Axion photon oscillations and some astrophysical consequences

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In this report, we look at axion photon oscillations. We consider some simple models of axion/photon oscillations – the single domain model, the multi-domain conventional model and the recently devised helical model. Along the way, we look at different approaches to the formalism and also investigate Stokes parameters to study the polarization states of photons. Based on the models, we discuss some astrophysical and laboratory consequences and compare them with known observational results, using them to impose constraints on relevant parameters.

CONTENTS

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I. INTRODUCTION

a. The strong CP problem and the Peccei Quinn mechanism Axions are hypothetical particles that first appeared as a way to resolve the strong CP problem in the Standard Model. Over time, they have been looked at as answers to other problems as well - for example, as dark matter candidates - and their properties can be looked at from various perspectives.
QED predicts accurate answers for fermionic magnetic dipole moments, $\vec{\mu}$, which have been experimentally verified. No electric dipole moment, $\vec{d}$, have been experimentally detected however. Specifically, the responsible term in the QCD Lagrangian is of the form [1],

$$\mathcal{L}_\theta \ \propto \ (\Theta + \Theta_{QCD}) G_b^{\mu\nu} \tilde{G}_{b\mu\nu} \ \equiv \ (\Theta + \Theta_{QCD}) \vec{E}_{colour} \cdot \vec{B}_{colour}$$

Here, G is the colour field strength tensor. The Θ_{QCD} term arises from non-perturbative QCD alone, while the Θ term arises entirely due to interactions. Since $\vec{E}$ is a true vector and $\vec{B}$ is a pseudo-vector, these terms are CP odd. They lead to a neutron electric dipole moment of $|\bar{\Theta}|(0.2 - 0.004)\ \mu_N$ nuclear magnetons with $\bar{\Theta} = \Theta + \Theta_{QCD}$. The experimentally observed small neutron moment, however, provides an upper bound of $|\bar{\Theta}| < 10^{-9}$ which is very small and unexpected. Thus, we expect the two terms, Θ and Θ_{QCD}, to cancel each other out finely. However, on the other hand, the two terms are supposed to be completely unrelated to each other.
In order to reconcile this issue, we resort to the Peccei Quinn mechanism, introducing a new pseudoscalar field, the axion, $a(x^\mu)$, that couples in exactly the same manner with the colour fields except for Θ being replaced by $-a/f_a$, where f_a is an appropriate energy scale (the Peccei Quinn energy scale). Due to a being a pseudo-scalar, this interaction is CP even. This kind of an interaction term ensures that the axion term can effectively cancel out the $\bar{\Theta}$ coupling to the colour field and account for the small neutron electric dipole moment.
While these axions are considered massless (as the Lagrangian is considered invariant under the shift $a \to a + a_0$), they do acquire an effective mass due to their interaction with gluons. The coupling to gluons effectively couples the axion to the neutral pions through which it gets coupled to the electromagnetic field - the effective interaction term is proportional to $aF^{\mu\nu}\tilde{F}_{\mu\nu}$. This coupling in the essence of this report - assuming that such a coupling exists, we investigate the associated consequences.

b. Outline In this report, as mentioned, we begin with the assumption that such an axion exists - by an axion, we refer to any light, pseudo-scalar particle. Given this, our axion necessarily couples to the photon field via a coupling constant, $g_{a\gamma\gamma}$ or $1/M$ - this interaction leads to the phenomena of axion photon oscillations in a background magnetic field in which the two particles mix with each other. This leads to a number of effects that can be observed in a laboratory or in nature, particularly concerning the polarization states of the photon field. We look into these effects in detail and consider several models of magnetic fields, analysing the observable effects in each. This is the content of section II to section V.
In section VI, we mostly concern ourselves with the practical effects of such phenomena - observations. We take a look at multiple beam path experiments and conclude that these experiments, while correct in principle, are difficult to implement physically. Turning to astrophysical phenomena, we look at how the polarization states change and compare our results to known experimental observations. This allows us to impose a constraint on the coupling constant in the problem. We summarise the report and mark out potential future avenues for work in section VII.

II. LAGRANGIAN DENSITY AND FIELD EQUATIONS

By an axion, as per [2], we mean *any real, pseudo-scalar and light* particle - we denote the axion field by a. Then, the Lagrangian density of such an axion field in the presence of an electromagnetic field A^μ, is given by,

$$\mathcal{L}_D = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{1}{2} \left[\partial_\mu a \partial^\mu a - m^2 a^2 \right] + \frac{a}{4M} \tilde{F}^{\mu\nu} F_{\mu\nu} + E.H. \tag{1}$$

Here, the Faraday tensor and it's dual are defined as,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu; \ \tilde{F}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta} \tag{2}$$

Note that our metric is the pseudo Lorentzian flat metric, $\eta_{\mu\nu}$, given by the mostly negative convention, i.e., $\eta = diag(1, -1, -1, -1)$.
In (1), the first term denotes the kinetic energy term for the photon while the second term is the density for any real (pseudo)scalar field. The third term captures the interaction between the axion and the photon. The

$E.H.$ denotes the additional Euler - Heisenberg terms given by,

$$E.H. = \frac{\alpha^2}{90m_e^4}\Big((F^{\mu\nu}F_{\mu\nu})^2 + \frac{7}{4}(\tilde{F}^{\mu\nu}F_{\mu\nu})^2\Big)$$

These are terms that we do not include in this analysis.
The action functional corresponding to the density in (1) is $S = S[A^\mu, a]$ and as such must be varied with respect to both the fields in order to arrive at the field equations. We start by varying the action with respect to the axion field,

$$\delta_a S = 0 \quad\Rightarrow\quad \Big[\partial_\mu a(\partial^\mu \delta a) - m^2 a\delta a\Big] + \frac{\delta a}{4M}\tilde{F}^{\mu\nu}F_{\mu\nu} = 0$$

$$\Rightarrow\quad \Big[-\partial^\mu\partial_\mu a - m^2 a + \frac{1}{4M}\tilde{F}^{\mu\nu}F_{\mu\nu}\Big]\delta a + \partial^\mu(\partial_\mu a\delta a) = 0$$

Ignoring the total derivative term, we arrive at,

$$\partial^\mu\partial_\mu a + m^2 a \;=\; (\Box + m^2)a \;=\; \frac{1}{4M}\tilde{F}^{\mu\nu}F_{\mu\nu}$$

Similarly, we vary the action with respect to the vector potential,

$$\delta_{A^\mu} S = 0 \;\Rightarrow\; -\frac{2}{4}F^{\mu\nu}\delta F_{\mu\nu} + \frac{a}{4M}\delta(\tilde{F}^{\mu\nu}F_{\mu\nu}) = 0$$

Now, due to the antisymmetry of the Faraday tensor,

$$F^{\mu\nu}\delta F_{\mu\nu} = 2F^{\mu\nu}\partial_\mu(\delta A_\nu) = 2\partial_\mu(F^{\mu\nu}\delta A_\nu) - 2\delta A_\nu\partial_\mu F^{\mu\nu}$$

Similarly, as the dual to the Faraday tensor is antisymmetric as well,

$$\delta(\tilde{F}^{\mu\nu}F_{\mu\nu}) = \delta(\tilde{F}^{\mu\nu})F_{\mu\nu} + \tilde{F}^{\mu\nu}\delta(F_{\mu\nu}) = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}\delta(F_{\alpha\beta})F_{\mu\nu} + 2\tilde{F}^{\mu\nu}\partial_\mu(\delta A_\nu) = \tilde{F}^{\alpha\beta}\delta(F_{\alpha\beta}) + 2\tilde{F}^{\mu\nu}\partial_\mu(\delta A_\nu)$$

Or,

$$a\delta(\tilde{F}^{\mu\nu}F_{\mu\nu}) = 4a\tilde{F}^{\mu\nu}\partial_\mu(\delta A_\nu) = 4\partial_\mu(a\tilde{F}^{\mu\nu}\delta A_\nu) - 4\delta A_\nu\partial_\mu(a\tilde{F}^{\mu\nu})$$

Therefore, ignoring the total derivative terms,

$$\delta_{A^\mu} S = 0$$

Or,

$$\delta A_\nu\Big[\partial_\mu F^{\mu\nu} - \frac{1}{M}\partial_\mu(a\tilde{F}^{\mu\nu})\Big] = 0 \Rightarrow \partial_\mu F^{\mu\nu} \;=\; \frac{1}{M}\partial_\mu(a\tilde{F}^{\mu\nu}) \;=\; \frac{1}{M}\tilde{F}^{\mu\nu}\partial_\mu a$$

Here, in the last step, we have used the Bianchi identity, $\partial_\mu\tilde{F}^{\mu\nu} = 0$.
Thus, our field equations can be summarised as,

$$\boxed{(\partial^\mu\partial_\mu + m^2)a = \frac{1}{4M}\tilde{F}^{\mu\nu}F_{\mu\nu}} \tag{3}$$

$$\boxed{\partial_\mu F^{\mu\nu} = \frac{1}{M}\tilde{F}^{\mu\nu}\partial_\mu a} \tag{4}$$

Written in terms of the electric and magnetic fields, these are the same equations as given in [3].

III. THE SINGLE DOMAIN MODEL FOR HOMOGENEOUS FIELDS

A. Linearized equations in presence of homogeneous external field

1. *Strong field approximation*

Now, in presence of a *strong* external magnetic field, $\vec{B}_e$, we can approximate $\tilde{F}^{\mu\nu} \approx \tilde{F}^{\mu\nu}_{external}$ which only has non-zero magnetic field components. Therefore, eqn. (3) gives us,

$$(\Box + m^2)a = -\frac{1}{M}\vec{E}\cdot\vec{B}_e = \frac{B_e}{M}\frac{\partial A_{||}}{\partial t} \tag{5}$$

Similarly, eqn. (4), gives us,

$$\partial_\mu F^{\mu\nu} = \Box A^\nu = \frac{1}{M}\tilde{F}^{\mu\nu}_{external}\partial_\mu a$$

Breaking the photon field into components parallel and perpendicular to the external magnetic field, this gives us two equations,

$$\Box A_{||} = \frac{B_e}{M}\frac{\partial a}{\partial t} \tag{6}$$

$$\Box A_\perp = 0 \tag{7}$$

2. *Mixing matrix for pure vacuum*

We Fourier transform the time component of eqns. (5)-(7) to the energy basis. This gives us,

$$(-\omega^2 - \nabla^2 + m^2)a = \frac{i\omega}{M}B_e A_{||}$$

$$(-\omega^2 - \nabla^2)A_{||} = \frac{i\omega}{M}B_e a$$

$$(-\omega^2 - \nabla^2)A_\perp = 0$$

We write these three equations in a matrix form as, for propagation only along the z-direction as,

$$\left[\omega^2 + \partial_z^2 - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{i\omega}{M}B_e \\ 0 & \frac{-i\omega}{M}B_e & m^2 \end{pmatrix}\right]\underbrace{\begin{pmatrix} A_\perp \\ A_{||} \\ a \end{pmatrix}}_{|a\rangle} = |0\rangle \tag{8}$$

Here, $|0\rangle = (0,0,0)^T$. We can see that the perpendicular component of the photon field has completely decoupled from the other two modes; from this, we define the 2 dimensional mixing matrix,

$$\mathcal{X} = \begin{pmatrix} 0 & i\omega\mu \\ -i\omega\mu & m^2 \end{pmatrix}$$

Here, $\mu = B_e/M$. This matrix has eigenvalues given by,

$$\lambda_\pm = \frac{1}{2}\left(m^2 \pm \sqrt{m^4 + 4\mu^2\omega^2}\right)$$

This is the mixing matrix that the authors have worked with in [4].

