Testing the electroweak phase transition with future collider experiments and gravitational wave observations,

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Abstract

In this report, we take a look at some aspects of spontaneous symmetry breaking and the Higgs mechanism in the context of both the standard model and the two Higgs doublet model. We consider the baryon asymmetry issue and look at electroweak phase transition induced baryogenesis in the two Higgs doublet model as a possible explanation for the problem. Further, we broadly consider two approaches to observationally detecting the viability of such an explanation - first, we calculate the leading loop corrections to the trilinear lightest Higgs self-coupling, λ_{hhh}, due to the heavier Higgs fields that can be probed for at colliders, and, second, we considered the possibility of generating primordial gravitational waves as a result of the bubble collisions during the electroweak phase transition which might be detected by gravitational wave detectors today.

Contents

1 Introduction

Symmetry breaking was first proposed in Anderson in condensed matter systems and was introduced into particle physics by Yoichiro Nambu. In the beginning, the topic was relatively untouched because of the appearance of the massless Nambu Goldstone modes in the theory - such massless scalars had not been observed. In 1964, 3 landmark papers were published simultaneously in the same issue of the *PRL* - these were by Brout and Englert, Guralnik, Hagen and Kibble, and, finally, by Higgs. All three of them discussed the same issue through different perspectives - how symmetry breaking through an additional scalar in gauge theories can lead to mass generation of the associated gauge fields while simultaneously absorbing the massless Nambu Goldstone modes. This new scalar field came to be known as the Higgs field and it was finally discovered at the LHC in 2012 providing closure to the theory of spontaneous symmetry breaking and the Higgs mechanism.

The standard model of particle physics is heavily dependent on the Higgs mechanism. The Glashow - Salam - Weinberg model is capable of explaining the generation of leptonic and gauge boson masses via a single Higgs particle through the spontaneous breakdown of a $SU(2) \times U(1)$ symmetry of the Lagrangian. While extremely successful, the standard model is still incomplete, however. There are several observations that it cannot explain - the most notable of these is the issue of small neutrino masses. There are other issues in particle physics as well - baryogenesis, the baryon asymmetry issue, the hierarchy problem and so on. As a new age of GW astronomy and more sharply honed future colliders rolls around, we will be able to probe at even higher energy scales and, quite possibly, discover even more phenomena outside the reach of the standard model as is. All of the current theories that try to explain our current open problems seek to extend the standard model in one way or the other. Some of these theories are extremely popular - supersymmetric models, extra dimensional theories and so on. Another way is to extend the Higgs sector by introducing additional Higgs fields.

In this report, we look into some areas of the Higgs sector and the extended Higgs sector, specifically the two Higgs doublet model where one introduces only a single additional Higgs doublet. We start by looking at symmetries in quantum field theories in Section 2. In Section 3, we detail spontaneous symmetry breaking and the Higgs mechanism in both Abelian field theories and Yang Mills ones. Section 4 provides a review of the Glashow - Salam - Weinberg model, i.e., the standard model of particle physics. We look into the two Higgs doublet model (THDM) in Section 5 and compute the masses of the additional Higgs scalars. Using the Coleman Weinberg effective potentials, we compute the loop corrections to the trilinear Higgs coupling, λ_{hhh}, arising out of the additional heavier Higgs particles and the top quark. The deviation to these couplings in the THDM can be large compared to the standard model and this provides a viable way to detecting the validity of such models in collider experiments. We look at the baryon asymmetry issue in Section 6 and specifically discuss how electroweak phase transition in the standard model and the THDM can be used to tackle the issue. We follow this up by discussing a second way to probe for heavier Higgs particles and electroweak phase transition induced baryogenesis in Section 7 - through the generation of GWs during bubble collisions and plasma turbulence in these theories at the critical temperature. Finally, Section 8 presents a summary of the report and an overall discussion of the contents.

2 Symmetries in field theories and Noether's Theorem

Symmetries in usual particle mechanics are usually defined as transformations that change a system's Lagrangian at most by a total time derivative. That is, for a transformation, $\mathcal{S}$, to be a symmetry, it must change the Lagrangian as,

$$\mathcal{S} : \mathcal{L} \to \mathcal{L} + \frac{dF}{dt} \tag{1}$$

Associated with these continous symmetries are conservation laws - for each such continuous symmetry, we can find a corresponding conserved quantity. For example, if the system is invariant under time translations, the conserved quantity is the total energy. This is Noether's Theorem.

This holds in the case of field theories as well. According to [1], a symmetry transformation can be defined in the following way. Consider a transformation, $\mathcal{S}$, dependent on a continuous parameter, λ, on the fields in the theory:

$$\mathcal{S} : \varphi^a(x) \to \varphi^a(x, \lambda) \tag{2}$$

We can estimate this change by defining,

$$D\varphi^a = \frac{\partial \varphi^a}{\partial \lambda}|_{\lambda=0} \tag{3}$$

Analogous to the particle case, this transformation is defined as a symmetry only if,

$$\mathcal{S} : D\mathcal{L} = \partial_\mu F^\mu \tag{4}$$

This means that a symmetry transformation can at most change the Lagrangian density by a total 4-divergence. The action remains unchanged under such a transformation assuming proper fall off conditions at the boundary - therefore, the field equations remain invariant as expected.

Noether's theorem can be derived from such a definition by constructing a conserved quantity explicitly. From the above equation,

$$D\mathcal{L} = \frac{\partial \mathcal{L}}{\partial \varphi^a} D\varphi^a + \pi^\mu_a \partial_\mu (D\varphi^a) \tag{5}$$

Here,

$$\pi^\mu_a = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi^a)} \tag{6}$$

The $\mu = 0$ components of the above field are the usual canonical momenta. The usual field equations (or, the Euler Lagrange equations) are given by,

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi^a)} \right) - \frac{\partial \mathcal{L}}{\partial \varphi^a} = \partial_\mu \pi^\mu_a - \frac{\partial \mathcal{L}}{\partial \varphi^a} = 0 \tag{7}$$

Given these quantities, we can construct a current, J^μ, as follows,

$$J^\mu = \pi^\mu_a D\varphi^a - F^\mu \tag{8}$$

Now, if we take the 4-divergence of this quantity,

$$\partial_\mu J^\mu = \left(\partial_\mu \pi_a^\mu\right) D\varphi^a + \pi_a^\mu \partial_\mu (D\varphi^a) - \partial_\mu F^\mu = \frac{\partial \mathcal{L}}{\partial \varphi^a} D\varphi^a + \pi_a^\mu \partial_\mu (D\varphi^a) - \partial_\mu F^\mu \tag{9}$$

From Eq. 5,

$$\partial_\mu J^\mu = D\mathcal{L} - \partial_\mu F^\mu = 0 \tag{10}$$

That is, for a continuous symmetry transformation, we can always construct a conserved current in this way. This is Noether's theorem in the field theoretic case. Now, given this conserved current, we can define the corresponding conserved Noether charge, Q,

$$Q(t) = \int d^3x J^0 \tag{11}$$

If we take the time derivative of the Noether charge, and assume that the current vanishes at the boundaries,

$$\frac{dQ}{dt} = \frac{d}{dt} \int d^3x J^0 = \int d^3x \partial_0 J^0 = -\int d^3x \partial_i J^i \to 0 \tag{12}$$

As expected, the Noether charge is conserved.

2.1 The complex scalar field

Now, these definitions apply not only to spacetime symmetries (like Lorentz transformations), but also to the internal symmetries in the theory. For example, we can look at a complex scalar field. The Lagrangian is given as,

$$\mathcal{L} = \left(\partial_\mu \varphi\right)^* (\partial^\mu \varphi) - m^2 \varphi^* \varphi \tag{13}$$

This Lagrangian has a global $U(1)$ symmetry. Therefore, the field equations are invariant under the following transformation,

$$\varphi \to e^{-i\alpha} \varphi \tag{14}$$

Here, $\alpha \in \mathbb{R}$. As per the earlier definitions, we have, $D\varphi = -i\varphi$ and $D\varphi^* = i\varphi^*$. Since the Lagrangian remains completely unchanged, $D\mathcal{L} = 0 = \partial_\mu F^\mu$ - in the simplest case, we can just put F^μ to 0. Further, $\pi_\mu = \left(\partial_\mu \varphi\right)^*$ and $\pi_\mu^* = \partial_\mu \varphi$. Therefore, as per the earlier definitions, we have,

$$J^\mu = \pi_a^\mu D\varphi^a - F^\mu = (\partial^\mu \varphi)^* (-i\varphi) + (\partial^\mu \varphi)(i\varphi^*) = i[\varphi^* \partial^\mu \varphi - (\partial^\mu \varphi^*)\varphi] \tag{15}$$

The complex scalar field can also be decomposed as,

$$\varphi = \frac{1}{\sqrt{2}}[\varphi_1 + i\varphi_2] \tag{16}$$

Where $\varphi_{1,2}$ are real scalar fields. The Lagrangian becomes,

$$\mathcal{L} = \frac{1}{2}\left[\left(\partial_\mu \varphi_1\right)^2 + \left(\partial_\mu \varphi_2\right)^2\right] - \frac{1}{2} m^2 [\varphi_1^2 + \varphi_2^2] \tag{17}$$

The $U(1)$ symmetry is still manifest here - but in the form of a $SO(2)$ symmetry, i.e., a rotation in the internal $\varphi_1 - \varphi_2$ plane. This means that the Lagrangian is invariant under the following transformation,

$$\begin{pmatrix} \varphi_1' \\ \varphi_2' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \tag{18}$$

We can do the same calculations here:

$$D\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix} = \begin{pmatrix}\varphi_2\\ -\varphi_1\end{pmatrix} \tag{19}$$

Similarly,

$$\pi^{\mu}_{1,2} = \partial^{\mu}\varphi_{1,2} \tag{20}$$

Or,

$$J^{\mu} = (\partial^{\mu}\varphi_1)\varphi_2 - (\partial^{\mu}\varphi_2)\varphi_1 \tag{21}$$

This is exactly the same expression we get if we substitute the decomposition for the complex φ into the earlier expression for the conserved current.

2.2 Localising the $U(1)$ symmetry

The $U(1)$ symmetry mentioned in the last subsection can be promoted to a local version of the same symmetry. In the global case, the parameter α is considered a constant independent of spacetime - when localising the symmetry, we make α a spactime dependent real function. That is, the new transformation is,

$$\varphi \to e^{i\alpha(x)}\varphi \tag{22}$$

The original complex scalar field Lagrangian is not invariant under such a transformation, however; the derivative now also acts on the transformation parameter. As a different example, consider a Dirac field. The original Lagrangian is,

$$\mathcal{L} = \bar{\psi}[i\partial\!\!\!/ - m]\psi \tag{23}$$

This Lagrangian also has a global $U(1)$ symmetry. When we promote the global symmetry to a local version, we end up with,

$$\bar{\psi}\partial\!\!\!/\psi \to \bar{\psi}e^{-i\alpha(x)}\partial\!\!\!/\big(e^{i\alpha(x)}\psi\big) = \bar{\psi}\partial\!\!\!/\psi + i(\partial\!\!\!/\alpha)\bar{\psi}\psi \tag{24}$$

The second term violates the symmetry (since the mass term is still invariant) and causes the local $U(1)$ symmetry to be lost. The way to remedy this is through the **minimal coupling prescription** [2] - we introduce the gauge covariant derivative with an additional vector field to exactly cancel off the second term when the transformation occurs. The covariant derivative is defined as,

$$D_{\mu}\psi \to e^{i\alpha(x)}D_{\mu}\psi \tag{25}$$

Only such a transformation will preserve the symmetry. We choose the ansatz,

$$D_{\mu} = \partial_{\mu} - ieA_{\mu} \tag{26}$$

Here, A_{μ} is the additional vector field we need to introduce to preserve the symmetry. Under a local $U(1)$ transformation, the fields now change like,

$$\psi \to e^{i\alpha(x)}\psi;\ A_{\mu} \to A_{\mu} + \frac{1}{e}\partial_{\mu}\alpha \tag{27}$$

This implies,

$$D_{\mu}\psi \to D'_{\mu}\big(e^{i\alpha(x)}\psi\big) = \big(\partial_{\mu} - ieA_{\mu} - i\partial_{\mu}\alpha\big)\big(e^{i\alpha(x)}\psi\big) = e^{i\alpha(x)}\big(\partial_{\mu} - ieA_{\mu}\big)\psi \tag{28}$$

Therefore, the Lagrangian remains invariant under the local $U(1)$ transformation. Since now we have an additional gauge field in our Lagrangian, we must introduce an appropriate kinetic term to take care of its dynamics as well. Therefore, the complete Lagrangian will be,

$$\mathcal{L} = \bar{\psi}[i\not{D} - m]\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \tag{29}$$

With $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Since the Lagrangian must be invariant under the gauge transformations described above, the gauge field cannot have a mass term (therefore, terms like $m^2 A_\mu A^\mu$ are forbidden due to gauge invariance).

2.3 Yang Mills theories

A similar method can be used to impose gauge invariance on non Abelian gauge theories. Consider a Dirac field ψ. Let us assume that ψ transforms under some representation, U, of a gauge group, G. That is, we assume,

$$\psi \to U\psi \tag{30}$$

Where U is any appropriate representation of G. Let the generators of G be $\{T^a\}$ where $a = 1, 2, ...$ depending on the number of paramters the gauge group depends on. We assume that U is a unitary representation, implying that the generators, T^a, must be Hermitian. The usual Lie algebra holds for the generators,

$$[T^a, T^b] = if^{abc}T^c \tag{31}$$

Here, f^{abc} is the structure constant. For Hermitian generators (and the Cartan metric $\mathrm{Tr}\big(T^a T^b\big) = k\delta^{ab}, k > 0$), the structure constants will be antisymmetric in their indices. We express U in the usual exponential representation of G,

$$U(\alpha) = e^{-i\alpha^a(x)T^a} \tag{32}$$

The summation over a is implied and $\alpha \equiv \alpha^a$ are real parameters. Under this transformation, the field transforms as,

$$\psi \to e^{-i\alpha^a(x)T^a}\psi \tag{33}$$

The Dirac Lagrangian is not invariant under this transformation. Therefore, we define the gauge covariant derivative again,

$$D_\mu\psi \to U\big(D_\mu\psi\big) = e^{-i\alpha^a(x)T^a}D_\mu\psi \tag{34}$$

The ansatz is,

$$D_\mu = \partial_\mu - igA_\mu \tag{35}$$

Therefore,

$$D_\mu\psi \to \big(\partial_\mu - igA'_\mu\big)(U\psi) \tag{36}$$

Equating this to Eq. 34,

$$\partial_\mu U - igA'_\mu U = -igUA_\mu \tag{37}$$

Or,

$$A'_\mu = UA_\mu U^{-1} - \frac{i}{g}(\partial_\mu U)U^{-1} \tag{38}$$

This is how the gauge field needs to transform for a non Abelian theory in order to preserve the local gauge symmetry. For infinitesimal transformations, we can write $U \approx 1 - i\alpha^a(x)T^a$. Substituting in the above expression for the gauge field's transformation,

$$A'_\mu = (1 - i\alpha^a T^a)A_\mu\big(1 + i\alpha^b T^b\big) - \frac{i}{g}\big(-i\partial_\mu \alpha^a T^a\big)\big(1 + i\alpha^b T^b\big) \tag{39}$$

Neglecting all the $\mathcal{O}(\alpha^2)$ terms, we get,

$$A'_\mu = A_\mu + i\alpha^a\left[A_\mu, T^a\right] - \frac{1}{g}(\partial_\mu \alpha^a)T^a \tag{40}$$

We express $A_\mu = A^a_\mu T^a$ in terms of the generators. This leads to,

$$A'^a_\mu T^a = A^a_\mu T^a + i\alpha^b A^c_\mu [T^c, T^b] - \frac{1}{g}(\partial_\mu \alpha^a)T^a = \left[A^a_\mu - \alpha^b A^c_\mu f^{cba} - \frac{1}{g}\big(\partial_\mu \alpha^a\big)\right]T^a \tag{41}$$

Or,

$$A'^a_\mu = A^a_\mu + f^{abc}\alpha^b A^c_\mu - \frac{1}{g}\partial_\mu \alpha^a \tag{42}$$

Therefore, the complete (infinitesimal) transformation is given by,

$$\psi \to (1 - i\alpha^a(x)T^a)\psi;\ A^a_\mu \to A^a_\mu + f^{abc}\alpha^b A^c_\mu - \frac{1}{g}\partial_\mu \alpha^a \tag{43}$$

Or, equivalently,

$$\psi \to U\psi;\ A_\mu \to UA_\mu U^{-1} - \frac{i}{g}(\partial_\mu U)U^{-1} \tag{44}$$

This keeps the Dirac Lagrangian invariant,

$$\mathcal{L}_D = \bar{\psi}[i\not{D} - m]\psi \tag{45}$$

The kinetic portion for the Yang Mills fields are,

$$\mathcal{L}_{YM} = -\frac{1}{2}\mathrm{Tr}\big(F^{\mu\nu}F_{\mu\nu}\big) = -\frac{1}{2}F^{\mu\nu}_a F^b_{\mu\nu}\mathrm{Tr}\big(T^a T^b\big) = -\frac{1}{4}F^{\mu\nu}_a F^a_{\mu\nu} \tag{46}$$

Here, we choose $k = \frac{1}{2}$. The total Lagrangian is given by,

$$\mathcal{L} = \bar{\psi}[i\not{D} - m]\psi - \frac{1}{4}F^{\mu\nu}_a F^a_{\mu\nu} \tag{47}$$

As before, the gauge bosons are massless yet again.

