Non-linear oscillators, Resonance and the Elastic Pendulum

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This project takes a look at a certain double well potential at the outset and some features of the same are highlighted. This is followed by an analysis of resonance in driven oscillators (with a periodic driving force). The behaviour of such oscillators around their resonant frequencies is analysed, followed by an analysis of a variable frequency oscillator using basic perturbative ideas. In conclusion, an analysis of an elastic pendulum is performed and it is shown that the system exhibits resonance when the frequencies of the oscillations matches a certain ratio.

CONTENTS

I. INTRODUCTION

The aim of this project is to look at two very related and important phenomena that take place in nature - oscillations and resonance. We encounter infinities and diverging parameters along the way and develop an idea about how divergent behaviour depends on the system itself and can be interesting as a topic all on its own. In particular, resonance describes a quite broad range of behaviours and a diverging system or parameter can blow up to infinity at different rates and with different behaviours. Finally, the tools that are used to analyse systems that diverge and systems that exhibit resonance come together in an actual physical system, the elastic pendulum. Using whatever is developed in the earlier sections, it is shown that the elastic pendulum exhibits a very unique sort of resonance where no external driving force is required and wherein the system will exhibit resonance by itself depending on a particular ratio of its parameters.

The project starts by analysing a double well potential and it is shown that the time period of a particle in this sort of a potential will go to infinity in a very specific fashion provided that the total energy of the particle matches a certain critical value. The behaviour of this diverging time period is analysed. This is followed by an analysis of driven oscillators and resonance that occurs if the natural and intrinsic frequency of the oscillator matches. The next step was to look into an oscillator with a variable frequency, but this time analysis of the resonant behaviour required the usage of basic perturbative ideas. Ultimately, an analysis of the elastic pendulum is performed and using the mathematical ideas and techniques used throughout, it is shown that the system exhibits a peculiar sort of resonance wherein its amplitude of oscillations diverges without any external driving force.

II. A DOUBLE WELL POTENTIAL

Consider a double well potential of the form,

$$V(x) \;=\; \frac{\lambda}{4}x^4 \;-\; \frac{\omega^2}{2}x^2 \tag{1}$$

For $\lambda = 4,\ \omega = \sqrt{2}$, the potential curve is shown in Fig(1). The depth of this well is $\omega^4/8\lambda$ and the minimas (symmetric around the $y = 0$ line) occur at $\pm\omega/\sqrt{2\lambda}$. From energy conservation, assuming the system to have a fixed total energy, E,

$$\frac{p^2}{2m} \;+\; V(x) \;=\; E \tag{2}$$

$$\Rightarrow p^2 \;=\; E \;-\; V(x)$$

Or,

$$\Rightarrow p^2 \;=\; mx^2(\omega^2 - \lambda x^2) + 2mE$$

This leads to the following 3 cases.

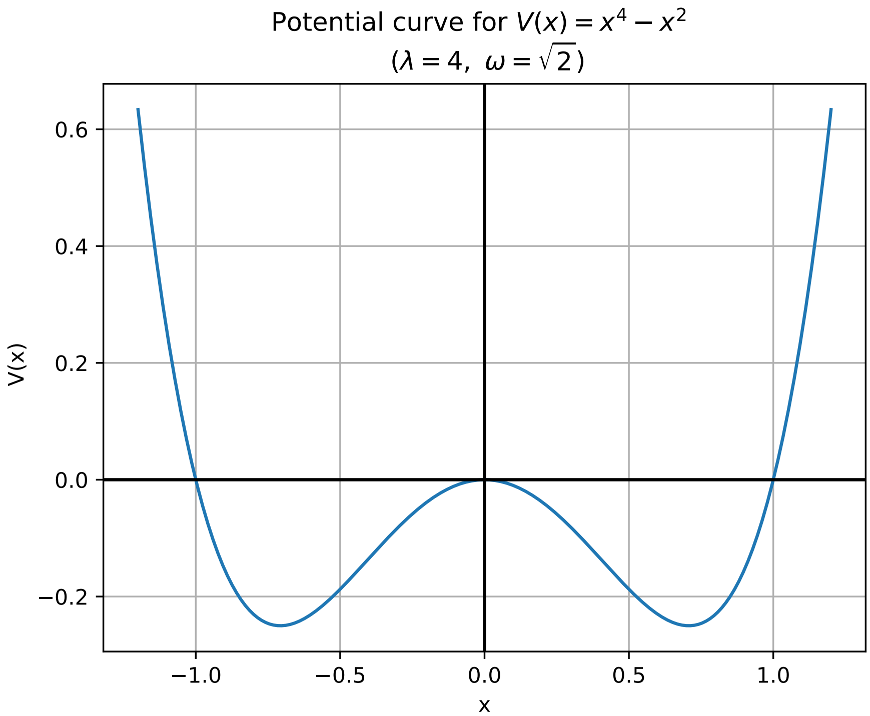

FIG. 1 *A double well potential of the form of equation (1) with* $\lambda = 4,\ \omega = \sqrt{2}$

A. Case 1: $E = 0$

$$p^2 = mx^2(\omega^2 - \lambda x^2) \tag{3}$$

This equation has to have a positive left hand side (since it is a square of some number), and thus,

$$x^2 - \frac{\omega^2}{\lambda} \le 0$$

$$\Rightarrow x \in [-\frac{\omega}{\sqrt{\lambda}}, \frac{\omega}{\sqrt{\lambda}}]$$

From the nature of the curve (as shown in Fig(2), the trajectory in the momentum position plane is closed, indicative of a closed or periodic behaviour.