B. Mixing matrix in more realistic scenarios

1. *Linearized equations*

In [2] and [1], the authors claim that there will be QED effects even in vacuum which will lead to eqn. (8) taking the following form,

$$\left[\omega^2 + \partial_z^2 + \begin{pmatrix} Q_\perp & 0 & 0 \\ 0 & Q_{||} & \frac{\omega}{M}B_e \\ 0 & \frac{\omega}{M}B_e & -m^2 \end{pmatrix}\right]|a\rangle \;=\; |0\rangle \tag{9}$$

Here, $Q_{\perp,||} = 2\omega^2(n_{\perp,||} - 1)$ arising out of the previously neglected Euler - Heisenberg terms. In the presence of an external magnetic field, light splits up into two components - one transverse to the field and the other, perpendicular. This is leads to a birefringence effect with each component having a different refractive index given by,

$$n_{||} = 1 + \frac{7}{2}\xi sin^2\Theta;\; n_\perp = 1 + \frac{4}{2}\xi sin^2\Theta$$

Here, Θ is the angle between the photon momentum and the external field, while ξ $(= \alpha/45\pi(B/B_c)^2$ with the critical field, $B_c = m_e^2/e)$ is a parameter characterising the phenomenon. In [1], the author further includes mixing terms between the parallel and perpendicular components with a refractive index, n_R, in the 21 and 12 entries.
Now, in order to linearize eqn. (9), we consider a dispersion relation of the form $k \sim \omega$. We also consider *ultra-relativistic* axions, implying that $\omega_a = \sqrt{m_a^2 + |\vec{k}|^2} \approx |\vec{k}| = \omega$. With this, we end up with in the leading order (Fourier transforming the spatial components back and forth),

$$\omega^2 + \partial_z^2 = (\omega + i\partial_z)(\omega - i\partial_z) \equiv (\omega + k)(\omega - k) \approx 2\omega(\omega - k) \equiv 2\omega(\omega - i\partial_z)$$

Then we have eqn. (9) in linearized form as,

$$\left[\omega - i\partial_z + \begin{pmatrix} \Delta_\perp & 0 & 0 \\ 0 & \Delta_{||} & \Delta_M \\ 0 & \Delta_M & \Delta_a \end{pmatrix}\right]|a\rangle \;=\; |0\rangle \tag{10}$$

Here,

$$\Delta_{||} = \frac{7}{2}\omega\xi \sin^2\Theta;\; \Delta_\perp = \frac{4}{2}\omega\xi \sin^2\Theta;\; \Delta_a = -\frac{m^2}{2\omega};\; \Delta_M = \frac{B_e}{2M}\sin\Theta$$

2. *Diagonalization*

Now, in order to diagonalize the above mixing matrix, we first note that we can work in the 2 dimensional invariant subspace, separate from $A_\perp$, without any loss of generality. Further, the matrix is real and symmetric and can be diagonalized by an appropriately chosen orthogonal matrix given as,

$$U \;=\; \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}$$

Then, we have in the diagonalised form, with $\mathcal{S}$ only including the lower right 2×2 block in (10),

$$\mathcal{S}' \;=\; \begin{pmatrix} \Delta'_{||} & \Delta'_M \\ \Delta'_M & \Delta'_a \end{pmatrix}$$

Since the matrix is diagonal, by definition,

$$\Delta'_M = 0 \;\Rightarrow\; \Delta_M \cos 2\phi + \frac{\Delta_a - \Delta_{||}}{2}\sin 2\phi = 0 \;\Rightarrow\; \frac{1}{2}\tan 2\phi = \frac{\Delta_M}{\Delta_{||} - \Delta_a} \tag{11}$$

This characterizes the strength of the mixing. We use this relation to simplify the diagonal terms,

$$\Delta'_{||} \;=\; \Delta_{||}\cos^2\phi + \Delta_a \sin^2\phi + \Delta_M \sin 2\phi$$

Substituting for Δ_M from (11), we get,

$$\Delta'_{||} = \Delta_{||}(\cos^2\phi - \frac{1}{2}\cos 2\phi) + \Delta_a(\sin^2\phi + \frac{1}{2}\cos 2\phi) + \frac{\Delta_{||} - \Delta_a}{2\cos 2\phi}$$

Thus, if we repeat the steps for the other diagonal term as well,

$$\Delta'_{||,a} = \frac{\Delta_{||} + \Delta_a}{2} \pm \frac{\Delta_{||} - \Delta_a}{2\cos 2\phi} \tag{12}$$

3. Solutions

From the decoupling of the orthogonal A^μ mode from the other two modes, we can easily write down its solution,

$$\Big(\omega - i\partial_z + \Delta_\perp\Big)A_\perp = 0 \Rightarrow A_\perp(z,t) = e^{i(\omega t - k_\perp z)}A_\perp(0,0) \tag{13}$$

Here, Fourier transforming the first equation in the spatial sense gives us the dispersion relation,

$$\omega - k_\perp + \Delta_\perp = 0 \Rightarrow k_\perp = \omega(1 + \frac{4}{2}\xi\sin^2\Theta) = \omega n_\perp \tag{14}$$

Similarly, the other modes are decoupled in the primed eigenbasis, and their solutions can be written down as,

$$A'_{||}(z,t) = e^{i(\omega t - k'_{||}z)}A'_{||}(0,0) \tag{15}$$

$$a'(z,t) = e^{i(\omega t - k'_a z)}a'(0,0) \tag{16}$$

Here, $k'_{||} = \omega + \Delta'_{||}$, $k'_a = \omega + \Delta'_a$. We see that the term $e^{i\omega(t+z)}$ is common to all the three solutions - for computational convenience, we take this common phase out of all three and measure the phases relative to $\Delta_{||}$. This means we take out a total common phase of $e^{i(\omega t - \omega z - \Delta_{||}z)}$ from all three terms. Therefore, our solution stands as,

$$A_\perp(z) = e^{-i(\Delta_\perp - \Delta_{||})z}A_\perp(0) \tag{17}$$

$$A'_{||}(z) = e^{-i(\Delta'_{||} - \Delta_{||})z}A'_{||}(0) \tag{18}$$

$$a'(z) = e^{-i(\Delta'_a - \Delta_{||})z}a'(0) \tag{19}$$

Transforming back to the unprimed coordinates, we have the full solution,

$$\begin{bmatrix} A_{||}(z) \\ a(z) \end{bmatrix} = \underbrace{\begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}\begin{pmatrix} e^{-i(\Delta'_{||} - \Delta_{||})z} & 0 \\ 0 & e^{-i(\Delta'_a - \Delta_{||})z} \end{pmatrix}\begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}}_{\text{The mixing matrix: } \mathcal{M}(z)}\begin{bmatrix} A_{||}(0) \\ a(0) \end{bmatrix} \tag{20}$$

C. Transition probabilities

Now that we have calculated the mixing matrix given in (20), we first look at the axion-photon transition probabilities in two simple cases - weak mixing and strong mixing - given by the element $\mathcal{M}_{12}$. We also look at the probability for inhomogeneous fields and suchlike later.
We define the transition probability between ALPs and photons to be,

$$P(\gamma_{||} \to a) = |\mathcal{M}_{12}|^2 = \left|\frac{1}{2}\sin 2\phi(1 - e^{i(\Delta'_{||} - \Delta'_a)z})e^{-i(\Delta'_{||} - \Delta_{||})z}\right|^2$$

$$= \sin^2 2\phi \sin^2\left(\frac{\Delta'_{||} - \Delta'_a}{2} z\right) \tag{21}$$

Now,

$$\Delta'_{||} - \Delta'_a = \left(\frac{\Delta_{||} + \Delta_a}{2} + \frac{\Delta_{||} - \Delta_a}{2\cos 2\phi}\right) - \left(\frac{\Delta_{||} + \Delta_a}{2} - \frac{\Delta_{||} - \Delta_a}{2\cos 2\phi}\right) = \frac{\Delta_{||} - \Delta_a}{\cos 2\phi} \tag{22}$$

With $\cos 2\phi = \frac{\Delta_{||} - \Delta_a}{\sqrt{(\Delta_{||} - \Delta_a)^2 + (2\Delta_M)^2}}$, we define,

$$\Delta_{osc} = \Delta'_{||} - \Delta'_a = \sqrt{(\Delta_{||} - \Delta_a)^2 + (2\Delta_M)^2} \tag{23}$$

This gives us,

$$\boxed{P(\gamma_{||} \to a) = \sin^2 2\phi \sin^2\left(\frac{\Delta_{osc}}{2} z\right)} \tag{24}$$

1. Weak mixing

For the case of weak mixing, we assume the strength of the transitions, characterised by the mixing angle, ϕ, is small. This means, to $\mathcal{O}(\phi)$,

$$\frac{1}{2}\tan 2\phi \approx \phi$$

Or,

$$\mathcal{M}_{weak}(z) = \begin{pmatrix} 1 & -\phi \\ \phi & 1 \end{pmatrix} \begin{pmatrix} e^{-i(\Delta'_{||} - \Delta_{||})z} & 0 \\ 0 & e^{-i(\Delta'_a - \Delta_{||})z} \end{pmatrix} \begin{pmatrix} 1 & \phi \\ -\phi & 1 \end{pmatrix} \tag{25}$$

From eqn. (24), it is evident that,

$$P(\gamma_{||} \to a) = \sin^2 2\phi \sin^2\left(\frac{\Delta_{osc}}{2} z\right) \approx 4\phi^2 \sin^2\left(\frac{\Delta_{osc}}{2} z\right) \tag{26}$$

Now, for weak mixing,

$$\Delta'_{||} - \Delta'_a = \left(\frac{\Delta_{||} + \Delta_a}{2} + \frac{\Delta_{||} - \Delta_a}{2\cos 2\phi}\right) - \left(\frac{\Delta_{||} + \Delta_a}{2} - \frac{\Delta_{||} - \Delta_a}{2\cos 2\phi}\right) \approx \Delta_{||} - \Delta_a \equiv \Delta_{osc} \tag{27}$$

Therefore, we have the weak mixing axion-photon transition probability,

$$\boxed{P_{weak}(\gamma_{||} \to a) = 4\phi^2 \sin^2\left(\frac{\Delta_{osc}}{2} z\right)} \tag{28}$$

We define the oscillation length as,

$$l_{osc} = \frac{2\pi}{\Delta_{osc}} \tag{29}$$

If the distance is an integral multiple of this quantity, the transition probability vanishes.

2. Strong mixing

For strong mixing, we assume $\phi = \pi/4$ radians, i.e., $\tan 2\phi$ diverges. This can be achieved by $\Delta_{||} = \Delta_a$ (with $\Delta_{||} < 0$ since Δ_a is defined with a negative sign). In order to look at this case, first, we consider (11),

$$\frac{1}{2}\tan 2\phi = \frac{\cos 2\phi}{2\sin 2\phi} = \frac{\Delta_M}{\Delta_{osc}} \Rightarrow \frac{\Delta_{osc}}{2\cos 2\phi} = \frac{\Delta_M}{\sin 2\phi} \tag{30}$$

This means, for $\phi = \pi/4$ radians,

$$\Delta'_{||,a} \ = \ \Delta_{||} \ \pm \ \Delta_M \ = \ \Delta_a \ \pm \ \Delta_M \tag{31}$$

The mixing matrix in this case is given by,

$$\mathcal{M}_{strong}(z) \ = \ \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} e^{-i\Delta_M z} & 0 \\ 0 & e^{i\Delta_M z} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \tag{32}$$

This means that the corresponding transition probability is,

$$\boxed{P_{strong}(\gamma_{||} \to a) \ = \ \sin^2(\Delta_M z)} \tag{33}$$

D. Axion induced optical effects

1. *Axion induced phase shift*

In the absence of axions, the phase shift between the two components of the photon field (in the presence of the external magnetic field) is given by, $\phi = (\Delta_\perp - \Delta_{||})z = \phi_{QED} + \phi_{CM}$, where the subscript QED denotes the QED induced phase shift and CM denotes the phase shift due to the Cotton-Mouton effect (the splitting of the photon field in gases or liquids in the presence of an external magnetic field - an external magnetic field induced birefringence in gases or liquids). These phases are described as,

$$\phi_{QED}(z) \ = \ \frac{2\alpha^2 B_e^2}{15 m_e^4} z, \ \phi_{CM} \ = \ 2\pi C B_e^2 z \tag{34}$$

Here, C is the Cotton-Mouton constant. Both these phases exhibit a linear relationship in z and cause a change in the polarisation of the initial photon field - from linear to elliptical, for example, as in the references cited in [2].
In presence of axions, there may be a further phase shift unrelated to the two given above. This phase shift is defined as,

$$\phi_a(z) \ = \ -\mathrm{Im}(\mathcal{M}_{11}) \tag{35}$$

From the form of the mixing matrix given in (20), we have,

$$\mathcal{M}_{11}(z) \ = \ e^{-i(\Delta'_{||} - \Delta_{||})z} \cos^2\phi + e^{-i(\Delta'_a - \Delta_{||})z} \sin^2\phi \tag{36}$$