3 Spontaneous symmetry breaking and the Higgs mechanism

In some cases, the ground state of a system might not be invariant under a certain symmetry even though it is manifest in the Lagrangian. For example, consider an infinite chain of spins with the interaction $-\Sigma_{\langle ij\rangle} s_i s_j \cos\theta_{ij}$ where s_i describes the spin of the i^{th} site and θ_{ij} is the angle in between spins i and j. The notation $\langle ij\rangle$ refers to nearest neighbours. In this case, the ground state, the state of minimal energy, will have all the spins pointing up. The interaction term is rotationally invariant (i.e. rotating all the spins by the same angle changes nothing). However, if we were to live on this chain of spins with no knowledge of the outside (that is, the dimensions of the observer are much smaller to that of the chain), we would think that there is a particular "special" direction in space, the direction in which all the spins point. In other words, we would not be able to discern the rotationally invariant nature of the Lagrangian simply because the ground state seemingly violates it. The ground state spins could point in any direction whatsover - all that is required is that all of them must point in the same direction. But once the ground state settles into one specific direction, it appears as if the symmetry is broken. This is what spontaneous symmetry breaking refers to.

This can occur in field theory as well. For example, consider a real scalar field with a φ^4 interaction. This system has $\mathbb{Z}_2$ symmetry, that is, it is invariant under the transformation $\varphi \to -\varphi$. The Lagrangian is given by,

$$\mathcal{L} = \frac{1}{2}\Big[\big(\partial_\mu\varphi\big)^2 - \mu^2\varphi^2\Big] - \frac{\lambda}{4}\varphi^4 \tag{48}$$

We want our potential to be bounded from below - therefore, $\lambda > 0$ always. The minima then occurs at,

$$\frac{dV(\varphi)}{d\varphi} = 0 \Rightarrow \varphi(\mu^2 + \lambda\varphi^2) = 0 \Rightarrow \langle\varphi\rangle = \pm\sqrt{-\frac{\mu^2}{\lambda}} = \pm a \quad \text{(say)} \tag{49}$$

Here, $\langle\varphi\rangle$ denotes the ground state (the vacuum expectation value in quantum field theory). From the above expression we can see that for $\mu^2 > 0$, we have exactly one solution - $\mu^2 = 0$ - because of the square root while there are no such problems if $\mu^2 < 0$. Spontaneous symmetry breaking will occur if the ground state is degenerate - in this case, this implies that $\mu^2 < 0$ for spontaneous symmetry breaking. We usually choose the positive solution as the minima of the system though both choices are equivalent.

Now, in perturbation theory, we usually expand about the minima of the system - in this case, we shift the field by defining $\varphi' = \varphi - \langle\varphi\rangle$. This doesn't change the kinetic term (since $\langle\varphi\rangle$ is a constant), but it modifies the potential term:

$$V(\varphi') = \frac{1}{2}(\varphi' + a)^2\left[-a^2\lambda + \frac{\lambda}{2}(\varphi' + a)^2\right] = \frac{\lambda}{4}\left[\varphi'^2(\varphi' + 2a)^2 - a^4\right] \tag{50}$$

If we consider only the quadratic terms, the mass of the scalar field comes out to be $m^2 = \frac{\lambda}{2}(4a^2) = 2\lambda a^2 = -2\mu^2$ - therefore, the mass has shifted from the earlier expectation of $\lambda a^2 = -\mu^2$. Therefore, the spontaneous breaking of the $\mathbb{Z}_2$ symmetry shifts the mass of the scalar field.

3.1 Spontaneous breaking of a continuous symmetry

Let us consider the complex scalar field with the manifest global $U(1)$ symmetry. Equivalently, when expressed in the form of two real fields, the theory has $SO(2)$ symmetry as seen in the previous section. we have,

$$V(\varphi^\dagger\varphi) = \mu^2(\varphi^\dagger\varphi) + \lambda(\varphi^\dagger\varphi)^2 \tag{51}$$

Here, as before, $\lambda > 0,\ \mu^2 < 0$. In terms of the real fields, the potential takes the form,

$$V(\varphi_1, \varphi_2) = \frac{1}{2}(\varphi_1^2 + \varphi_2^2)\left[\mu^2 + \frac{\lambda}{2}(\varphi_1^2 + \varphi_2^2)\right] \tag{52}$$

The minimal state is defined on the circle,

$$\langle\varphi_1^2\rangle + \langle\varphi_2^2\rangle = a^2 = -\frac{\mu^2}{\lambda} \tag{53}$$

We make the choice $\langle\varphi_1\rangle = a, \langle\varphi_2\rangle = 0$ breaking the degeneracy of the ground state. We now perturbatively expand around this minima by using,

$$\varphi = \frac{1}{\sqrt{2}}(a + \eta(x) + i\xi(x)) \tag{54}$$

In terms of these new fields, the potential becomes,

$$\begin{aligned} V(\eta, \xi) &= \frac{1}{2}\Big((a+\eta)^2 + \xi^2\Big)\left[-a^2\lambda + \frac{\lambda}{2}\Big((a+\eta)^2 + \xi^2\Big)\right] \\ &= \frac{\lambda}{4}(\eta^2 + a^2 + 2\eta a + \xi^2)(\eta^2 - a^2 + 2\eta a + \xi^2) \end{aligned} \tag{55}$$

Comparing the quadratic terms, we find,

$$m_\eta^2 = 2a^2\lambda,\ m_\xi^2 = 0 \tag{56}$$

Therefore, one of the scalars becomes massless as a result of the spontaneous symmetry breaking - these massless modes that arise due to the breaking of a symmetry in the ground state are referred to as **Nambu-Goldstone bosons**.

In general, let us assume that we have a set of real scalar fields, $\Phi = (\varphi_1, \varphi_2, ..., \varphi_n)^T$, that transforms under some group, G, defined by N real parameters, $\{\lambda^k\}$. Then,

$$\Phi \to e^{-i\lambda^k T^k}\Phi \tag{57}$$

This is a global symmetry - therefore, the kinetic term, $\partial_\mu\Phi^\dagger\partial^\mu\Phi$, will remain invariant. Since we are assuming that the Lagrangian remains invariant under this transformation, so must the potential function. Let $\langle\Phi\rangle$ be the minima of the potential. In general, the minima is not invariant under the above transformation, but let us assume that there is a subgroup, H, which does leave the vacuum unchanged. Therefore,

$$U(H)\langle\Phi\rangle = \langle\Phi\rangle \tag{58}$$

Where U is the representation. The generators of H are the unbroken generators while the rest are the spontaneously broken generators. Let us order the generators of G as follows:

$$G : \{T^1, ..., T^m, T^{m+1}, ..., T^N\},\ H : \{T^1, ..., T^m\} \tag{59}$$

That is, the first m generators are the generators common to H and G. Then, by definition,

$$e^{-i\lambda^k T^k}\Phi = \Phi,\ \ k < m+1 \tag{60}$$

Or,

$$\sum_{k=1}^{m} \lambda^k T^k \Phi = 0 \Rightarrow \lambda^k = 0 \tag{61}$$

This means that there are $(n-m)$ directions in which changing the field does not affect the potential at all (since $\lambda^k T^k \Phi = 0$). Therefore, the potential must be flat in these directions - this implies that the second derivative of the potential (which gives the coefficient of the quadratic term) must vanish in these direction. Since the number of directions is $(n-m)$, the number of vanishing masses is also the same - in other words, there are $(n-m)$ massless bosons generated due to spontaneous symmetry breaking. In the previous example, our symmetry group was $SO(2)$ - the number of generators of this group is 1. The number of Goldstone bosons will, therefore, also be 1 - as we saw. In the case of $SO(3)$, the number of generators is 3. The minima is characterised by the equation,

$$\langle\varphi_1^2\rangle + \langle\varphi_2^2\rangle + \langle\varphi_3^2\rangle = a^2 = -\frac{\mu^2}{\lambda} \tag{62}$$

Of the 3 generators of the group, only 1 leaves the vacuum unaffected - therefore, the dimension of H this case is 1. The number of Goldstone bosons is therefore 2 and there's 1 massive scalar in this theory.

3.2 The Higgs Mechanism

Let us consider the complex scalar field again - but this time with a local $U(1)$ symmetry. As seen earlier, the Lagrangian gets modified to include the gauge covariant derivative.

$$\mathcal{L} = \left(D_\mu\varphi\right)^*(D^\mu\varphi) - \mu^2(\varphi^*\varphi) - \lambda(\varphi^*\varphi)^2 - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \tag{63}$$

As before, we have $\lambda > 0, \mu^2 < 0$. The ground state is chosen as earlier, $\langle\varphi\rangle = a$, and the perturbation is given by $\varphi = \frac{1}{\sqrt{2}}(a + \eta(x) + i\xi(x))$. The potential term generates a massive scalar, $\eta(x)$, and a Goldstone boson, $\xi(x)$. The gauge covariant derivative acts as,

$$D_\mu\varphi = \frac{1}{\sqrt{2}}\left[\partial_\mu\eta + i\partial_\mu\xi - ieA_\mu(a+\eta+\xi)\right] \tag{64}$$

If we look at the kinetic term,

$$\begin{aligned}&\left(D_\mu\varphi\right)^*(D^\mu\varphi) = (\partial^\mu + ieA^\mu)\varphi^*(\partial_\mu - ieA_\mu)\varphi = \\ &\frac{1}{2}\Big[\left(\partial_\mu\eta\right)^2 + \left(\partial_\mu\xi - eaA_\mu\right)^2 + e^2A^\mu A_\mu(\xi+\eta)^2\Big]\end{aligned} \tag{65}$$

It looks like the gauge boson has gained a mass but there is an additional cross term. The cross term can be removed if we instead start from the vacuum state, $\varphi = \frac{1}{\sqrt{2}}(a + \eta(x))e^{i\xi/a}$. In this case, the derivative acts as,

$$\begin{aligned} D_\mu\varphi &= \frac{e^{i\xi/a}}{\sqrt{2}}\left[\partial_\mu\eta - ieA_\mu(a+\eta) + \frac{i}{a}(a+\eta)\partial_\mu\xi\right] \\ &= \frac{e^{i\xi/a}}{\sqrt{2}}\left[\partial_\mu\eta - ie(a+\eta)\left(A_\mu - \frac{1}{ae}\partial_\mu\xi\right)\right] \end{aligned} \tag{66}$$

Since this is a gauge invariant theory, we can transform to $\varphi \to e^{-i\xi/a}\varphi$ - this is a specific choice of gauge. By definition,

$$D_\mu\varphi \to e^{-i\xi/a}D_\mu\varphi = \frac{1}{\sqrt{2}}\left[\partial_\mu\eta - ie(a+\eta)\left(A_\mu - \frac{1}{ae}\partial_\mu\xi\right)\right] \tag{67}$$

The gauge field also transforms as $A_\mu \to A_\mu - \frac{1}{ae}\partial_\mu\xi = B_\mu$ (say) - this is exactly the second term in the expression above. Then, we get,

$$\mathcal{L} = \left(D_\mu\varphi\right)^*(D^\mu\varphi) - \mu^2(\varphi^*\varphi) - \lambda(\varphi^*\varphi)^2 - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \tag{68}$$

With $D_\mu = \partial_\mu + ieB_\mu$. The kinetic term can be decomposed to give,

$$(D^\mu\varphi)^*\left(D_\mu\varphi\right) = \ldots + \frac{1}{2}e^2a^2B_\mu B^\mu + \ldots \tag{69}$$

The field, $\xi(x)$, is completely removed from the Lagrangian - it has been absorbed into the gauge field by fixing the gauge. On the other hand, the gauge field itself is now massive with $m_B^2 = e^2a^2$. This is the **Higgs mechanism** - by breaking the symmetry of the ground state, we generated a Goldstone boson, but coupling it to the gauge field and appropriately fixing the gauge makes the Goldstone boson vanish, leaving behind a massive gauge field in turn.

3.3 Symmetry breaking in Yang Mills theories

The steps above can be extended to non Abelian gauge theories as well. We have the Yang Mills Lagrangian along with the scalar part,

$$\mathcal{L} = -\frac{1}{4}F_a^{\mu\nu}F_{\mu\nu}^a + (D^\mu\Phi)^\dagger\left(D_\mu\Phi\right) - \mu^2\left(\Phi^\dagger\Phi\right) - \lambda\left(\Phi^\dagger\Phi\right)^2 \tag{70}$$

As before, we now have a set of real scalar fields rather than one. The gauge covariant derivative is defined as before and the ground state is $\langle\Phi\rangle$. As earlier, we arrange the generators of the gauge group such that the unbroken generators are ordered first and are then followed by the spontaneously broken ones. That is,

$$G = \left\{T^1, T^2, \ldots, T^m, T^{m+1}, \ldots, T^N\right\} \tag{71}$$

And,

$$T^k\langle\Phi\rangle = 0 \quad \text{for } k < m+1 \tag{72}$$

While,

$$T^j\langle\Phi\rangle \neq 0 \quad \text{for } j > m \tag{73}$$

The transformations are represented as $\Phi \to e^{-i\lambda^aT^a}\Phi$. For the ground state, this becomes, $\langle\Phi\rangle \to e^{-i\lambda^jT^j}\langle\Phi\rangle$. For infinitesimal transformations, this is just $\langle\Phi\rangle \to \left(1 - i\lambda^jT^j\right)\langle\Phi\rangle$. In order to completely eliminate the Goldstone bosons from the theory, we impose a gauge choice called the unitary gauge choice. This can be expressed as [1],

$$\Phi' \cdot T^b \langle \Phi \rangle = 0 \tag{74}$$

Here, the primed fields are the gauge transformed fields. The unitary gauge removes all the unphysical degrees of freedom (the Goldstone bosons) and leaves us with massive gauge bosons.

For example, let us consider a theory with local $SU(2)$ symmetry. The Lagrangian is,

$$\mathcal{L} = -\frac{1}{4} F_a^{\mu\nu} F_{\mu\nu}^a + (D^\mu \Phi)^\dagger (D_\mu \Phi) - \mu^2 (\Phi^\dagger \Phi) - \lambda (\Phi^\dagger \Phi)^2 \tag{75}$$

Here, Φ is an $SU(2)$ spin-$\frac{1}{2}$ representation (or a doublet),

$$\Phi = \begin{pmatrix} \varphi_\alpha \\ \varphi_\beta \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \varphi_1 + i\varphi_2 \\ \varphi_3 + i\varphi_4 \end{pmatrix} \tag{76}$$

The transformation law is: $\Phi \to e^{-i\sigma^a \alpha^a(x)/2} \Phi$. Here, σ^a are the usual Pauli matrices. The covariant derivative is given by,

$$D_\mu = \partial_\mu - ig\frac{\sigma^a}{2} W_\mu^a \tag{77}$$

The vacuum is given by,

$$\langle \varphi_1^2 \rangle + \langle \varphi_2^2 \rangle + \langle \varphi_3^2 \rangle + \langle \varphi_4^2 \rangle = a^2 = -\frac{\mu^2}{\lambda} \tag{78}$$

We choose $\langle \varphi_1 \rangle = \langle \varphi_2 \rangle = \langle \varphi_4 \rangle = 0$ and $\langle \varphi_3 \rangle = a$. The unitary gauge is given by,

$$\frac{1}{4} (\varphi_A', \varphi_B')^* \sigma^a \begin{pmatrix} 0 \\ a \end{pmatrix} + \text{h.c.} = 0 \tag{79}$$

This gives us $\text{Re}(\varphi_A') = \text{Im}(\varphi_A') = 0$ and $\text{Re}(\varphi_B') = 0$. The theory has a global $U(1)$ symmetry as well, so we can choose the $U(1)$ gauge to ensure that the only non zero field is $\text{Re}(\varphi_B')$. In other words, the field just away from the minima can be written as,

$$\Phi = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ a + h(x) \end{pmatrix} \tag{80}$$

Now, the gauge boson masses are generated from the kinetic term - specifically the one coupling to $\langle \Phi \rangle$,

$$\left(ig\frac{\sigma^a}{2} W_\mu^a \langle \Phi \rangle \right)^\dagger \left(ig\frac{\sigma^b}{2} W_b^\mu \langle \Phi \rangle \right) = \frac{g^2 a^2}{8} W_\mu^a W_b^\mu \sigma^a \sigma^b \tag{81}$$

Now, $\sigma^a \sigma^b = \delta^{ab} + i\varepsilon^{abc} \sigma^c$.

$$\left(ig\frac{\sigma^a}{2} W_\mu^a \langle \Phi \rangle \right)^\dagger \left(ig\frac{\sigma^b}{2} W_b^\mu \langle \Phi \rangle \right) = \frac{g^2 a^2}{8} W_\mu^a W_a^\mu \tag{82}$$

The second term goes to zero since $W_\mu^a W_b^\mu$ is symmetric in indices (a, b) while ε^{abc} is not. This means that the gauge bosons get a mass of $m_W^2 = \frac{1}{4} g^2 a^2$. All three gauge bosons gain mass - there are no leftover Goldstone bosons in the theory.

The Higgs mechanism can also generate fermion masses if there is a Yukawa kind of interaction in the theory. The Yukawa term is like,

$$\mathcal{L}_{\text{Yukawa}} \propto -g\bar{\psi}\psi\varphi \tag{83}$$

When the field is shifted to the vacuum state after symmetry breaking, we get,

$$\mathcal{L}_{\text{Yukawa}} \propto -g\bar{\psi}\psi(\varphi' + \langle\varphi\rangle) = -g\langle\varphi\rangle\bar{\psi}\psi - g\bar{\psi}\psi\varphi' \tag{84}$$

As the ground state value is a constant for the scalar field, the first term in the above Lagrangian is essentially a fermion mass term. Therefore, the Higgs mechanism can simultaneously account for both the masses of gauge bosons and fermions.