B. Case 2: $E \in (-\omega^4/8\lambda, 0)$

Taking $E = -|E|$,

$$p^2 = mx^2(\omega^2 - \lambda x^2) - 2m|E| \tag{4}$$

Solving for the roots of the right hand side (which has to be greater than 0 always because the left hand side consists of a positive number),

$$mx^2(\omega^2 - \lambda x^2) = 2m|E|$$

Solving this for x^2,

$$x^2 = \frac{\omega^2}{2\lambda} \pm \frac{1}{2}\sqrt{\frac{\omega^4}{\lambda^2} - \frac{8|E|}{\lambda}} = \frac{\omega^2}{2\lambda} \pm B$$

Here, B includes the terms underneath the square root. Solving the above equation leads us to 4 roots which are,

$$x = \pm\frac{1}{\sqrt{2}}\sqrt{\frac{\omega^2}{\lambda} + B} \tag{5}$$

And,

$$x = \pm\frac{1}{\sqrt{2}}\sqrt{\frac{\omega^2}{\lambda} - B} \tag{6}$$

The trajectories traced out for this case are also shown in Fig(2).

C. Case 3: $E > 0$

In this case,

$$p^2 = mx^2(\omega^2 - \lambda x^2) + 2mE \tag{7}$$

Proceeding as in case 2, we will have 4 roots for the equation.

$$x = \pm\frac{1}{\sqrt{2}}\sqrt{\frac{\omega^2}{\lambda} - A} \tag{8}$$

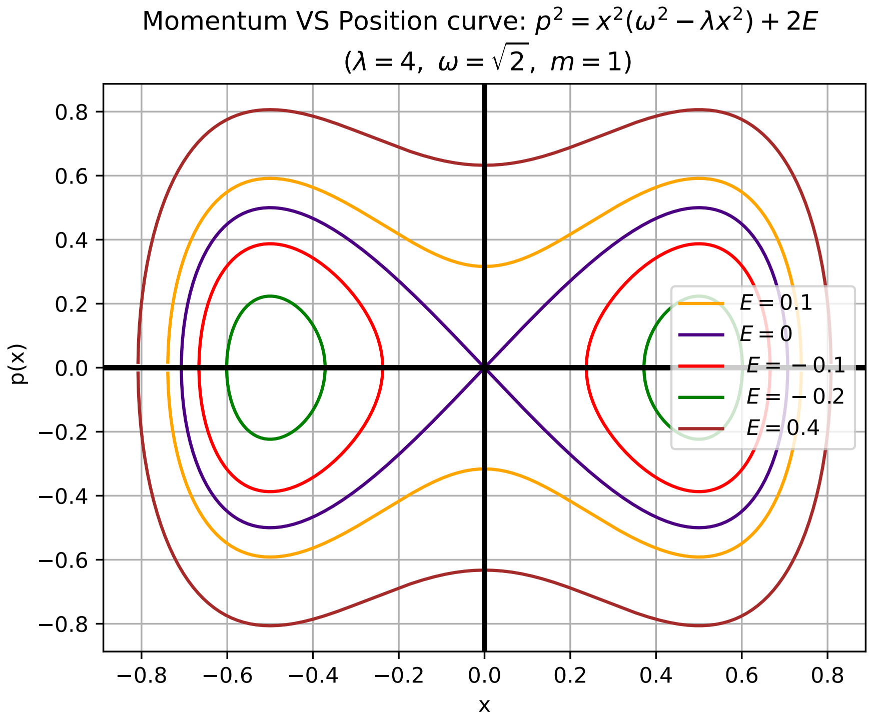

FIG. 2 *Various trajectories traced out in the phase space for different values of total energy* $(\lambda = 4,\ \omega = \sqrt{2})$

$$x \;=\; \pm\frac{1}{\sqrt{2}}\sqrt{\frac{\omega^2}{\lambda} + A} \tag{9}$$

Here,

$$A \;=\; \sqrt{\frac{\omega^4}{\lambda^2} + \frac{8E}{\lambda}}$$

Now, as,

$$\left(\frac{\omega^2}{\lambda}\right)^2 < \left(\frac{\omega^2}{\lambda}\right)^2 + \frac{8E}{\lambda}$$

$$\Rightarrow A > \frac{\omega^2}{\lambda}$$

This makes 2 of the roots above imaginary and the other 2 real. The trajectories traced out for this case are also present in Fig(2).

III. THE CRITICAL CASE TIME PERIOD

From the previous section and Fig(2), it can be seen that for a particular critical value of the total energy, the system displays a special kind of motion by having the origin, (0,0), as one of it's fixed points. In the case of this double well potential, this particular value of energy happens to be $E = 0$.

In order to find the time period for this periodic motion, consider,

$$T \;=\; \int_0^T dt \;=\; 2\int_0^A \frac{m}{p}\,dx \tag{10}$$

In the above integral, A represents the amplitude (or the maximum distance covered) and the limits have been modified by using the symmetry of the momentum function (it is an even function).
For the critical case, $E = 0$, $A = \omega/\sqrt{\lambda}$, and,

$$p^2 \;=\; mx^2(\omega^2 - \lambda x^2)$$

Or,

$$T \;=\; \int_0^A \frac{2\sqrt{m}}{x\sqrt{\omega^2 - \lambda x^2}}\,dx$$

Taking $x^2\lambda = \omega^2\sin^2\theta$,

$$T \;=\; \frac{2\sqrt{m}}{\omega}\ln\left(\frac{1-\cos\theta}{\sin\theta}\right)\Big|_0^{\pi/2} \to \infty$$

As we can see, this time period diverges to infinity and the system never really returns to it's starting point.

