

HomeWork 3

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Q1

With the scales of the latent variables properly fixed, is the following model identified?

Express each parameter in terms of the elements of Σ

We have a covariance matrix

$$\Sigma = \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \sigma_{13} & \sigma_{14} \\ \sigma_{21} & \sigma_2^2 & \sigma_{23} & \sigma_{24} \\ \sigma_{31} & \sigma_{32} & \sigma_3^2 & \sigma_{34} \\ \sigma_{41} & \sigma_{42} & \sigma_{43} & \sigma_4^2 \end{bmatrix} \quad (1)$$

, four major equations

$$x_1 = \lambda_1 \xi_1 + \delta_1; x_2 = \lambda_2 \xi_1 + \delta_2; x_3 = \lambda_3 \xi_2 + \delta_3; x_4 = \lambda_4 \xi_2 + \delta_4 \quad (2)$$

, and the covariance matrix between two ξ s

$$\Phi = \begin{bmatrix} 1 & \phi_{12} \\ \phi_{21} & 1 \end{bmatrix} \quad (3)$$

where $\phi_{12} = \phi_{21}$, and we define them as ϕ . We also let $Var(\xi_1) = Var(\xi_2) = 1$. Therefore, we have 10 data points and 9 parameters, thereby leading to 1 degree of freedom in this case. Then we can derive the equations for each σ_{ij} in the Σ in terms of the parameters.

$$\sigma_1^2 = \lambda_1^2 + Var(\delta_1) \quad (4)$$

$$\sigma_2^2 = \lambda_2^2 + Var(\delta_2) \quad (5)$$

$$\sigma_3^2 = \lambda_3^2 + Var(\delta_3) \quad (6)$$

$$\sigma_4^2 = \lambda_4^2 + Var(\delta_4) \quad (7)$$

$$\sigma_{12} = \lambda_1 \lambda_2 \quad (8)$$

$$\sigma_{13} = \lambda_1 \lambda_3 \phi \quad (9)$$

$$\sigma_{14} = \lambda_1 \lambda_4 \phi \quad (10)$$

$$\sigma_{23} = \lambda_2 \lambda_3 \phi \quad (11)$$

$$\sigma_{24} = \lambda_2 \lambda_4 \phi \quad (12)$$

$$\sigma_{34} = \lambda_3 \lambda_4 \quad (13)$$

Using those equations, we can further derive the formulations of the parameters. We let (9) divided by (10) and (9) divided by (11). Will have

$$\lambda_3 = \frac{\sigma_{13}}{\sigma_{14}} \lambda_4; \lambda_4 = \frac{\sigma_{14}}{\sigma_{13}} \lambda_3; \lambda_1 = \frac{\sigma_{13}}{\sigma_{23}} \lambda_2; \lambda_2 = \frac{\sigma_{23}}{\sigma_{13}} \lambda_1 \quad (14)$$

and let (14) fill into (8) and (13). Then we obtain

$$\lambda_1 = \pm \sqrt{\frac{\sigma_{12}\sigma_{13}}{\sigma_{23}}}; \lambda_2 = \pm \sqrt{\frac{\sigma_{12}\sigma_{23}}{\sigma_{13}}}; \lambda_3 = \pm \sqrt{\frac{\sigma_{34}\sigma_{13}}{\sigma_{14}}}; \lambda_4 = \pm \sqrt{\frac{\sigma_{34}\sigma_{14}}{\sigma_{13}}} \quad (15)$$

and we substitute (15) into (4), (5), (6) and (7). We can derive

$$Var(\delta_1) = \sigma_1^2 - \frac{\sigma_{12}\sigma_{13}}{\sigma_{23}}; Var(\delta_2) = \sigma_2^2 - \frac{\sigma_{12}\sigma_{23}}{\sigma_{13}}; Var(\delta_3) = \sigma_3^2 - \frac{\sigma_{34}\sigma_{13}}{\sigma_{14}}; Var(\delta_4) = \sigma_4^2 - \frac{\sigma_{34}\sigma_{14}}{\sigma_{13}} \quad (16)$$

Also, substitute λ_1 & λ_3 into (9), leading to a equation of

$$\phi = \sqrt{\frac{\sigma_{23}\sigma_{14}}{\sigma_{12}\sigma_{34}}} \quad (17)$$

In (15), (16), and (17), we have expressed each parameter in terms of the σ s.

Apply the two-step rule for model identification

We can consider the CFA part and assume that all the parameters have non-zero values. By applying the two-indicator rule, it is found that there are two non-zero loadings for each factor and all of the observed variables (x) are only influenced by one factor. Also, there is no zero element in Φ , suggesting these two factors somehow have the relationship between each other. Likewise, each variance of δ is not correlated with others. According to these criteria, we have found the the CFA part is identified. Since the present model does not have the structural part, we can claim that this model is identified.

How about if $\xi_1 \rightarrow \xi_2$ instead? Repeat 1 & 2.

