Interference aided finite resonant response in an undamped forced oscillator

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We apply perturbative techniques to a driven undamped oscillator. We look at the behaviour at resonance and find that the finite resonant behaviour of the oscillator can be attributed to alternating signs of amplitudes, with long term divergences, in various perturbative orders giving rise to an interference like effect. We compare our solutions to the exact ones and show that, in general, our approximate solution matches the exact solution over a timescale that increases with the order of the perturbation theory. Notably, we find that such a finite answer exists regardless of friction or air resistance, contrary to what one would expect.

Keywords: Nonlinear oscillators, resonance, parametric resonance

I. INTRODUCTION

Perturbation theory at or near resonance generally implies computation of small changes in the resonance frequency or calculations of small changes in the response functions at resonance because of small deformations in the assumed ideal shape of the resonating object. A detailed study of this for deformed dielectric spheres has been carried out recently in [1] and [2]. Similar examples are found in optical setups (see [3]) and in the deformation of Helium drops (see [4]). Nonlinear vibrations in the context of strings has been studied in [5].
In this work, we deal with an undamped nonlinear system at resonance where the response at each order is quite large but alternating signs create an interference like effect keeping things controlled, i.e., the resonance exists at linear order and gets larger at every higher order but conspires to give a finite answer for the full nonlinear problem. We discuss forced resonance in an undamped pendulum with the Hamiltonian, $\mathcal{H} = \frac{p^2}{2m} - mgl\cos\theta$, where θ describes the angle with the vertical and g, the uniform gravitational field.
For the oscillator described by $\ddot{x} + \omega^2 x = F\cos\omega t$, the particular integral describing its solution is given by $\frac{F}{2\omega}t\sin\omega t$. If the system under consideration is a real pendulum with length, l, and in a uniform gravitational field, g, then the forced system takes the form,

$$\ddot{\theta} + \omega^2 \sin\theta = F\cos\omega t \tag{1}$$

Numerical integration of the above equation to find $\theta(t)$ yields a perfectly well behaved and bounded solution shown in Fig. (1) (note that throughout the text we use the notation $\dot{\theta}_0 = \dot{\theta}(t=0) = \omega_0$). We question whether it is possible to devise a perturbation series solution for the system described by the Eq. (1) that can capture the answer described by the plot above. We find that the n^{th} order perturbation theory is capable of giving a reasonable approximation for $\theta(t)$ for $t < \tau_n$, where τ_n is a time scale that increases with increasing n. This is a non trivial agreement because the oscillation amplitudes at the n^{th} order increase in magnitude with increasing n but an alternating sign change at various orders allows the perturbative solution (almost miraculously) to keep a reasonable track of the exact answer for the interval τ_n. Contrary to expectations, such a finite answer therefore exists even without friction - the significance is that even in the absence of damping

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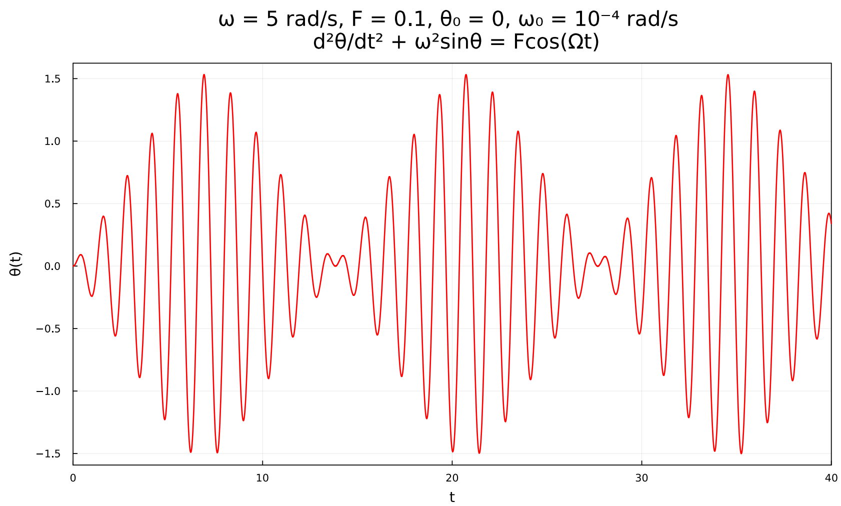