IV. ANALYSIS OF THE TIME PERIOD IN TERMS OF A SMALL PARAMETER

In the previous section, it was concluded that at the critical energy, the system undergoes a periodic motion,

but with a time period that diverges. In order to analyse the behaviour of the time period for this case, consider a total energy that differs from the actual critical energy by a very small amount, ϵ.
Therefore,

$$p^2 \ = \ mx^2(\omega^2 - \lambda x^2) - 2m\epsilon \tag{11}$$

This particular value of energy implies that the system now behaves as in case 2 (with 4 real roots). Thus, the amplitudes are $\pm A_1$ and $\pm A_2$ where,

$$A_1 = \frac{1}{\sqrt{2}}\sqrt{\frac{\omega^2}{\lambda} - \sqrt{\frac{\omega^4}{\lambda^2} - \frac{8\epsilon}{\lambda}}}$$

$$A_2 = \frac{1}{\sqrt{2}}\sqrt{\frac{\omega^2}{\lambda} + \sqrt{\frac{\omega^4}{\lambda^2} - \frac{8\epsilon}{\lambda}}}$$

Now, for small δ, the following series expansion is valid,

$$(1+\delta)^{\frac{1}{2}} \ = \ 1 + \frac{\delta}{2} + O(\delta^2)$$

Here, $O(\delta^2)$ represents orders of δ^2 and above.
This allows us to write,

$$(1 - \frac{8\epsilon\lambda}{\omega^4})^{\frac{1}{2}} \ \approx \ 1 - \frac{4\epsilon\lambda}{\omega^4}$$

This means,

$$A_1 \ \approx \ \frac{\sqrt{2\epsilon}}{\omega} \tag{12}$$

$$A_2 \ \approx \ \frac{1}{\omega\sqrt{\lambda}}\sqrt{\omega^4 - 2\epsilon\lambda} \tag{13}$$

Moving on, this leads to,

$$T \ = \ \int_{A_1}^{A_2} \frac{2\sqrt{m}}{\sqrt{x^2(\omega^2 - \lambda x^2) - 2\epsilon}}\, dx$$

$$= \frac{1}{\sqrt{2\epsilon}}\int_{A_1}^{A_2} \frac{2\sqrt{m}}{\sqrt{\frac{x^2(\omega^2-\lambda x^2)}{2\epsilon} - 1}}\, dx$$

Taking $\epsilon \ll 1$,

$$T \approx \int_{A_1}^{A_2} \frac{2\sqrt{m}}{x\sqrt{\omega^2 - \lambda x^2}}\, dx$$

$$= \ \frac{2\sqrt{m}}{\omega}\ln\left(\frac{\frac{\omega}{\sqrt{\lambda}} - \sqrt{\frac{\omega^2}{\lambda} - A_2^2}}{\frac{\omega}{\sqrt{\lambda}} - \sqrt{\frac{\omega^2}{\lambda} - A_1^2}} \times \frac{A_1}{A_2}\right)$$

Simplifying the above expression by using the approximate values of the limits of integration,

$$T \ = \ \frac{2\sqrt{m}}{\omega}\ln\left(\frac{\omega^2 + \sqrt{\omega^4 - 2\epsilon\lambda}}{\sqrt{2\epsilon\lambda}} \times \sqrt{\frac{\omega^2 - \sqrt{2\epsilon\lambda}}{\omega^2 + \sqrt{2\epsilon\lambda}}}\right) \tag{14}$$

In the limit of ϵ tending to 0,

$$\lim_{\epsilon\to\ 0} T \ = \ \frac{2\sqrt{m}}{\omega}\ln\left(\frac{\omega^2\sqrt{2}}{\sqrt{\lambda\epsilon}}\right)$$

From the expression above, we can see that the time period near the critical value of energy for this particular set of trajectories is very large (since ϵ is a very small quantity) and that the behaviour follows a particular relationship,

$$T \ \propto \ \ln\left(\frac{1}{\sqrt{\epsilon}}\right)$$

This behaviour is shown in Fig(3). From the curve, we can see that as ϵ tends to zero, the time period, T, diverges and goes to infinity. This is as was concluded in the previous section. As the value of the deviation increases, the time period varies as is outlined.