Now, up to $\mathcal{O}(\phi^2)$, we have,

$$\Delta'_{||,a} \ = \ \frac{\Delta_{||} + \Delta_a}{2} \pm \frac{\Delta_{osc}}{2\cos 2\phi} \ \approx \ \frac{\Delta_{||} + \Delta_a}{2} \pm \frac{\Delta_{osc}}{2}(1 + 2\phi^2) \tag{37}$$

Or,

$$\Delta'_{||} \ = \ \Delta_{||} + \Delta_{osc}\phi^2, \ \Delta'_a \ = \ \Delta_a - \Delta_{osc}\phi^2 \tag{38}$$

This means, to $\mathcal{O}(\phi^2)$,

$$e^{-i(\Delta'_{||} - \Delta_{||})z} \cos^2\phi \ \approx \ e^{-i\Delta_{osc}\phi^2 z}(1 - \phi^2)^2 \ \approx \ (1 - \phi^2) - i\phi^2\Delta_{osc} z \tag{39}$$

This is because $e^{i\theta} = \cos\theta + i\sin\theta$ and we further expand both the cosine and the sine to appropriate orders. Similarly,

$$e^{-i(\Delta'_a - \Delta_{||})z} \sin^2\phi \ \approx \ \phi^2 e^{-i(1+\phi^2)\Delta_{osc} z} \ \approx \ \phi^2 e^{i\Delta_{osc} z}(1 + i\phi^2\Delta_{osc}) \tag{40}$$

This means,

$$\mathcal{M}_{11} \ = \ (1 - \phi^2 - i\phi^2\Delta_{osc} z) + \phi^2 e^{i\Delta_{osc} z}(1 + \phi^2\Delta_{osc} z) \tag{41}$$

Or, the axion induced photon phase shift is given by,

$$\boxed{\phi_a(z) \ = \ -\mathrm{Im}(\mathcal{M}_{11}) \ = \ \phi^2(\Delta_{osc} z - \sin(\Delta_{osc} z))} \tag{42}$$

2. Rotation of the plane of polarization

In the previous section, we saw that the axion field can induce an additional phase shift into the photon field. However, the axion-photon coupling could have another consequence - the parallel component of the photon field will have its amplitude reduced by an amount proportional to $\mathrm{Re}(\mathcal{M}_{11})$ $(= 1 - \epsilon(z))$. A reduction in the amplitude of one of the components will cause the total photon amplitude vector to rotate - that is, the ellipticity of the polarization ellipse will change (this is caused by the previously discussed phase shift). Here, ϵ contributes to a further effect in which the plane of polarization of the photon rotates due to the axion-photon coupling.
Note that both these effects are second order in ϕ - these are beyond the linear order theory. This means that the order of magnitude estimates of these effects are quite low and it might be difficult to prove the existence of axions conclusively from these measurements (as we will see later).

E. Level crossing

In section III C 2, a condition we imposed for strong mixing to occur was $\Delta_a = \Delta_{||}$ and $\Delta_{||} < 0$. However, such a condition is not satisfied or true in most realistic physical scenarios - therefore, we consider the more common case where there is a gradient in $\Delta_{||}$. This means that we have a region that initially has $\Delta_{||} < \Delta_a$ which changes over distance to $\Delta_{||} = \Delta_a$ and further, $\Delta_{||} > \Delta_a$. We assume that in the region of validity of the strong mixing analysis, the change in $\Delta_{||}$ is very slow (adiabatic - the gradient must be slower than that of the oscillation frequency).
This implies the inequality,

$$\left|\frac{d\phi}{dz}\right| \ < \ \frac{2\pi}{l_{osc}} \ = \ 2\Delta_M \tag{43}$$

Now, we had,

$$\frac{1}{2}\tan 2\phi = \frac{\Delta_M}{\Delta_{osc}} \ \Rightarrow \ \frac{d\phi}{dz} = \frac{1}{4\Delta_M \sin^2 2\phi}\frac{d\Delta_{||}}{dz} \tag{44}$$

This is as $\frac{\Delta_{osc}}{2\cos 2\phi} = \frac{\Delta_M}{\sin 2\phi}$. For the strong mixing case, $\phi = \pi/4$ radians, and therefore, (43) takes the form,

$$\frac{d\phi}{dz} = \frac{1}{4\Delta_M}\frac{d\Delta_{||}}{dz} \ \Rightarrow \ \left|\frac{d\Delta_{||}}{dz}\right| = 8\Delta_M^2 \tag{45}$$

In astrophysical objects, we need to consider both the vacuum and gaseous contributions to $\Delta_{||}$ $(= \Delta_{||}^{vac} + \Delta_{||}^{g})$.
Here, we assume that,

$$\left|\frac{d\Delta_{||}^{g}}{dz}\right| \ >> \ \left|\frac{d\Delta_{||}^{vac}}{dz}\right| \tag{46}$$

Now, let us assume that this contribution is proportional to the local electron density, that is, $\Delta_{||}^{g} = \beta N_e$, where N_e is the local electron density and β is the constant of proportionality. We have no idea of this constant or its order of magnitude - therefore, we will rewrite the condition in (45) in a β-independent way. To this end, consider,

$$\Delta_{||} \approx \Delta_a \text{ (condition for level crossing) } \Rightarrow \ |\Delta_{||}^{g}| \approx |\Delta_{||}^{vac} - \Delta_a| \tag{47}$$

Now, for the solely vacuum weak mixing case, $\Delta_{||}^{vac} - \Delta_a = \Delta_M/\phi^{vac}$. Therefore,

$$|\Delta_{||}^{g}| \ = \ \beta N_e \ = \ \frac{\Delta_M}{\phi} \ \Rightarrow \ \beta = \frac{\Delta_M}{\phi^{vac} N_e} \tag{48}$$

This gives us, from (45),

$$\left|\frac{d\Delta_{||}^{g}}{dz}\right| = \beta\left|\frac{dN_e}{dz}\right| < 8\Delta_M^2 \tag{49}$$

Or,

$$\boxed{\left|\frac{d\ln N_e}{dz}\right| \ < \ 8\Delta_M\phi^{vac} \ = \ \frac{8\pi\phi^{vac}}{l_{osc}} \ = \ \frac{180\pi B_c^2}{7\alpha\omega M^2}} \tag{50}$$

IV. THE SINGLE DOMAIN MODEL FOR INHOMOGENEOUS FIELDS

Up to this point, we have considered only homogeneous fields. Now, we relax that assumption and consider fields which vary in the z-direction - note that the direction of the magnetic field does *not* change under this assumption and only the magnitude does. Therefore, as earlier, only the lower 2×2 part of the mixing matrix will contribute.
To do this, we exploit the Schrodinger-like nature of the linearized axion-photon mixing equations. We have,

$$i\frac{d|a(z)\rangle}{dz} \;=\; (H_0 + H_1)|a(z)\rangle \tag{51}$$

With,

$$H_0 = \omega + \begin{pmatrix} \Delta_\perp(z) & 0 & 0 \\ 0 & \Delta_{||}(z) & 0 \\ 0 & 0 & \Delta_a \end{pmatrix};\; H_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \Delta_M(z) \\ 0 & \Delta_M(z) & 0 \end{pmatrix} \tag{52}$$

The spatial variation in the magnetic field is carried through H_1. Now, for $M \to \infty$, $\Delta_M \to 0$, giving us the "unperturbed" solution,

$$|a(z)\rangle \;=\; U(z)|a(0)\rangle \;=\; \exp\left(-i\int_0^z H_0(\xi)d\xi\right)|a(0)\rangle \tag{53}$$

We move to the interaction picture by defining the interaction picture states and operator as,

$$|a\rangle_i \;=\; U^\dagger(z)|a\rangle;\; H_i \;=\; U^\dagger(z)H_1U(z) \tag{54}$$

Note that by definition, $|a(0)\rangle_i = |a(0)\rangle$. The interaction Schrodinger equation is simply $i\partial_t|\Psi\rangle_i = \mathcal{H}_i|\Psi\rangle_i$. Therefore, we have,

$$i\frac{d|a(z)\rangle_i}{dz} = H_i|a(z)\rangle_i \;\Rightarrow\; |a(z)\rangle_i = |a(0)\rangle - i\int_0^z dz' H_i(z')|a(0)\rangle + ... \tag{55}$$

In the first order (considering only the lower 2×2 part of $U(z)$ which contributes to the mixing), we have,

$$\int_0^z dz' H_i(z') \;=\; \int_0^z dz'\; U^\dagger(z')H_1(z')U(z')$$

$$= \int_0^z dz' \begin{pmatrix} e^{i\int_0^{z'} dz''\Delta_{||}(z'')} & 0 \\ 0 & e^{i\Delta_a z'} \end{pmatrix} H_1(z') \begin{pmatrix} e^{-i\int_0^{z'} dz''\Delta_{||}(z'')} & 0 \\ 0 & e^{-i\Delta_a z'} \end{pmatrix} \tag{56}$$

Now, we consider the off-diagonal term in the above matrix - only these terms will contribute to the mixing probability. Thus, we have,

$$\boxed{P(\gamma_{||} \to a) \;=\; \left|\int_0^z dz'\; \Delta_M(z')\exp\left(i\Delta_a z' - i\int_0^{z'} \Delta_{||}(z'')dz''\right)\right|^2} \tag{57}$$

As before, the axion induced phase shift occurs in a higher order. Expanding to second order, we get,

$$\boxed{\phi_a(z) \;=\; \mathrm{Im}\left(\int_0^z dz' \int_0^{z'} dz''\; \Delta_M(z')\Delta_M(z'')\exp\left(i\Delta_a(z'-z'') - i\int_{z'}^{z''} d\xi\Delta_{||}(\xi)\right)\right)} \tag{58}$$

A. Periodic fields

Let us consider spatially periodic magnetic fields, $B_e = B_0\cos(\Delta_0 z)$. The field strength is assumed sufficiently weak so that $|\Delta_a z'| > |\int_0^{z'} dz''\Delta_{||}(z'')|$ in eqns. (57) and (58). Consider the probability derived earlier,

$$P(\gamma_{||} \to a) = \left|\int_0^z dz'\; \frac{B_0\cos(\Delta_0 z')}{2M}e^{i\Delta_a z'}\right|^2 = \frac{1}{4}\Big(\frac{B_0}{2M}\Big)^2\left|\int_0^z dz'\Big(e^{i(\Delta_0+\Delta_a)z} + e^{i(\Delta_a-\Delta_0)z}\Big)\right|^2 \tag{59}$$

Here, we only take into account the potentially resonant part of the above expression (only the second term). Therefore,

$$P(\gamma_{||} \to a) = \frac{4B_0^2 \sin^2(\frac{\Delta_a - \Delta_0}{2} z)}{4(2M)^2(\Delta_a - \Delta_0)^2} = \left(\frac{B_0 z}{2M}\right)^2 g(\xi) \tag{60}$$

Where,

$$g(\xi) = \frac{1}{\xi^2} \sin^2(\xi/2);\ \xi = \frac{\Delta_a - \Delta_0}{2} z \tag{61}$$

We can extend this calculation to include the effects on the axion induced phase shift as well. From eqn. (58), we have (considering only potentially resonant terms),

$$\phi_a(z) = \mathrm{Im}\left(\left(\frac{B_0}{2M}\right)^2 \int_0^z dz' \int_0^{z'} dz'' \cos(\Delta_0 z') \cos(\Delta_0 z'') e^{i\Delta_a (z' - z'')}\right)$$

$$= \frac{1}{2}\left(\frac{B_0}{2M}\right)^2 \mathrm{Im}\left(\int_0^z dz' \cos(\Delta_0 z') e^{i\Delta_a z'} \frac{e^{i(\Delta_0 - \Delta_a)z'} - 1}{i(\Delta_0 - \Delta_a)}\right) \approx \frac{1}{4}\left(\frac{B_0 z}{2M}\right)^2 f(\xi) \tag{62}$$

Here,

$$f(\xi) \;=\; \frac{1}{\xi} - \frac{1}{\xi^2} \sin\xi;\ \xi \;=\; \frac{\Delta_a - \Delta_0}{2} z \tag{63}$$

From this, the resonant limit is evident,

$$\lim_{\xi \to 0}\ g(\xi) \;=\; \frac{1}{4};\ \lim_{\xi \to 0}\ f(\xi) \;\sim\; \frac{1}{6}\xi \tag{64}$$

Or, at resonance,

$$P(\gamma_{||} \to a) \;=\; \frac{1}{4}\left(\frac{B_0 z}{2M}\right)^2;\ \phi_a(z) \;\to\; 0 \;\Leftarrow\; \text{At resonance} \tag{65}$$

Therefore, in the limit $\Delta_0 \to \Delta_a$, we see an enhancement in the transition probability while the phase shift vanishes.