FIG. 1: The forced oscillator

a particular variety of nonlinear terms (infinite in number) can make the total response finite. We deal with this in detail in section 2. Boundedness of such nonlinear differential equations and resonant behaviour have been studied more rigorously earlier in [6] and [7]. A brief discussion is given in section 3.

II. THE RESONANT FORCED OSCILLATOR

Perturbation theory is usually applicable in modelling systems that include a small deviation from an original, integrable system. In this case, the original solution corresponds to the linear approximation to (1); therefore, we begin by rescaling the problem to make F dimensionless and write,

$$\ddot{\theta} \ + \ \omega^2 \sin\theta \ = \ \omega^2 F \cos\omega t \tag{2}$$

This enables us to use F as the regulator in the problem; the original solution to the problem is given by the solution of the equation,

$$\ddot{\theta}_0 \ + \ \omega^2 \theta_0 \ = \ \omega^2 \cos\omega t \tag{3}$$

We introduce the small correction to the above as,

$$\Lambda \ = \ \omega^2(\sin\theta \ - \ \theta) \tag{4}$$

In order for this to be small (and for perturbation theory to be applicable), we require that θ be small and in order to achieve this we expand the variable in a specific manner assuming that F is very small. From (2), after scaling θ as $F\theta$,

$$\ddot{\theta} + \omega^2\theta + \Lambda \ = \ \ddot{\theta} + \omega^2 \sin\theta \ = \omega^2 F \cos\omega t \tag{5}$$

Now, we expand θ as $F\theta_0 + F^2\theta_1 + F^3\theta_2 + ...$, yielding us in the zeroth order (or $\mathcal{O}(F)$) with $\sin\theta \approx \theta = F\theta_0$,

$$\ddot{\theta}_0 \ + \ \omega^2\theta_0 \ = \ \omega^2 \cos\omega t \tag{6}$$

This is the desired leading order equation that gives us the original solution to the problem to which the corrected solution will be added. The complete solution to this equation is,

$$\theta_0(t) \ = \ A\cos\omega t + (B + \frac{\omega t}{2})\sin\omega t \ = \ (\alpha + \frac{\omega t}{2})\sin\omega t \tag{7}$$

Here we impose the initial conditions $\theta_0(t = 0) = 0;\ \dot{\theta}_0(t = 0) = \omega\alpha$.

In the next higher order, $\mathcal{O}(F^2)$, we match powers of F and end up with,

$$\ddot{\theta}_1 \; + \; \omega^2\theta_1 \; = \; 0 \tag{8}$$

The solutions to this are very well known; however, we equal the initial velocity and displacements (in this order) to 0, effectively giving us no contribution from this order. In the next order, $\mathcal{O}(F^3)$, we get a more interesting equation,

$$\ddot{\theta}_2 \; + \; \omega^2\theta_2 \; = \; \frac{\omega^2}{6}\theta_0^3$$

$$= \; \frac{\omega^2}{24}\Big(\alpha + \frac{\omega t}{2}\Big)^3(3\sin\omega t - \sin 3\omega t) \tag{9}$$

This is the equation of a forced harmonic oscillator with the forcing term proportional to the zeroth order solution. As in [8, 9], we neglect the higher frequency term in this approximation and keep the largest contribution from the cubic term. Scaling t as $\tau = \omega t$ and using the double dot notation to denote a derivative with respect to the latter, we get,

$$\frac{d^2\theta_2}{d\tau^2} \; + \; \theta_2 \; = \; \ddot{\theta}_2 \; + \; \theta_2 \; = \; \frac{1}{64}\tau^3\sin\tau \; = \; G(\tau) \tag{10}$$