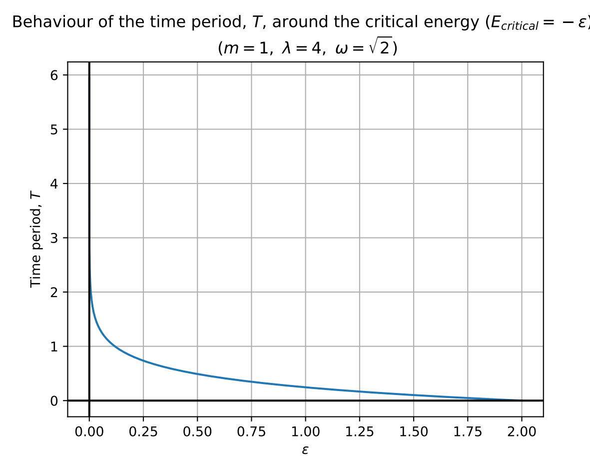

FIG. 3 *Behaviour of the time period, T, around the critical energy in terms of the small deviation, ϵ*

V. DRIVEN OSCILLATORS AND RESONANCE

Consider a driven oscillator with a periodic external driving force of the form $F\cos\Omega t$.
Then, the equation of motion is of the form,

$$\ddot{x} \ + \ \omega^2 x \ = \ F\cos\Omega t \tag{15}$$

In order to solve this particular differential equation, one can make use of the Fourier transform. The Fourier transform is defined as,

$$\mathcal{F}[f(x)] \ = \ \hat{f}(\alpha) \ = \ \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-\iota\alpha x}\, dx$$

It has an associated inverse Fourier transform as well,

$$ {}_{inv}\mathcal{F}[\hat{f}(\alpha)] \;=\; f(x) \;=\; \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(\alpha)e^{\iota\alpha x}\,d\alpha $$

An interesting property of the Fourier transform is as follows,

$$ \hat{f}'(\alpha) \;=\; \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\mathrm{d}f}{\mathrm{d}x}e^{-\iota\alpha x}\,dx \;=\; \iota\alpha\hat{f}(\alpha) $$

Using the above in equation (15),

$$ \mathcal{F}[\ddot{x}] \;+\; \mathcal{F}[\omega^2 x] \;=\; \mathcal{F}[F\cos\Omega t] \;=\; \hat{f}\ (say) $$

Or,

$$ (-\alpha^2 \;+\; \omega^2)\hat{x} \;=\; \hat{f} $$

$$ \hat{x} \;=\; \frac{\hat{f}}{\omega^2-\alpha^2} $$

This gives us,

$$ {}_{inv}\mathcal{F}[\hat{x}] \;=\; x \;=\; {}_{inv}\mathcal{F}\Big[\frac{\hat{f}}{\omega^2-\alpha^2}\Big] \tag{16} $$

Now,

$$ \hat{f} \;=\; \frac{F}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-\iota\alpha t}\cos\Omega t\,dt $$

$$ = \frac{F}{2\sqrt{2\pi}}\int_{-\infty}^{\infty}(e^{\iota\Omega t}+e^{-\iota\Omega t})e^{-\iota\alpha t}\,dt $$

$$ = \frac{F}{2\sqrt{2\pi}}\Big[\int_{-\infty}^{\infty}e^{\iota(\Omega-\alpha)t}\,dt+\int_{-\infty}^{\infty}e^{-\iota(\Omega+\alpha)t}\,dt\Big] $$

Using the definition of the Dirac Delta function in the above equation,

$$ \hat{f} \;=\; \frac{2F\pi}{2\sqrt{2\pi}}(\delta(\Omega-\alpha)+\delta(\Omega+\alpha)) $$

This leads to,

$$ x \;=\; \frac{F}{2}\int_{-\infty}^{\infty}e^{\iota\alpha t}\frac{\delta(\Omega-\alpha)\;+\;\delta(\Omega+\alpha)}{\omega^2-\alpha^2}\,d\alpha $$

Now,

$$ \int_{-\infty}^{\infty}f(x)\delta(x-a)\,dx \;=\; f(a) $$

This gives us,

$$ x(t) \;=\; \frac{F}{2}\times\frac{e^{\iota\Omega t}+e^{-\iota\Omega t}}{\omega^2-\Omega^2} \;=\; \frac{F\cos\Omega t}{(\omega^2-\Omega^2)} \tag{17} $$

From the above solution, we can see that when the external driving force has a frequency that matches the frequency of the system, the denominator of the amplitude becomes undefined. That is,

$$ \lim_{\Omega\to\omega}x(t) \;=\; F\cos\omega t\times\lim_{\Omega\to\omega}\Big(\frac{1}{\omega^2-\Omega^2}\Big)\to\infty $$

This particular behaviour is called resonance, and it is often described by saying that the amplitude of the oscillations diverges to infinity.
Here we utilise a similar method to section V in order to better analyse the behaviour of this oscillator close to resonance. For this purpose we can take,

$$ \Omega = \omega+\epsilon $$

Here ϵ is a very small parameter that measures the difference in between the two frequencies. At $\epsilon = 0$, the system shows resonance.
Using the above substitution,

$$ x(t) \;=\; \frac{-F}{\epsilon(2\omega+\epsilon)}\cos{(\omega+\epsilon)t} $$

$$ = \frac{-F}{\epsilon(2\omega+\epsilon)}[\cos\omega t\cos\epsilon t \;-\; \sin\omega t\sin\epsilon t] $$

$$ = \;=\; \frac{F}{\epsilon(2\omega+\epsilon)}[\;\sin\omega t\sin\epsilon t \;-\; \cos\omega t\cos\epsilon t] $$

In the limit that $\epsilon\to 0$,

$$ \lim_{\epsilon\to 0}x(t) \;=\; \frac{F}{2\omega}\Big[\sin\omega t\lim_{\epsilon\to 0}\Big(\frac{\sin\epsilon t}{\epsilon t}\Big)t \;-\; \lim_{\epsilon\to 0}\frac{\cos\omega t\cos\epsilon t}{\epsilon}\Big] $$