4 The Glashow - Salam - Weinberg Model

The Glashow - Salam - Weinberg (GSW) model presents a theory of the electroweak interaction which is renormalizable and explains the observed gauge boson numbers and properties. As a model of our world, it contains fermionic and scalar fields. Since there are no Goldstone bosons observed in the real world, we must also have the Higgs mechanism and spontaneous symmetry breaking incorporated into it to get massive gauge bosons and remove massless scalar modes.

The GSW model assumes that the complete Lagrangian describing the electroweak interaction is invariant under the action of the gauge group $SU(2)_L \times U(1)_Y$. The first group is called the **weak isospin** with generators T^a $(a = 1, 2, 3)$ while the second group is the Abelian phase transformation group called the **weak hypercharge**. The associated generator is called Y. The generators satisfy the Gellmann - Nishijima relation,

$$Q = T^3 + \frac{1}{2}Y \tag{85}$$

Here, Q is the charge of the fields. There is a further global $U(1)$ symmetry satisfied by the Dirac fields - this is associated with the conservation of lepton number.

4.1 Fields

We have a four parameter group and so, will have four gauge bosons. The isospin generators give rise to W^a_μ while the hypercharge gives us B_μ. The coupling constants associated with the isospin generators is g while that of the hypercharge is $g'/2$. The other fields in the theory are as follows.

To start with, we have the Higgs field, Φ. We make a specific choice of the Higgs scalar - specifically, it is assumed to be a doublet of $SU(2)$,

$$\Phi = \begin{pmatrix} \varphi^+ \\ \varphi^0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \varphi_1 + i\varphi_2 \\ \varphi_3 + i\varphi_4 \end{pmatrix} \tag{86}$$

The charges (as seen above) are assumed to be positive or zero - this gives us $Y_{\text{Higgs}} = 2(Q - T^3) = 1$. The real world consists of one massless boson - the photon - and the model must give rise to it. Therefore, the ground state is assumed to be,

$$\langle \Phi \rangle = \frac{1}{\sqrt{2}} \binom{0}{v} \tag{87}$$

As before, $-v^2\lambda = \mu^2$ with $\lambda > 0, \mu^2 < 0$. This choice of ground state means that the vacuum is invariant under the charge operation, that is, $Q\langle \Phi \rangle = 0$. This means that the vacuum preserves the $U(1)_{EM}$ symmetry and we will have a massless photon as a result.

We must also have fermion fields - these will be leptons and quarks. The left handed and right handed components of fermion fields are obtained as follows,

$$\psi = \psi_L + \psi_, \ \psi_L = \frac{1}{2}(1 - \gamma_5)\psi, \ \psi_R = \frac{1}{2}(1 + \gamma_5)\psi \tag{88}$$

This implies,

$$\bar{\psi}_L = \frac{1}{2}\bar{\psi}(1 + \gamma_5), \ \bar{\psi}_R = \frac{1}{2}\bar{\psi}(1 - \gamma_5) \tag{89}$$

This means that the fermion mass term can be written as,

$$\bar{\psi}\psi = (\bar{\psi}_L + \bar{\psi}_R)(\psi_L + \psi_R) = \bar{\psi}_L\psi_L + \bar{\psi}_R\psi_R + \bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L \tag{90}$$

Since,

$$\bar{\psi}_L\psi_L + \bar{\psi}_R\psi_R = \frac{1}{4}\bar{\psi}[(1+\gamma_5)(1-\gamma_5) + (1-\gamma_5)(1+\gamma_5)]\psi = 0 \tag{91}$$

The mass term only has mixed components,

$$\bar{\psi}\psi = \bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L \tag{92}$$

On the other hand,

$$\bar{\psi}\gamma^\mu\psi = \bar{\psi}_L\gamma^\mu\psi_L + \bar{\psi}_R\gamma^\mu\psi_R \tag{93}$$

We choose a left handed lepton doublet and a right handed singlet,

$$\chi_L^l = \begin{pmatrix}\nu_e \\ e\end{pmatrix}_L, \ \chi_R^l = e_R \tag{94}$$

From the above, $Y_{\chi_L} = -1, \ Y_{e_R} = -2$. The quarks will however include two right handed singlets,

$$\chi_L^q = \begin{pmatrix}u \\ d\end{pmatrix}_L, \ \chi_R^{q(1,2)} = \{u_R, d_R\} \tag{95}$$

The weak hypercharge for the quark doublet is $1/3$ while for the other two it is $4/3, \ 2/3$ respectively. Such choices of the fermion fields will lead to appropriate couplings and mass terms for the fermions and will keep the neutrino massless. Now, since the Lagrangian has to be hypercharge invariant, there are specific hypercharge preserving interaction terms we can include. Specifically,

$$\mathcal{L} = ... - f_l\bar{\chi}_L^l\Phi\chi_R^l - f_d\bar{\chi}_L^q\Phi d_R - f_u\bar{\chi}_L^q\Phi^C u_R + \text{h.c.} + ... \tag{96}$$

Here, the charge conjugated Higgs is given by,

$$\Phi^C = -i\sigma^2\Phi^* = \begin{pmatrix}-\varphi^{*0} \\ \varphi^-\end{pmatrix} \tag{97}$$

The minima of this is at $(v,0)/\sqrt{2}$. This has hypercharge -1 and helps the third term preserve hypercharge symmetry.

Therefore, the complete Lagrangian of the model is given by,

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}F^a_{\mu\nu}F_a^{\mu\nu} + (D^\mu\Phi)^\dagger\big(D_\mu\Phi\big) - \mu^2\Phi^\dagger\Phi - \lambda\big(\Phi^\dagger\Phi\big)^2 \\ &+\bar{\chi}_L^l(i\not{D})\chi_L^l + \bar{\chi}_R^l(i\not{D})\chi_R^l + \bar{\chi}_L^q(i\not{D})\chi_L^q + \bar{\chi}_R^{q1}(i\not{D})\chi_R^{q1} + \bar{\chi}_R^{q2}(i\not{D})\chi_R^{q2} \\ &-\big(f_l\bar{\chi}_L^l\Phi\chi_R^l + f_d\bar{\chi}_L^q\Phi d_R + f_u\bar{\chi}_L^q\Phi^C u_R + \text{h.c.}\big)\end{aligned} \tag{98}$$

4.2 Generation of masses

The gauge covariant derivative acting of the Higgs scalar is expressed as,

$$D_\mu\Phi = \left(\partial_\mu - \frac{1}{2}igW_\mu^a\sigma^a - \frac{1}{2}ig'YB_\mu\right)\Phi \tag{99}$$

Since the Higgs scalar has hypercharge 1, this becomes,

$$D_\mu \Phi = \left(\partial_\mu - \frac{1}{2} i g W^a_\mu \sigma^a - \frac{1}{2} i g' B_\mu\right)\Phi \tag{100}$$

The standard perturbation around the minima is carried out,

$$\Phi = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v + h(x) \end{pmatrix} \tag{101}$$

The unitary gauge is used to remove all the Goldstone degrees of freedom above and leave only the Higgs part. The gauge boson masses are generated from,

$$M^2 = \left(\left(-\frac{1}{2} i g W^a_\mu \sigma^a - \frac{1}{2} i g' B_\mu\right)\langle\Phi\rangle\right)^\dagger \left(\left(-\frac{1}{2} i g W^b_\mu \sigma^b - \frac{1}{2} i g' B_\mu\right)\langle\Phi\rangle\right) \tag{102}$$

Now,

$$\begin{aligned}(g\sigma^a W^a_\mu + g' B_\mu)\langle\Phi\rangle &= \frac{1}{\sqrt{2}}\begin{pmatrix} gW^3_\mu + g'B_\mu & g(W^1_\mu - iW^2_\mu) \\ g(W^1_\mu + iW^2_\mu) & gW^3_\mu - g'B_\mu \end{pmatrix}\begin{pmatrix} 0 \\ v \end{pmatrix} \\ &= \frac{1}{\sqrt{2}}\begin{pmatrix} g(W^1_\mu - iW^2_\mu) \\ gW^3_\mu - g'B_\mu \end{pmatrix}\end{aligned} \tag{103}$$

Thus,

$$M^2 = \frac{v^2}{8}\left[g^2(W^1_\mu + iW^2_\mu)(W^1_\mu - iW^2_\mu) + (gW^3_\mu - g'B_\mu)^2\right] \tag{104}$$

Let us define the new fields, $W^\pm_\mu = \frac{1}{\sqrt{2}}(W^1_\mu \pm iW^2_\mu)$. These are charged gauge bosons. This can be seen by noting that the hypercharge of these bosons is zero (since they belong to disjoint groups) and therefore, $QT^\pm = T^3 T^\pm = \pm T^\pm$. Thus, $W^\pm_\mu$ have equal and opposite charges. Now, in terms of these new charged bosons,

$$\begin{aligned}M^2 &= \frac{g^2 v^2}{4} W^+_\mu W^\mu_- + \frac{v^2}{8}(W^3_\mu, B_\mu)\begin{pmatrix} g^2 & -gg' \\ -gg' & g'^2 \end{pmatrix}\begin{pmatrix} W^3_\mu \\ B_\mu \end{pmatrix} \\ &= \frac{g^2 v^2}{8} W^+_\mu W^\mu_- + \frac{v^2}{8}(W^3_\mu, B_\mu) M^2_G \begin{pmatrix} W^3_\mu \\ B_\mu \end{pmatrix}\end{aligned} \tag{105}$$

The eigenvalues of M^2_G are given by,

$$m^2_G = 0, g^2 + g'^2 \tag{106}$$

One of the eigenvalues is zero - therefore, one of the gauge modes is massless. This is what was required to explain the photon. We can find the eigenvectors of the matrix to find the physically realisable states - the orthonormal eigenvectors are,

$$\frac{1}{\sqrt{g^2 + g'^2}}\begin{pmatrix} g \\ -g' \end{pmatrix}, \ \frac{1}{\sqrt{g^2 + g'^2}}\begin{pmatrix} g' \\ g \end{pmatrix} \tag{107}$$

The diagonalising matrix for M^2_G is then given by,

$$\frac{1}{\sqrt{g^2+g'^2}}\begin{pmatrix} g & g' \\ -g' & g \end{pmatrix} \tag{108}$$

Therefore, the physically realisable fields are,

$$\begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix} = \frac{1}{\sqrt{g^2+g'^2}}\begin{pmatrix} g & -g' \\ g' & g \end{pmatrix}\begin{pmatrix} W_\mu^3 \\ B_\mu \end{pmatrix} = \frac{1}{\sqrt{g^2+g'^2}}\begin{pmatrix} gW_\mu^3 - g'B_\mu \\ g'W_\mu^3 + gB_\mu \end{pmatrix} \tag{109}$$

In the diagonal form,

$$M^2 = \frac{g^2v^2}{4}W_\mu^+W_-^\mu + \frac{1}{8}v^2\left(g^2+g'^2\right)Z^\mu Z_\mu + \frac{1}{8}\times 0 \times A_\mu A^\mu \tag{110}$$

The masses can be read off from the coefficients of the corresponding quadratic terms,

$$m_W^2 = \frac{g^2v^2}{4}, m_Z^2 = \frac{1}{4}v^2\left(g^2+g'^2\right), m_A^2 = 0 \tag{111}$$

Therefore, the theory gives us two charged massive bosons, one neutral massive boson and a massless neutral boson as well. This is in agreement with the real world.

Now, for the the Dirac mass terms. After spontaneous symmetry breaking, the Yukawa interaction for leptons can be written as,

$$\mathcal{L}^l_{\text{Yukawa}} = -f_l\left(\bar{\chi}^l_L\langle\Phi\rangle e_R + \bar{e}_R\langle\Phi\rangle\chi^l_L\right) = -\frac{f_l v}{\sqrt{2}}[\bar{e}_L e_R + \bar{e}_R e_L] \tag{112}$$

This is a mass term for the leptons - therefore,

$$m_l = \frac{vf_l}{\sqrt{2}} \tag{113}$$

Masses for the quarks can also be generated from the Higgs scalar in the same way,

$$\mathcal{L}^q_{\text{Yukawa}} = -f_d\bar{\chi}^q_L\langle\Phi\rangle d_R - f_u\bar{\chi}^q_L\langle\Phi^C\rangle u_R + \text{h.c.} \tag{114}$$

From this, we can read off the quark masses,

$$m_u = \frac{vf_u}{\sqrt{2}},\ m_d = \frac{vf_d}{\sqrt{2}} \tag{115}$$

In this manner, the model can account for both the lepton and the quark masses as well.

4.2.1 Weinberg mixing angle

We can also define the define a parameter called the Weinberg mixing angle for the gauge bosons. This is defined as,

$$\cos\theta_W = \frac{g}{\sqrt{g^2+g'^2}}, \sin\theta_W = \frac{g'}{\sqrt{g^2+g'^2}} \tag{116}$$

This enables us to write,

$$\begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix} = \frac{1}{\sqrt{g^2+g'^2}}\begin{pmatrix} gW_\mu^3 - g'B_\mu \\ g'W_\mu^3 + gB_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_W W_\mu^3 - \sin\theta_W B_\mu \\ \sin\theta_W W_\mu^3 + \cos\theta_W B_\mu \end{pmatrix} \tag{117}$$

We can further define the parameter, ρ, which is a measure of the relative strength of the weak neutral and charged current processes,

$$\rho = \frac{m_W^2}{m_Z^2 \cos^2\theta_W} \tag{118}$$

In the GWS model, this parameter is exactly equal to unity,

$$\rho = \frac{m_W^2}{m_Z^2 \cos^2\theta_W} = \frac{g^2 v^2}{v^2\big(g^2 + {g'}^2\big)} \times \frac{{g'}^2 + g^2}{g^2} = 1 \tag{119}$$

The experimentally determined value of ρ is quite close to 1 as well. Now, since the Gellmann - Nishijima relation is satisfied,

$$j_\mu^{EM} = j_\mu^3 + \frac{1}{2} j_\mu^Y \tag{120}$$

We can also consider the fermion and boson interaction terms. In general, these are of the form,

$$\bar{\psi}\left[-\frac{i}{2} g T^a \not{W}^a - \frac{i}{2} g' y \not{B}\right]\psi \tag{121}$$

The complete interaction terms are,

$$\begin{aligned} &\bar{\chi}_L^l\left[-\frac{i}{2} g \sigma^a \not{W}^a + \frac{i}{2} g' \not{B}\right]\chi_L^l + \bar{\chi}_L^q\left[-\frac{i}{2} g\sigma^a \not{W}^a - \frac{i}{6} g' \not{B}\right]\chi_L^q \\ &+\bar{e}_R(ig'\not{B})e_R + \bar{u}_R\left(-2\frac{i}{3} g'\not{B}\right)u_R + \bar{d}_R\left(\frac{i}{3}g'\not{B}\right)d_R \end{aligned} \tag{122}$$

Consider only the W_μ^3 and B_μ bosons. We can write the above as,

$$\begin{aligned} &-\frac{i}{2} g\big(\bar{\chi}_L^l \sigma^3 \gamma^\mu \chi_L^l + \bar{\chi}_L^q \sigma^3 \gamma^\mu \chi_L^q\big) W_\mu^3 \\ &+ig'\left(\frac{1}{2}\bar{\chi}_L^l \gamma^\mu \chi_L^l - \frac{1}{6}\bar{\chi}_L^q \gamma^\mu \chi_L^q + \bar{e}_R \gamma^\mu e_R - \frac{2}{3}\bar{u}_R \gamma^\mu u_R + \frac{1}{3}\bar{d}_R \gamma^\mu d_R\right) B_\mu \\ &= -ig j_\mu^3 W_3^\mu - \frac{1}{2} i g' j_\mu^Y B^\mu \end{aligned} \tag{123}$$

From the earlier expression for Z_μ, A_μ in terms of the Weinberg mixing angle, the current interaction can be written as,

$$\begin{aligned} -igj_\mu^3 W_3^\mu - \frac{1}{2} i g' j_\mu^Y B^\mu &= -i\left(g\sin\theta_W j_\mu^3 + g'\cos\theta_W \frac{j_\mu^Y}{2}\right)A^\mu - i\left(g\cos\theta_W j_\mu^3 - g'\sin\theta_W \frac{j_\mu^Y}{2}\right)Z^\mu \\ &= -iej_\mu^{EM} A^\mu - i\left(g\cos\theta_W j_\mu^3 - g'\sin\theta_W\frac{j_\mu^Y}{2}\right)Z^\mu \end{aligned} \tag{124}$$

This implies,

$$e = g\sin\theta_W = g'\cos\theta_W \tag{125}$$

This relation can be used to look at the renormalization group equations for different interactions (electroweak and strong) and determine the appropriate unification scale of a "unified strong-electroweak" interaction, guiding experimental probes into the relevant energy scales.

5 The extended Higgs sector

In the standard model of particle physics, the Higgs boson is introduced as an $SU(2)$ doublet with hypercharge unity. The vacuum expectation value of the Higgs spontaneously breaks the $SU(2)_L \times U(1)_Y$ symmetry of the Lagrangian and leads to Nambu Goldstone modes. These modes are then "absorbed" by the gauge bosons generating the symmetry which, consequently, become massive. In this manner, the theory leads to three massive gauge bosons (two charged and one neutral) and one massless boson (the photon - also neutral). Through various Yukawa couplings, the Higgs scalar also leads to massive fermions (leptons and quarks) in the standard model.

While very successful, the standard model cannot explain every observation - for example, we know from experiments at the Kamiokande and SNO that the solar neutrino deficit observed in these detectors is a sign of neutrino oscillations, as postulated by Bruno Pontecorvo. Neutrino oscillations, however, require neutrinos to be massive as well - in the standard model, however, neutrinos are massless. There are other areas where the standard model is not completely correct as well. One way to tackle these issues is by introducing an extra Higgs particle - this is often called the extended Higgs sector. One such model in the extended Higgs sector is the two Higgs doublet model (THDM).