V. GENERALISATIONS OF THE SINGLE DOMAIN MODEL

A. Multi-domain conventional model

Up to this point, we have looked at external magnetic fields that might or might not vary in magnitude, but do not change their direction. This is clearly not a very general case, therefore, as a first step towards generalisation, we would like to consider the question of magnetic fields that do change their directions. For this we divide the total length traversed by the photon into N "domains" of coherence length, L - in each domain, the magnetic field direction remains unchanged, but when we move to the next domain, the direction changes. In the conventional model, we only consider a change in direction from domain to domain while the magnitude of the field remains fixed throughout all the domains. We follow the treatment in [5].

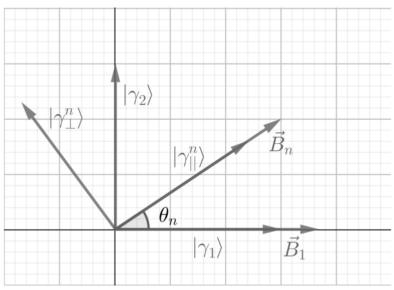

FIG. 1: Orientation of the n^{th} domain fields

The situation in the n^{th} domain with respect to the first domain is depicted in fig. 1. We have a magnetic field, $\vec{B}_1$, in the first domain and we orient our axes to ensure that the initial state is,

$$|\Psi\rangle_{initial} \equiv |\Psi(0)\rangle = c_1(0)|\gamma_1\rangle + c_2(0)|\gamma_2\rangle + c_a(0)|a\rangle \tag{66}$$

Here, $|\gamma_{1,2}\rangle$ are mutually perpendicular photon field states in domain 1. The initial intensities are,

$$I_\gamma(0) = |c_1(0)|^2 + |c_2(0)|^2, \; I_a(0) = |c_a(0)|^2 \tag{67}$$

In the n^{th} domain, we have,

$$\overbrace{|\Psi(\underbrace{(n-1)L}_{\equiv z_n})\rangle}^{\text{beginning of } n^{th} \text{ domain}} = c_n^{||}(z_n)|\gamma_{||}^n\rangle + c_n^{\perp}(z_n)|\gamma_\perp^n\rangle + c_n^a(z_n)|a\rangle \tag{68}$$

From the figure, it is clear that,

$$|\gamma_{||}^n\rangle = c_n|\gamma_1\rangle + s_n|\gamma_2\rangle, \; |\gamma_\perp^n\rangle = -s_n|\gamma_1\rangle + c_n|\gamma_2\rangle \tag{69}$$

Here, $c_n = \cos\theta_n, \; s_n = \sin\theta_n$. Therefore,

$$|\Psi(z_n)\rangle = c_1(z_n)|\gamma_1\rangle + c_2(z_n)|\gamma_2\rangle + c_a(z_n)|a\rangle \tag{70}$$

Thus,

$$c_n^{||}(z_n) = c_n c_1(z_n) + s_n c_2(z_n), \; c_n^{\perp}(z_n) = c_n c_2(z_n) - s_n c_1(z_n) \tag{71}$$

Now, the intensity at the end of the n^{th} domain will be,

$$I_\gamma(nL) = I_\gamma(z_{n+1}) = I_\gamma(z_n) + P_{\gamma a}(\overbrace{I_a(z_n)}^{a\to\gamma_{||}} - \underbrace{I_\gamma(z_n)}_{\gamma_{||}\to a}) \tag{72}$$

Here, $P_{\gamma a}$ denotes the transition probability in a single domain (calculated earlier). Now,

$$I_\gamma(z_n) = |c_1(z_n)|^2 + |c_2(z_n)|^2 = |c_n^{||}(z_n)|^2 + |c_n^{\perp}(z_n)|^2 \tag{73}$$

And,

$$I_{\gamma_{||}^n} = |c_n^{||}(z_n)|^2 = c_n^2|c_1(z_n)|^2 + s_n^2|c_2(z_n)|^2 + 2s_nc_n\mathrm{Re}(c_1^*(z_n)c_2(z_n)) \tag{74}$$

$$I_a(z_n) = |c_a(z_n)|^2 \tag{75}$$

Thus,

$$I_\gamma(z_n) = (1 - P_{\gamma a}c_n^2)|c_1(z_n)|^2 + (1 - P_{\gamma a}s_n^2)|c_2(z_n)|^2 + P_{\gamma a}|c_a(z_n)|^2 - 2P_{\gamma a}s_nc_n\mathrm{Re}(c_1^*(z_n)c_2(z_n)) + ... \tag{76}$$

Here, the ... part represents linear powers of the sines and cosines. Similarly, for the axion intensity, we have,

$$I_a(z_n) = (1 - P_{\gamma a})|c_a(z_n)|^2 + P_{\gamma a}(c_n^2|c_1(z_n)|^2 + s_n^2|c_2(z_n)|^2 + 2s_nc_n\mathrm{Re}(c_1^*(z_n)c_2(z_n))) + ... \tag{77}$$

The mean intensity in each domain can be calculated by noting that the mean of the sines/cosines is 0 while that of the squared terms is 1/2. Thus,

$$I_\gamma(z_{n+1}) = (1 - \frac{1}{2}P_{\gamma a})I_\gamma(z_n) + P_{\gamma a}I_a(z_n) \tag{78}$$

$$I_a(z_{n+1}) = (1 - P_{\gamma a})I_a(z_n) + \frac{1}{2}P_{\gamma a}I_\gamma(z_n) \tag{79}$$

Or,

$$\begin{pmatrix} I_\gamma \\ I_a \end{pmatrix}(z_{n+1}) = \begin{pmatrix} 1 - \frac{1}{2}P_{\gamma a} & P_{\gamma a} \\ \frac{1}{2}P_{\gamma a} & 1 - P_{\gamma a} \end{pmatrix}\begin{pmatrix} I_\gamma \\ I_a \end{pmatrix}(z_n) = \mathcal{W}\begin{pmatrix} I_\gamma \\ I_a \end{pmatrix}(z_n) = \mathcal{W}^n\begin{pmatrix} I_\gamma \\ I_a \end{pmatrix}(z_1) \tag{80}$$

Now,

$$\mathcal{W}^n = \frac{1}{3}\begin{pmatrix} 2+\alpha^n & 2(1-\alpha^n) \\ 1-\alpha^n & 1+2\alpha^n \end{pmatrix} \Rightarrow \lim_{n\to\inf} \mathcal{W}^n = \frac{1}{3}\begin{pmatrix} 2+e^{-\beta z_{n+1}} & 2(1-e^{-\beta z_{n+1}}) \\ 1-e^{-\beta z_{n+1}} & 1+2e^{-\beta z_{n+1}} \end{pmatrix} \tag{81}$$

Here, $\alpha = 1 - 3P_{\gamma a}/2$, and,

$$\lim_{n\to\infty} \alpha^n = \lim_{n\to\infty} (1 - \frac{3}{2}P_{\gamma a})^n = \lim_{n\to\infty} e^{\frac{3}{2}nP_{\gamma a}} = \lim_{n\to\infty} e^{\frac{3}{2}P_{\gamma a}(nL)/L} = \lim_{n\to\infty} e^{\frac{3}{2L}P_{\gamma a}z_{n+1}} \equiv e^{-\beta z} \tag{82}$$

Thus, $\beta = \frac{3P_{\gamma a}}{2L}$, and dropping the domain labels since we now are in a continuum,

$$\begin{pmatrix} I_\gamma \\ I_a \end{pmatrix}(z) \;=\; \frac{1}{3}\begin{pmatrix} 2+e^{-\beta z} & 2(1-e^{-\beta z}) \\ 1-e^{-\beta z} & 1+2e^{-\beta z} \end{pmatrix}\begin{pmatrix} I_\gamma \\ I_a \end{pmatrix}(0) \tag{83}$$

Or, redefining $P_{\gamma a}$ as $P_{\gamma a} = \frac{1}{3}(1 - e^{-\beta z})$,

$$\boxed{I_\gamma(z) \;=\; I_\gamma(0) - P_{\gamma a}\Big(I_\gamma(0) - 2I_a(0)\Big)}$$

$$\boxed{I_a(z) = I_a(0) + P_{\gamma a}\Big(I_\gamma(0) - 2I_a(0)\Big)} \tag{84}$$

Note the abuse of notation - $I_{\gamma,a}(z)$ is actually the mean value, $\langle I_{\gamma,a}(z)\rangle$. We see that for the weak mixing scenario with small Δ_{osc} (or, large l_{osc}), the single domain probability takes the form, $P(\gamma_{||} \to a) \approx \phi^2\Delta^2_{osc}L^2 = \Delta^2_M L^2$. The decay length is defined as,

$$L_{decay} \;=\; \frac{1}{\beta} \;=\; \frac{2L}{3P(\gamma_{||} \to a)} \xrightarrow{\text{weak mixing with large } l_{osc}} L_{decay} = \frac{2}{3\Delta^2_M L} \tag{85}$$

We see that as $z \to \infty$, $P_{\gamma a} = 1/3$ - therefore, on an average, a third of the initial photons get converted to axions.

B. The density matrix formalism

A useful thing about the linearised field eqns. (10) is its Schrodinger like nature - this implies most of the techniques used in non-relativistic quantum mechanics can be borrowed over after making superficial changes (like $t \to z$). This leads us to think about introducing a density matrix as,

$$\rho \;=\; |a\rangle \otimes \langle a| \;=\; \begin{pmatrix} A_\perp \\ A_{||} \\ a \end{pmatrix} \otimes \begin{pmatrix} A^*_\perp & A^*_{||} & a^* \end{pmatrix} \tag{86}$$

An alternative approach to constructing and looking at the conventional model is through the density matrix. Later, we shall also use it to look at the polarisation states of the involved photons through the Stokes parameters. Here we shall look into the conventional model again, following [6].

Consider a magnetic field, $\vec{B} \equiv (B_1, B_2, B_3)$. For propagation along the z-direction, we consider the components perpendicular to the propagation direction, $\vec{B}_\perp = \vec{B} - B_3\hat{z}$. The mixing matrix (in each domain) will depend upon the angle θ defined as $\cos\theta = \vec{B}_\perp \cdot \hat{x}$.