Using,

$$ \lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1 $$

We have,

$$ \lim_{\epsilon\to 0}x(t) \;=\; \frac{F}{2\omega}\Big[t\sin\omega t \;-\; \frac{\cos\omega t}{\epsilon}\Big] $$

This allows us to write the general solution as,

$$ \lim_{\epsilon\to 0}x(t) \;=\; A\cos\omega t\;+\;B\sin\omega t\;+\;\frac{F}{2\omega}\Big[t\sin\omega t\;-\;\frac{\cos\omega t}{\epsilon}\Big] $$

$$ = A'\cos\omega t \;+\; B\sin\omega t \;+\; \frac{F}{2\omega}t\sin\omega t \tag{18} $$

The interesting part about this equation is the last term that it contains. A' diverges as ϵ tends to 0, a feature that had been concluded from equation (17) already, but it is the last term that features a behaviour that was not present in the original solution. The last term shows that as the frequency of the external periodic driving

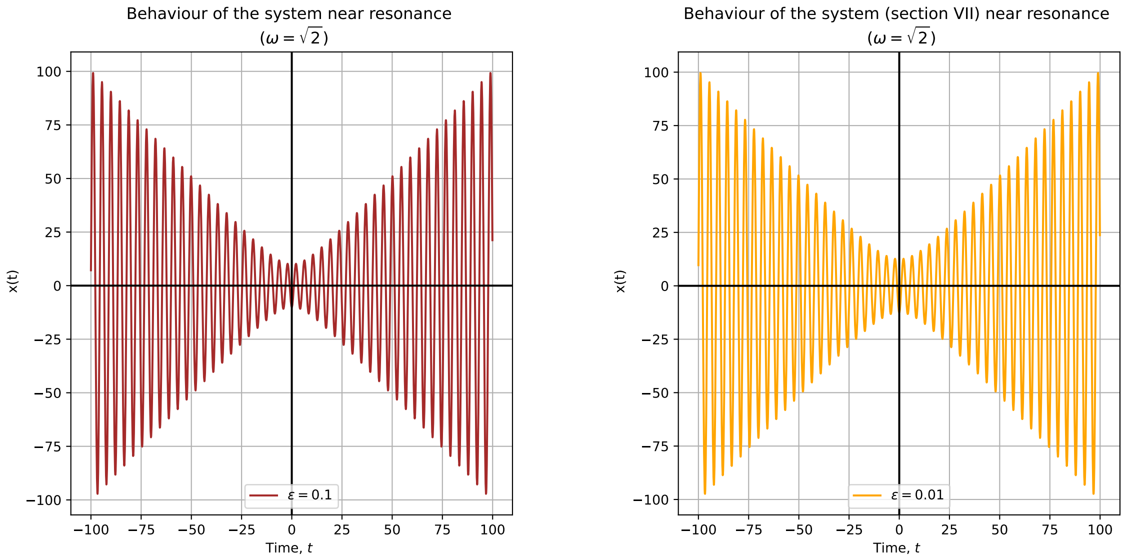

FIG. 4 *Behaviour of the system in section VI near resonance. As one can see, the amplitude increases with time.*

force approaches that of the system, the amplitude of the oscillations starts to increase linearly with time (the amplitude begins to increase with time). Resonance is characterised by this sort of a behaviour (an amplitude that depends linearly on time) in this particular case. As the frequencies equal each other, the amplitude diverges to infinity. This behaviour is shown for two different values of ϵ in Fig(4).

VI. AN OSCILLATOR WITH VARIABLE FREQUENCY

In this section, a different type of system is taken - one with a variable frequency.

$$\ddot{x} \;+\; \omega^2(1 \;+\; f\cos\Omega t)x \;=\; 0 \tag{19}$$

In order to analyse this particular equation, f is assumed to be small. This allows us to write a perturbative series in f for the solution as follows,

$$x \;=\; x_0 \;+\; fx_1 \;+\; f^2x_2 \;+\; ...$$

The zero order solution (when $f = 0$ or negligible) is,

$$\ddot{x}_0 \;+\; \omega^2 x_0 \;=\; 0 \;\Rightarrow\; x_0 \;=\; A\cos\omega t \;+\; B\sin\omega t$$

The first order equation in f will be,

$$\ddot{x}_1 \;+\; \omega^2 x_1 \;=\; -\omega^2 x_0\cos\Omega t \tag{20}$$

This gives us,

$$\ddot{x}_1 \;+\; \omega^2 x_1 \;=\; -\omega^2(A\cos\omega t \;+\; B\sin\omega t)\cos\Omega t$$

$$= \;-\frac{\omega^2}{2}[A(\sin\theta t + \sin\phi t) + B(\cos\theta t + \cos\phi t)]$$

Here, $\theta \;=\; \omega + \Omega$ and $\phi \;=\; \omega - \Omega$.
If we compare the above to the analysis we used in section VI, then we can guess that the form of the solution to the above equation will have terms like,

$$x_1(t) \propto \frac{1}{\omega^2 - \theta^2}, \;\frac{1}{\omega^2 - \phi^2}$$

In this case, as $\omega^2 - \theta^2 \;=\; -\Omega(2\omega + \Omega)$ and $\omega^2 - \phi^2 \;=\; \Omega(2\omega - \Omega)$, resonance will occur at $\Omega \;=\; 2\omega$.
In fact, if the entire procedure is carried out (using the Fourier transform as in section VI), the solution for the first order in f comes out to be,

$$x_1(t) \;=\; \frac{\omega^2}{2\Omega}\Big(\frac{A\sin\phi t - B\cos\phi t}{2\omega - \Omega} - \frac{A\sin\theta t - B\cos\theta t}{2\omega + \Omega}\Big) \tag{21}$$