The Higgs boson was discovered at the LHC in 2012. The introduction of additional Higgs particles can modify the existing standard model couplings and masses at higher orders and affect cross sections and scattering amplitudes - in this manner, we can experimentally look for indirect signals of the other Higgs scalars aside from directly detecting these particles in colliders. Here, following [3], we look at the masses of the various Higgs modes in the THDM and how they can affect the different couplings in the theory.

5.1 The two Higgs doublet model

The two Higgs doublet model consists of two complex $SU(2)$ Higgs doublets with hypercharge 1. There is a further symmetry - the Lagrangian remains invariant under $\Phi_1 \to \Phi_1$, $\Phi_2 \to -\Phi_2$. There are two further divisions - in Model I, Φ_2 is solely responsible for generating quark and charged lepton masses while in Model II, Φ_1 generates the masses of down quarks and charged leptons while Φ_2 generates up quark masses. As in [3], only Model II is considered.

The potential is given by,

$$\begin{aligned} V_{\text{THDM}} = m_1^2 \, |\Phi_1|^2 + m_2^2 \, |\Phi_2|^2 + \frac{\lambda_1}{2} \, |\Phi_1|^4 + \frac{\lambda_2}{2} \, |\Phi_2|^2 - m_3^2 \Big(\Phi_1^\dagger \Phi_2 + \text{h.c.}\Big) \\ + \lambda_3 \, |\Phi_1|^2 \, |\Phi_2|^2 + \lambda_4 \, |\Phi_1^\dagger \Phi_2|^2 + \frac{\lambda_5}{2} \left[\Big(\Phi_1^\dagger \Phi_2\Big)^2 + \text{h.c.}\right] \end{aligned} \tag{126}$$

The eight parameters in the above expression are all assumed to be real (though, in general, m_3^2 and λ_5 can be complex - a redefinition of the field varibales can only absorb one of those phases and so, one physical phase always remains in the problem). The vacuum expectation values of the Higgs fields are assumed to be v_i with $i = 1, 2$ (we assume that both are non-zero). The vacuum expectation value of the Higgs field is around 246 [GeV], so we assume that $v = \sqrt{v_1^2 + v_2^2} \approx 246$ [GeV]. When we expand perturbatively, we can parametrise the Higgs fields as,

$$\Phi_i = \begin{pmatrix} w_i^+ \\ \frac{1}{\sqrt{2}}(v_i + h_i + i z_i) \end{pmatrix} \tag{127}$$

The upper fields are charged complex fields, while the lower ones are neutral. Now, we know that at the minima, the derivative of the potential must vanish. This gives us two equations,

$$m_1^2 = m_3^2\frac{v_2}{v_1} - \frac{\lambda_1}{2}v_1^2 - \frac{1}{2}\lambda v_2^2 \tag{128}$$

$$m_2^2 = m_3^2\frac{v_1}{v_2} - \frac{\lambda_2}{2}v_2^2 - \frac{1}{2}\lambda v_1^2 \tag{129}$$

Here, $\lambda = \lambda_3 + \lambda_4 + \lambda_5$. These equations allow us to eliminate $m_{1,2}$ from the problem. Now, in order to differentiate between the massive modes in the theory and the Nambu Goldstone modes, we have to diagonalise the mass matrix for all the fields. For this, we need to rewrite the potential function in Eq. 126 in terms of the fields in Eq. 127. The contributions of the various terms are roughly,

$$\boldsymbol{m_i^2} \to |\Phi_i|^2 = |w_i|^2 + \frac{1}{2}\Big[(v_i+h_i)^2 + z_i^2\Big] \tag{130}$$

$$\frac{\boldsymbol{\lambda_i}}{\mathbf{2}} \to |\Phi_i|^4 = |w_i|^4 + \frac{1}{4}\Big[(v_i+h_i)^2 + z_i^2\Big]^2 + |w_i|^2\ \Big[(v_i+h_i)^2 + z_i^2\Big] \tag{131}$$

$$\boldsymbol{m_3^2} \to \Phi_1^\dagger\Phi_2 + \text{h.c.} = (w_1^* w_2 + w_2^* w_1) + [(v_1+h_1)(v_2+h_2) + z_1 z_2] \tag{132}$$

$$\begin{aligned}\boldsymbol{\lambda_3} \to |\Phi_1|^2\ |\Phi_2|^2 &= |w_1|^2\ |w_2|^2 + \frac{1}{4}\Big[(v_1+h_1)^2 + z_1^2\Big]\Big[(v_2+h_2)^2 + z_2^2\Big] \\ &+\frac{1}{2}\ |w_1|^2\ \Big[(v_2+h_2)^2 + z_2^2\Big] + \frac{1}{2}\ |w_2|^2\ \Big[(v_1+h_1)^2 + z_1^2\Big]\end{aligned} \tag{133}$$

$$\begin{aligned}\boldsymbol{\lambda_4} \to |\Phi_1^\dagger\Phi_2|^2 &= \Big[w_1^* w_2 + \frac{1}{2}(v_1+h_1-iz_1)(v_2+h_2+iz_2)\Big] \\ &\times\Big[w_2^* w_1 + \frac{1}{2}(v_2+h_2-iz_2)(v_1+h_1+iz_1)\Big]\end{aligned} \tag{134}$$

$$\begin{aligned}\frac{\boldsymbol{\lambda_5}}{\mathbf{2}} \to \left(\Phi_1^\dagger\Phi_2\right)^2 + \text{h.c.} &= \Big[(w_1^* w_2)^2 + (w_2^* w_1)^2\Big] + \frac{1}{4}(v_1+h_1-iz_1)^2(v_2+h_2+iz_2)^2 \\ &+\frac{1}{4}(v_2+h_2-iz_2)^2(v_1+h_1+iz_1)^2 + w_1^* w_2(v_1+h_1-iz_1)(v_2+h_2+iz_2) \\ &+w_2^* w_1(v_2+h_2-iz_2)(v_1+h_1+iz_1)\end{aligned} \tag{135}$$

Let us start with (z_1, z_2). The relevant quadratic terms in the Lagrangian then are,

$$\mathcal{L}^{(1)} = \frac{1}{2}(m_3^2 - v_1 v_2\lambda_5)(z_1, z_2)\begin{pmatrix} v_2/v_1 & -1 \\ -1 & v_1/v_2\end{pmatrix}\begin{pmatrix} z_1 \\ z_2\end{pmatrix} \tag{136}$$

The eigenvalues of this matrix are $(v_1^2+v_2^2)/(v_1 v_2)$ and 0; therefore, there we get one massless mode from this matrix. In order to diagonalise this matrix, we can use any two dimensional rotation matrix, $\mathcal{R}(\beta)$,

$$M^2_{\text{diag}} = \mathcal{R}^{-1}(\beta)M^2\mathcal{R}(\beta) \Rightarrow \text{Put off diagonal term to 0} \tag{137}$$

This gives us $\tan\beta = v_2/v_1$. This is angle that diagonalises the (z_1, z_2) mass matrix. With this, we also have,

$$\sin\beta = \frac{v_2}{\sqrt{v_1^2+v_2^2}}, \quad \cos\beta = \frac{v_1}{\sqrt{v_1^2+v_2^2}} \tag{138}$$

The eigenvectors (the physical fields) are given by,

$$\begin{pmatrix} z \\ A \end{pmatrix} = \mathcal{R}^{-1}(\beta)\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \tag{139}$$

The masses of the physical fields are,

$$\begin{aligned} m_A^2 &= (m_3^2 - v_1 v_2 \lambda_5) \times \frac{v_1^2+v_2^2}{v_1 v_2} = m_3^2 \frac{\sqrt{v_1^2+v_2^2}}{v_1}\frac{\sqrt{v_1^2+v_2^2}}{v_2} - v^2\lambda_5 \\ &= \frac{m_3^2}{\sin\beta\cos\beta} - v^2\lambda_5 = M^2 - v^2\lambda_5 \end{aligned} \tag{140}$$

$$m_z^2 = 0 \tag{141}$$

Here, $M^2 = m_3^2/\sin\beta\cos\beta$. We can similarly diagonalise the other fields. The complex charged fields have quadratic terms of the form,

$$\mathcal{L}^{(2)} = \left[m_3^2 - \frac{v_1 v_2}{2}(\lambda_4+\lambda_5)\right](w_1^-, w_2^-)\begin{pmatrix} v_2/v_1 & -1 \\ -1 & v_1/v_2 \end{pmatrix}\begin{pmatrix} w_1^+ \\ w_2^+ \end{pmatrix} \tag{142}$$

Therefore, as the matrix itself is identical upto a multiplicative factor, the same angle, β, diagonalises these fields as well. As before, one of the eigenvalues vanishes and therefore, we get another massless mode. The physical fields this time are,

$$\begin{pmatrix} w^+ \\ H^+ \end{pmatrix} = \mathcal{R}^{-1}(\beta)\begin{pmatrix} w_1^+ \\ w_2^+ \end{pmatrix} \tag{143}$$

The non vanishing mass is given by,

$$m_{H^\pm}^2 = \left[m_3^2 - \frac{v_1 v_2}{2}(\lambda_4+\lambda_5)\right] \times \frac{v_1^2+v_2^2}{v_1 v_2} = M^2 - \frac{1}{2}v^2\lambda_{45} \tag{144}$$

Here, $\lambda_{45} = \lambda_4 + \lambda_5$. Finally, let us consider the (h_1, h_2) fields. The relevant quadratic terms in the Lagrangian then are,

$$\mathcal{L}^{(3)} = \frac{1}{2}(h_1, h_2)\begin{pmatrix} m_3^2\frac{v_2}{v_1} + \lambda_1 v_1^2 & -m_3^2 + v_1 v_2 \lambda \\ -m_3^2 + v_1 v_2\lambda & m_3^2\frac{v_1}{v_2} + \lambda_2 v_2^2 \end{pmatrix}\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \tag{145}$$

To diagonalise these fields, we would need to introduce a second angle, α. We can rotate the fields directly by α or we can do it in steps by first rotating by β and then by $\xi = \alpha - \beta$. The answers will be the same since,

$$\mathcal{R}(\alpha-\beta)\mathcal{R}(\beta) = \mathcal{R}(\alpha-\beta+\beta) = \mathcal{R}(\alpha) \tag{146}$$

Following [3], we adopt the second approach. If we rotate the fields by β, the mass matrix becomes,

$$M_\beta^2 = \begin{pmatrix} M_{11}^2 & M_{12}^2 \\ M_{12}^2 & M_{22}^2 \end{pmatrix} \tag{147}$$

Where,

$$M_{11}^2 = v^2[\lambda_1 \cos^4\beta + \lambda_2 \sin^4\beta + 2\lambda \sin^2\beta \cos^2\beta] \tag{148}$$

$$M_{22}^2 = M^2 + v^2 \sin^2\beta \cos^2\beta[\lambda_1 + \lambda_2 - 2\lambda] = M^2 + \frac{v^2}{8}(1 - \cos 4\beta)[\lambda_1 + \lambda_2 - 2\lambda] \tag{149}$$

$$M_{12}^2 = v^2 \cos\beta \sin\beta[-\lambda_1 \cos^2\beta + \lambda_2 \sin^2\beta + \lambda \cos 2\beta] \tag{150}$$

This doesn't diagonalise the (h_1, h_2) mass matrix. Therefore, we apply a second rotation using $\mathcal{R}(\alpha - \beta)$,

$$\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \mathcal{R}(\beta)\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \mathcal{R}(\alpha)\begin{pmatrix} H \\ h \end{pmatrix} \tag{151}$$

Therefore, the mass matrices transform as,

$$M_{h_1h_2}^2 \to \mathcal{R}^{-1}(\beta) M_{\varphi_1\varphi_2}^2 \mathcal{R}(\beta) \to \mathcal{R}^{-1}(\alpha - \beta) M_{Hh}^2 \mathcal{R}(\alpha - \beta) \tag{152}$$

Putting the off diagonal terms to zero, we have (with $\xi = \alpha - \beta$),

$$\tan 2\xi = \frac{2M_{12}^2}{M_{11}^2 - M_{22}^2} \tag{153}$$

The eigenvalues of this matrix give us the masses for the fields,

$$m_H^2 = M_{11}^2 \cos^2\xi + M_{22}^2 \sin^2\xi + M_{12}^2 \sin 2\xi \tag{154}$$

$$m_h^2 = M_{11}^2 \sin^2\xi + M_{22}^2 \cos^2\xi - M_{12}^2 \sin 2\xi \tag{155}$$

These two fields are defined so that $m_h \lesssim m_H$. Now, as $M^2 \to \infty$, only M_{22} diverges, or, $\tan 2\xi \to 0$. This implies that $\sin\xi \to 0$ or ± 1. We choose $\sin\xi \to -1$ (which means $\cos\xi \to 0$). In this situation, $m_h^2 \to M_{11}^2$ and $m_H^2 \to M_{22}^2 \approx M^2$ (as it goes to infinity). This is called "decoupling" - in the sense that the total Lagrangian separates into a standard model part and a beyond-standard-model part. In this case, we recover standard model values for the different observables. For small M^2 (i.e. $M^2 \lesssim \lambda_i v^2$), however, we can have a "non-decoupling" limit where the additional Higgs fields can contribute to the measured observables, allowing for indirect signals.

Now, to summarise, the masses of the fields are,

$$m_A^2 = M^2 - v^2\lambda_5 \tag{156}$$

$$m_{H^\pm}^2 = M^2 - \frac{1}{2}v^2[\lambda_4 + \lambda_5] \tag{157}$$

$$m_H^2 = M_{11}^2 \cos^2\xi + M_{22}^2 \sin^2\xi + M_{12}^2 \sin 2\xi \tag{158}$$

$$m_h^2 = M_{11}^2 \sin^2\xi + M_{22}^2 \cos^2\xi - M_{12}^2 \sin 2\xi \tag{159}$$

In the end, we get massive charged scalars ($H^\pm$), a massive neutral pseudo-scalar (A) and two massive neutral scalars (H and h). As in the minimal Higgs model, we have massless charged scalars and a massless pseudo-scalar which are the Nambu Goldstone modes - these will give mass to the gauge bosons in the theory. Now, we can rewrite the coupling constants in terms of these physical measurable parameters as,

$$\lambda_1 = \frac{1}{v^2 \cos^2\beta}\left[m_H^2 \cos^2\alpha + m_h^2 \sin^2\alpha - M^2 \sin^2\beta\right] \tag{160}$$

$$\lambda_2 = \frac{1}{v^2 \sin^2\beta}\left[m_H^2 \sin^2\alpha + m_h^2 \cos^2\alpha - M^2 \cos^2\beta\right] \tag{161}$$

$$\lambda_3 = \frac{1}{v^2}\left[2m_{H^\pm}^2 - M^2 + \frac{\sin 2\alpha}{\sin 2\beta}(m_H^2 - m_h^2)\right] \tag{162}$$

$$\lambda_4 = \frac{1}{v^2}\left[M^2 + m_A^2 - 2m_{H^\pm}^2\right] \tag{163}$$

$$\lambda_5 = \frac{1}{v^2}\left[M^2 - m_A^2\right] \tag{164}$$

This allows us to determine the coupling constants provided we have the measured values of the mass scales and mixing angles.

5.2 The effective potential method

Using the effective potential method, we can determine the how the coupling constant for the hhh coupling looks like in the standard model. The effective potential is given by [3,4],

$$V_{\text{eff}}[\varphi] = V_{\text{tree}}[\varphi] + \frac{1}{64\pi^2}N_c N_s(-1)^{2s} M_f^4[\varphi]\left[\ln\left(\frac{M_f^2}{Q^2}\right) - \frac{3}{2}\right] \tag{165}$$

Here, $\varphi = v + \langle h \rangle$, N_c is the color number, s is the spin and N_s is the spin degree of freedom of the field f. $M_f[\varphi]$ is a field dependent mass term for f and Q is an arbitrary mass scale.

The conditions we can impose are as follows,

$$\frac{\partial V_{\text{eff}}}{\partial \varphi}\ |_{\varphi=v} = 0 \Rightarrow \text{condition for extrema} \tag{166}$$

$$\frac{\partial^2 V_{\text{eff}}}{\partial \varphi^2}\ |_{\varphi=v} = m_h^2 \Rightarrow \text{condition for mass} \tag{167}$$

$$\frac{\partial^3 V_{\text{eff}}}{\partial \varphi^3}\ |_{\varphi=v} = \lambda_{hhh} \Rightarrow \text{condition for coupling} \tag{168}$$

We look at two cases - the contribution of the top quark (standard model like) and further, the contribution of the heavier Higgs particles (non standard model like) to the trilinear lightest Higgs coupling, λ_{hhh}.