Consider the probability in each domain,

$$P_{\gamma a} = \sin^2 2\phi \sin^2\Big(\frac{\Delta_{osc}L}{2}\Big) \tag{87}$$

Here, in general,

$$\Delta_{osc} = \sqrt{(\Delta_a - \Delta_{||})^2 + 4\Delta^2_M} = 2\Delta_M\sqrt{1 + \Big(\frac{\omega_c}{\omega}\Big)^2}, \;\; \frac{\omega_c}{\omega} = \frac{\big|\Delta_a - \Delta_{||}\big|}{2\Delta_M} = |\sec 2\phi| \tag{88}$$

Here, $\Delta_M = |\vec{B}_\perp|/2M$. In case of dominant plasma effect, $\Delta_{||} = \Delta_\perp \approx \Delta_{pl} = -\omega_{pl}^2/2\omega$, where ω_{pl} is the plasma frequency, and,

$$\omega_c = \frac{|\Delta_a - \Delta_{||}|}{2\Delta_M}\omega = \frac{|m_a^2 - \omega_{pl}^2|}{4\Delta_M} \tag{89}$$

We specifically consider the high energy regime where $\omega >> \omega_c$ or $\Delta_{osc} \approx 2\Delta_M$ and maximal mixing takes place ($\phi \sim \pi/4$). Assuming $\Delta_M L << 1$, we have $P_{\gamma a} \approx \Delta_M L$. We neglect the diagonal contributions in the matrix in eqn. (9) and write,

$$\mathcal{H} = \Delta - iD \tag{90}$$

Here,

$$\Delta = \Delta_M \begin{pmatrix} 0 & 0 & \cos\theta \\ 0 & 0 & \sin\theta \\ \cos\theta & \sin\theta & 0 \end{pmatrix}, \; D = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{91}$$

Here, D is an absorption matrix with the absorption factor, Γ_γ. In every domain, the density matrix is now defined as,

$$\rho \;=\; \begin{pmatrix} A_1 \\ A_2 \\ a \end{pmatrix} \otimes \begin{pmatrix} A_1^* & A_2^* & a^* \end{pmatrix} \tag{92}$$

(The indices $1, 2$ denote directions)
Since the equation is Schrodinger like, we can define a "translation" operator, $e^{-iH_n L}$, that translates the states through every domain,

$$e^{-iH_n L}|a_n\rangle = |a_{n+1}\rangle \tag{93}$$

Thus,

$$\rho_n = |a_n\rangle \otimes \langle a_n| = e^{-iH_{n-1}L}\rho_{n-1}e^{iH_{n-1}^\dagger L} \tag{94}$$

Since there are infinite choices for θ in each domain, we take a mean and define the ensemble average of the density matrix as,

$$\bar{\rho}_{n+1} = \langle e^{-iH_n L}\rho_n e^{iH_n^\dagger L}\rangle_{1,\ldots,n} = \langle e^{-iH_n L} \ldots \underbrace{e^{-iH_1 L}\rho_1 e^{iH_1^\dagger L}}_{\bar{\rho}_1} \ldots e^{iH_n^\dagger L} = \langle e^{-iH_n L}\bar{\rho}_n e^{iH_n^\dagger L}\rangle_n \tag{95}$$

This is because the field in every domain depends only on that domain - in other words, we assume that the magnetic fields are uncorrelated in every domain. We expand the exponential to second order to get,

$$\bar{\rho}_{n+1} = \bar{\rho}_n - iL\Big(\langle H_n\rangle_n\bar{\rho}_n - \bar{\rho}_n\langle H_n^\dagger\rangle_n\Big) + L^2\langle H_n\bar{\rho}_n H_n^\dagger\rangle_n - \frac{1}{2}L^2\Big(\langle H_n^2\rangle_n\bar{\rho}_n + \bar{\rho}_n\langle H_n^{\dagger^2}\rangle_n\Big) \tag{96}$$

We have (assuming equipartition of the fields in all directions),

$$\langle\Delta_M^2\rangle_n = \frac{1}{4M}\langle\vec{B}_{\perp,n}^2\rangle_n = \frac{1}{4M}\langle\vec{B}_n^2 - B_{3n}^2\rangle_n = \frac{1}{4M}\Big(\underbrace{\frac{2}{3}\langle\vec{B}_n^2\rangle_n}_{\equiv B_{eff}^2}\Big) = \bar{\Delta}_M^2 \tag{97}$$

Thus,

$$\langle H_n\rangle_n = \underbrace{\langle\Delta_n\rangle_n}_{=0} - i\langle D_n\rangle_n = -iD_n \tag{98}$$

Further, since $\langle D_n\bar{\rho}_n\Delta_n\rangle_n = \langle\Delta_n\bar{\rho}_n D_n\rangle_n = 0$,

$$\langle H_n\bar{\rho}_n H_n\rangle_n = \langle\Delta_n\bar{\rho}_n\Delta_n\rangle_n = \frac{1}{2}\bar{\Delta}_M^2\begin{pmatrix} \rho_{aa} & 0 & \rho_{a1} \\ 0 & \rho_{aa} & \rho_{a2} \\ \rho_{1a} & \rho_{2a} & \rho_{11} + \rho_{22} \end{pmatrix} \tag{99}$$

Similarly,

$$\langle \Delta_n^2 \rangle_n = \bar{\Delta}_M^2 \begin{pmatrix} \cos^2\theta_n & \sin\theta_n\cos\theta_n & 0 \\ \sin\theta_n\cos\theta_n & \sin^2\theta_n & 0 \\ 0 & 0 & 1 \end{pmatrix} = \bar{\Delta}_M^2 \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{100}$$

Combining it all together, we finally get,

$$\bar{\rho}_{n+1} - \bar{\rho}_n = -L\{D_n, \bar{\rho}_n\} + L^2\langle \Delta_n \bar{\rho}_n \Delta_n \rangle_n - \frac{1}{2}L^2\{\langle \Delta_n^2 \rangle_n, \bar{\rho}_n\} \tag{101}$$

Define,

$$P = \bar{\Delta}_M^2 L^2,\ \alpha = \Gamma_\gamma L/P_{\gamma a},\ \mu = \alpha + 1/2,\ \nu = \alpha/2 + 3/4 \tag{102}$$

Taking the limit $n \to \infty$ and writing $\bar{\rho}_{n+1} - \bar{\rho}_n = L\partial_z \bar{\rho}(z)$, we have,

$$\boxed{\partial_z \bar{\rho}(z) \ = \ \frac{P}{L} \begin{pmatrix} \frac{1}{2}\bar{\rho}_{aa} - \mu\bar{\rho}_{11} & -\bar{\rho}_{12} & \frac{1}{2}\bar{\rho}_{a1} - \nu\bar{\rho}_{1a} \\ -\mu\bar{\rho}_{21} & \frac{1}{2}\bar{\rho}_{aa} - \mu\bar{\rho}_{22} & \frac{1}{2}\bar{\rho}_{a2} - \nu\bar{\rho}_{2a} \\ \frac{1}{2}\bar{\rho}_{1a} - \nu\bar{\rho}_{a1} & \frac{1}{2}\bar{\rho}_{2a} - \nu\bar{\rho}_{a2} & \frac{1}{2}(\bar{\rho}_{11} + \bar{\rho}_{22}) - \bar{\rho}_{aa} \end{pmatrix}} \tag{103}$$

With $T_\gamma = \bar{\rho}_{11} + \bar{\rho}_{22},\ T_a = \bar{\rho}_{aa}$, this gives us

$$\partial_z \begin{pmatrix} T_\gamma \\ T_a \end{pmatrix} = \frac{P}{L} \begin{pmatrix} -\mu & 1 \\ 1/2 & -1 \end{pmatrix} \begin{pmatrix} T_\gamma \\ T_a \end{pmatrix} \tag{104}$$

Or, with $\xi = Pz/L$,

$$\partial_\xi T_\gamma = -\mu T_\gamma + T_a,\ \partial_\xi T_a = \frac{1}{2}T_\gamma - T_a \tag{105}$$

Combining these equations,

$$\partial_\xi^2 T_\gamma + \Big(\alpha + \frac{3}{2}\Big)\partial_\xi T_\gamma + (\alpha + \partial_\xi \alpha)T_\gamma = 0 \tag{106}$$

For a constant α and with the initial conditions as earlier ($T_\gamma(0) = 1,\ T_a(0) = 0$), we get the solution ($k = \sqrt{\nu^2 - \alpha}$),

$$T_\gamma(\xi) = e^{-\nu\xi}\Big(\cosh(k\xi) + \frac{1-2\alpha}{4k}\sinh(k\xi)\Big) = \begin{cases} \frac{2}{3} + \frac{1}{3}e^{-\frac{3}{2}\xi}, & \text{no absorption: } \alpha = 0 \\ \frac{1}{2\alpha^2}e^{-\xi}, & \text{high absorption: } \alpha >> 1 \end{cases} \tag{107}$$

This matches with what we derived earlier.

C. Helical model - single and multi-domain

The helical model (as in [7]) is an extension of the previously discussed multi-domain model that incorporates an inhomogeneous magnetic field in every domain. We consider in every domain a field of the form,

$$\vec{B}_n = B\Big(\cos(\theta_n z)\hat{x} + \sin(\theta_n z)\hat{y}\Big) \tag{108}$$

Here, θ_n is randomly selected in every domain and the field evolves in each domain separately while keeping its total magnitude fixed. The field equations in every domain (in the linearized sense) take the form,

$$i\partial_z \begin{pmatrix} A_x \\ A_y \\ a \end{pmatrix} \equiv i\partial_z|a'\rangle = \begin{pmatrix} \Delta_{xx} & \Delta_{xy} & \Delta_M\cos(\theta_n z) \\ \Delta_{yx} & \Delta_{yy} & \Delta_M\sin(\theta_n z) \\ \Delta_M\cos(\theta_n z) & \Delta_M\sin(\theta_n z) & \Delta_a \end{pmatrix}|a'\rangle \tag{109}$$

Here,

$$\Delta_M = \frac{B}{2M},\ \Delta_a = -\frac{m_a^2}{2\omega},\ \Delta_{xy} = \Delta_{yx} = (\Delta_{||} - \Delta_\perp)\sin(\theta_n z)\cos(\theta_n z)$$

$$\Delta_{xx} = \Delta_{||}\cos^2(\theta_n z) + \Delta_\perp \sin^2(\theta_n z),\ \Delta_{yy} = \Delta_{||}\sin^2(\theta_n z) + \Delta_\perp \cos^2(\theta_n z) \tag{110}$$

And,

$$|a'\rangle = \begin{pmatrix} A_x \\ A_y \\ a \end{pmatrix} e^{-i\omega z} \tag{111}$$

Now, eqn. (109) is difficult to diagonalise in general - a general closed form expression for the various quantities does not occur so easily due to the "random" nature of θ_n. In this case, we resort to numerical simulations of the equation to compare with the analytic expression in (84).
We choose the coherent length of each domain to be 1 $[Mpc]$ which is a reasonable choice for IGMF (intergalactic magnetic fields). The magnitude of the field is chosen to be 1 $[nG]$, which is an upper limit on the IGMF as calculated from observations of the Faraday rotation effect. We choose θ_n randomly in each domain in $[-\pi, \pi]$. The values of the rest of the quantities are mentioned in each plot - we consider different parameters in order to compare the results obtained here with the ones given in [7]. The simulation and plotting is performed on Julia with a Jupyterlab front. The results are shown in Fig. 2 for a total of 5000 domains.

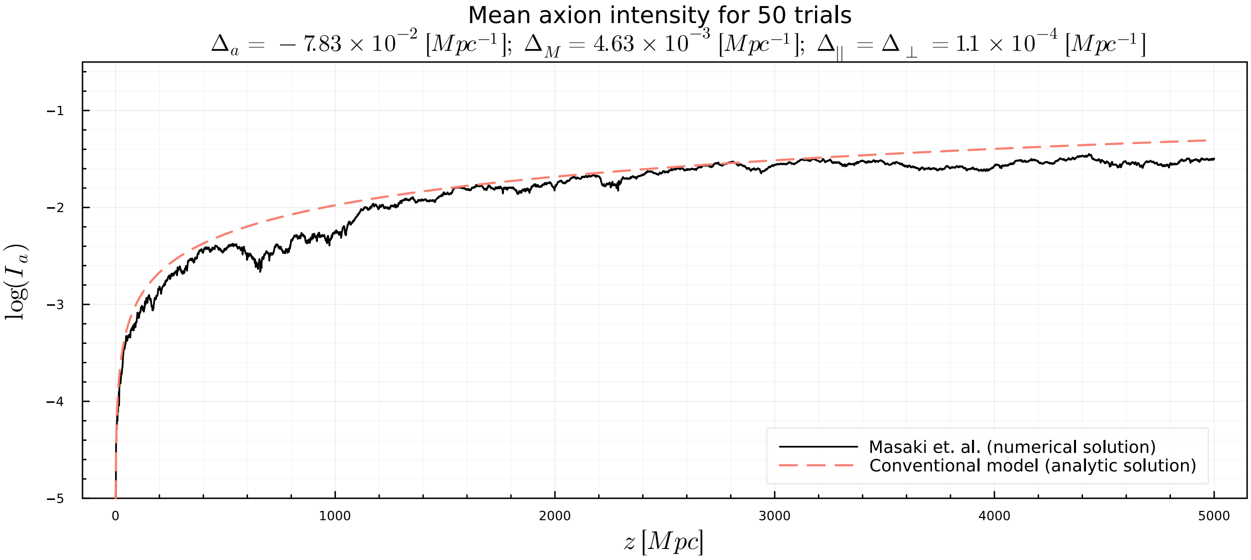

FIG. 2: The variation of the mean axion intensity in the multi-domain conventional model and the helical model

As we can see here, the helical model more or less mimics the previous model with no significant deviations - the analytical expression obtained in eqn. (84) makes for a strict envelope. The axion intensity in both cases is quite low, however. The plot given here matches the one in [7] quite well.