The particular solution to the above forced oscillator is given by,

$$\theta_2(\tau) \; =$$

$$\sin\tau\int_0^\tau G(\xi)\cos\xi\, d\xi - \cos\tau\int_0^\tau G(\xi)\sin\xi\, d\xi$$

$$= \; I_1\sin\tau - I_2\cos\tau \tag{11}$$

Here we have chosen the homogenous solutions to be $\sin\tau, \cos\tau$ such that the Wronskian of the differential equation is -1. Then, we have,

$$I_1 \; = \; \frac{1}{512}\Big((3\tau - 2\tau^3)\cos 2\tau + \frac{3}{2}(2\tau^2 - 1)\sin 2\tau\Big) \tag{12}$$

$$I_2 \; = \; \frac{1}{512}\Big(\tau^4 + (3\tau - 2\tau^3)\sin 2\tau + (\frac{3}{2} - 3\tau^2)\cos 2\tau - \frac{3}{2}\Big) \tag{13}$$

Taking the largest term, we get,

$$\theta_2(\tau) \sim -\frac{1}{512}\tau^4\cos\tau$$

The negative sign tells us that in this order the amplitude drops rapidly, in constrast to the leading order where the signs to the solution were positve denoting a growing contribution (more accurately, this is a phase change). In a similar fashion, we move to the next order, $\mathcal{O}(F^4)$,

$$\ddot{\theta}_3 \; + \; \omega^2\theta_3 \; = \; \frac{1}{2}\theta_0^2\theta_1 \tag{14}$$

As with θ_1, this amplitude clearly goes to 0 as well. Therefore, we go to the next order,

$$\ddot{\theta}_4(\tau) \; + \; \theta_4(\tau) \; = \; \frac{1}{2}\theta_0^2\theta_2 \; = \; \frac{-\tau^4}{4096}\Big(\alpha + \frac{\tau}{2}\Big)^2\cos\tau$$

$$= \; \tilde{G}(\tau) \tag{15}$$

We approximate and keep the largest term, $\tilde{G}(\tau) \sim -a_1\tau^6\cos\tau$, and as above, we have (a_1 is the constant numerical prefactor in the expression),

$$\theta_4(\tau) \; =$$

$$\sin\tau\int_0^\tau \tilde{G}(\xi)\cos\xi\, d\xi - \cos\tau\int_0^\tau \tilde{G}(\xi)\sin\xi\, d\xi$$

$$= \; J_1\sin\tau - J_2\cos\tau \tag{16}$$

The integrals yield (keeping only the largest term),

$$J_1 \; \sim \; -\frac{8a_1}{112}\tau^7; \; J_2 \; \sim \; \frac{a_1}{4}\tau^6\cos 2\tau \tag{17}$$

Or,

$$\theta_4(\tau) \; \sim \; -\frac{a_1}{14}\tau^7\sin\tau \tag{18}$$

The sign of the amplitude is still negative and exponentiation of time in this case is much larger than the previous and the increase in amplitude, therefore, is much faster. The next order again provides no contribution like $\theta_1,\ \theta_3$, but the next order features a significant change. The equation in the next contributing order, $\mathcal{O}(F^7)$, is,

$$\ddot{\theta}_6(\tau) \ + \ \theta_6(\tau) \ = \ \frac{1}{2}\theta_0\theta_2^2 \ + \ \frac{1}{2}\theta_0^2\theta_4 \ =$$

$$\chi_1(\tau) \ + \ \chi_2(\tau) \ \sim \ a_2\tau^9\sin\tau \ = \ \bar{G}(\tau) \quad (19)$$

As before, this equation leads us to,

$$\theta_6(\tau) \ = \ K_1\sin\tau - K_2\cos\tau$$

The largest term from the above integrals is due to $K_2 \ \sim \ \frac{1}{20}a_2\tau^{10}$. This leads to,

$$\theta_6(\tau) \ \sim \ -\frac{a_2}{20}\tau^{10}\cos\tau \quad (20)$$

The interesting part about this solution is its sign - the amplitude can grow with time *if* the numerical prefactor (which captures the difference between successive order amplitudes in a certain sense) is negative. If we go back to (19), a_2 is like the difference in the 2 different forcing terms and its sign depends on which of the forcing terms is larger.