We can clearly see that the \sum is still the same but a new equation is generated to indicate the direct effect from $\xi_1 \rightarrow \xi_2$. Indeed, ξ_2 should be represented as η_1 under the representation of the JKW model. (Note that we still use x rather than y for consistent demonstrations). Thus, we have

$$\eta_1 = \gamma_{11}\xi_1 + \zeta_1 \quad (18)$$

and Φ become a vector $\Phi=\phi=1$. Also, $\Psi = \psi = Var(\zeta_1)$. Since we fix the $Var(\xi_1) = Var(\eta_1) = 1$ for comparisons (usually we fix the loadings for endogenous variables), we have a fixed $Var(\zeta_1)$ and it can be expressed as $1 - \gamma_{11}^2$ as well.

The representations of the σ in the new model are the same, except for (9), (10), (11), and (12). These four equations have been changed since the covariance between ξ_1 and ξ_2 is replaced by the direct effect γ_{11} . The equations are shown as follow.

$$\sigma_{13} = \lambda_1 \lambda_3 \gamma_{11} \quad (19)$$

$$\sigma_{14} = \lambda_1 \lambda_4 \gamma_{11} \quad (20)$$

$$\sigma_{23} = \lambda_2 \lambda_3 \gamma_{11} \quad (21)$$

$$\sigma_{24} = \lambda_2 \lambda_4 \gamma_{11} \quad (22)$$

In the same vein, the representation of parameters in term of σ are still the same as (15) and (16). The only difference is that

$$\gamma_{11} = \sqrt{\frac{\sigma_{23}\sigma_{14}}{\sigma_{12}\sigma_{34}}} \quad (23)$$

That is, we can express each parameter in the new model in terms of the elements of Σ using (15), (16), and (23).

Furthermore, we also apply the two-step rule for identification of the new model. In the CFA model, we can use the aforementioned two-indicator rule to show that this measurement part is identified. About the structural part, we can show that the model is identified by using null B rule and recursive rule due to the reason that we have only one η in our model.

Are these two models equivalent?

In summary, these two models are both identified. And, they are indeed equivalent. As shown in (15), (16), the expressions of the parameters in these two models are roughly the same. The different expressions are shown in (17) and (23). However, the elements to represent those parameters are exactly the same. In other words, these two model (model 1 & model 2) can give identical covariance matrix, although they use different parameters :

$$\sum(\hat{\theta}_1) = \sum(\hat{\theta}_2) \quad (24)$$

, where $\hat{\theta}_1$ and $\hat{\theta}_2$ are different parameters in model 1 & 2. As shown, we can conclude that these two model are equivalent.

Q2

With the scales of the latent variables properly fixed, is the following model identified?

Express each parameter in terms of the elements of Σ

We have a covariance matrix

$$\Sigma = \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \sigma_{13} & \sigma_{14} & \sigma_{15} & \sigma_{16} \\ \sigma_{21} & \sigma_2^2 & \sigma_{23} & \sigma_{24} & \sigma_{25} & \sigma_{26} \\ \sigma_{31} & \sigma_{32} & \sigma_3^2 & \sigma_{34} & \sigma_{35} & \sigma_{36} \\ \sigma_{41} & \sigma_{42} & \sigma_{43} & \sigma_4^2 & \sigma_{45} & \sigma_{46} \\ \sigma_{51} & \sigma_{52} & \sigma_{53} & \sigma_{54} & \sigma_5^2 & \sigma_{56} \\ \sigma_{61} & \sigma_{62} & \sigma_{63} & \sigma_{64} & \sigma_{65} & \sigma_6^2 \end{bmatrix} \quad (25)$$

, six major equations for the measurement part

$$y_1 = \lambda_1 \eta_1 + \epsilon_1; y_2 = \lambda_2 \eta_1 + \epsilon_2; y_3 = \lambda_3 \eta_1 + \epsilon_3; y_4 = \lambda_4 \eta_2 + \epsilon_4; y_5 = \lambda_5 \eta_2 + \epsilon_5; y_6 = \lambda_6 \eta_2 + \epsilon_6 \quad (26)$$

, two equations for the structural part

$$\eta_1 = \gamma_{11} \xi_1 + \zeta_1; \eta_2 = \gamma_{21} \xi_1 + \zeta_2 \quad (27)$$

, and the covariance matrix between two ζ s

$$\Psi = \begin{bmatrix} \psi_{11} & 0 \\ 0 & \psi_{22} \end{bmatrix} \quad (28)$$

Furthermore, we let $Var(\xi_1) = \lambda_1 = \lambda_4 = 1$ to fix the scale of the latent variables. Then we can try to derive the equations for each σ_{ij} in the Σ in terms of the parameters.