5.2.1 Top quark contribution

We want to determine the higher order contributions to the coupling constant in the standard model due to the top quark, for example. For the top quark, $N_c = 3$ and $N_s = 2$. The mass term becomes,

$$M_t[\varphi] = y_t \frac{\varphi}{\sqrt{2}} \tag{169}$$

We expand the effective potential about the minima - after writing $\varphi = (v+h)/\sqrt{2}$.

$$\begin{aligned} V_{\text{eff}} &= -\frac{\mu^2}{2}(v+h)^2 + \frac{\lambda}{4}(v+h)^4 - \frac{1}{64\pi^2}N_c N_s \frac{y_t^4 (v+h)^4}{4}\left[\ln\left(\frac{y_t^2(v+h)^2}{2Q^2}\right) - \frac{3}{2}\right] \\ &= -\frac{\mu^2}{2}(v+h)^2 + \frac{\Lambda}{4}(v+h)^4 - \frac{N_c N_s}{64\pi^2}\frac{y_t^4 v^4}{2}\left(1+\frac{h}{v}\right)^4 \ln\left(1+\frac{h}{v}\right) \end{aligned} \tag{170}$$

Here,

$$\Lambda = \lambda - \frac{N_c N_s}{64\pi^2} y_t^4 \left[\ln\left(\frac{y_t^2 v^2}{2Q^2}\right) - \frac{3}{2}\right] \tag{171}$$

Now, we expand,

$$\begin{aligned} &\left(1+\frac{h}{v}\right)^4 \ln\left(1+\frac{h}{v}\right) \\ &= \left[1 + 4\frac{h}{v} + 6\left(\frac{h}{v}\right)^2 + 4\left(\frac{h}{v}\right)^3 + \left(\frac{h}{v}\right)^4\right]\left[\frac{h}{v} - \frac{1}{2}\left(\frac{h}{v}\right)^2 + \frac{1}{3}\left(\frac{h}{v}\right)^3 + ...\right] \\ &= \frac{h}{v} + \frac{7}{2}\left(\frac{h}{v}\right)^2 + \frac{13}{3}\left(\frac{h}{v}\right)^3 + ... \end{aligned} \tag{172}$$

Therefore,

$$V_{\text{eff}} = -\frac{\mu^2}{2}(v+h)^2 + \frac{\Lambda}{4}(v+h)^4 - \frac{N_c N_s}{64\pi^2}\frac{y_t^4 v^4}{2}\left[\frac{h}{v} + \frac{7}{2}\left(\frac{h}{v}\right)^2 + \frac{13}{3}\left(\frac{h}{v}\right)^3 + ...\right] \tag{173}$$

The derivatives will be,

$$\frac{\partial V_{\text{eff}}}{\partial h} = -\mu^2(v+h) + \Lambda(v+h)^3 - \frac{N_c N_s}{64\pi^2}\frac{y_t^4 v^3}{2}\left[1 + 7\frac{h}{v} + 13\left(\frac{h}{v}\right)^2 + ...\right] \tag{174}$$

$$\frac{\partial^2 V_{\text{eff}}}{\partial h^2} = -\mu^2 + 3\Lambda(v+h)^2 - \frac{N_c N_s}{64\pi^2}\frac{y_t^4 v^2}{2}\left[7 + 26\left(\frac{h}{v}\right) + ...\right] \tag{175}$$

$$\frac{\partial^3 V_{\text{eff}}}{\partial h^3} = 6\Lambda(v+h) - \frac{N_c N_s}{64\pi^2}\frac{y_t^4 v}{2}[26 + ...] \tag{176}$$

From the first two equations, we have when $h = 0$,

$$-\mu^2 v + \Lambda v^3 - \frac{N_c N_s}{64\pi^2}\frac{y_t^4 v^3}{2} = 0 \Rightarrow \mu^2 = \Lambda v^2 - \frac{N_c N_s}{64\pi^2}\frac{y_t^4 v^2}{2} \tag{177}$$

And,

$$-\mu^2 + 3\Lambda v^2 - 7\frac{N_c N_s}{64\pi^2}\frac{y_t^4 v^2}{2} = m_h^2 \Rightarrow \Lambda v = \frac{1}{2}\left[\frac{m_h^2}{v} + 6\frac{N_c N_s}{64\pi^2}\frac{y_t^4 v^2}{2}\right] \tag{178}$$

Substituting these in the third equation gives us,

$$\frac{\partial^3 V_{\text{eff}}}{\partial h^3}\ |_{h=0} = 3\Lambda v - 26\frac{N_c N_s}{64\pi^2}\frac{y_t^4 v^2}{2} = \frac{3m_h^2}{v}\left[1 - \frac{N_c N_s}{64\pi^2 m_h^2 v^2}\frac{y_t^4 v^4}{4} \times \frac{16}{3}\right] \tag{179}$$

For $h = 0$, $m_t = y_t v/\sqrt{2}$. Thus, we have,

$$\frac{\partial^3 V_{\text{eff}}}{\partial h^3}\, |_{h=0} = \frac{3m_h^2}{v}\left[1 - \frac{N_c N_s m_t^4}{64\pi^2 m_h^2 v^2} \times \frac{16}{3}\right] = \lambda_{hhh} \Rightarrow \lambda_{hhh} = \frac{3m_h^2}{v}\left[1 - \frac{N_c N_s m_t^4}{12\pi^2 v^2 m_h^2}\right] \tag{180}$$

Therefore, there is a contribution from the top quark mass in the higher orders - this is a feature in the standard model of particle physics and in theories beyond the standard model we discussed here, specifically in the extended Higgs sector. However, next we shall look at an exclusively beyond the standard model contribution.

5.2.2 Heavier Higgs contribution

The contribution from the Heavier Higgs particles are more complicated, particularly because of the way the field dependent mass involves the lightest Higgs boson. For simplicity, we assume that $m_1 = m_2$ and $\lambda_1 = \lambda_2$. The vacuum is chosen to be,

$$\langle \Phi_1 \rangle = \langle \Phi_2 \rangle = \frac{1}{2}\begin{pmatrix} 0 \\ \varphi \end{pmatrix} \tag{181}$$

This corresponds to the case where $\sin\xi = -1$ and $\tan\beta = 1$. The effective potential can be written as,

$$V_{\text{eff}}[\varphi] = -\frac{1}{2}m^2\varphi^2 + \frac{1}{4}\lambda'\varphi^4 + \frac{1}{64\pi^2}\sum_f N_c N_s (-1)^{2s} M_f^4[\varphi]\left[\ln\left(\frac{M_f^2}{Q^2}\right) - \frac{3}{2}\right] \tag{182}$$

Here, $m^2 = m_1^2 - m_3^2$ and $\lambda' = (\lambda_1 + \lambda_3 + \lambda_4 + \lambda_5)/4$. Further, let us consider the contribution from only one of the heavy Higgs particles, H - the rest will be the same as this since all of the field dependent masses are roughly of the same form,

$$m_\Phi^2[\varphi] = M^2 + \bar{\lambda}\varphi^2 \tag{183}$$

Here, $\Phi = H, A, H^\pm$ while $\bar{\lambda}$ will be some linear combination of $\{\lambda_i\}$. Specifically,

$$m_A^2[\varphi] = M^2 - \lambda_5\varphi^2 \tag{184}$$

$$m_{H^\pm}^2[\varphi] = M^2 - \frac{1}{2}[\lambda_4 + \lambda_5]\varphi^2 \tag{185}$$

$$m_H^2[\varphi] = M^2 + \frac{1}{2}(\lambda_1 - \lambda)\varphi^2 \tag{186}$$

The physical masses all are defined by $m_\Phi^2[v]$, where v is the vacuum expectation value of the Higgs, as has been experimentally measured. Now, to start with, let us consider only H,

$$V_{\text{eff}}^H[\varphi] = -\frac{1}{2}m^2\varphi^2 + \frac{1}{4}\lambda'\varphi^4 + A_H M_H^4[\varphi]\left[\ln\left(\frac{M_H^2}{Q^2}\right) - \frac{3}{2}\right] \tag{187}$$

Here, A_H denotes the constant numerical pre-factors. Now, we have,

$$V_{\text{eff}}^H[\varphi] = -\frac{1}{2}m^2\varphi^2 + \frac{1}{4}\lambda'\varphi^4 + A_H M_H^4[\varphi]\left[\ln\left(\frac{M_H^2}{Q^2}\right) - \frac{3}{2}\right] \tag{188}$$

This can be simplified as,

$$\begin{aligned} V_{\text{eff}}^H[\varphi] &= -\frac{1}{2}m^2\varphi^2 + \frac{1}{4}\lambda'\varphi^4 + A_H(M^2+\lambda_H\varphi^2)^2\left[\ln\left(\frac{M^2+\lambda_H\varphi^2}{Q^2}\right) - \frac{3}{2}\right] \\ &= -\frac{1}{2}m^2\varphi^2 + \frac{1}{4}\lambda'\varphi^4 + A_H M^4\left(1+\frac{\lambda_H}{M^2}\varphi^2\right)^2\left[\ln\left(1+\frac{\lambda_H}{M^2}\varphi^2\right) + \ln\left(\frac{M^2}{Q^2}\right) - \frac{3}{2}\right] \\ &= -\frac{1}{2}\mu^2\varphi^2 + \frac{1}{4}\Lambda\varphi^4 + A_H\lambda_H\varphi^2[2M^2+\lambda_H\varphi^2]\ln\left(1+\frac{\lambda_H}{M^2}\varphi^2\right) \end{aligned} \tag{189}$$

Here,

$$\mu^2 = m^2 + 4A_H M^2\lambda_H\left[\ln\left(\frac{M^2}{Q^2}\right) - \frac{3}{2}\right],\ \Lambda = \lambda' + 4A_H\lambda_H^2\left[\ln\left(\frac{M^2}{Q^2}\right) - \frac{3}{2}\right] \tag{190}$$

Now, we differentiate our effective potential and impose the conditions on the derivatives,

$$\begin{aligned} \frac{\partial V_{\text{eff}}^H}{\partial\varphi} &= -\mu^2\varphi + \Lambda\varphi^3 + 4A_H\lambda_H\varphi[M^2+\lambda_H\varphi^2]\ln\left(1+\frac{\lambda_H}{M^2}\varphi^2\right) \\ &\quad + \frac{2A_H\lambda_H^2\varphi^3}{M^2+\lambda_H\varphi^2}[2M^2+\lambda_H\varphi^2] \end{aligned} \tag{191}$$

$$\begin{aligned} \frac{\partial^2 V_{\text{eff}}^H}{\partial\varphi^2} &= -\mu^2 + 3\Lambda\varphi^2 + 4A_H\lambda_H[M^2+3\lambda_H\varphi^2]\ln\left(1+\frac{\lambda_H}{M^2}\varphi^2\right) \\ &\quad + \frac{2A_H\lambda_H^2\varphi^2}{M^2+\lambda_H\varphi^2}[10M^2+9\lambda_H\varphi^2] - \frac{4A_H\lambda_H^3\varphi^4}{(M^2+\lambda_H\varphi^2)^2}[2M^2+\lambda_H\varphi^2] \end{aligned} \tag{192}$$

$$\begin{aligned} \frac{\partial^3 V_{\text{eff}}^H}{\partial\varphi^3} &= 6\Lambda\varphi + 24A_H\lambda_H^2\varphi\ln\left(1+\frac{\lambda_H}{M^2}\varphi^2\right) + \frac{48A_H\lambda_H^2\varphi}{M^2+\lambda_H\varphi^2}[M^2+2\lambda_H\varphi^2] \\ &\quad - \frac{12A_H\lambda_H^3\varphi^3}{(M^2+\lambda_H\varphi^2)^2}[6M^2+5\lambda_H\varphi^2] + \frac{16A_H\lambda_H^4\varphi^5}{(M^2+\lambda_H\varphi^2)^3}[2M^2+\lambda_H\varphi^2] \end{aligned} \tag{193}$$

Now, we impose the previous conditions on the derivatives,

$$\frac{\partial V_{\text{eff}}^H}{\partial\varphi}|_{\varphi=v} = 0 \Rightarrow \mu^2 = \Lambda v^2 + 4A_H\lambda_H m_H^2\ln\left(1+\frac{\lambda_H}{M^2}v^2\right) + \frac{2A_H\lambda_H^2v^2}{m_H^2}[M^2+m_H^2] \tag{194}$$

Here, we have used $m_H^2 = M^2 + \lambda_H v^2$. Now, moving on,

$$\begin{aligned} &\frac{\partial^2 V_{\text{eff}}^H}{\partial\varphi^2}|_{\varphi=v} = m_h^2 \\ \Rightarrow 6\Lambda v &= \frac{3m_h^2}{v} - 48A_H\lambda_H^2 v - 24A_H\frac{\lambda_H}{v}[m_H^2-M^2]\ln\left(1+\frac{\lambda_H}{M^2}v^2\right) + \frac{12A_H\lambda_H^3v^3}{m_H^4}[m_H^2+M^2] \end{aligned} \tag{195}$$

We substitute the above relations in the third derivative evaluated at $\varphi = v$ in order to determine the relevant coupling term,

$$\frac{\partial^3 V_{\text{eff}}^H}{\partial\varphi^3}|_{\varphi=v} = \frac{3m_h^2}{v} - 48A_H\frac{\lambda_H^3v^3}{m_H^2} - 48A_H\frac{\lambda_H^2v^2}{m_H^2}M^2 + 16A_H\frac{\lambda_H^4v^5}{m_H^6}(m_H^2+M^2) \tag{196}$$

This can be further simplified,

$$\frac{\partial^3 V_{\text{eff}}^H}{\partial \varphi^3}\Big|_{\varphi=v} = \frac{3m_h^2}{v}\left[1 - 16A_H\frac{\lambda_H^3 v^6}{m_h^2 m_H^2 v^2} - 16A_H\frac{\lambda_H^2 v^4}{m_h^2 m_H^2 v} + 16A_H\frac{\lambda_H^4 v^8}{3m_h^2 m_H^6 v^2}(m_H^2 + M^2)\right] \tag{197}$$

Now, $\lambda_H v^2 = m_H^2 - M^2$. Substituting this in the above expression, we get,

$$\begin{aligned}\frac{\partial^3 V_{\text{eff}}^H}{\partial \varphi^3}\Big|_{\varphi=v} &= \frac{3m_h^2}{v}\left[1 - \frac{16A_H m_H^4}{m_h^2 v^2}\left(1 - \frac{M^2}{m_H^2}\right)^3 - \frac{16A_H m_H^2}{m_h^2 v^2}\left(1 - \frac{M^2}{m_H^2}\right)^2\right.\\ &\left. + \frac{16A_H m_H^4}{3m_h^2 v^2}\left(1 - \frac{M^4}{m_H^4}\right)\left(1 - \frac{M^2}{m_H^2}\right)^3\right]\end{aligned} \tag{198}$$

The first correction term is the largest of the three - therefore, we neglect the other two and concentrate on the third only. Further,

$$A_H = (-1)^{2s}\frac{N_c N_s}{64\pi^2} = -\frac{1}{32\pi^2} = A_A = \frac{1}{2}A_{H^\pm} \tag{199}$$

Or, finally, with all the other fields also taken into account,

$$\begin{aligned}\lambda_{\text{hhh}}^{\text{THDM}} = \frac{3m_h^2}{v}&\left[1 + \frac{m_H^4}{2\pi^2 m_h^2 v^2}\left(1 - \frac{M^2}{m_H^2}\right)^3 + \frac{m_A^4}{2\pi^2 m_h^2 v^2}\left(1 - \frac{M^2}{m_A^2}\right)^3\right.\\ &\left. + \frac{m_{H^\pm}^4}{\pi^2 m_h^2 v^2}\left(1 - \frac{M^2}{m_{H^\pm}^2}\right)^3 - \frac{N_c N_s m_t^4}{12\pi^2 v^2 m_h^2}\right] + \text{higher order terms}\end{aligned} \tag{200}$$

Therefore, the mass of the heavier Higgs particle enters the expression with power four - this can serve as an indirect way to search for the additional Higgs particles since the deviation can be very large.

6 The baryon asymmetry problem

Baryon asymmetry is one of the many open problems currently present in theoretical physics. The basic premise is that the observed number density of baryonic matter and anti-baryonic matter in the observable universe is not equal. Specifically,

$$\frac{\Delta n_B}{s} \sim 10^{-10} \tag{201}$$

Here, Δn_B is the difference in the number densities of baryonic and anti-baryonic matter and s is the entropy density. While the difference is small, it is distinctly non zero. The standard model of particle physics is unable to explain this imbalance in its current formulation and, according to current theories, the big bang should have produced equal amounts of matter and anti-matter in the universe.

Broadly, there are three distinct conditions that need to be satisfied for the observed baryon asymmetry. These are the Sakharov conditions given by,

1. Violation of baryon number, B
2. Violation of both C-symmetry and CP symmetry
3. Thermal non-equilibrium

CP violation has already been observed (in kaons, for example). Baryon number violation would produce unequal amounts of baryon and anti-baryon while the symmetry violations would prevent counter interactions that could, in theory, produce more anti-baryons than baryons. We also need a departure from thermal equilibrium to ensure that the reactions are driven forward so that the imbalance remains throughout the process.

Electroweak symmetry breaking and the associated phase transition satisfy all three of the conditions and can lead to a baryon asymmetry, in principle. In the standard model, however, this asymmetry is very small - smaller than the observed number. The extended Higgs sector can play a viable part in the baryogenesis program and might be able to explain the observed asymmetry in the universe. Following [5], we concentrate on the two Higgs doublet model introduced in Section 5.

6.1 Electroweak phase transition and the THDM

Spontaneous symmetry breaking in the electroweak interaction in the context of the THDM can lead to baryon asymmetry during the evolution of the universe. For this, we need to consider a finite temperature field theory since during the Big Bang, temperatures and densities were extremely high compared to the current scenario. There are two things we need for generating the asymmetry - one is that the phase transition must be first order so as to ensure that sphaleron processes are avoided since they can convert baryons to anti-leptons and second, CP violation must occur at these finite temperatures.

In the very beginning (during and after the big bang), the electroweak potential was of a different form than what we have discussed - as temperatures cooled down, at a specific critical temperature the form of the potential changed [6] to the one we have already discussed previously - after this, spontaneous symmetry breaking occured and the Nambu Goldstone bosons gave mass to the gauge bosons. This is shown in Figure 1.