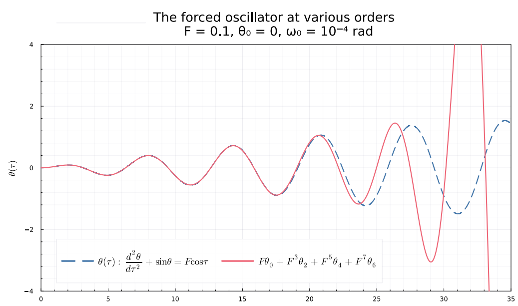

FIG. 3: Solution at $\mathcal{O}(F^7)$. There is a striking match with the original solution for a small initial amount of time.

The above plots depict the solutions obtained using the perturbative approach and compares them with the original solution obtained through numerical integration. Fig. 3 is especially striking and shows a remarkable match between the 2 solutions for a small initial amount of time. In hindsight, this is expected; for smaller times, the solution obtained via the perturbative method will be dominated by the leading order term which is obtained by approximating $sin(\theta) \sim \theta$. A more interesting feature of the solution is highlighted in fig. 2 below. This plot compares the actual solution to the perturbation theory results obtained above. From the figure, it is clear that successively adding higher order terms causes modifications to the amplitude of the total solution. This is especially clear around the $t \sim 5.5$ and $t \sim 6.4$ points, where one can see that adding the $\mathcal{O}(F^7)$ term to the solution causes, in one case, a maxima to become a minima and, in the other, suppresses the magnitude of the $\mathcal{O}(F^7)$ term itself. This shows that successive higher order terms can cause suppression of the amplitude of the total system and may be able to damp the diverging behaviour of the individual solutions themselves even in the absence of external damping mechanisms.

A. Bounds

As a further check, one can look at an easy bound of the solution we have obtained. In general, the n^{th} order solutions are zero if n is odd and non zero if even. In particular, the n^{th} order amplitude can be written as (for $n = 0, 2, 4...$),

$$|A_n| \ = \ \tau^{n/2}(\tau F)^{n+1}\Big|\sin(\tau + \frac{n\pi}{4})\Big| \quad (21)$$

The total amplitude will be bound from above by,

$$S(\tau) = \sum_{n=0}^{\infty}|A_{2n}| = \sum_{n=0}^{\infty}\tau^n(\tau F)^{2n+1}\Big|\sin(\tau + \frac{n\pi}{2})\Big| \quad (22)$$