Equation (21) shows that the amplitude of the first term diverges at $\Omega \;=\; 2\omega$ and therefore, resonance occurs when the two frequencies get related in this manner.
In order to analyse the resonant behaviour of this system, we follow the same procedure as in the previous section and let $\Omega \;=\; 2\omega \;+\; \epsilon$ for a very small number, ϵ. Skipping over the trigonometric simplifications and algebra, we end up with the following,

$$x_1(t) \;=\; \frac{\omega}{4}\Big(\frac{A\sin\omega t + B\cos\omega t}{\epsilon} - \frac{A\sin 3\omega t - B\cos 3\omega t}{4\omega}$$

$$+\; t(A\cos\omega t \;-\; B\sin\omega t)\Big)$$

The problem with this equation lies in the first term that diverges as $\epsilon \to \infty$. In the previous section, we clubbed a similar term together with the original solution in order to get an extra term in the special case; however, in this case, the first term itself is a new extra term that was not present in our original solution. Hence, this procedure for analysing the resonant behaviour of the system is not fruitful in this particular case.
In order to analyse the resonant behaviour, we adopt a new technique. We assume a solution to the original equation (equation (19)) as follows,

$$x(t) \; = \; A(t)\cos\left(\omega t + \theta(t)\right) \; + \; B(t)\sin\left(\omega t + \theta(t)\right)$$

$$= \; A\cos\phi \; + \; B\sin\phi$$

This is basically a Fourier series solution, but we assume that the coefficients (or amplitudes) of the higher order terms (2ϕ, 3ϕ etc) are very small and can be ignored. Apart from this, we assume that the amplitudes are slowly varying functions of time such that derivatives of second order or higher and powers of first derivative equal to or above the second degree can be ignored as too small. θ is also assumed to be slowly varying and similar considerations hold for it as well.
We then have,

$$\dot{x}(t) \; = \; (\dot{A} + B\dot{\phi})\cos\phi \; + \; (\dot{B} - A\dot{\phi})\sin\phi$$

$$\ddot{x}(t) = (\ddot{A}+B\ddot{\phi}+2\dot{B}\dot{\phi}-A\dot{\phi}^2)\cos\phi+(\ddot{B}-A\ddot{\phi}-2\dot{A}\dot{\phi}-B\dot{\phi}^2)\sin\phi$$

$$\omega^2 x(t) \; = \; \omega^2 A\cos\phi \; + \; \omega^2 B\sin\phi$$

$$\omega^2 f\cos\Omega t x(t) \; = \; \frac{\omega^2 f}{2}\Big(A\cos\left(\phi + \Omega t\right) \; + \; B\sin\left(\phi + \Omega t\right)$$

$$+ \; A\cos\left(\phi - \Omega t\right) \; + \; B\sin\left(\phi - \Omega t\right)\Big)$$

Since we want to look at the resonant behaviour, we take $\Omega \; = \; 2\omega$, which gives us the following equation (after ignoring the higher derivatives and powers and expanding ϕ as $\phi \; = \; \omega t + \theta(t)$),

$$(\dot{B} - A\dot{\theta})\cos\phi \; - \; (\dot{A} + B\dot{\theta})\sin\phi \; + \; \frac{\omega^2 f}{4}\Big(A\cos\left(3\phi - 2\theta\right)$$

$$+A\cos\left(\phi - 2\theta\right) + B\sin\left(3\phi - 2\theta\right) - B\sin\left(\phi - 2\theta\right)\Big) \; = \; 0$$

In the special case that $\dot{\theta} = 0$ and $\theta = 0$, we can match the coefficients of $\cos\phi$ and $\sin\phi$ and put them equal to zero as the two are linearly independent. In this case, we ignore the terms with 3ϕ since we had already ignored them in the original solution and thus, they cannot appear at this stage.
We get,

$$\dot{B} \; + \; \frac{\omega f}{4}A \; = \; 0$$

$$\dot{A} \; + \; \frac{\omega f}{4}B \; = \; 0$$

This means,

$$\ddot{B} \; = \; \left(\frac{\omega f}{4}\right)^2 B$$

Or,

$$B(t) \; = \; C_1 e^{\frac{\omega f}{4}t}$$

Similarly,

$$A(t) \; = \; C_2 e^{\frac{\omega f}{4}t}$$

Therefore, in this particular system, resonant behaviour is exponential rather than linear. At $\Omega \; = \; 2\omega$, the amplitude diverges in a characteristic fashion as time goes to infinity. This behaviour is plotted in Fig(5).