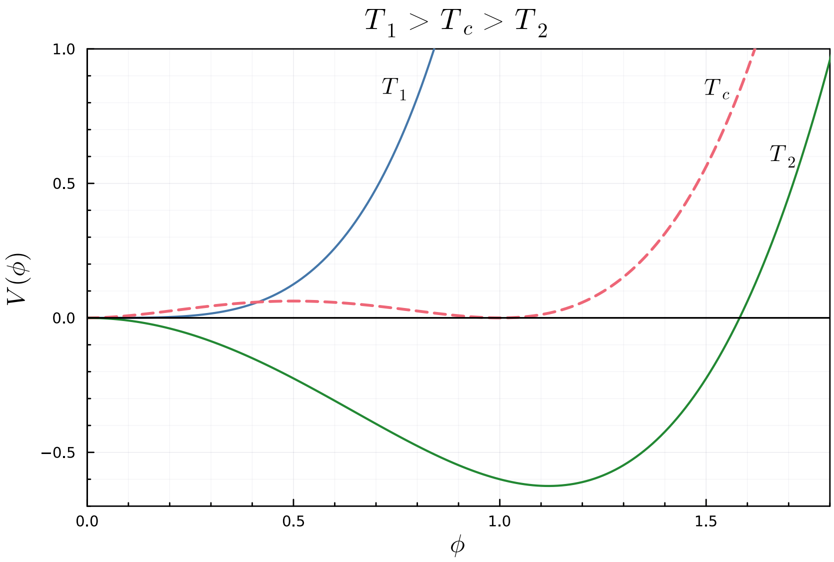


Figure 1: The electroweak phase transition [6]

Initially, we start with a potential with a single minima at $\varphi = 0$, but at some temperature, the curvature of the potential changes and it develops an inflection point. Temperatures went down further and eventually, reached the critical point. At the critical temperature, the two minima of the potential become degenerate, i.e., $V(0) = V(\varphi_c)$. After this, as the temperatures went down further, distinct minima (apart from the one at the origin) developed and finally, the potential took the form of the double well we have discussed earlier.

The usual (tree level) potential in the THDM is given by,

$$\begin{aligned} V_{\text{tree}} = m_1^2\ |\Phi_1|^2 + m_2^2\ |\Phi_2|^2 + \frac{\lambda_1}{2}\ |\Phi_1|^4 + \frac{\lambda_2}{2}\ |\Phi_2|^2 - m_3^2\left(\Phi_1^\dagger\Phi_2 + \text{h.c.}\right) \\ +\lambda_3\ |\Phi_1|^2\ |\Phi_2|^2 + \lambda_4\ |\Phi_1^\dagger\Phi_2|^2 + \frac{\lambda_5}{2}\left[\left(\Phi_1^\dagger\Phi_2\right)^2 + \text{h.c.}\right] \end{aligned} \tag{202}$$

For simplicity, we again assume that all eight parameters are real. We have $\tan\beta = v_2/v_1$, where v_i $(i = 1, 2)$ are the vacuum expectation values of the two Higgs fields. We have a second diagonalising angle, α, as well. There are two specific regimes here - one is if $M^2 = m_3^2/\sin\beta\cos\beta \gg v^2$. In this case, all the heavy Higgs fields have the same mass, M, and the theory becomes similar to the usual standard model with the lightest Higgs, h. This is called decoupling. A second situation arises if $M^2 << \lambda v_i^2$ where the mass contributions to the heavier Higgs bosons are primarily from the vacuum expectation values and these additional Higgs fields also contribute significantly to coupling constants and masses in the theory at higher orders, providing us with the scope to indirectly probe for their existence. This is called non-decoupling. Following [5], we assume that $m_1 = m_2$ and $\lambda_1 = \lambda_2$ for simiplicity. We choose the case where $\sin(\alpha - \beta) = -1$ and $\tan\beta = 1$ for baryogenesis.

Now, the effective potential for this theory at a finite temperature is given by,

$$V_{\text{eff}}(\varphi, T) = V_{\text{tree}}(\varphi) + \Delta V(\varphi) + \Delta V^T(\varphi, T) \tag{203}$$

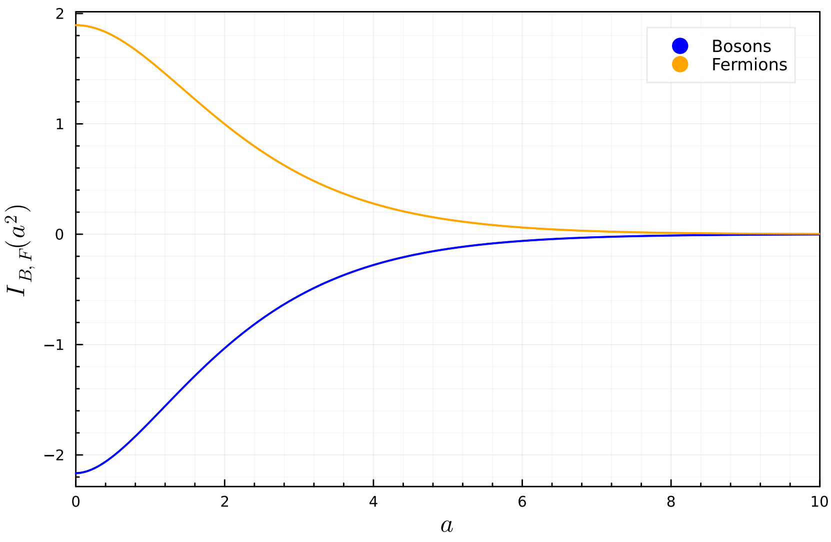


Figure 2: The integrals in Eq. 208

Here,

$$\Delta V(\varphi) = \frac{1}{64\pi^2} N_c N_s (-1)^{2s} M_f^4[\varphi] \left[\ln\left(\frac{M_f^2}{Q^2}\right) - \frac{3}{2}\right] \tag{204}$$

As before, N_c is the color number, s is the spin and N_s is the spin degree of freedom of the field f. $M_f[\varphi]$ is a field dependent mass term for f and Q is an arbitrary mass scale. The last term is the finite temperature correction term, given by,

$$\Delta V^T(\varphi, T) = \frac{T^4}{2\pi^2}\left[\sum_{\text{bosons}}^{i} n_i I_B(a_i^2) + n_t I_F(a_t^2)\right] \tag{205}$$

Here, n_i is the degree of freedom of the particle under consideration. So, for example,

$$n_h = n_H = n_A = \underbrace{2\times 0+1}_{\text{Spin DOF}} = 1, n_{H^\pm} = \underbrace{[2\times 0+1]}_{\text{Spin DOF}} \times \underbrace{2}_{\text{Charge}} = 2 \tag{206}$$

Similarly,

$$n_Z = 3,\ n_W = 6,\ n_t = 12 \tag{207}$$

Further, we define $a_i = m_i(\varphi)/T$, where $m_i(\varphi)$ is the field dependent mass (as defined earlier). The functions, $I_{B,F}$, define the following integrals,

$$I_{B,F}(a_i^2) = \int_0^\infty dx\ x^2 \log\left(1 \mp e^{-\sqrt{x^2+a_i^2}}\right) \tag{208}$$

These are depicted in Figure 2. We can use a high temperature expansion to look at some features of the potential analytically. For temperatures $T \gg m_i$, we can expand the integrals and approximate the answers. We assume $m_\Phi^2 \gg m_h^2, M^2$ $(\Phi = H, A, H^\pm)$ so that $m_\Phi^2(\varphi) \approx m_\varphi^2\varphi^2/v^2$. With these assumptions, the effective potential becomes,

$$V_{\text{eff}}(\varphi, T) \approx D(T^2 - T_0^2)\varphi^2 - ET\ |\varphi|^3 + \frac{\lambda_T}{4}\varphi^4 + ... \tag{209}$$

The various parameters are defined as follows,

$$D = \frac{1}{24v^2}\left[6m_W^2 + 3m_Z^2 + 6m_t^2 + m_H^2 + m_A^2 + 2m_{H^\pm}^2\right] \tag{210}$$

$$T_0 = \frac{1}{D}\left[\frac{1}{4}m_h^2 - \frac{1}{32\pi^2 v^2}\left(6m_W^4 + 3m_Z^4 - 12m_t^4 + m_H^4 + m_A^4 + 2m_{H^\pm}^4\right)\right] \tag{211}$$

$$E = \frac{1}{12\pi v^3}\left[6m_W^3 + 3m_Z^3 + m_H^3 + m_A^3 + 2m_{H^\pm}^3\right] \tag{212}$$

$$\begin{aligned}\lambda_T = \frac{m_h^2}{2v^2}\Bigg[1 - \frac{1}{8\pi^2 v^2 m_h^2}\Bigg(&6m_W^4 \log\left(\frac{m_W^2}{\alpha_B T^2}\right) + 3m_Z^4 \log\left(\frac{m_Z^2}{\alpha_B T^2}\right) - 12m_t^4 \log\left(\frac{m_t^2}{\alpha_F T^2}\right) \\ &+ m_H^4 \log\left(\frac{m_H^2}{\alpha_B T^2}\right) + m_A^4 \log\left(\frac{m_A^2}{\alpha_B T^2}\right) + 2m_{H^\pm}^4 \log\left(\frac{m_{H^\pm}^2}{\alpha_B T^2}\right)\Bigg)\Bigg]\end{aligned} \tag{213}$$

Here, $\log \alpha_B = 2\log 4\pi - 2\gamma_E$ and $\log \alpha_F = 2\log \pi - 2\gamma_E$ with γ_E being the Euler - Mascheroni constant. The expansion is derived in the next section. The phase transition that is required occurs due to the cubic term that has appeared in the effective potential. Now, as mentioned earlier, at the critical point, the potential has two degenerate minima. Therefore,

$$V(0) = V(\varphi_c) = 0 \Rightarrow \varphi_c^2\left[D(T_c^2 - T_0^2) - E\varphi_c + \frac{1}{4}\lambda_{T_c}\varphi_c^2\right] = 0 \tag{214}$$

This quadratic equation must have only one distinct root - the discriminant has to be equal to zero. This can be used to determine the critical temperature,

$$E^2T_c^2 - \lambda_{T_c} D(T_c^2 - T_0^2) = 0 \Rightarrow T_c^2 = \frac{T_0^2}{1 - \frac{E^2}{\lambda_{T_c} D}} \tag{215}$$

And, the minima occurs at,

$$\varphi_c = \frac{2ET_c}{\lambda_{T_c}} \tag{216}$$

Now, baryogenesis in this model requires the suppression of sphaleron processes - from numerical studies, the condition for this has been found out to be [5],

$$\frac{\varphi_c}{T_c} = \frac{2E}{\lambda_{T_c}} \gtrsim 1 \tag{217}$$

Therefore, first order phase transitions and baryogenesis can take place in the THDM if these conditions are satisfied - therefore, the THDM provides a viable answer to the baryon asymmetry problem. It is interesting to note that Eq. 212 tells us that even in the standard model, there will be a cubic term in the high temperature limit,

$$E = \frac{1}{12\pi v^3}\left[6m_W^3 + 3m_Z^3\right] \tag{218}$$

Now, we know from observations that $v \approx 246$ [GeV] while $m_Z \approx 91.19$ [GeV] and $m_W \approx 80.38$ [GeV]. This means that,

$$E \sim 0.0031 \tag{219}$$

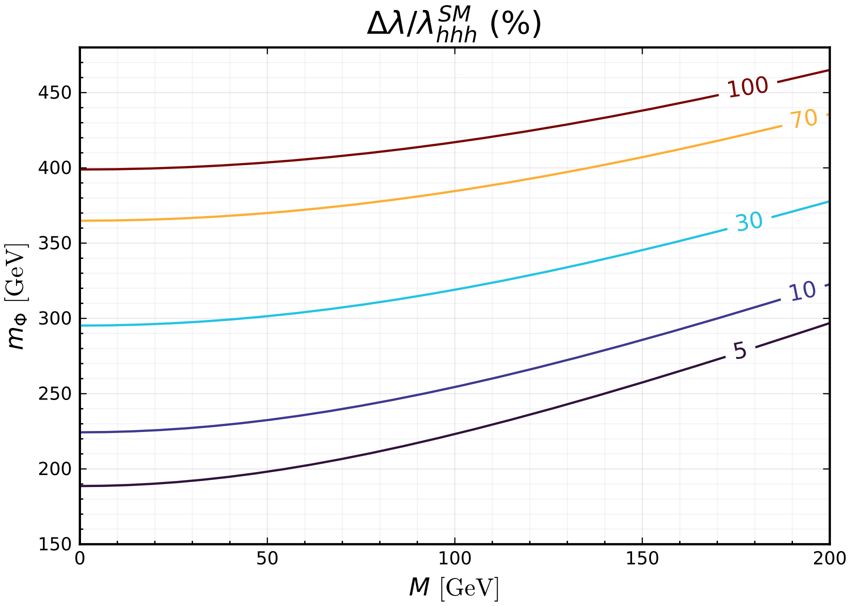


Figure 3: Percentage deviation in the trilinear Higgs coupling in the THDM

This is a very small number - meaning that the first order transition is itself affected. The baryogenesis condition in Eq. 217 cannot be satisfied in this case - in fact, it turns out that in this scenario, no proper first order transition occurs at all. The phase transition is what is called a “smooth crossover” - the order parameter and its first derivative are continuous at the critical temperature.

On the other hand, at and above the required condition for baryogenesis by electroweak phase transition, the THDM can lead to large deviations in the Higgs trilinear coupling strength, λ_{hhh} [5]. The deviations are depicted in Figure 3 according to the derivation shown in Section 5 - these ranges can be probed at future collider experiments and they can provide indirect signals about the feasibility of the THDM and the possibility of electroweak phase transition induced baryogenesis in this model.

6.2 The high temperature expansion

The calculation here mostly follows [7]. Let us consider the bosonic integral,

$$I_B(a^2) = \int_0^\infty dx\ x^2 \log\left(1 - e^{-\sqrt{x^2+a^2}}\right) \tag{220}$$

We had,

$$a^2 = \frac{m(\varphi)^2}{T^2} \tag{221}$$

At high temperatures, thus, the parameter a is very small. As $T \to \infty$, $a \to 0$. In this case, we determine the high temperature expansion by a process of double integration - we first look at its derivatives upto the second order, approximate the double derivative to $\mathcal{O}(a^2)$ and then re-integrate. This gives us an approximate form for the potential function in this limit to the required order. To start with,

$$I_B(0) = \int_0^\infty dx\ x^2 \log(1 - e^{-x}) = \frac{-\pi^4}{45} \tag{222}$$

Further,

$$\frac{\partial I_B}{\partial a^2} = \frac{1}{2}\int_0^\infty dx\, \frac{x^2}{\sqrt{x^2+a^2}\left(e^{\sqrt{x^2+a^2}}-1\right)} \tag{223}$$

At $a = 0$,

$$\frac{\partial I_B}{\partial a^2}\mid_{a=0} = \frac{1}{2}\int_0^\infty dx\, \frac{x}{e^x-1} = \frac{\pi^2}{12} \tag{224}$$

This gives us two boundary conditions - once we begin to re-integrate the second derivative, we will use these two conditions to eliminate the constants of integration. For the next term,

$$\frac{\partial^2 I_B}{\partial a^2 \partial a^2} = -\frac{1}{4}\int_0^\infty dx\, \frac{x^2}{(x^2+a^2)\left(e^{\sqrt{x^2+a^2}}-1\right)}\left[\frac{e^{\sqrt{x^2+a^2}}}{e^{\sqrt{x^2+a^2}}-1} + \frac{1}{\sqrt{x^2+a^2}}\right] \tag{225}$$

Now,

$$\begin{aligned}
\int_0^\infty dx\, \frac{x^2 e^{\sqrt{x^2+a^2}}}{(x^2+a^2)\left(e^{\sqrt{x^2+a^2}}-1\right)^2} &= -\int_0^\infty dx\, \frac{x}{\sqrt{x^2+a^2}}\frac{d}{dx}\left[\frac{1}{e^{\sqrt{x^2+a^2}}-1}\right] \\
= -\left[\frac{x^2 e^{\sqrt{x^2+a^2}}}{(x^2+a^2)\left(e^{\sqrt{x^2+a^2}}-1\right)^2}\right]_0^\infty &+ \int_0^\infty dx\, \frac{1}{e^{\sqrt{x^2+a^2}}-1}\frac{d}{dx}\left[\frac{x}{\sqrt{x^2+a^2}}\right] \\
&= \int_0^\infty dx\, \frac{a^2}{(x^2+a^2)^{\frac{3}{2}}\left(e^{\sqrt{x^2+a^2}}-1\right)}
\end{aligned} \tag{226}$$

Or, substituting into Eq. 225,

$$\frac{\partial^2 I_B}{\partial a^2 \partial a^2} = -\frac{1}{4}\int_0^\infty dx\, \frac{1}{\sqrt{x^2+a^2}\left(e^{\sqrt{x^2+a^2}}-1\right)} \tag{227}$$

In order to estimate this integral, we look at,

$$I_\varepsilon(a) = \int_0^\infty dx\, \frac{x^{-\varepsilon}}{\sqrt{x^2+a^2}\left(e^{\sqrt{x^2+a^2}}-1\right)} \tag{228}$$

We recover our original integral in the limit $\varepsilon \to 0$. For convergence purposes, we assume that $\varepsilon \in (0,1)$. Now, we can write,

$$\begin{aligned}
I_\varepsilon(a) &= \int_0^\infty dx\, \frac{x^{-\varepsilon}}{\sqrt{x^2+a^2}\left(e^{\sqrt{x^2+a^2}}-1\right)} \\
&= \frac{1}{2}\int_0^\infty dx\, \frac{x^{-\varepsilon}}{\sqrt{x^2+a^2}}\left[\frac{e^{\sqrt{x^2+a^2}}+1-\left(e^{\sqrt{x^2+a^2}}-1\right)}{\left(e^{\sqrt{x^2+a^2}}-1\right)}\right] \\
&= \frac{1}{2}\int_0^\infty dx\, \frac{x^{-\varepsilon}}{\sqrt{x^2+a^2}}\left[\coth\left(\frac{\sqrt{x^2+a^2}}{2}\right)-1\right] = I_\varepsilon^{(1)}(a) + I_\varepsilon^{(2)}(a)
\end{aligned} \tag{229}$$

