Exercise 4.6. Carry out the ket recipe with $H=\frac{\hbar\omega}{2}\sigma_z$, final observable σ_x, initial state $|u\rangle=\begin{pmatrix}1\\0\end{pmatrix}$, and observable σ_y measured at time *t*. Find the possible outcomes and their probabilities of occurrence.

Solution. This exercise is the culmination of a problem started in Exercise 3.4 and continued in 4.4 and 4.5.

There is an unfortunate choice of wording in the 4.6 problem statement. We should not confuse "initial state" in the problem statement with "initial state" in step 2 of the ket recipe. They are quite different.

The term "initial state" in step 2 is simply the state vector that corresponds to the selected observable. However, the phrase "initial state $|u\rangle=\begin{pmatrix}1\\0\end{pmatrix}$" in the wording of this problem simply means that as developed in chapter 2, formula (2.13), we should use the following additional formulas:

$$|d\rangle=\begin{pmatrix}0\\1\end{pmatrix},\ |i\rangle=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix},\ |o\rangle=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix},\ |r\rangle=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix},\ |l\rangle=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix},$$
$$|\sigma_x\rangle=\begin{pmatrix}0&1\\1&0\end{pmatrix},\ |\sigma_y\rangle=\begin{pmatrix}0&-i\\i&0\end{pmatrix},\ \text{and}\ |\sigma_z\rangle=\begin{pmatrix}1&0\\0&-1\end{pmatrix}.$$

Step 1. Find *H*. $H=\frac{\hbar\omega}{2}\sigma_z$

Step 2. Prepare a state vector. The state vector that corresponds to the observable σ_x is

$$|\Psi(0)\rangle=|r\rangle\ .$$

Step 3. Find the energy eigenvalues and eigenvectors of *H*. From Exercise 4.5 we learned that for a generic direction $\hat{n}=\begin{pmatrix}n_x\\n_y\\n_z\end{pmatrix}=\begin{pmatrix}\sin\theta\cos\phi\\\sin\theta\sin\phi\\\cos\theta\end{pmatrix}$ that $H=\frac{\hbar\omega}{2}\sigma_n$ has the following energy eigenvalues and corresponding eigenvectors:

$$E_1 = \frac{\hbar\omega}{2},\ \ E_2 = -\frac{\hbar\omega}{2},\ \ \left|E_1\right\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}\, e^{i\phi} \end{pmatrix},\ \text{ and } \left|E_2\right\rangle = \begin{pmatrix} \sin\frac{\theta}{2} \\ -\cos\frac{\theta}{2}\, e^{i\phi} \end{pmatrix}.$$

$H = \frac{\hbar\omega}{2}\sigma_z$, and for $\sigma_n = \sigma_z$ we get that $\hat{n} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix} \Rightarrow \theta = 0$. Hence

$$\left|E_1\right\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}\, e^{i\phi} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \left|u\right\rangle \ ✔$$

$$\left|E_2\right\rangle = \begin{pmatrix} \sin\frac{\theta}{2} \\ -\cos\frac{\theta}{2}\, e^{i\phi} \end{pmatrix} = \begin{pmatrix} 0 \\ -e^{i\phi} \end{pmatrix}.$$

$\left|E_2\right\rangle$ is a unit vector for any value of ϕ. However, in Exercise 4.5 it was shown that $\left|E_1\right\rangle = \left|\lambda_1\right\rangle$ and $\left|E_2\right\rangle = \left|\lambda_2\right\rangle$ where $\left|\lambda_1\right\rangle$ and $\left|\lambda_2\right\rangle$ are the eigenvectors of $\sigma_n = \sigma_z$. Since $\left|\lambda_1\right\rangle = \left|u\right\rangle$ and $\left|\lambda_2\right\rangle = \left|d\right\rangle$ we have that

$\left|E_2\right\rangle = \left|d\right\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ (and consequently $\phi = \pi$).

Step 4. Calculate $\alpha_j(0) = \left\langle E_j \middle| \Psi(0)\right\rangle$.

$$\alpha_1(0) = \left\langle E_1 \middle| r\right\rangle = \left(\left\langle E_1\right|\right)\left(\left|r\right\rangle\right) = (1\ \ 0)\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}$$

$$\alpha_2(0) = \left\langle E_2 \middle| r\right\rangle = (0\ \ 1)\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}.$$

Step 5. $\left|\Psi(0)\right\rangle = \sum_{j=1}^{2}\alpha_j(0)\left|E_j\right\rangle$.

$$\left|\Psi(0)\right\rangle = \sum_{j=1}^{2}\alpha_j(0)\left|E_j\right\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Sanity check: $\left|\Psi(0)\right\rangle = \left|r\right\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\left|\Psi(0)\right\rangle$ in Step 2 is same as in Step 5 ✔