VII. A SECOND ANALYSIS OF DRIVEN OSCILLATORS AROUND RESONANT FREQUENCY

In this section we will once again look into the resonant behaviour of the driven oscillator, but using the procedure shown in section VII. As such, let us assume a solution of the form,

$$x(t) \; = \; A\cos\omega t \; + \; B\sin\omega t$$

Note that here we do not include any phase factor and the amplitudes are once again taken to be slowly varying. Then, we have (differentiating $x(t)$),

$$\ddot{x} \; = \; 2\omega(\dot{B}\cos\omega t - \dot{A}\sin\omega t) \; - \; \omega^2 x(t)$$

This implies that equation (15) becomes,

$$\dot{B}\cos\omega t - \dot{A}\sin\omega t) \; = \; \frac{F}{2\omega}\cos\Omega t$$

Taking $\Omega \; = \; \omega \; + \; \delta$,

$$\dot{B}\cos\omega t - \dot{A}\sin\omega t) \; = \; \frac{F}{2\omega}\Big(\cos\omega t\cos\delta t - \sin\Omega t\sin\delta t\Big)$$

Matching the terms on both sides,

$$\dot{B} \; = \; \frac{F}{2\omega}\cos\delta t \; \Rightarrow \; B(t) \; = \; C_1 \; + \; \frac{F}{2\omega\delta}\sin\delta t$$

In the limit $\delta \to 0$,

$$\lim_{\delta\to 0} B(t) \; = \; C_1 \; + \; \frac{F}{2\omega}t$$

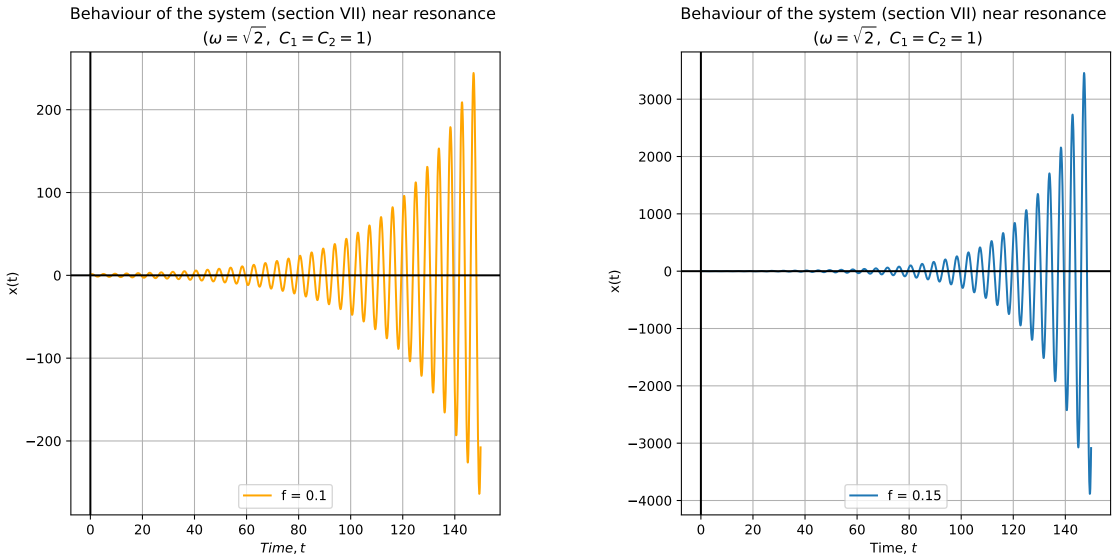

FIG. 5 *Behaviour of the system in section VII near resonance. As one can see, the amplitude increases with time, but this time, exponentially.*

Similarly,

$$\lim_{\delta\to 0} A(t) \;=\; C_2 \;-\; \frac{F}{2\omega\delta}$$

This means, under resonance conditions,

$$x(t) \;\propto\; t\sin\omega t$$

This result matches the results we obtained through a different means in section VI and the resonant behaviour is shown in Fig(4).

VIII. THE ELASTIC PENDULUM

In this section we will consider the dynamics of an elastic pendulum, one that can oscillate as a normal pendulum but is attached to a spring with a spring constant, k.

For this pendulum (considering 0 gravitational potential energy at the point of suspension and measuring all angles with respect to the vertical), we have,

$$x \;=\; r\sin\theta;\; y \;=\; -(l+\Delta)\cos\theta$$

Writing out the kinetic and potential energies,

$$KE \;=\; \frac{m}{2}(\dot{x}^2 \;+\; \dot{y}^2)$$

$$V \;=\; \frac{k}{2}\Delta^2 \;-\; mg(l+\Delta)\cos\theta$$

Here, we have taken the natural length to be l, the extension to be Δ and the gravitational potential energy to be zero at the point of suspension.

Then, the Lagrangian for the system is,

$$\mathcal{L} \;=\; \frac{m}{2}\Big(\dot{r}^2 + r^2\dot{\theta}^2\Big) \;-\; \frac{k}{2}\Delta^2 \;+\; mg(l+\Delta)\cos\theta$$

The system has 2 degrees of freedom, (r,θ) (or, (Δ,θ)). Therefore, Lagrange's equations give us,

$$\frac{d}{dt}\Big(\frac{\partial\mathcal{L}}{\partial\dot{r}}\Big) \;-\; \frac{\partial\mathcal{L}}{\partial r} \;=\; 0$$

$$\frac{d}{dt}\Big(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\Big) \;-\; \frac{\partial\mathcal{L}}{\partial\theta} \;=\; 0$$

This gives us,

$$\ddot{\Delta} \;=\; (l+\Delta)\dot{\theta}^2 + g\cos\theta - \frac{k}{m}\Delta$$

And,

$$\ddot{\theta} \;=\; \frac{-g}{l+\Delta}\sin\theta - \frac{2\dot{\Delta}\dot{\theta}}{l+\Delta}$$