Here,

$$I_\varepsilon^{(1)} = \frac{1}{2}\int_0^\infty dx\ \frac{x^{-\varepsilon}}{\sqrt{x^2+a^2}}\coth\left(\frac{\sqrt{x^2+a^2}}{2}\right),\ I_\varepsilon^{(2)} = -\frac{1}{2}\int_0^\infty dx\ \frac{x^{-\varepsilon}}{\sqrt{x^2+a^2}} \tag{230}$$

Now, we have the following series representation,

$$\sum_{n=1}^\infty \frac{x}{x^2+n^2} = -\frac{1}{2x}+\frac{\pi}{2}\coth(\pi x) \tag{231}$$

Consider,

$$\begin{aligned}\sum_{n\in\mathbb{Z}\ |\ n=-\infty}^{\infty} \frac{1}{x^2+a^2+4\pi^2n^2} &= \frac{1}{x^2+a^2}+\frac{1}{2\pi^2}\sum_{n=1}^\infty\frac{1}{\frac{x^2+a^2}{4\pi^2}+n^2}\\ &= \frac{1}{2\sqrt{x^2+a^2}}\coth\left(\frac{\sqrt{x^2+a^2}}{2}\right)\end{aligned} \tag{232}$$

Therefore,

$$I_\varepsilon^{(1)} = \int_0^\infty dx\ \frac{x^{-\varepsilon}}{2\sqrt{x^2+a^2}}\coth\left(\frac{\sqrt{x^2+a^2}}{2}\right) = \int_0^\infty dx\ x^{-\varepsilon}\sum_{0,\pm1,\dots}\frac{1}{x^2+a^2+4\pi^2n^2} \tag{233}$$

Now, we apply the transformation $x^2 \to x^2(a^2+4\pi^2n^2)$. This simplifies our integral to give us,

$$I_\varepsilon^{(1)}(a) = \sum_n \frac{1}{(a^2+4\pi^2n^2)^{\frac{1+\varepsilon}{2}}}\int_0^\infty dx\ \frac{x^{-\varepsilon}}{1+x^2} \tag{234}$$

Now, we have,

$$\int_0^\infty dx\ \frac{x^{-\varepsilon}}{1+x^2} = \frac{\pi}{2\cos\frac{\varepsilon\pi}{2}} \tag{235}$$

Therefore,

$$I_\varepsilon^{(1)}(a) = \frac{\pi}{2\cos\frac{\varepsilon\pi}{2}}\sum_n\frac{1}{(a^2+4\pi^2n^2)^{\frac{1+\varepsilon}{2}}} = \frac{\pi}{2\cos\frac{\varepsilon\pi}{2}}\left[\frac{1}{a^{1+\varepsilon}}+2\sum_{n=1}^\infty\frac{1}{(a^2+4\pi^2n^2)^{\frac{1+\varepsilon}{2}}}\right] \tag{236}$$

We can simplify,

$$\begin{aligned}2\sum_{n=1}^\infty\frac{1}{(a^2+4\pi^2n^2)^{\frac{1+\varepsilon}{2}}} &= 2\sum_{n=1}^\infty\frac{1}{(2\pi n)^{1+\varepsilon}}\left[\frac{1}{\left(1+\frac{a^2}{4\pi^2n^2}\right)^{\frac{1+\varepsilon}{2}}}\right]\\ &= 2\sum_{n=1}^\infty\frac{1}{(2\pi n)^{1+\varepsilon}}+2\sum_{n=1}^\infty\frac{1}{(2\pi n)^{1+\varepsilon}}\left[\frac{1}{\left(1+\frac{a^2}{4\pi^2n^2}\right)^{\frac{1+\varepsilon}{2}}}-1\right]\\ &\approx 2\sum_{n=1}^\infty\frac{1}{(2\pi n)^{1+\varepsilon}}+2\sum_{n=1}^\infty\underbrace{\frac{1}{(2\pi n)^{1+\varepsilon}}\left[-\frac{1+\varepsilon}{8\pi^2n^2}a^2+\mathcal{O}(a^4)\right]}_{\mathcal{O}(a^2)}\end{aligned} \tag{237}$$

In the limit $\varepsilon \to 0$, we then have,

$$I_\varepsilon^{(1)}(a) = \frac{\pi}{2a}+\frac{1}{\pi^\varepsilon 2^{1+\varepsilon}}\zeta(1+\varepsilon)+\mathcal{O}(a^2) \tag{238}$$

Here, $\zeta(1+\varepsilon)$ is the usual Riemann zeta function that can be expanded as,

$$\zeta(1+\varepsilon) = 2^{\varepsilon}\pi^{\varepsilon}\left[\frac{1}{\varepsilon} + \gamma_E - \log(2\pi) + \mathcal{O}(\varepsilon)\right] \tag{239}$$

Substituting, we get,

$$I_{\varepsilon}^{(1)}(a) = \frac{\pi}{2a} + \frac{1}{2\varepsilon} + \frac{1}{2}[\gamma_E - \log(2\pi)] + \mathcal{O}(\varepsilon) + \mathcal{O}(a^2) \tag{240}$$

This gives us the first term. Now for the second -

$$I_{\varepsilon}^{(2)} = -\frac{1}{2}\int_0^{\infty} dx\ \frac{x^{-\varepsilon}}{\sqrt{x^2+a^2}} \tag{241}$$

We make the transformation $x \to ax$ to get,

$$I_{\varepsilon}^{(2)} = -\frac{a^{-\varepsilon}}{2}\int_0^{\infty} dx\ \frac{x^{-\varepsilon}}{\sqrt{1+x^2}} \tag{242}$$

Now,

$$\int_0^{\infty} dx\ \frac{x^{-\varepsilon}}{\sqrt{1+x^2}} = \frac{1}{2\sqrt{\pi}}\Gamma\left(\frac{1-\varepsilon}{2}\right)\Gamma\left(\frac{\varepsilon}{2}\right) \tag{243}$$

We have the following approximations for the Gamma function,

$$\Gamma\left(\frac{\varepsilon}{2}\right) \approx \frac{2}{\varepsilon} - \gamma_E + \mathcal{O}(\varepsilon) \tag{244}$$

$$\Gamma\left(\frac{1-\varepsilon}{2}\right) \approx \sqrt{\pi}\left[1 + \frac{\varepsilon}{2}(\gamma_E + 2\log 2)\right] + \mathcal{O}(\varepsilon^2) \tag{245}$$

Therefore, we find,

$$\Gamma\left(\frac{1-\varepsilon}{2}\right)\Gamma\left(\frac{\varepsilon}{2}\right) \approx 2\sqrt{\pi}\left[\frac{1}{\varepsilon} + \log 2\right] + \mathcal{O}(\varepsilon) \tag{246}$$

Further, we apply the standard expansion used in regularization,

$$a^{-\varepsilon} = e^{-\varepsilon\log a} \approx 1 - \varepsilon\log a + \mathcal{O}(\varepsilon^2) \tag{247}$$

This gives us,

$$\begin{aligned} I_{\varepsilon}^{(2)} &= -\frac{1}{2}[1 - \varepsilon\log a + \mathcal{O}(\varepsilon^2)]\left[\frac{1}{\varepsilon} + \log 2 + \mathcal{O}(\varepsilon)\right] \\ &= -\frac{1}{2\varepsilon} + \frac{1}{2}\log a - \frac{1}{2}\log 2 + \mathcal{O}(a^2) + \mathcal{O}(\varepsilon) = -\frac{1}{2\varepsilon} + \frac{1}{2}\log\frac{a}{2} + \mathcal{O}(a^2) + \mathcal{O}(\varepsilon) \end{aligned} \tag{248}$$

Combining Eq. 240 and Eq. 248 together and taking the limit $\varepsilon \to 0$, we finally have,

$$I_0(a) = \frac{\pi}{2a} + \frac{1}{2}\gamma_E + \frac{1}{2}\log\left(\frac{a}{4\pi}\right) + \mathcal{O}(a^2) \tag{249}$$

This implies,

$$\frac{\partial^2 I_B}{\partial a^2 \partial a^2} = -\frac{1}{4} I_0(a) = -\frac{1}{4}\left[\frac{\pi}{2a} + \frac{1}{2}\gamma_E + \frac{1}{2}\log\left(\frac{a}{4\pi}\right)\right] + \text{higher order terms} \tag{250}$$

The above expression has been truncated up to the desired order - we now integrate it twice to get back the approximate form of our potential. We eliminate the constants of integration using the earlier mentioned boundary conditions Eq. 222 and Eq. 223. Therefore,

$$\begin{aligned} I_B(a^2) &\approx \int da^2 \int da^2 \, -\frac{1}{4}\left[\frac{\pi}{2a} + \frac{1}{2}\gamma_E + \frac{1}{2}\log\left(\frac{a}{4\pi}\right)\right] \\ &= -\frac{\pi^4}{45} + \frac{\pi^2}{12}a^2 - \frac{\pi}{6}a^3 - \frac{1}{32}a^4\log(a^2) + \frac{1}{64}(3 + 4\log(4\pi) - 4\gamma_E)a^4 \end{aligned} \tag{251}$$

Now, every term in the potential contributes as,

$$V = \frac{T^4}{2\pi^2}\left[\sum_{\text{bosons}}^{i} n_i I_B(a_i^2) + n_t I_F(a_t^2)\right] \tag{252}$$

With the appropriate pre-factor, the integral contributes to the potential function as,

$$\frac{T^4}{2\pi^2} I_B(a^2) = -\frac{\pi^2}{90}T^4 + \frac{m^2}{24}T^2 - \frac{m^3}{12\pi}T - \frac{m^4}{64\pi^2}\log\left(\frac{m^2}{T^2}\right) + \frac{C}{64\pi^2}m^4 \tag{253}$$

Here, $C = 3/2 + 2\log(4\pi) - 2\gamma_E$ and $m \equiv m(\varphi)$. The third term is proportional to m^3 and under the previously mentioned assumptions, it gives rise to the φ^3 contribution responsible for the phase transition as we saw previously. If we add all the terms together (for all the different fields) along with their relevant degrees of freedom as given in the previous section, we find the expression given in Eq. 209.

A similar derivation can be performed for the fermionic integral as well - the procedure is exactly similar. The final integral is slightly different because of the differing signs. In particular, for the fermionic case, we have,

$$I_\varepsilon(a) = \int_0^\infty dx \, \frac{x^{-\varepsilon}}{\sqrt{x^2 + a^2}\left(e^{\sqrt{x^2+a^2}} + 1\right)} = I_\varepsilon^{(1)}(a) + I_\varepsilon^{(2)}(a) \tag{254}$$

This time the series representation of the first integral is slightly different,

$$I_\varepsilon^{(1)}(a) = -\int_0^\infty dx \, x^{-\varepsilon} \sum_n \frac{1}{x^2 + a^2 + (2n+1)^2\pi^2}, \text{with } n = 0, \pm 1 ... \tag{255}$$

The integrals are simplified and approximated as in the bosonic case, giving us,

$$I_0(a) = -\frac{1}{2}\log\left(\frac{a}{\pi}\right) - \frac{1}{2}\gamma_E + \text{higher order terms} \tag{256}$$

Double integrating this will give us the required expression.

7 Electroweak phase transition and GWs

As we saw, electroweak phase transition can induce baryogenesis and account for the unequal amounts of matter and anti-matter in the universe provided a few conditions are satisfied. The standard model of particle physics leads to a smooth cross-over kind of phase transition which cannot explain the baryon asymmetry issue - in contrast, the THDM can lead to the required first order phase transition that will drive the system away from thermal equilibrium and prevent matter and anti-matter from balancing out.

There can be a few broad ways to test for this hypothesis. The first one has been already discussed - future collider experiments can help us correlate the trilinear Higgs coupling with the standard model and the THDM allowing us to compare the predictions of each. However, the technological know-how for this is still some way off into the future. There is a second method as well - primordial GWs can be used to indirectly probe for signatures of the first order transition in the early universe serving as a way to look into both the extended Higgs sector and the electroweak phase transition [8,9].

7.1 The model

To start with, we consider a potential of the form,

$$V_0(\Phi, S) = V_{\rm SM}(\Phi) + \frac{\mu_s^2}{2}\ |S|^2 + \frac{\lambda_s}{4}\ |S|^4 + \frac{\lambda_{\Phi S}}{2}\ |\Phi^2|\ |S|^2 \tag{257}$$

Here, we have added an additional set of singlet scalars, $S = (S_1, S_2, ..., S_N)^T$, which are invariant under an $O(N)$ symmetry. These additional scalars do not undergo spontaneous symmetry breaking - only the Higgs does. The field dependent mass of the scalars is [8,9],

$$M_s^2[\Phi] = \mu_s^2 + \frac{\lambda_{\Phi S}}{2}\Phi^2 \tag{258}$$

As before, the effective potential at a finite temperature is given by,

$$V_{\rm eff}(\varphi, T) = V_0(\varphi) + V_1(\varphi) + V^T(\varphi, T) \tag{259}$$

Here, V_1 is the effective potential including loop corrections we discussed in Section 5 and V^T is the temperature correction we saw in Section 6. Further, the field dependent masses are to be replaced by the thermally corrected ones as,

$$M_s^2 \to (m_s^2 - \mu_s^2)\frac{\varphi^2}{v^2} + \mu_s^2 + \frac{T^2}{12\pi^2}[(N+2)\lambda_s v^2 + 4(m_s^2 - \mu_s^2)] \tag{260}$$

Similar to what we did in the THDM in Section 5, the derivatives of the effective potential give us the different conditions and differentiating thrice gives us the coupling strength,

$$\lambda_{hhh}^{O(N)} = \frac{3m_h^2}{v}\left[1 - \frac{m_t^4}{m_h^2 v^2 \pi^2} + \frac{N m_s^4}{12 m_h^2 v^2 \pi^2}\left(1 - \frac{\mu_s^2}{m_s^2}\right)^3\right] \tag{261}$$

As earlier, the necessary condition for a strongly first order transition which is required for baryogenesis is that,

$$\frac{\varphi_c}{T_c} \gtrsim 1 \tag{262}$$

This has to be satisfied here as well.

7.1.1 Classical scale invariant theories

In some cases, we might consider these additional singlet scalars to be scale invariant as well. In these cases, there will be quadratic mass terms in the tree level in the potential. The masses will arise due to spontaneous symmetry breaking via the Coleman Weinberg mechanism. The potential in this case would be,

$$V_0(\Phi, S) = V_{\rm SM}(\Phi) + \frac{\mu_s^2}{2}\,|S|^2 + \frac{\lambda_s}{4}\,|S|^4 + \frac{\lambda_{\Phi S}}{2}\,|\Phi^2|\,|S|^2 \tag{263}$$

We take the Higgs vacuum expectation to be,

$$\langle\Phi\rangle = \frac{1}{\sqrt{2}}\binom{0}{v} \tag{264}$$

The effective potential at zero temperature contributes to the Higgs coupling. In these scale invariant models, the field dependent mass has a simpler form, $M[\varphi]^2 = m^2\varphi^2/v^2$. The effective potential can be rewritten as,

$$V_{\rm eff} = A\varphi^4 + B\varphi^4\log\left(\frac{\varphi^2}{Q^2}\right) \tag{265}$$

Here, we have,

$$A = \sum_{i=W^\pm,Z,\gamma,t,b,S}\frac{n_i m_i^4}{64\pi^2 v^4}\left[\log\left(\frac{m_i^2}{v^2}\right) - \frac{3}{2}\right],\ B = \sum_{i=W^\pm,Z,\gamma,t,b,S}\frac{n_i m_i^4}{64\pi^2 v^4} \tag{266}$$

Now, we extract the derivatives from the effective potential. We have,

$$\frac{\partial V_{\rm eff}}{\partial\varphi} = \left[4A + 2B + 8B\log\left(\frac{\varphi}{Q}\right)\right]\varphi^3 \tag{267}$$

We set this equal to zero at $\varphi = v$ to get,

$$2A + B + 4B\log\left(\frac{v}{Q}\right) = 0 \Rightarrow \frac{\partial V_{\rm eff}}{\partial\varphi} = 8B\varphi^3\log\left(\frac{\varphi}{v}\right) \tag{268}$$

Now,

$$\frac{\partial^2 V_{\rm eff}}{\partial\varphi^2} = 24B\varphi^2\log\left(\frac{\varphi}{v}\right) + 8B\varphi^2 \tag{269}$$

We know,

$$\frac{\partial^2 V_{\rm eff}}{\partial\varphi^2}|_{\varphi=v} = m_h^2 \Rightarrow m_h^2 = 8Bv^2 \tag{270}$$

Or,

$$\frac{\partial^2 V_{\rm eff}}{\partial\varphi^2} = \frac{m_h^2}{v^2}\varphi^2\left[1 + 3\log\left(\frac{\varphi}{v}\right)\right] \tag{271}$$

Thus, finally,

$$\lambda_{\rm hhh} = \frac{\partial^3 V_{\rm eff}}{\partial\varphi^3}|_{\varphi=v} = 5\frac{m_h^2}{v} \tag{272}$$

The deviation is thus,

$$\frac{\Delta\lambda_{\text{hhh}}}{\lambda_{\text{hhh}}^{\text{SM}}} = \frac{2}{3} \tag{273}$$

This is a characteristic feature of models with scale invariance that regardless of the inner details, the deviation is always 2/3. As before, this is a large deviation and can be tested for in colliders.