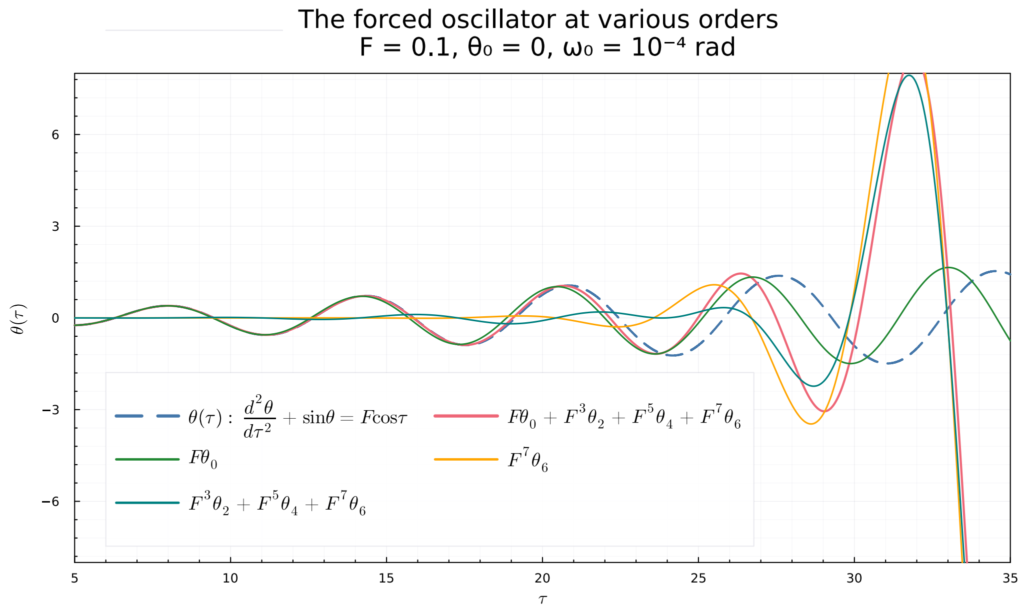

FIG. 2: Comparison of the obtained solutions at various orders with the actual solution.

This is easily evaluated since it is a geometric series,

$$S(\tau) = \tau F\left[|\sin\tau|\sum_{m=0}^{\infty} a^{2m} + |\cos\tau|\sum_{m=0}^{\infty} a^{2m+1}\right] \tag{23}$$

Here, $a = \tau^3 F^2 < 1$ for validity. Thus,

$$S(\tau) \;=\; \tau F\frac{|\sin\tau| + \tau^3 F^2|\cos\tau|}{1-(\tau^3 F^2)^2} \tag{24}$$

Eq. (24) is plotted with respect to τ in Fig. 4 - As one can see, the expression is quite small and rises slowly till the divergence point. The calculations above are only valid up to the point of divergence - however, from Eq. (24) it is clear that F determines the divergence point and the smaller the driving force, the later is the divergence. This means that the validity of our perturbative approach increases (and the upper bound remains small and finite) for longer times as the driving force becomes smaller and smaller.

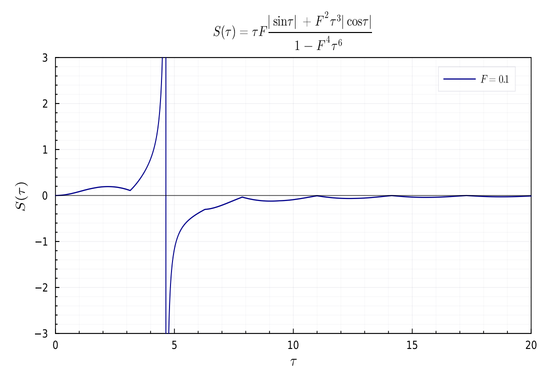

FIG. 4: The upper bound on the answer

For instance, for $F \sim 0.001$, such a bound remains finite and small for around $\tau \sim 100$ - quite a long period of time. It should be noted that the above calculations are for an upper bound (since we cannot concretely talk about the phase differences in every order) - therefore, the divergence point above bears no real meaning for the actual oscillator. However, the upper bound being small and finite before the divergence point does imply that the real answer must be small and finite as well.

III. DISCUSSION

Perturbation theory has usually only been applied to systems around resonance; we have tried to extend the idea to a system at resonance by assuming a small forcing amplitude. This leads to a series of equations with various forcing terms for different orders. We have investigated the first few orders in detail, noting that there is a possibility for different orders to differ in phases. We believe that this can result in a final bounded solution despite the fact that individual solutions at each order may diverge despite the absence of external resistive forces. This idea is strikingly similar to the idea of destructive interference where differing phases between different waves can cause a suppression of the total amplitude (intensity, for light).
Interference is a ubiquitous idea in physics. If we consider the famous double slit experiment (in both optical and quantum mechanical contexts), we find alternating light and dark bands. For light, these represent the variation of intensity and the phenomena is described using wave mechanics where different light waves superpose on each other and either add up to an increased intensity (constructive interference) or subtract to zero intensity (destructive interference). The basic idea lies in the different phases of the superposing waves - two sine waveshalf a wavelength out of phase cancel each other out. But such a sine wave is $\sin(\theta + \pi) = -\sin(\theta)$; therefore, there is a sign difference in the two waves. Something similar is seen here with differing signs in alternative orders - the solutions in different orders are out of phase leading to a cancellation (or, at the very least, a suppression) of the indiidual diverging amplitudes keeping the total amplitude bounded. The exact physical significance of such a phase difference is not very clear to us at the moment. As in section 2, we require the following for the amplitude change in the last order,