VI. EXPERIMENTAL OBSERVATIONS IN ASTROPHYSICAL AND OTHER CONTEXTS

A. Multiple beam path experiment

As discussed in [2], an easy way to search for axions is to reflect a beam of light multiple times between two mirrors with a background magnetic field. This would lead to an accumulation of the axion induced effects over a large distance which might be detectable. As pointed out by the authors, however, such an idea is futile because axions are not reflected by ordinary mirrors and therefore, every time a reflection occurs, the axion component goes to 0. Therefore, if we assume N reflections with a distance l between the mirrors, then the total axion-induced phase is $\phi_a^{total} = N\phi_a(l) \neq \phi_a(Nl)$ (since $\phi_a(z)$ is not linear in z). Note that this is in contrast to the QED induced phase shift which is, on the other hand, linear.
For this case, we have with $Nl = L$,

$$\phi_{QED}^{total} = \frac{2\alpha^2 B^2}{15 m_e^4}\omega L \tag{112}$$

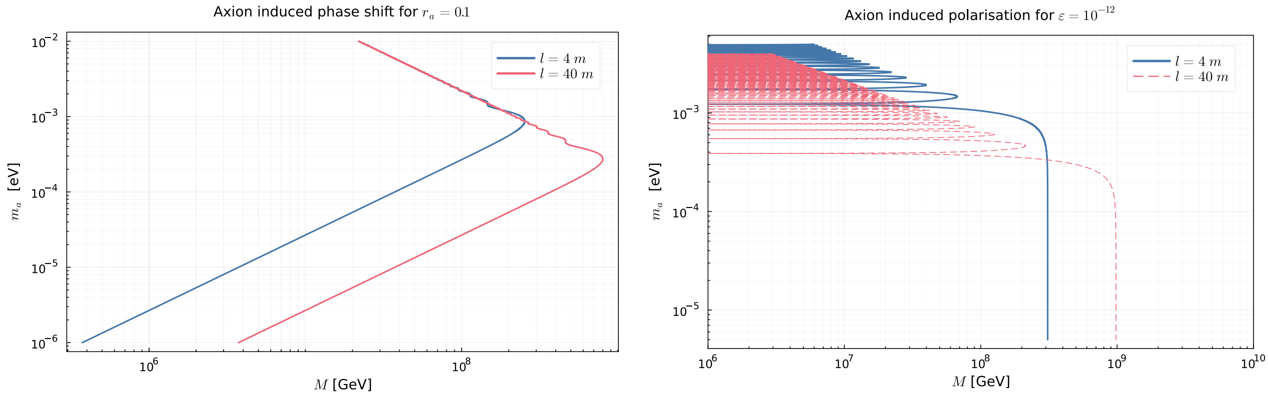

FIG. 3: Constraining the parameter region in the (M, m_a) plane with optical effects

$$\phi_a^{total} = N\phi_a(l) = N\phi^2(\Delta_{osc}l - \sin\Delta_{osc}l) = \Delta_{osc}L\phi^2\Big(1 - \frac{\sin\Delta_{osc}l}{\Delta_{osc}l}\Big) \tag{113}$$

$$\epsilon^{total} = 2N\phi^2\sin^2\frac{\Delta_{osc}l}{2} \tag{114}$$

In the case of small axion masses ($\Delta_{osc}l << 1$) and negligible contributions from the QED effects, we may write the above equations as,

$$\phi_a^{total} = \Delta_{osc}L\phi^2\Big(1 - \frac{\sin\Delta_{osc}l}{\Delta_{osc}l}\Big) = N\Big(\frac{\Delta_M}{\Delta_{osc}}\Big)^2\frac{\Delta_{osc}^3 l^3}{6} = \frac{Nl^3}{48\omega}\Big(\frac{Bm_a}{M}\Big)^2 \tag{115}$$

And, similarly,

$$\epsilon^{total} = \frac{N}{8}\Big(\frac{Bl}{M}\Big)^2 \tag{116}$$

The first thing we note from these equations is that in the massless case (applicable for a particle like the graviton), both the effects vanish making these particles undetectable with these methods. For arbitrary $\Delta_{osc}l$, however, and for $\phi_a^{total} > \phi_{QED}^{total}$, we can define the ratio between them as,

$$r_a = \frac{\phi_a^{total}}{\phi_{QED}^{total}} = \frac{15m_e^4}{(2\alpha M m_a)^2}\Big(1 - \frac{2\omega}{m_a^2 l}\sin\frac{m_a^2 l}{2\omega}\Big) \tag{117}$$

This ratio *does not* depend on either N or L, but only on the distance between the two mirrors, l. In part, this is a problem, since the idea of accumulating the effect through multiple reflections is not as feasible. The authors in [2] state that even $r_a > 0.1$ can be measured with accuracy - this corresponds to a $\phi_a^{total} \sim 10^{-12}$. For a laboratory field of 10^5 $[G]$ and photons with $\omega = 2.4$ $[eV]$, we can plot the above equation with $r_a = 0.1$ in order to constrain the parameter set in the (m_a, M) plane. This has been plotted in Fig. 3 - the plot on the right is similar to the one on the left, except that it imposes the constraint that $\epsilon^{total} \sim 10^{-12}$. The plots, as we see, depict large axion mass values on the y axis - this typically does not happen for axions (for example, in [7], the constraint is imposed for $m_a < 10^{-14}$ $[eV]$ which is quite a narrow range).
As we see from the expression for r_a, the maximum value it can take occurs for $m_a^2 = 2\pi\omega/l$ (since the argument of the sine term cannot be negative in this case) - this typically gives us values of l in the $[km]$ range if we want to look at $M > 10^{10}$ $[GeV]$ (a current bound). This is the case for the rotation of the plane of polarisation as well - as we can see from this discussion, due to the huge distance between the mirrors, its not a very feasible technique to look for axions. In the following section, we look at another way to impose a constraint on axions - using Stokes parameters as a possible technique to estimate the circular polarisation caused by the mixing effect.

B. Stokes parameters in different models

A brief introduction to Stokes parameters in given in appendix A in which the general expressions for the parameters are derived. Here, we take a look at these parameters in the single domain model, multi-domain conventional model and the helical model following [7]. In the previous section, we dealt with the total phase shift induced by the axion/photon oscillation mechanism - Stokes parameters allow us to characterise parts of this effect by providing a simple way to estimate the degrees of linear and circular polarisation that we expect as a result of this phenomenon.

1. *Single domain model*

In general, the time evolution of the fields in the single domain model is given by (see [4], [7]),

$$a(z) = (\cos^2\phi e^{-i\lambda_+ z} + \sin^2\phi e^{-i\lambda_- z})a(0) + \cos\phi\sin\phi(e^{-i\lambda_+ z} - e^{-i\lambda_- z})A_{||}(0)$$

$$A_{||}(z) = (\sin^2\phi e^{-i\lambda_+ z} + \cos^2\phi e^{-i\lambda_- z})A_{||}(0) + \cos\phi\sin\phi(e^{-i\lambda_+ z} - e^{-i\lambda_- z})a(0)$$

$$A_\perp(z) = A_\perp(0) \tag{118}$$

Here, $\lambda_\pm = \frac{1}{2}(\Delta_a \pm \Delta_{osc})$ are the eigenvalues of the mixing matrix (2×2 ;pwer block). Note that the solutions here are redefined with respect to a phase factor of $e^{i(kz-\omega t)}$. We further impose another phase factor of $e^{-i\Delta_{pl}z}$ assuming that in the case under consideration the plasma contribution is much higher than the QED/CM contribution and $\Delta_{||} = \Delta_\perp = \Delta_{pl}$. This does not affect the form of the Stokes parameters (they are scalar products), but this changes the mixing matrix:

$$\begin{pmatrix} \Delta_\perp & 0 & 0 \\ 0 & \Delta_{||} & \Delta_M \\ 0 & \Delta_M & \Delta_a \end{pmatrix} \Longrightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \Delta_M \\ 0 & \Delta_M & \Delta_a - \Delta_{pl} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \Delta_M \\ 0 & \Delta_M & \Delta_a' \end{pmatrix} \tag{119}$$

From here on, we drop the prime on Δ_a (choose $\Delta_{pl} = 0$ and substitute $\Delta_a \to \Delta_a - \Delta_{pl}$ when required). Let us consider the case where $a(0) = 0,\ V(0) = 0$ for convenience. Then,

$$|A_{||}(z)|^2 = A_{||}^2(0)\Big(1 - \frac{1}{2}\sin^2 2\phi(1 - \cos(\Delta_{osc}z))\Big) \tag{120}$$

This gives us,

$$I_\gamma(z) = A_\perp^2(0) + A_{||}^2(0)\Big(1 - \frac{1}{2}\sin^2 2\phi(1 - \cos(\Delta_{osc}z))\Big)$$

$$= \ I_\gamma(0) - (I_\gamma(0) - Q(0))\Big(\frac{\Delta_M}{\Delta_{osc}}\Big)^2(1 - \cos(\Delta_{osc}z)) \tag{121}$$

Similarly,

$$Q(z) = |A_{||}(z)|^2 - |A_\perp(z)|^2 = Q(0) - \frac{1}{2}\sin^2 2\phi(1 - \cos(\Delta_{osc}z))A_{||}^2(0)$$

$$= \ Q(0) - (I_\gamma(0) + Q(0))\Big(\frac{\Delta_M}{\Delta_{osc}}\Big)^2(1 - \cos(\Delta_{osc}z)) \tag{122}$$

Further,

$$U(z) = A_{||}(0)A_\perp(0)\Big(\sin^2\phi(e^{-i\lambda_+ z} + e^{-i\lambda_+ z}) + \cos^2\phi(e^{-i\lambda_- z} + e^{-i\lambda_- z})\Big)$$

$$= \ 2A_{||}(0)A_\perp(0)\Big(\sin^2\phi\cos(\lambda_+ z) + \cos^2\phi\cos(\lambda_- z)\Big) \tag{123}$$

We can write this in a form that matches [7]. We have, by definition, $\lambda_+\lambda_- = \Delta_a^2 - \Delta_{osc}^2$. Now,

$$\cos 2\phi = \cos^2\phi - \sin^2\phi = \frac{\Delta_a}{\Delta_{osc}} \xrightarrow{\cos^2\phi + \sin^2\phi = 1} \cos^2\phi = \frac{\Delta_{osc} + \Delta_a}{2\Delta_{osc}} = -\frac{2\Delta_M^2}{\Delta_{osc}\lambda_-} \tag{124}$$

Similarly, $\sin^2\phi = \frac{2\Delta_M^2}{\Delta_{osc}\lambda_+}$. Therefore, we have,

$$U(z) \;=\; U(0)\frac{2\Delta_M^2}{\Delta_{osc}}\Big(\frac{\cos(\lambda_+ z)}{\lambda_+} - \frac{\cos(\lambda_- z)}{\lambda_-}\Big) \tag{125}$$

And,

$$V(z) \;=\; U(0)\frac{2\Delta_M^2}{\Delta_{osc}}\Big(\frac{\sin(\lambda_+ z)}{\lambda_+} - \frac{\sin(\lambda_- z)}{\lambda_-}\Big) \tag{126}$$

This shows us that, in general, even in the simplest case of a single domain model, we can expect a non trivial value for the degree of circular polarization. As we shall see, this idea is extended to the other two models as well.