7.2 Bubble dynamics and GWs

The idea is as follows. As explained in Section 6, we start with a potential with a single minimum at the origin right after the Big Bang. As the temperatures cool down, this potential develops inflection points and further curvature, eventually having two degenerate minima - one at the origin and another not at the origin. This occurs at the critical temperature - this is where the phase transition takes place. The phase transition leads to the potential developing the usual double well form that then gets symmetry broken, following which the Higgs mechanism accounts for the masses of the gauge bosons.

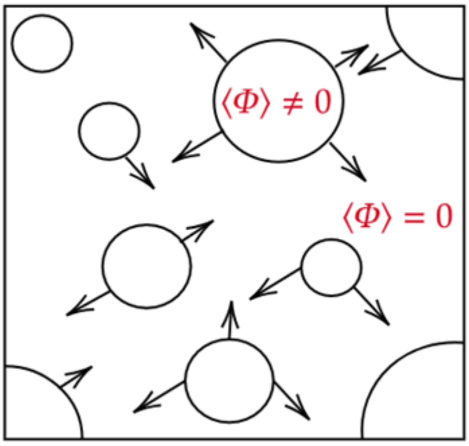


Figure 4: A schematic representation of the bubble dynamics at $T = T_c$

During the phase transition, however, at the critical temperature, both phases sort of co-exist. Areas with vacuum expectations not equal to zero and those with vacuum expectations equal to zero. It is a “bubble” like picture of the universe with bubbles of non zero vacuum expectation values scattered around other areas with zero vacuum expectation values as often happens during such phase transitions at the critical temperature. These bubbles can “expand” - i.e., more and more areas assume a non zero vacuum expectation - and collisions between the bubbles and their expansion can lead to the generation of gravitational waves. The primordial GWs would still be able to be detected today - and this is what we can use to indirectly probe for signs of electroweak phase transition induced baryogenesis. This is schematically depicted in Figure 4.

Now, as mentioned earlier, during the phase transition, the potential develops degenerate minima - the phase transition occurs via tunnelling from one of the minima to the other. The transition leads to the release of latent heat - part of it goes into raising the temperature of the surrounding plasma, while the rest of it is converted into the kinetic energy of the bubble wall [10].

7.2.1 Bubble dynamics parameters

The dynamics of the bubbles can be essentially parametrized by two parameters, α and β. These two parameters are sufficient to describe the spectra of GWs generated as a result of bubble collisions. To begin with, we introduce the nucleation rate per unit volume per unit time for the bubbles,

$$\Gamma(t) = \Gamma_0(t) e^{-S_E(t)} \tag{274}$$

Here, S_E is proportional to the Euclidean action for the bubbles and is given by,

$$S_E(t) = \frac{1}{T} \int d^3x \left[\frac{1}{2}(\nabla\varphi)^2 + V_{\text{eff}}(\varphi, T)\right] \tag{275}$$

Also, the pre-factor $\Gamma_0(t) \sim T^4$. Now, let the time of the phase transition be t_c. We can define a quantity, β, as follows,

$$\beta = -\frac{dS_E}{dt}|_{t=t_c} \approx \frac{1}{\Gamma}\frac{d\Gamma}{dt}|_{t=t_c} \tag{276}$$

This is assuming that Γ_0 varies very slowly with time. Now, β tells us how the nucleation rate of the bubbles varies with time - its inverse roughly describes the time taken by the phase transition. Further,

$$\beta = -\frac{dS_E}{dt}|_{t=t_c} = -\frac{dS_E}{dT}|_{T=T_c}\ \frac{dT}{dt}|_{t=t_c} \tag{277}$$

The Hubble parameter, H, can be expressed as $HT = -dT/dt$, then,

$$\beta = H_c T_c \frac{dS_E}{dT}|_{T=T_c} \tag{278}$$

This allows us to define a dimensionless parameter, $\tilde{\beta} = \beta/H_c$. The condition for which the bubbles spread throughout the universe and the phase transition is completed is,

$$\frac{\Gamma}{H^4}\ |_{T=T_c} \sim 1 \tag{279}$$

This means,

$$\ln\left(\frac{\Gamma(t_c)}{H_c^4}\right) = \ln\left(\frac{\Gamma_0(t_c)}{H_c^4}\right) - S_E(t_c) \approx 0 \Rightarrow S_E(t_c) \approx 4\ln\left(\frac{T_c}{H_c}\right) = 140 - 150 \tag{280}$$

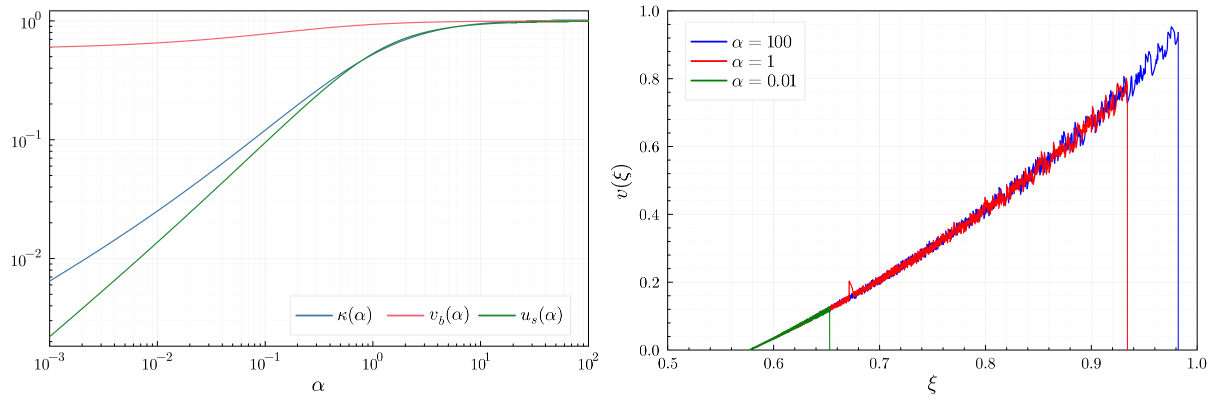
A second important parameter is the latent heat available for the expansion of the bubbles. This has two contributions - one is the difference between the two minima which vanishes at the critical point and another from the entropy contribution,

$$\varepsilon(T) = -\Delta V_{\text{eff}} - T\Delta S = -\Delta V_{\text{eff}}(\varphi, T) - T\frac{\partial V_{\text{eff}}}{\partial T} \tag{281}$$

We define a new parameter,

$$\alpha = \frac{\varepsilon(T_c)}{\rho(T_c)} \tag{282}$$

Here, $\rho(T)$ is the radiation energy density given by $\rho = (\pi^2 g/30)T^4$ with g being the relativistic degrees of freedom in the plasma. With α and β, we can now describe the GW spectra.



(a) Variation of the parameters with α (b) Velocity profile of the fluid ($\xi = r/t$)

Figure 5: Graphical description of the system [11]

The bubble dynamics can be described by looking into the hydrodynamics of a sphere of fluid [11]. In particular, we can consider a spherical "combustion" of the fluid. In this case, the system can be described in terms of a parameter, $\xi = r/t$, where r is the radial distance from the centre of the bubble while t is the time since the nucleation process began. The velocity of the fluid in this case, v, is a function of ξ and ξ itself can be considered to be measure of the outward velocity of the bubble wall - the bubble wall velocity. The fluid velocity satisfies the following equation in terms of ξ,

$$\left[\left(\frac{\mu}{c_s}\right)^2 - 1\right]\frac{dv}{d\xi} = \frac{2v}{\xi(1-v\xi)}\gamma^{-2} \tag{283}$$

Here, $\gamma^{-2} = 1 - v^2$ and $\mu = (\xi - v)/(1 - v\xi)$ while c_s is a constant parameter in the problem. Now, usually, there can be two cases - one is "detonation" and other is "deflagration". In the former case, the velocity of fluid velocity in the symmetric (non-broken) phase is greater than the fluid velocity in the broken phase and the latter is the opposite. In this case, we consider the first kind only. Further we take the relativistic limit in which c_s (which usually describes the speed of sound) is taken to be $1/\sqrt{3}$. First, let us define the "efficiency factor" of the process,

$$\kappa(\alpha) = \frac{1}{1 + 0.715\alpha}\left(0.715\alpha + \frac{4}{27}\sqrt{\frac{3\alpha}{2}}\right) \tag{284}$$

This is a measure of how much of the vacuum energy (or, latent heat) goes into the kinetic energy of the fluid, rather than thermal energy. In the case of detonation, the velocity of the bubble wall is given in terms of α by,

$$\xi_d \equiv v_b(\alpha) = \frac{1/\sqrt{3} + \sqrt{\alpha^2 + 2\alpha/3}}{1+\alpha} \tag{285}$$

The variation of these quantities with α is shown in Figure 5. The velocity profile of the fluid is obtained by integrating Eq. 283 from $\xi = c_s$ to $\xi = v_b$ with the boundary condition being $v(v_b) = (v_b - c_s)/(1 - v_b c_s)$. The integration for various values of α has been performed in Figure 5. The quantity $dv/d\xi$ diverges as $\xi \to v_b$ - therefore, the numerical integration was not performed till exactly v_b. Rather, the upper limit of the integral was shifted by a small amount, ε, to $(v_b - \varepsilon)$. Further, the turbulent fluid velocity in the plasma (the fluid velocities corresponding to the largest length scales on which turbulence in the plasma generates GWs) is defined by,

$$u_s(\alpha) \approx \sqrt{\frac{\kappa\alpha}{\frac{4}{3}+\kappa\alpha}} \tag{286}$$

This is a separate quantity and has nothing to do with the bubble wall velocity.

7.2.2 The GW spectra

We are mostly concerned with the energy density of the generated gravitational waves. There are two basic contributions to this - one from the bubble collisions and another from plasma turbulence. Thus, the total energy density of the GWs with frequency, f, is roughly,

$$\Omega_{\text{GW}}(f)h^2 = \Omega_{\text{bubble}}(f)h^2 + \Omega_{\text{plasma}}(f)h^2 \tag{287}$$

Here, $\Omega(f)h^2$ gives us the required energy density. The individual contributions are,

$$\Omega_{\text{bubble}}(f)h^2 = \bar{\Omega}_{\text{bubble}}(f_{\text{bubble}})h^2 \times \begin{cases} \left(\frac{f}{f_{\text{bubble}}}\right)^{2.8} \text{ for } f < f_{\text{bubble}} \\ \left(\frac{f}{f_{\text{bubble}}}\right)^{-1} \text{ for } f > f_{\text{bubble}} \end{cases} \tag{288}$$

Here,

$$\bar{\Omega}_{\text{bubble}}(f_{\text{bubble}})h^2 = c\kappa^2\left(\frac{H_c}{\beta}\right)^2\left(\frac{\alpha}{1+\alpha}\right)^2\left(\frac{v_b^3}{0.24+v_b^3}\right)\left(\frac{100}{g}\right)^{\frac{1}{3}} \tag{289}$$

For $f = f_{\text{bubble}}$, $c = 1.1\times10^{-6}$. The peak frequency is given by,

$$f_{\text{bubble}} = 5.2\times10^{-3}\text{ mHz}\left(\frac{\beta}{H_c}\right)\left(\frac{T_c}{100\text{ GeV}}\right)\left(\frac{g}{100}\right)^{\frac{1}{6}} \tag{290}$$

Similarly, the plasma contribution is,

$$\Omega_{\text{plasma}}(f)h^2 = \bar{\Omega}_{\text{plasma}}(f_{\text{plasma}})h^2 \times \begin{cases} \left(\frac{f}{f_{\text{plasma}}}\right)^{2} \text{ for } f < f_{\text{plasma}} \\ \left(\frac{f}{f_{\text{plasma}}}\right)^{-3.5} \text{ for } f > f_{\text{plasma}} \end{cases} \tag{291}$$

Here,

$$\bar{\Omega}_{\text{plasma}}(f_{\text{plasma}})h^2 = 1.4\times10^{-4}\ u_s^5 v_b^2\left(\frac{H_c}{\beta}\right)^2\left(\frac{100}{g}\right)^{\frac{1}{3}} \tag{292}$$

And,

$$f_{\text{plasma}} = 3.4\times10^{-3}\text{ mHz}\ \frac{u_s}{v_b}\left(\frac{\beta}{H_c}\right)\left(\frac{T_c}{100\text{ GeV}}\right)\left(\frac{g}{100}\right)^{\frac{1}{6}} \tag{293}$$

Now, the bubble solutions can be used to determine the parameters α and β - with these two parameters, we can describe the spectra of the GWs generated during electroweak phase transition. In [8], it has been shown that in these cases, the GW spectra can be significant and experimentally observable. It has also been shown that the signal strength increases for larger N in theories with additional $O(N)$ symmetric scalars. There can be further numerical improvements - there can be a third contribution to the GW generation from sound waves in the plasma following collisions [9].

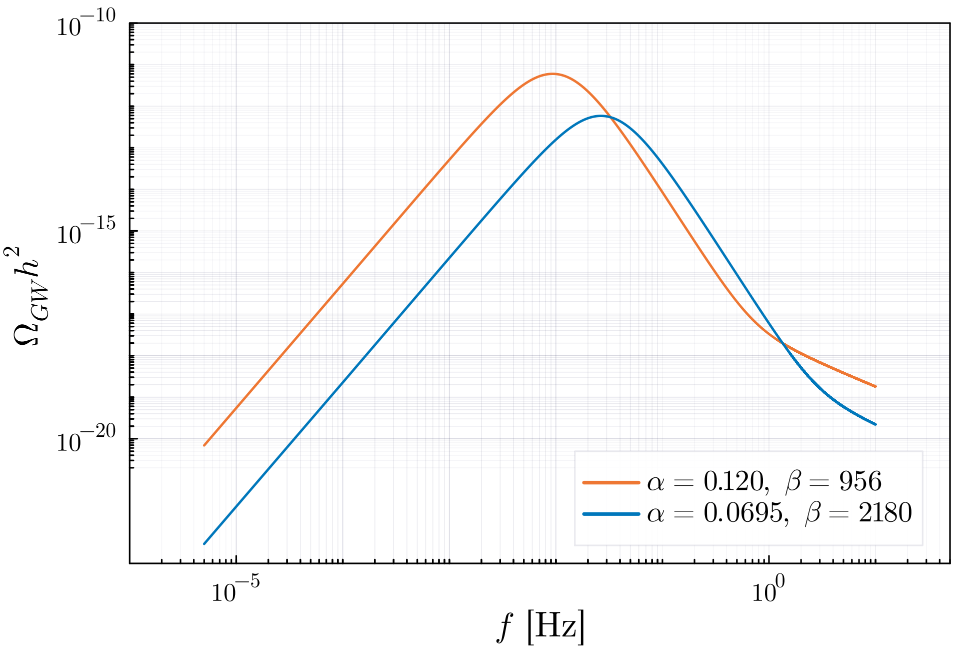


Figure 6: GW spectra for different parameter sets

Figure 6 depicts such GW spectra for two different parameter sets - these sets were taken from [9] for the $N = 4$ case. We also include contributions from the sound wave compression in the plasma as outlined in [9]. The above figure is not the most detailed, however. Here, we have mentioned only one type of detonatation solutions here (Eq. 285) - there are a larger class of solutions for which the equation does not necessarily hold. In improved numerical studies, v_b is considered to be a free parameter. However, the bottomline remains the same - GW generation through these processes can be significant. Therefore, searching for such primordial GWs can serve as a viable alternative when it comes to looking for signatures of the electroweak phase transition induced baryogenesis process.

8 Discussion

The Higgs particle was discovered at the Large Hadron Collider (LHC) at CERN, Geneva in 2012 firmly establishing the veracity of the standard model of particle physics and, in particular, the ideas of spontaneous symmetry breaking and the Higgs mechanism for mass generation of various particles. Gravitational waves were first observed in 2015 by LIGO. Even more technological upgrades are being carried out at the LHC (for the HL-LHC) and plans are underway for future colliders like the ILC, FCC and even muon colliders. If and when such experiments start collecting data, we are sure to find even more phenomena and results that disagree with the standard model.

In this report, we have looked at the baryon asymmetry problem which is still an open issue in modern particle physics. The Sakharov conditions seem to imply the existence of a strong first order electroweak phase transition during the early stages of the universe after the big bang, but such a transition is not possible within the standard model which only allows for a smooth crossover. We have considered the possibility of this transition taking place in the context of the two Higgs doublet model and approached the feasibility of the THDM in two ways.

The first is the usual particle physics way - collider phenomenology. We have calculated loop corrections and deviations to the trilinear Higgs self coupling, λ_{hhh}, in the THDM due to the additional Higgs fields present in the theory. The calculations are performed in Section 5 using the effective potential approach in [4,7] and it is shown that the corrections can be large. The relative deviation of this quantity in the THDM is graphically depicted in Figure 3 and it is expected that such significant deviations can be experimentally observed at the relevant energy scales in future collider experiments.

The second is the newer way - gravitational wave astronomy. Ever since LIGO first detected GWs in 2015, GW astronomy has undergone rapid growth. As discussed in Section 7, electroweak phase transition can lead to the generation of primordial GWs owing to various contributions - bubble collisions, plasma turbulence and sound waves in the plasma. These GWs can be modelled and characterised using two basic parameters, α and β. As shown in [8,9], the GW spectra resulting from these sources can be significant as well - well within the reaches of future GW detectors like LISA and DECIGO. Probing for such primordial GWs can, thus, be another fruitful way to looking for signals of the electroweak phase transition.

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10 Numerical evaluation

All of the plots and numerical computations performed over the course of the project were carried out using the SciML ecosystem on Julia. I would like to thank members of the online Julia community for their help in resolving a couple of issues I ran into.

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