To take into account the change in natural length due to gravity (otherwise, we'll have a g hanging around pointlessly), we take $\Delta \;=\; \frac{mg}{k} \;+\; x$. This gives us, for small θ and Δ $(L = l + mg/k)$,

$$\ddot{x} \;+\; \frac{k}{m}x \;=\; (L+x)^2\dot{\theta}^2 \;-\; \frac{g}{2}\theta^2 \tag{22}$$

$$\ddot{\theta} \;+\; \frac{g}{L}\theta \;=\; \frac{g}{L^2}x\theta \;-\; \frac{2\dot{x}\dot{\theta}}{L}\Big(1 \;-\; \frac{x}{L}\Big) \tag{23}$$

In order to analyse the solutions of this equation, perturbations are introduced. In this case though, since both θ and x are considered small, we assume they are of the order of the small perturbation parameter, ϵ. We have,

$$x \ = \ x_0 \ + \ \epsilon x_1 \ + \ \epsilon^2 x_2 \ + \ ...$$

$$\theta \ = \ \theta_0 \ + \ \epsilon\theta_1 \ + \ \epsilon^2\theta_2 \ + \ ...$$

Therefore,

$$\sum_{i=0}\Big(\ddot{x_i} \ + \ \frac{k}{m}x_i\Big) \ = \ [L\dot{\theta_0}^2 - \frac{g}{2}\theta_0^2] \ + \ \mathcal{O}(\epsilon^2)$$

And,

$$\sum_{i=0}\Big(\ddot{\theta_i} \ + \ \frac{g}{L}\theta_i\Big) \ = \ (\frac{g}{L}x_0\theta_0 - \frac{2}{L}\dot{x_0}\dot{\theta_0}) \ + \ \mathcal{O}(\epsilon^2)$$

Therefore, at the zero order,

$$\ddot{x_0} \ + \ \frac{k}{m}x_0 \ = \ 0$$

$$\ddot{\theta_0} \ + \ \frac{g}{L}\theta_0 \ = \ 0$$

Taking $\omega_1^2 \ = \ k/m$ and $\omega_2^2 \ = \ g/L$,

$$x_0(t) \ = \ A\cos\omega_1 t \ + \ B\sin\omega_1 t$$

$$\theta_0(t) \ = \ C\cos\omega_2 t \ + \ D\sin\omega_2 t$$

For the first order in ϵ,

$$\ddot{x_1} \ + \ \omega_1^2 x_1 \ = \ L\dot{\theta_0}^2 \ - \ \frac{g}{2}\theta_0^2$$

$$= \ g\Big[(D - \frac{C}{\sqrt{2}})\cos\omega_2 t + (-C - \frac{D}{\sqrt{2}})\sin\omega_2 t\Big]\times$$

$$\Big[(D + \frac{C}{\sqrt{2}})\cos\omega_2 t + (\frac{D}{\sqrt{2}} - C)\sin\omega_2 t\Big]$$

From our previous analysis of similar equations, we can deduce that since this equation contains terms (once expanded and resolved using proper trigonometric identities) of the form $\sin 2\omega_2 t, \ \cos 2\omega_2 t$, it will have diverging amplitudes at $\omega_1 \ = \ 2\omega_2$.
If we move to the equation for θ,

$$\ddot{\theta_1} \ + \ \omega_2^2\theta_1 \ = \ \frac{g}{L}x_0\theta_0 - \frac{2}{L}\dot{x_0}\dot{\theta_0}$$

$$= \ \omega_2^2(A\cos\omega_1 t + B\sin\omega_1 t)(C\cos\omega_2 t \ + \ D\sin\omega_2 t)$$

$$+ \ \frac{2\omega_1\omega_2}{L}(A\sin\omega_1 t - B\cos\omega_1 t)(D\cos\omega_2 t - C\sin\omega_2 t)$$

We, once again, anticipate from the form of the equation that it will contain terms with frequencies like $\omega_1 \ + \ \omega_2$ and $\omega_1 - \omega_2$. This will lead to diverging solutions when $\omega_1 \ = \ 2\omega_2$. Therefore, the elastic pendulum will show resonance in first order if the following condition is satisfied,

$$\omega_1 \ = \ 2\omega_2 \ \Rightarrow \ 2\sqrt{\frac{g}{L}} \ = \ \sqrt{\frac{k}{m}}$$

From the above 2 equations we conclude that the elastic pendulum exhibits a peculiar sort of resonance when the ratio of the 2 frequencies takes on a specific value. This resonance is novel because it arises due to the coupling between the 2 different types of oscillating behaviours that occur within the system. In other words, while we required an external driving force in section VI and an inherently variable frequency contributing to non-linearity in section VII, in this case, we have 2 linear oscillators which get coupled together in a way that makes the system behave in a non-linear fashion with coupling terms in the equations of motion (in the Euler-Lagrange equations written earlier). This coupling contributes to the non-linearity in this case and leads to a situation where the system exhibits resonance by itself provided that a specific parameter matches a certain value (the ratio ω_1/ω_2 must equal 2).

IX. CONCLUSION

The basic idea behind this project is three fold. First, it displays the quite diverse range of behaviours that can clubbed together under the term "diverging". The uniqueness within these behaviours is discussed in details and a better idea of resonance comes through. Second, the analysis of such resonance behaviour, apart from providing a much clearer idea into the actual phenomenon, also provides a basic idea of perturbations and how they are useful in physics or any related branch of science. Finally, the ideas developed are applied to a real, physical system and a new kind of resonance comes across, showing that what one needs for resonance is just a transfer of energy between different modes of the system.

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