangle + |dd\rangle + |uu\rangle - |dd\rangle) = |uu\rangle$ ✓

6, 7) $|\Psi(t)\rangle \overset{(6)}{=} \alpha_{T_2}(t)|T_2\rangle + \alpha_{T_3}(t)|T_3\rangle \overset{(7)}{=} \frac{1}{\sqrt{2}}e^{-\frac{i}{\hbar}\frac{\hbar\omega}{2}(1)t}(|T_2\rangle + |T_3\rangle)$

$\boxed{|\Psi(t)\rangle = \frac{1}{\sqrt{2}}e^{-\frac{i\omega t}{2}}(|T_2\rangle + |T_3\rangle)}$

Sanity check: $|\Psi(t)\rangle = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}e^{-\frac{i\omega t}{2}}(|uu\rangle + |dd\rangle + |uu\rangle - |dd\rangle) = \boxed{e^{-\frac{i\omega t}{2}}|uu\rangle}$

$t = 0 \Rightarrow |\Psi(0)\rangle = |uu\rangle$ ✓

6.10 p.2

(4)

2) $|\Psi(0)\rangle = |ud\rangle$

4) $\alpha_{sing}(0) = \langle sing|\Psi(0)\rangle = \frac{1}{\sqrt{2}}(\langle ud| - \langle du|)|ud\rangle = \frac{1}{\sqrt{2}}$

$\alpha_{T_1}(0) = \langle T_1|\Psi(0)\rangle = \frac{1}{\sqrt{2}}(\langle ud| + \langle du|)|ud\rangle = \frac{1}{\sqrt{2}}$

$\alpha_{T_2}(0) = \alpha_{T_3}(0) = 0$ since $\langle uu|ud\rangle = \cancel{\langle u|u\rangle}\cancel{\langle u|d\rangle}^{0} = 0$ and $\langle du|ud\rangle = 0$

5) $|\Psi(0)\rangle = \frac{1}{\sqrt{2}}(|sing\rangle + |T_1\rangle)$. Sanity ck: $|ud\rangle = |\Psi(0)\rangle = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}(|ud\rangle - |du\rangle + |ud\rangle + |du\rangle) = |ud\rangle$ ✓

6,7) $|\Psi(t)\rangle = \frac{1}{\sqrt{2}}\left(e^{-\frac{i}{\hbar}(-3)\frac{\hbar\omega}{2}t}|sing\rangle + e^{-\frac{i}{\hbar}(1)\frac{\hbar\omega}{2}t}|T_1\rangle\right) = \boxed{\frac{1}{\sqrt{2}}e^{\frac{1}{2}i\omega t}\left(e^{i\omega t}|sing\rangle + e^{-i\omega t}|T_1\rangle\right)}$

$= \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}e^{\frac{1}{2}i\omega t}\{2\cos\omega t|ud\rangle + 2i\sin\omega t|du\rangle\}$ $= \boxed{e^{\frac{1}{2}i\omega t}(\cos\omega t|ud\rangle + i\sin\omega t|du\rangle)}$

2) $|\Psi(0)\rangle = |du\rangle$

4) $\alpha_{T_1}(0) = \frac{1}{\sqrt{2}}(\langle ud| + \langle du|)|du\rangle = \frac{1}{\sqrt{2}}$

$\alpha_{sing}(0) = \frac{1}{\sqrt{2}}(\langle ud| - \langle du|)|du\rangle = -\frac{1}{\sqrt{2}}$

$\alpha_{T_2}(0) = \alpha_{T_3}(0) = 0$

5) $|\Psi(0)\rangle = \frac{1}{\sqrt{2}}(-|sing\rangle + |du\rangle)$. Sanity ck: $|du\rangle = |\Psi(0)\rangle = \frac{1}{2}(-|ud\rangle + |du\rangle + |ud\rangle + |du\rangle)$ ✓

6,7) $|\Psi(t)\rangle = \boxed{\frac{1}{\sqrt{2}}\left(-e^{\frac{3}{2}i\omega t}|sing\rangle + e^{-\frac{1}{2}i\omega t}|T_1\rangle\right)}$ $= -ie^{\frac{1}{2}i\omega t}(\sin\omega t|ud\rangle + i\cos\omega t|du\rangle)$

$\boxed{= e^{\frac{1}{2}i\omega t}(\cos\omega t|du\rangle - i\sin\omega t|ud\rangle)}$

2) $|\Psi(0)\rangle = |dd\rangle$

4) $\alpha_{T_2}(0) = \frac{1}{\sqrt{2}}$, $\alpha_{T_3}(0) = -\frac{1}{\sqrt{2}}$, $sing(0) = T_1(0) = 0$

5) $|\Psi(0)\rangle = \frac{1}{\sqrt{2}}(|T_2\rangle - |T_3\rangle)$ Sanity ck: $|dd\rangle = |\Psi(0)\rangle = \frac{1}{2}(|uu\rangle + |dd\rangle - |uu\rangle + |dd\rangle)$ ✓

6,7) $|\Psi(t)\rangle = \boxed{\frac{1}{\sqrt{2}}e^{-\frac{i\omega t}{2}}(|T_2\rangle - |T_3\rangle)}$

Also, $|\Psi(t)\rangle = \frac{1}{\sqrt{2}}e^{-\frac{i\omega t}{2}}\left(\frac{1}{\sqrt{2}}(|uu\rangle + |dd\rangle - |uu\rangle + |dd\rangle\right) =$

$\boxed{|\Psi(t)\rangle = e^{-\frac{i\omega t}{2}}|dd\rangle}$