Birth and Death Processes(Sup)

Hyunwoo Gu

1 Birth-death Markov Chains

(6.4. from *Gallager*)

A *birth-death Markov chain* is a Markov chain where the state space is the set of nonnegative integers. A transition from i to $i+1$ is regarded as a birth and $i+1$ to i is regarded as a death. Let us denote $p_i := P_{i,i+1}$ and $q_i := P_{i,i-1}$, thus $P_{ii} = 1 - p_i - q_i$. So what is the steady-state probabilities of these BD chains?

Note that the number of transitions from i to $i+1$ differs by at most 1 from the number of transitions

Thus if we visualize renewal-reward process with renewals on occurrences of state i and unit reward on transitions from i to $i+1$, the limiting time-average number of transitions per unit time is $\pi_i p_i$. Similarly, the limiting time-average number of transitions per unit time from $i+1$ to i is $\pi_{i+1} q_{i+1}$. Thus by equating in limit,

$$\pi_i p_i = \pi_{i+1} q_{i+1}, \quad \forall i \geq 0$$

It is convenient to define ρ_i as p_i / q_{i+1}. Then we have $\pi_{i+1} = \rho_i \pi_i$, and iterating this,

$$\pi_i = \pi_0 \prod_{j=0}^{i-1} \rho_j, \quad \pi_0 := \frac{1}{1 + \sum_{i=1}^{\infty} \prod_{j=0}^{i-1} \rho_j}$$

If $\sum_{i=1}^{\infty} \prod_{j=0}^{i-1} \rho_j < \infty$, then π_0 is positive and **all the states are positive recurrent**. If this sum of products is infinite, then no state is positive recurrent.

2 General pure birth processes

2.1 Postulates for the Poisson process

In order to define more general processes of a similar kind, let us point out various further properties that the Poisson possesses. In particular, it is a Markov process on the nonnegative integers which has the following properties :

$$P\{X(t+h) - X(t) = 1 | X(t) = x\} = \lambda h + o(h), h \to 0_+, (x = 0, 1, 2, \cdots)$$

$$\Leftrightarrow \lim_{h \to 0+} \frac{P\{X(t+h) - X(t) = 1 | X(t) = x\}}{h} = \lambda$$

Notice that the right-hand side is independent of x.

$$P\{X(t+h) - X(t) = 0 | X(t) = x\} = 1 - \lambda h + o(h), h \to 0_+$$

$$X(0) = 0$$

2.2 Pure birth process

A natural generalization of the Poisson process is to permit the chance of an event occurring at a given instant of time to depend upon the number of events which have already occurred. For example, **the probability of a birth at a given instant is proportional to the population size at that time**, which is known as the **Yule process**.

The characteristic function of S_n is given by

$$\phi_n(w) = \mathbb{E}(exp(iwS_n)) = \prod_{k=0}^{n-1} \mathbb{E}(exp(iwT_k)) = \prod_{k=0}^{n-1} \frac{\lambda_k}{\lambda_k - iw}$$

2.3 Yule process

The **Yule process** is an example of a **pure birth process** that arises in physics and biology. Assume that each member in a population has a probability $\beta h + o(h)$ of giving birth to a new member in an interval of time length $h(\beta > 0)$. Furthermore assume that there are $X(O) = N$ members present at time 0. Assuming independence and no interaction among members of the population, the binomial theorem gives

$$\begin{aligned} Pr[X(t+h) - X(t) = 1 | X(t) = n] &= \binom{n}{1} [\beta h + o(h)] (1 - \beta h + o(h))^{n-1} \\ &= n\beta h + o_n(h) \end{aligned}$$

where $\lambda_n = n\beta$. Thus

$$P_n'(t) = -\beta[nP_n(t) - (n-1)P_{n-1}(t)], \quad n = 1, 2, \cdots$$

under BV : $P_1(0) = 1, P_n(0) = 0, \quad n = 2, 3, \cdots$.

The solution is

$$P_n(t) = e^{-\beta t}(1 - e^{-\beta t})^{n-1}$$

The generating function is

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty} P_n(t)s^n \\ &= se^{-\beta t} \sum_{n=1}^{\infty} \left[(1 - e^{-\beta t})s\right]^{n-1} \\ &= \frac{se^{-\beta t}}{(1 - (1 - e^{-\beta t})s)} \end{aligned}$$

Letting $P_{N_n}(t) := Pr[X(t) = n | X(0) = N]$ and $f_N(s) = \sum_{n=N}^{\infty} P_{N_n}(t)s^n$, we have

$$\begin{aligned} f_N(s) &= [f(s)]^N \\ &= \left[\frac{se^{-\beta t}}{1 - (1 - e^{-\beta t})s}\right]^N \\ &= (se^{-\beta t})^N \sum_{m=0}^{i} nfty \binom{m + N - 1}{m} (1 - e^{-\beta t})^m s^m \end{aligned}$$

T : the waiting time of $X(t)$ in the state i.

Letting $G_i(t) := P(T_i \geq t)$,

$$G_i(t+h) = G_i(t)G_i(h) = G_i(t)\left[P_{ii}(h) + o(h)\right]$$

$$= G_i(t)[1 - (\lambda_i + \mu_i)h] + o(h)$$

$$\Leftrightarrow \frac{G_i(t+h) - G_i(t)}{h} = -(\lambda_i + \mu_i)G_i(t) + o(1)$$

$$G_i'(t) = -(\lambda_i + \mu_i)G_i(t)$$

where the last line corresponds to the IVP of with $G_i(0) = 1$

2.4 Birth and Death Processes

To generalize the pure birth processes, we can permit $X(t)$ to decrease as well as increase, for example, by the death of members. This can be regarded as the continuous time analogs of random walks.