Step 6. Expand $\left|\Psi(t)\right\rangle = \sum_{j=1}^{2} \alpha_j(t)\left|E_j\right\rangle$ in terms of $\{\alpha_j(t)\}$

$$\left|\Psi(t)\right\rangle = \sum_{j=1}^{2} \alpha_j(t)\left|E_j\right\rangle = \alpha_1(t)\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \alpha_2(t)\begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} \alpha_1(t) \\ -\alpha_2(t) \end{pmatrix}.$$

Step 7. Replace $\alpha_j(t)$ in (6) with $\alpha_j(0)e^{-\frac{i}{\hbar}E_j t}$.

$$\left|\Psi(t)\right\rangle = \begin{pmatrix} \alpha_1(0)e^{-\frac{i}{\hbar}E_1 t} \\ -\alpha_2(0)e^{-\frac{i}{\hbar}E_2 t} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}}e^{-\frac{i\hbar\omega}{2\hbar}t} \\ -\frac{1}{\sqrt{2}}e^{\frac{i\hbar\omega}{2\hbar}t} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{-\frac{i\omega t}{2}} \\ -e^{\frac{i\omega t}{2}} \end{pmatrix}.$$

Step 8. Specify a new observable at time *t*, compute its eigenvalues $\{\lambda\}$ and eigenvectors $\{\left|\lambda\right\rangle\}$.

According to the problem statement, the observable σ_y is measured at time *t*.

Thus $\hat{n} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix} \Rightarrow \theta = \frac{\pi}{2} = \phi$. From Exercise 3.4, $\lambda_1 = +1$,

$$\left|\lambda_1\right\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}e^{i\phi} \end{pmatrix} = \begin{pmatrix} \cos\frac{\pi}{4} \\ \sin\frac{\pi}{4}e^{\frac{\pi}{2}i} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} = \left|i\right\rangle,\ \lambda_2 = -1 \text{, and}$$

$$\left|\lambda_2\right\rangle = \begin{pmatrix} \sin\frac{\theta}{2} \\ -\cos\frac{\theta}{2}e^{i\phi} \end{pmatrix} = \begin{pmatrix} \sin\frac{\pi}{4} \\ -\cos\frac{\pi}{4}e^{\frac{\pi}{2}i} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix} = \left|o\right\rangle.$$

Thus

$$P_1(t)=\langle i|\Psi(t)\rangle\langle\Psi(t)|i\rangle=\left\{\frac{1}{\sqrt{2}}(1\ \ {-i})\frac{1}{\sqrt{2}}\begin{pmatrix} e^{-\frac{1}{2}i\omega t} \\ e^{\frac{1}{2}i\omega t}\end{pmatrix}\right\}\left\{\frac{1}{\sqrt{2}}\left(e^{\frac{1}{2}i\omega t}\ \ e^{-\frac{1}{2}i\omega t}\right)\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i\end{pmatrix}\right\}$$

$$=\frac{1}{4}\left(e^{-\frac{1}{2}i\omega t}-ie^{-\frac{1}{2}i\omega t}\right)\left(e^{\frac{1}{2}i\omega t}+ie^{-\frac{1}{2}i\omega t}\right)=\frac{1}{2}\left[1+i\left(\frac{e^{-i\omega t}}{2}-\frac{e^{i\omega t}}{2}\right)\right]$$

$$=\frac{1}{2}(1+\sin\omega t)\quad \text{by De Moivre's Theorem}$$

$$P_{-1}(t)=\langle o|\Psi(t)\rangle\langle\Psi(t)|o\rangle=\left\{\frac{1}{\sqrt{2}}(1\ \ i)\frac{1}{\sqrt{2}}\begin{pmatrix} e^{-\frac{1}{2}i\omega t} \\ e^{\frac{1}{2}i\omega t}\end{pmatrix}\right\}\left\{\frac{1}{\sqrt{2}}\left(e^{\frac{1}{2}i\omega t}\ \ e^{-\frac{1}{2}i\omega t}\right)\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i\end{pmatrix}\right\}$$

$$=\frac{1}{4}\left(e^{-\frac{1}{2}i\omega t}+ie^{\frac{1}{2}i\omega t}\right)\left(e^{\frac{1}{2}i\omega t}-ie^{-\frac{1}{2}i\omega t}\right)=\frac{1}{2}\left[1-i\left(\frac{e^{-i\omega t}}{2}-\frac{e^{i\omega t}}{2}\right)\right]$$

$$=\frac{1}{2}(1-\sin\omega t)$$

Check: $P_1(t)+P_{-1}(t)=1$ ✔

Another check: Both probabilities vary between 0 and 1. ✔

The conclusion is that σ_y varies as a sinusoidal wave over time with a mean value of $1/2$.

We could also have shown a similar result for σ_x. Of course $\sigma_z=0$ for all *t*.