2. *Multi-domain conventional model*

In case of the conventional model, as in [5] and in section V A, we have already looked at the first Stokes parameter, the unobstructed intensity of the light, $I_\gamma(z)$. Here, we focus on the other three.
We can derive the rest of the parameters in the same manner as we did the intensity - or, we could use the previously derived results from the density matrix formalism. For the sake of completeness, here we derive $\langle Q(z)\rangle$ in the same manner we derived the mean intensity in this model.
Generalising the equations from the previous section (see [7]), we have for the n^{th} domain,

$$A^n_{||}(z_n) \;=\; (\sin^2\phi e^{-i\lambda_+ L} + \cos^2\phi e^{-i\lambda_- L})\Big[c_n A^{n-1}_{||}(z_{n-1}) + s_n A^{n-1}_{\perp}(z_{n-1})\Big]$$

$$+\; \cos\phi\sin\phi(e^{-i\lambda_+ L} - e^{-i\lambda_- L})a(z_{n-1})$$

$$A^n_{\perp}(z_n) \;=\; -s_n A^{n-1}_{||}(z_{n-1}) + c_n A^{n-1}_{\perp}(z_{n-1}) \tag{127}$$

Therefore,

$$\langle Q(z_n)\rangle \;=\; \langle A^{n-1}_{||}(z_{n-1})A^{n-1^*}_{||}(z_{n-1}) - A^{n-1}_{\perp}(z_{n-1})A^{n-1^*}_{\perp}(z_{n-1})\rangle \tag{128}$$

We retain only the squares of the sines and cosines - taking the mean sets the linear powers to 0 anyhow. We have,

$$\langle Q(z_n)\rangle \;=\; \langle P_{\gamma a}|a(z_{n-1})|^2 - P_{\gamma a}\Big(c_n^2|A^{n-1}_{||}(z_{n-1})|^2 + s_n^2|A^{n-1}_{\perp}(z_{n-1})|^2\Big)\rangle$$

$$=\; P_{\gamma a}|a(z_{n-1})|^2 - \frac{1}{2}P_{\gamma a}\Big(|A^{n-1}_{||}(z_{n-1})|^2 + |A^{n-1}_{\perp}(z_{n-1})|^2\Big) \;=\; P_{\gamma a}I_a(z_{n-1}) - \frac{1}{2}P_{\gamma a}I_\gamma(z_{n-1}) \tag{129}$$

Or,

$$\langle Q(z_n)\rangle \;=\; P_{\gamma a}\begin{pmatrix}-\frac{1}{2} & 1\end{pmatrix}\underbrace{\begin{pmatrix}I_\gamma\\ I_a\end{pmatrix}(z_{n-1})}_{=\mathcal{W}^{n-1}\begin{pmatrix}I_\gamma\\ I_a\end{pmatrix}(z_1)} \tag{130}$$

Substituting our previous results from (83),

$$\langle Q(z)\rangle = \frac{1}{3}P_{\gamma a}\begin{pmatrix}-\frac{1}{2} & 1\end{pmatrix}\begin{pmatrix}2+e^{-\beta z} & 2(1-e^{-\beta z})\\ 1-e^{-\beta z} & 1+2e^{-\beta z}\end{pmatrix}\begin{pmatrix}I_\gamma\\ I_a\end{pmatrix}(0) \tag{131}$$

Thus, we finally get,

$$\boxed{\langle Q(z)\rangle \;=\; P_{\gamma a}e^{-\beta z}\Big(I_a(0) - \frac{1}{2}I_\gamma(0)\Big)} \tag{132}$$

Note that $P_{\gamma a}$ is the single domain transition probability here - not the redefined quantity used in eqn. (84). As we can see, the mean value in this case, (i.e. for $\lim z \to \infty$), the parameter vanishes, $\langle Q(z)\rangle \to 0$.
The same result can be obtained from the density matrix formalism introduced in section V B. Consider eqn. (103). We have,

$$\partial_z\Big(\bar{\rho}_{11} - \bar{\rho}_{22}\Big) = \partial_z\langle U(z)\rangle = -\frac{\bar{P}_{\gamma a}}{2L}\langle U(z)\rangle \;\Rightarrow\; \langle U(z)\rangle \sim e^{-\beta z} \xrightarrow{\lim z\to\infty} 0 \tag{133}$$

And,

$$i\partial_z\Big(\bar{\rho}_{12} - \bar{\rho}_{21}\Big) = \partial_z\langle V(z)\rangle = -\frac{\bar{P}_{\gamma a}}{2L}\langle V(z)\rangle \;\Rightarrow\; \langle V(z)\rangle \sim e^{-\beta z} \xrightarrow{\lim z\to\infty} 0 \tag{134}$$

Here, $\bar{P}_{\gamma a}$ refers to the relevant quantity defined in that section. Thus, the asymptotic means of the rest of the parameters vanish. Next, we calculate the mean squared values, It is easiest to do this in the density matrix formulation. Since the original linearised equations are Schrodinger-like, we make the following analogy,

$$i\hbar\partial_t|\Psi\rangle = \hat{\mathcal{H}}|\Psi\rangle \;\Rightarrow\; i\hbar\partial_t\rho = [\hat{\mathcal{H}},\rho] \;\to\; i\partial_z|a\rangle = \mathcal{M}|a\rangle \;\Rightarrow\; i\partial_z\rho = [\mathcal{M},\rho] \tag{135}$$

The formal solution for ρ is,

$$\rho(z) \;=\; T(z,z_0)\rho(z_0)T^\dagger(z,z_0) \tag{136}$$

Here, $T(z,z_0)$ is the unitary *transfer function* (which replaces the traditional time evolution operator in non-relativistic quantum mechanics). Due to the unitarity,

$$\rho^2(z) = T(z,z_0)\rho(z_0)\underbrace{T^\dagger(z,z_0)T(z,z_0)}_{=\,\mathbb{I}}\rho(z_0)T^\dagger(z,z_0) = T(z,z_0)\rho^2(z_0)T^\dagger(z,z_0) \tag{137}$$

Therefore, ρ^2 and ρ have the same transfer functions - the limiting value in both cases must be the same. Now, in terms of the Stokes parameters, we can write,

$$\rho(z) \;=\; \begin{pmatrix} \frac{I(z)-Q(z)}{2} & \frac{U(z)+iV(z)}{2} & \frac{M(z)+iN(z)}{2} \\ \frac{U(z)-iV(z)}{2} & \frac{I(z)+Q(z)}{2} & \frac{K(z)+iL(z)}{2} \\ \frac{M(z)-iN(z)}{2} & \frac{K(z)-iL(z)}{2} & I_a(z) \end{pmatrix} \tag{138}$$

$$\rho^2(z) \;=\; \begin{pmatrix} \frac{M^2+N^2+U^2+V^2+(I_\gamma-Q)^2}{4} & - & - \\ - & \frac{K^2+L^2+U^2+V^2+(I_\gamma+Q)^2}{4} & - \\ - & - & I_a^2+\frac{K^2+L^2+M^2+N^2}{4} \end{pmatrix} \tag{139}$$

Here, we have defined $2aA^*_{||} = K+iL$, $2aA^*_{\perp} = M+iN$. We assume that the values are statistically independent and have 0 correlation; this means that the cross terms (which will contain terms like $\langle IQ\rangle = 0$) do not contribute. Further, we assume an equipartitiion of values for all the quantities except I_γ^2. Then, we have in the limit $z\to\infty$,

$$I_a^2 + \frac{K^2+L^2+M^2+N^2}{4} = \frac{1}{3},\; \frac{I_\gamma^2+Q^2+U^2+V^2}{2} + \frac{M^2+N^2+K^2+L^2}{4} = \frac{2}{3},$$

$$\frac{K^2+L^2-M^2-N^2}{4} \;=\; 0 \tag{140}$$

This gives us,

$$\langle I_a^2\rangle = \frac{1}{6} = \langle Q^2\rangle = \langle U^2\rangle = \langle V^2\rangle,\; \langle I_\gamma^2\rangle = \frac{1}{2} \tag{141}$$

Note that these are asymptotic values only. Further, we can calculate the asymptotic values of the degrees of circular polarization and linear polarization in this model as,

$$\boxed{\Pi_C = \sqrt{\frac{\langle V^2\rangle}{\langle I_\gamma^2\rangle}} = \frac{1}{\sqrt{3}} \approx 0.577,\; \Pi_L = \sqrt{\frac{\langle Q^2+U^2\rangle}{\langle I_\gamma^2\rangle}} \approx 0.816} \tag{142}$$

What this means is that for the conventional model, we expect a sizeable amount of circular polarization when these photon beams traverse cosmological distances and reach the earth - provided that axion/photon mixing occurs as discussed here. This provides us with a very nice way to detect possible axion activity through astrophysical phenomena.

3. Helical model

Finally, we move on to the helical model. As noted before, this model is challenging to tackle analytically - therefore, we numerically simulate the results. The conditions used here are the same as the ones used in section V C. We start with $A_{||}(0) = A_{\perp}(0) = 1/\sqrt{2},\ a(0) = 0$.

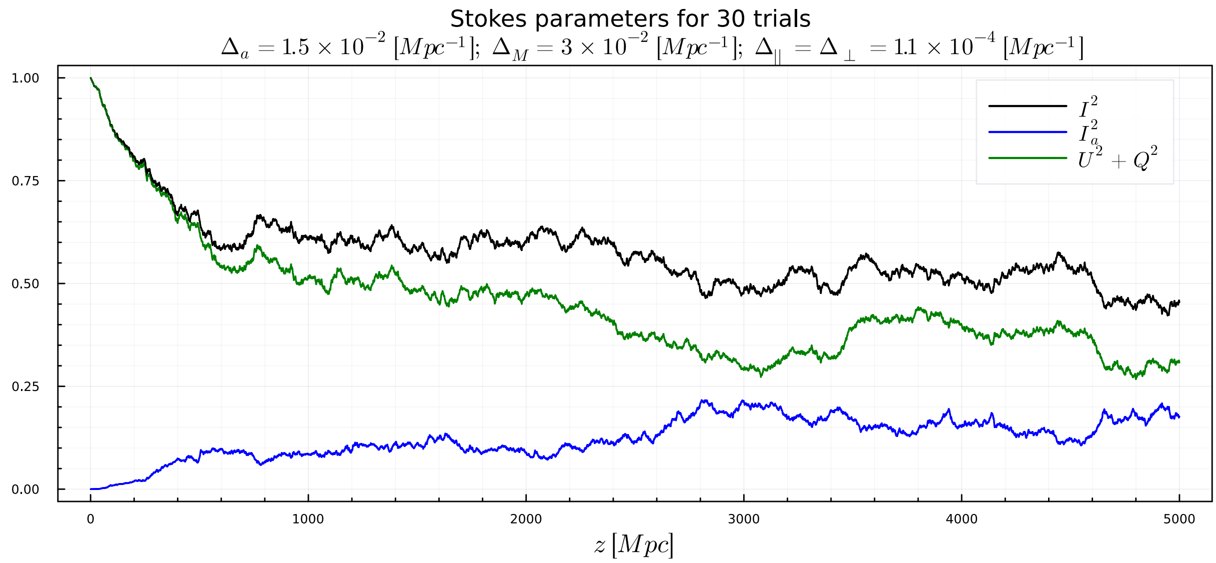

FIG. 4: Evolution of the Stokes parameters in the helical model

As we can see, the values seem to have a well-defined asymptote - in fact, in this case (as can be verified from the figure), these are the *exact* same values that we analytically obtained in the previous case for the multi-domain conventional model. This might seem surprising - but in the light of the discussion following Fig. 2, this might be expected. We have seen that the numerical results for the helical model were a milder approximation of the analytical expression for the conventional model in case of the axion intensity; in a similar fashion, we see the same result here - the results for both match (or, at the very least, are similar). Note however, that this occurs only for a specific range of parameter values - in [7], the authors call these values the "transient" values while dismissing the values that do not match with the asymptotic values from the conventional model as being on the extremes.
Note that only three of the Stokes parameters have been plotted here. As mentioned in the appendix, the four parameters are not completely independent of each other - only three are. The fourth one will be plotted later.

4. Constraining the parameter set with the Stokes parameters

Now that we have looked in detail at the Stokes parameters and their asymtotic values, the question should be why do we have to go to such lengths? One advantage of these parameters is the ease of measurement and another is the ease with which one can characterise the degree of polarization in a given light beam.
In Fig. 5, we have plotted the degree of circular polarization for the helical model - note that we use slightly restrictive conditions on the random angle as mentioned in the figure. From the discussions in case of the single domain and conventional models and from the figure here, it is evident that all three models have a common feature - the development of circular polarization in an initially unpolarized or linearly polarized wave due to the axion/photon mixing scenario. As such, this is a definitely strong observational effect and can be used to look for axions from photon signals generated in astrophysical contexts.
We have several such observations from astrophysics (see the references quoted in [7] for an overview). In particular, here we mention one instance. In a study, Hutsemekers *et. al.* showed that there is an unexpectedly high level of alignment of distant quasar polarizations - too high to be a statistical fluke (see [8]). In a later study, [9] showed that such quasar alignment can be explained by the axion/photon conversion in a multi-domain conventional model with correlated fields (i.e. the fields in the different domains are not entirely random but have a non vanishing correlation). However, as expected from the discussions above, such a mixing phenomena also gives rise to a circular polarization. In another study, however, [10] had shown that such a circular polarization does not exist in case of the quasars which imposes a significant bound on the coupling strength.

