Numerical Techiques in Physics

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§ WEEK 2 §

Problem 1: Solving second order ODEs

Resources:

- https://kyleniemeyer.github.io/ME373-book/content/second-order/numerical-methods.html

The Euler method can be used to solve second-order ODE by doing some variable changes. Using this technique, solve the following Damped-driven-harmonic oscillator.

$$y'' + 4y' + 3y = \sin x + 2\cos x \tag{1.1}$$

with the initial conditions $y(0) = 1$ and $y'(0) = -1$. Find the exact solution for the equation. Recall MA105!! Plot the Euler method solution and the exact solution on the same graph and compare your results. *Choose $h = 0.1$ & $h = 0.01$*

Solve the above equation using the RK4 method. Plot the RK4 solution and the exact solution and compare your results. *Choose $h = 0.1$*

Problem 2: Van der Pol Oscillator

Resources:

- https://en.wikipedia.org/wiki/Van_der_Pol_oscillator

Conceived by Balthasar van der Pol using vacuum tube circuits, this self-starting oscillator is used to model neuron activations and geological faults. This can be described using the following second-order DE where an SHM has variable anti-damping/ fixed self-driving but variable damping: $\frac{d^2x}{dt^2} - \mu(1-x^2)\frac{dx}{dt} + x = 0$ It can be transformed to the following two-variable system by the transformation $y = x - \frac{1}{3}x^3 - \frac{\dot{x}}{\mu}$:

$$\dot{x} = \mu\left(x - \frac{1}{3}x^3 - y\right) \tag{2.1}$$

$$\dot{y} = \frac{x}{\mu} \tag{2.2}$$

Implement the above for $\mu = 7$, using Euler's method. Use a timestep of 0.1. Make plots of y vs x, x vs t, and y vs t. In the y vs x plot, try plotting the $y = x - \frac{x^3}{3}$.

Now try the RK4 method with the same μ and timestep. Compare this with the Euler method. See if you can make an animation. Wikipedia has a nice animation, one with changing μ from 0.1 to 4. Try making that. Use RK4 for the animations, well you know the reason why.

Bonus: There is also an animation showing how randomly chosen initial conditions are attracted to a stable orbit.

Problem 3: Lorenz Attractor

Resources:

- https://en.wikipedia.org/wiki/Lorenz_system

The term "butterfly effect" in popular media may stem from the real-world implications of the Lorenz attractor, namely that tiny changes in initial conditions evolve to completely different trajectories. This underscores that chaotic systems can be completely deterministic and yet still be inherently impractical or even impossible to predict over longer periods of time. For example, even the small flap of a butterfly's wings could set the earth's atmosphere on a vastly different trajectory, in which for example a hurricane occurs where it otherwise would have not (see Saddle points). The shape of the Lorenz attractor itself, when plotted in phase space, may also be seen to resemble a butterfly.

In 1963, Edward Lorenz developed a simplified mathematical model for atmospheric convection. The model is a system of three ordinary differential equations now known as the Lorenz equations:

$$\frac{dx}{dt} = \sigma(y - x) \tag{3.1}$$

$$\frac{dy}{dt} = x(\rho - z) - y \tag{3.2}$$

$$\frac{dz}{dt} = xy - \beta z \tag{3.3}$$

Plot this system of equations using the Euler method and the RK4 method. Use $\rho = 28, \sigma = 10, \beta = \frac{8}{3}$. Try plotting it for $\rho = 14$ and $\rho = 99.96$

This system is highly sensitive to initial conditions. Use $\rho = 28, \sigma = 10, \beta = \frac{8}{3}$ and show the 3-D evolution of two trajectories in the Lorenz attractor starting at two initial points that differ only by 10^{-5} in the x-coordinate.