$$\sigma_1^2 = \gamma_{11}^2 + \psi_{11} + Var(\epsilon_1) \quad (29)$$

$$\sigma_2^2 = \lambda_2^2(\gamma_{11}^2 + \psi_{11}) + Var(\epsilon_2) \quad (30)$$

$$\sigma_3^2 = \lambda_3^2(\gamma_{11}^2 + \psi_{11}) + Var(\epsilon_3) \quad (31)$$

$$\sigma_4^2 = \gamma_{21}^2 + \psi_{22} + Var(\epsilon_4) \quad (32)$$

$$\sigma_5^2 = \lambda_5^2(\gamma_{21}^2 + \psi_{22}) + Var(\epsilon_5) \quad (33)$$

$$\sigma_6^2 = \lambda_6^2(\gamma_{21}^2 + \psi_{22}) + Var(\epsilon_6) \quad (34)$$

$$\sigma_{12} = \lambda_2(\gamma_{11}^2 + \psi_{11}) \quad (35)$$

$$\sigma_{13} = \lambda_3(\gamma_{11}^2 + \psi_{11}) \quad (36)$$

$$\sigma_{14} = \gamma_{11} \gamma_{21} \quad (37)$$

$$\sigma_{15} = \lambda_5 \gamma_{11} \gamma_{21} \quad (38)$$

$$\sigma_{16} = \lambda_6 \gamma_{11} \gamma_{21} \quad (39)$$

$$\sigma_{23} = \lambda_2 \lambda_3 (\gamma_{11}^2 + \psi_{11}) \quad (40)$$

$$\sigma_{24} = \lambda_2 \gamma_{11} \gamma_{21} \quad (41)$$

$$\sigma_{25} = \lambda_2 \lambda_5 \gamma_{11} \gamma_{21} \quad (42)$$

$$\sigma_{26} = \lambda_2 \lambda_6 \gamma_{11} \gamma_{21} \quad (43)$$

$$\sigma_{34} = \lambda_3 \gamma_{11} \gamma_{21} \quad (44)$$

$$\sigma_{35} = \lambda_3 \lambda_5 \gamma_{11} \gamma_{21} \quad (45)$$

$$\sigma_{36} = \lambda_3 \lambda_6 \gamma_{11} \gamma_{21} \quad (46)$$

$$\sigma_{45} = \lambda_5 (\gamma_{21}^2 + \psi_{22}) \quad (47)$$

$$\sigma_{46} = \lambda_6 (\gamma_{21}^2 + \psi_{22}) \quad (48)$$

$$\sigma_{56} = \lambda_5 \lambda_6 (\gamma_{21}^2 + \psi_{22}) \quad (49)$$

By using (37), (38), (39), (41), and (44), we can obtain

$$\lambda_2 = \frac{\sigma_{24}}{\sigma_{14}}; \lambda_3 = \frac{\sigma_{34}}{\sigma_{14}}; \lambda_5 = \frac{\sigma_{15}}{\sigma_{14}}; \lambda_6 = \frac{\sigma_{16}}{\sigma_{14}} \quad (50)$$

By substituting λ_2 & λ_3 into (35) and (47) respectively, we can derive that

$$\gamma_{11}^2 + \psi_{11} = \frac{\sigma_{12}\sigma_{14}}{\sigma_{24}} \gamma_{21}^2 + \psi_{22} = \frac{\sigma_{45}\sigma_{14}}{\sigma_{15}} \quad (51)$$

Moreover, via substituting (51) into (29) ~ (34), we will have

$$Var(\epsilon_1) = \sigma_1^2 - \frac{\sigma_{12}\sigma_{14}}{\sigma_{24}}; Var(\epsilon_2) = \sigma_2^2 - \frac{\sigma_{24}\sigma_{12}}{\sigma_{14}}; Var(\epsilon_3) = \sigma_3^2 - \frac{\sigma_{34}^2\sigma_{12}}{\sigma_{14}\sigma_{24}} \quad (52)$$

$$Var(\epsilon_4) = \sigma_4^2 - \frac{\sigma_{45}\sigma_{14}}{\sigma_{15}}; Var(\epsilon_5) = \sigma_5^2 - \frac{\sigma_{15}\sigma_{45}}{\sigma_{14}}; Var(\epsilon_6) = \sigma_6^2 - \frac{\sigma_{16}^2\sigma_{45}}{\sigma_{14}\sigma_{15}} \quad (53)$$

In (50), (52) and (53), we can express λ s and $Var(\epsilon)$ in terms of the σ s.

Discussion

Although we have expressed many available parameters using the elements in Σ as shown in (50), (52) and (53), there are still 4 parameters that we have yet derived (and cannot be derived in fact). Those are γ_{11}^2 , ψ_{11} , γ_{21}^2 and ψ_{22} . It seems that γ_{11}^2 & ψ_{11} are always combined and γ_{21}^2 & ψ_{22} are always combined. It is impossible to distinguish these four parameters in the present model, thus causing these four parameters to be unidentified. However, a model is identified if and only if all its parameters are identified. Therefore, this model is not identified. In another viewpoint, when we only consider the structural part of this model, the model can be regarded as a two indicator model (one factor with two indicators). It is obvious that a two indicator model will not be identified. In brief, we can conclude that this model is an unidentified one.