For our purposes, we take the message that the observed circular polarization from astrophysical sources is too small. Therefore, we take an upper bound on the degree of circular polarization to be 10^{-2} - this corresponds to the asymptote reached for $|\alpha| < \pi/180$ in Fig. 5. We choose standard values for the rest of the parameters (as mentioned in the plots) and constrain the parameter set in the (Δ_a, Δ_M) plane. We choose a standard value of $\omega = 100$ $[keV]$ and $m_a \leq 10^{-14}$ $[eV]$. This gives us the strongest constraints.

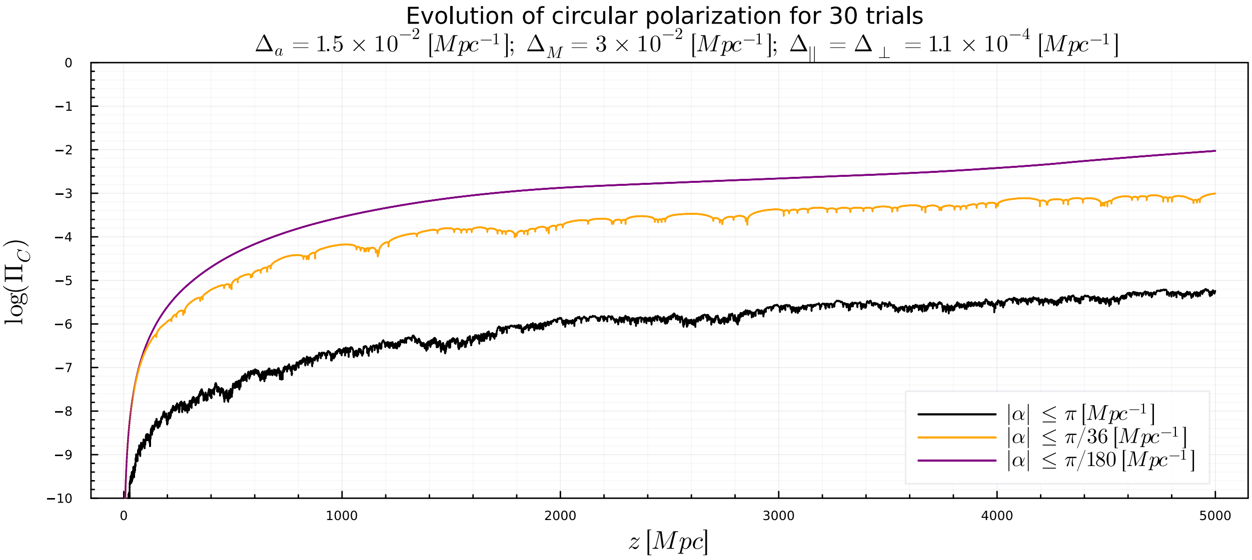

FIG. 5: Degree of polarization in the helical model

In this case, we find two constraints - one by fixing the value of the IGMF and the other by fixing the value of the coupling strength. If we choose $g_{a\gamma\gamma} \sim 10^{-11}$ $[GeV^{-1}]$, we find,

$$B \; < \; 10^{-11} \; [G] \tag{143}$$

Here, $g_{a\gamma\gamma} = 1/M$. This falls within the current bounds on the IGMF - $10^{-16} - 10^{-9}$ $[G]$. On the other hand, if we fix the IGMF as $B \sim 10^{-9}$ $[G]$, we get a strong constraint,

$$\boxed{g_{a\gamma\gamma} \; < \; 10^{-13} \; [GeV^{-1}]} \tag{144}$$

An important point to note here is that we do not know much about the structure of the IGMF - we know an upper bound from Faraday rotation measurements and a lower bound from γ ray measurements. In this case, the conventional and the helical models are generalisations and possible structures for the IGMF - but not exactly the one present. Thus, it depends crucially on furthering our current idea of the IGMF to impose stronger constraints on the coupling constant and detect axions.

VII. DISCUSSIONS AND SUMMARY

This project report is a detailed review of the axion photon coupling scenario and the expected astrophysical or other consequences of such a mixing. We consider the basic Lagrangian density of the fields and derive the field equations, generalising them by including QED corrections and look at plane wave solutions in the linearised limit. Based on the solutions, we estimate the transition probability of the mixing in different cases and also point out other axion induced effects that can be potentially detected. Further, we consider more general models of the mixing scenario for different structures of the inter-galactic magnetic field and compute the axion flux in those cases, noting how the other optical effects induced by axions also change - mostly through the use of Stokes parameters. We find that such photons must necessarily have a non-zero degree of circular polarisation regardless of their initial polarization. Building on this, we look at practical set ups and astrophysical observations to match our predictions and impose constraints on the coupling strength in the theory.

Further avenues of work would most likely be concerned with the particle physics aspect of the axion/photon coupling apart from generalising the mixing models further. One possible idea can be to introduce turbulent magnetic fields with a much lower coherent length in addition to the regular one we deal with here in the conventional and helical cases - it has been seen in other studies that in a certain scenario, this leads to an enhancement of the axion flux, so it would be interesting to see what happens in the specific case of the IGMF. Further, as mentioned earlier, in [9], the authors have shown that a correlated magnetic field might also enhance the axion flux - another idea would be to couple the correlated regular fields with additional turbulent ones and look at their combined effect.

VIII. ACKNOWLEDGEMENTS

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Appendix A: Jones calculus and Stokes parameters

In this appendix, we briefly introduce the Stokes parameters and derive a general form for them which is later used in section VI B. We mostly follow the treatment in [11], [12] and [13].
The (generally complex) electric field for an electromagnetic wave along the z direction (assuming tranversality holds) can be represented by,

$$|\mathbf{E}\rangle = \begin{pmatrix} E_{0x}e^{i\phi_x} \\ E_{0y}e^{i\phi_y} \\ 0 \end{pmatrix} e^{i(kz-\omega t)} \tag{A1}$$

The physically measured field will be the real part of this quantity. If we disregard the z component for now, we can define,

$$|\mathbf{E}\rangle = \underbrace{\begin{pmatrix} E_{0x}e^{i\phi_x} \\ E_{0y}e^{i\phi_y} \end{pmatrix}}_{\text{Jones vector } \equiv\, |\mathcal{E}\rangle} e^{i(kz-\omega t)} \tag{A2}$$

Usually, we normalise the Jones vector, i.e. $\langle\mathcal{E}|\mathcal{E}\rangle = 1$. Here, we will not assume the normalisation unless mentioned explicitly.
In Jones calculus, we usually work with Jones vectors that represent the states of the electromagnetic fields.

Alongside this, we use 2×2 matrices to represent various optical elements (analysers, polarisers, lenses etc) and combine them to formulate a more convenient approach to geometric optics.
As an example, if we have a linearly polarised wave along the x direction (i.e. the electric field is linearly polarised along the x direction), then a possible representation can be,

$$\underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\text{Polarization along } \hat{x}} \quad , \quad \underbrace{\begin{bmatrix} 0 \\ -1 \end{bmatrix}}_{\text{Polarization along } -\hat{y}} \tag{A3}$$

Here the magnitude of the field is normalised to unity. As another example, consider a circularly polarized wave,

$$\mathbf{E} = E_0 e^{i(kz-\omega t)}(\hat{x} + i\hat{y}) = E_0 e^{i(kz-\omega t)}(\hat{x} + e^{i\pi/2}\hat{y}) \tag{A4}$$

The above wave has a $\pi/2$ phase difference between the components and is thus circularly polarised (specifically, this is left circularly polarized). In this case, the Jones vector would be,

$$\mathbf{E} = E_0 e^{i(kz-\omega t)}(\hat{x} + e^{i\pi/2}\hat{y}) \;\Rightarrow\; |\mathcal{E}\rangle = \begin{bmatrix} 1 \\ i \end{bmatrix} \tag{A5}$$

Right circularly polarized light would be represented by $\begin{bmatrix} 1 \\ -i \end{bmatrix}$.
Let us now move on and define the Stokes parameters - these are basically specific quantities we define using a combination of light intensities after they pass through certain polarizers.
Consider a non-normalized Jones vector given by,

$$|\mathcal{A}\rangle = \begin{pmatrix} A_{||}(z) \\ A_{\perp}(z) \end{pmatrix} \tag{A6}$$

In case of no electrostatic fields, the time derivaitve of the above directly gives us the usual $|\mathcal{E}\rangle$. The first Stokes parameter is simply the intensity of unobstructed light, just the total intensity, I - also denoted by S_0,

$$S_0 \equiv I(z) = \langle\mathcal{A}|\mathcal{A}\rangle = |A_{||}|^2 + |A_{\perp}|^2 \tag{A7}$$

The second Stokes parameter, $Q(z)$, is the difference between two intensities - after the wave passes through a linear polarizer in the x direction and after it (separately) passes through a linear polarizer in the y direction. The directions x, y hold no special significance, of course - all that is necessary is that the two directions be orthogonal to each other. For convenience, we choose these specific directions. Then,

$$S_1 \equiv Q(z) = \langle\mathcal{A}| \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}_{\text{for } \hat{x}} - \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}_{\text{for } \hat{y}} |\mathcal{A}\rangle = |A_{||}|^2 - |A_{\perp}|^2 \tag{A8}$$

The third parameter, $U(z)$, is again a difference - this time we rotate the previously selected directions (for $Q(z)$) counterclockwise by $\pi/4$ radians. In our case, this amounts to linear polarizers along the $(\hat{x}+\hat{y})/\sqrt{2}$ and $(-\hat{x}+\hat{y})/\sqrt{2}$ directions (i.e. $\pi/4$ and $3\pi/4$ counterclockwise to the x axis). We have,

$$S_2 \equiv U(z) = \langle\mathcal{A}| \underbrace{\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}}_{\pi/4\ [rad]} - \underbrace{\frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}}_{3\pi/4\ [rad]} |\mathcal{A}\rangle = A_{||}A_{\perp}^* + A_{||}^* A_{\perp} \tag{A9}$$

Finally, the fourth parameter, $V(z)$ - instead of linear polarizers, we pass the light through a (left or right) circular polarizer and define $V(z)$ as the difference between twice of this intensity and the usual total intensity. If we choose a left circular polarizer (LCP), we have,

$$S_3 \equiv V(z) = 2\langle\mathcal{A}| \underbrace{\frac{1}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}}_{\text{LCP}} |\mathcal{A}\rangle - I(z) = i(A_{||}A_{\perp}^* - A_{||}^* A_{\perp}) \tag{A10}$$

Using these, we define the degree of linear and circular polarization as,

$$\Pi_L \;=\; \frac{\sqrt{U^2(z) + Q^2(z)}}{I(z)},\; \Pi_C \;=\; \frac{|V(z)|}{I(z)} \tag{A11}$$

The Stokes parameters also satisfy a constraint relation,

$$Q^2(z) + U^2(z) + V^2(z) = I^2(z) \tag{A12}$$

Therefore, all four are not independent - only three are. Often these parameters are defined in a dimensionless manner by dividing the original parameters by $I(z)$. In this case,

$$\bar{Q}^2(z) + \bar{U}^2(z) + \bar{V}^2(z) = 1 \tag{A13}$$

Here, the over-bar means that these quantities are dimensionless. These are the parameters used in the text (also, see [7]). Knowing the four Stokes parameters allows us to estimate the ellipticity and the orientation of the polarization ellipse, for example,

$$\tan 2\xi = \frac{U}{Q},\ Q = \frac{e^2 \cos 2\xi}{2 - e^2},\ U = \frac{e^2 \sin 2\xi}{2 - e^2} \tag{A14}$$

Here, e is the ellipticity of the polarization ellipse and ξ is the angle made by the semi-major axis of the ellipse with the x axis. The advantage of using Stokes parameters is that they help us characterise the polarization ellipse of the photon field much more easily, especially in an experimental context since these quantities are simply intensities passed through some polarizers. Additionally, these parameters enable us to estimate a degree of polarization of the photon field quite easily - as we saw in case of axion/photon mixing, the final photon states do have a non trivial degree of polarization which can be used to impose significant bounds on the parameters in the theory.