- $P_{i,i+1}(h) = \lambda_i h + o(h), h \to 0_+,\ i \geq 0$
- $P_{i,i-1}(h) = \mu_i h + o(h), h \to 0_+,\ i \geq 1$
- $P_{i,I}(h) = 1 - (\lambda_i + \mu_i)h + o(h), h \to 0_+,\ i \geq 0$
- $P_{ij}(0) = \delta_{ij}$
- $\mu_0 = 0,\ \lambda_0 > 0, \mu_i, \lambda_i > 0, i = 1, 2, \cdots$

The matrix A, the **infinitesimal generator** of the process,

$$A = \begin{bmatrix} -\lambda_0 & \lambda_0 & 0 & 0 & \cdots \\ \mu_1 & -(\lambda_1 + \mu_1) & \lambda_1 & 0 & \cdots \\ 0 & \mu_2 & -(\lambda_2 + \mu_2) & \lambda_2 & \cdots \\ 0 & 0 & \mu_3 & -(\lambda_3 + \mu_3) & \cdots \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

2.5 Differential Equations of Birth and Death Processes

As in the pure birth and Poisson processes, the transition probabilities P_{ij} satisfy a system of differential equations, known as *Kolmogorov differential equations.*

$$P'_{0j}(t) = -\lambda_0 P_{0j}(t) + \lambda_0 P_{ij}(t)$$

$$P'_{ij}(t) = \mu_i P_{i-1,j}(t) - (\lambda_i + \mu_i) P_{ij}(t) + \lambda_i P_{i+1,j}(t)$$

where the boundary condition $P_{ij}(0) = \delta_{ij}$.

2.6 Examples of Birth and Death Processes

2.6.1 Linear Growth with Immigration

A birth and death process is called a **linear growth process** if $\lambda_n = \lambda_n + a$ and $\mu_n = \mu n$ with $\lambda, \mu, a > 0$. Such

2.6.2 Queueing

2.7 Birth and Death Processes with Absorbing States

Probability of absorption into state 0

$$\mathbf{u}_i = \frac{\lambda_i}{\mu_i + \lambda_i}\mathbf{u}_{i+1} + \frac{\mu_i}{\mu_i + \lambda_i}\mathbf{u}_{i-1}$$

where $u_0 = 1$.

Probability of absorption into state 0

Theorem 7.1.. *Consider a BDP with birth and death parameters* λ_n *and* μ_n *with* $n \geq 1$*, where* $\lambda_0 = 0$ *so that* 0 *is an absorbing state.*

The probability of absorption into state 0 ***from the intial state*** m *is given as*

$$\begin{cases} \frac{\sum_{i=m}^{\infty}\left(\prod_{j=1}^{i}\mu_j/\lambda_j\right)}{1+\sum_{i=1}^{\infty}\left(\prod_{j=1}^{i}\mu_j/\lambda_j\right)} & \sum_{i=1}^{\infty}\left(\prod_{j=1}^{i}\mu_j/\lambda_j\right) < \infty \\ 1 & \sum_{i=1}^{\infty}\left(\prod_{j=1}^{i}\mu_j/\lambda_j\right) = \infty \end{cases}$$

The ***mean time to absorption*** *is*

Proof. ($\Rightarrow$)

2.8 Finite State Continuous Time Markov Chains

For the single-server process with $\lambda < \mu$ the stationary distribution is

$$\pi_n = \frac{\lambda_0\lambda_1\cdots\lambda_{n-1}}{\mu_1\mu_2\cdots\mu_n} = (\frac{\lambda}{\mu})^n$$

which, when normalized, results in

$$P_n = \frac{\mu-\lambda}{\mu}(\frac{\lambda}{\mu})^n, \quad n \geq 0$$

If the process has been going on a long time and $\lambda < \mu$, the probability of being served immediately upon arrival is

$$P_0 = (1 - \frac{\lambda}{\mu})$$

If an arriving customer finds n people in front of her, her total waiting time T, including his own service time, is the sum of service times of herself and those ahead, all distributed exponentially with param μ, thus

$$
\begin{aligned}
T|n \text{ ahead} &\equiv Gamma(n+1,\mu) \\
\Leftrightarrow \quad P\{T \leq t|n \text{ ahead}\} &= \int_0^t \frac{\mu^{n+1}\tau^n e^{-\mu t}}{\Gamma(n+1)} d\tau \\
\therefore \quad P\{T \leq t\} &= \sum_{n=0}^{\infty} P\{T \leq t|n \text{ ahead}\} \cdot (\frac{\lambda}{\mu})^n (1-\frac{\lambda}{\mu})
\end{aligned}
$$

since $(\frac{\lambda}{\mu})^n(1-\frac{\lambda}{\mu})$ is the probability that in the stationary case a customer on arrival will find n ahead in line.

$$P\{T \leq t\} =$$

$$
\begin{aligned}
M(t) &= \sum_{j=0}^{\infty} jP_{ij}(t) \\
M'(t) &= \lambda - \mu M(t) \\
M(t) &= \frac{\lambda}{\mu}(1-e^{-\mu t}) + ie^{-\mu t}
\end{aligned}
$$

If we let $t \to \infty$, then $M(t) \to \lambda/\mu$, which is the mean value of the stationary distribution given above.