$$|\theta_2||\theta_2| < |\theta_0||\theta_4| \tag{25}$$

The above inequality must hold if the amplitude is to flip signs (the modulus denotes the magnitudes of the variable amplitudes). The change of signs in the amplitude is essential for consistency; the system is known to be finite and alternating signs in various orders enable us to obtain solutions via perturbation theoretic methods which are non diverging. However, there seems to be no deep reason why the difference should be of a specific sign leading to the finite solutions.
As mentioned in the beginning, the time scale for which the approximate solution is valid will increase as the forcing term amplitude decreases. Fig. 5 shows two other cases with smaller values of F; it is evident that the timescale of validity has increased significantly.

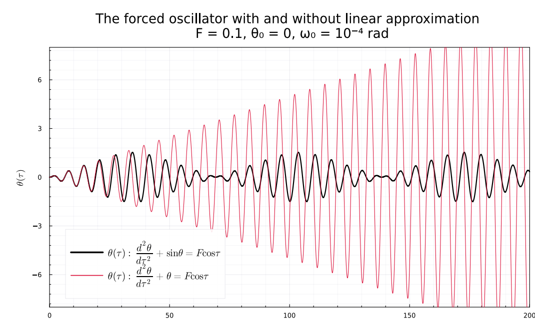

FIG. 6: The forced oscillator with and without the linear approximation

Further, the solutions for the forced oscillator equation (Eq. (1)) are quite different from the solution for the linear approximation case as in Fig. (6); the linearly approximated oscillator diverges, but the actual forced oscillator is bounded. Thus, the equation we have considered does indeed present a different case. One should also note that our approach is different to the standard approach when dealing with nonlinear oscillators like the forced Duffing oscillator (see [10]). We have straightaway

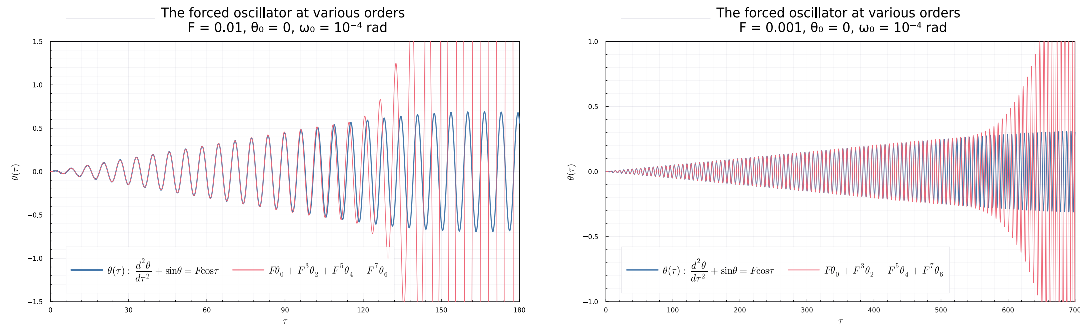

FIG. 5: Comparison of the exact and approximate solutions of the forced oscillator for different F

considered a perturbation in θ with the forcing amplitude acting as the regulator without considering the frequency of the oscillator or the forcing term in the expansion (no additional restraints on the frequency - amplitude relations in the system).

IV. ACKNOWLEDGEMENTS

This work was entirely funded by the authors' respective institutes. SH would like to acknowledge his KVPY fellowship provided by the DST on a yearly basis.

V. CONFLICT OF INTEREST

The authors have no relevant interests to declare.

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