Limit theorems for Markov Chains

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Theorem 1.2(The basic limit theorem of Markov chains).

(a) Consider a ***irreducible***, ***ergodic****(recurrent and aperiodic) Markov chain. Let P_{ii}^n the probability of entering state i at the nth transition, $n = 0, 1, 2, \cdots$, given that $X(O) = i$ (the initial state is i) . By our earlier convention $P_{ii}^0 = 1$. Let f_{ii}^n the probability of first returning to state i at the nth transition, $n = 0, 1, 2, \cdots$ where $f_{ii}^0 = 0$. Thus*

$$P_{ii}^n - \sum_{k=0}^{n} f_{ii}^{n-k} P_{ii}^k = \begin{cases} 1 & n = 0 \\ 0 & n > 0 \end{cases}$$

Then,

$$\lim_{n \to \infty} P_{ii}^n = \frac{1}{\sum_{n=0}^{\infty} n f_{ii}^n}$$

(b) Under the same conditions as in (a),

$$\lim_{n \to \infty} P_{ji}^n = \lim_{n \to \infty} P_{ii}^n.$$

Proof.

Under these circumstances we prove that $\lim_{n\to\infty}\gamma_n = c$. In fact,

$$\begin{aligned} y_n - c &= \sum_{k=0}^{n} a_{n-k}x_k - c\sum_{m=0}^{\infty} a_m \\ &= \sum_{k=0}^{n} a_{n-k}(x_k - c) - c\sum_{m=n+1}^{\infty} a_m \end{aligned}$$

For $\epsilon > 0$ prescribed we determine $K(\epsilon)$ so that $|x_k - c| < \epsilon/3$ for all $k \geq K(\epsilon)$.

$$y_n - c = \sum_{k=0}^{K(\epsilon)} a_{n-k}(x_k - c) + \sum_{k=K(\epsilon)+1}^{n} a_{n-k}(x_k - c) - c\sum_{m=n+1}^{\infty} a_m$$

and so

$$|y_n - c| \leq \sum_{k=0}^{K(\epsilon)} a_{n-k}(x_k - c) + \sum_{k=K(\epsilon)+1}^{n} a_{n-k}(x_k - c) - c\sum_{m=n+1}^{\infty} a_m$$

where

$$M = \max_{k\geq 0}|x_k - c|$$

We choose $N(\epsilon)$ so that $|c|\sum_{m=n+1}^{\infty} a_m < \epsilon/3$ and

$$\sum_{k=0}^{K(t)} a_{n-k} \equiv \sum_{m=n-K(t)}^{n} a_m < \frac{\epsilon}{3M} \quad n \geq N(\epsilon)$$

Then

Remark 1.3. Let C be a recurrent class. Then $P_{ij}^n = 0$, for $i \in C, j \notin C$, and every n. Hence, once in C it i snot possible to leave C.

Theorem 1.3. *In a* ***positive recurrent aperiodic*** *class with states* $j = 0, 1, 2, \cdots,$

$$\lim_{n\to\infty} P_{jj}^n = \pi_j = \sum_{i=0}^{\infty} \pi_i P_{ij}, \quad \sum_{i=0}^{\infty} \pi_i = 1$$

and the π*'s are uniquely determined by the set of equations*

0.1 Absorption Probabilities

Remark 3.1. If there are only a finite number of states, M, then there are no null states and not all states can be transient . In fact, since Lf 01 Pij == 1 for all n, it cannot happen that limn-+ oo Pij = 0 for all j. The same argument restricted to recurrent classes shows that there are no null states . Let C, C 1 , C2 , • . • denote recurrent classes . We define ni(C) as the probability that the process will be ultimately absorbed into the recurrent class C if the initial state is the transient state i. (Recall that once the process enters a recurrent class, it never leaves it.)

Theorem 1.3. *In a* ***positive recurrent aperiodic*** *class with states* $j = 0, 1, 2, \cdots,$

$$\lim_{n\to\infty} P_{jj}^n = \pi_j = \sum_{i=0}^{\infty} \pi_i P_{ij}, \quad \sum_{i=0}^{\infty} \pi_i = 1$$

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Proof. Clearly $\pi_i^n(C) = \sum_{k\in C} \pi_{ik}^n(C)$ where $\pi_{ik}^n(C)$ represents the probability starting from state i of being absorbed at the nth transition into class C at state k. We have

$$\pi_i(C) = \sum_{\nu=1}$$

Therefore $\forall \epsilon > 0$, $\exists$ a finite number of states $C' \subset C$ and an integer $N(\epsilon) = N$ such that

$$|\pi_i(C) - \sum_{\nu=1}^{n} \sum_{k \in C'} \pi_{ik}^{\nu}(C)| < \epsilon$$

i.e.

Combining these relations, we have

Example (The gambler's ruin on $n+1$ states).

$$P = \begin{bmatrix} 1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

0.2 Criteria for Recurrence

Let us prove theorems which will be useful in determining whether a given Markov chain is recurrent or transient and then we apply them to several examples

> ***Theorem 4.1.*** *Let $\mathcal{B}$ be an irreducible Markov chain whose state space is labeled by the nonnegative integers. Then a necessary and sufficient condition that $\mathcal{B}$ be transient (i.e. each state is a transient state) is taht the system of equations*

$$\sum_{j=0}^{\infty} P_{ij}\gamma_j = \gamma_i, \quad i \neq 0$$

have a bounded nonconstant solution.

Proof. For the sufficiency, let the transition matrix for $\mathcal{B}$ be

$$P = \|P_{ij}\| \dashv [$$

where the zero state has been converted into an absorbing barrier while the transition probabilities governing the motion among the other states are unchanged. Let us denote

If the chain is recurrent, then

$$\lim_{n\to\infty}\tilde{P}_{i0}^{n} = 1$$

and

where

Theorem 4.2. *In an irreducible Markov chain a sufficient condition for recurrence is that there exists a sequence* $\{\gamma_i\}$ *such that*

$$\sum_{j=0}^{\infty} P_{ij}\gamma_j \leq \gamma_i \quad i \neq 0, \gamma_i \to \infty$$

Proof. Using the same notation as in the previous theorem, we have

$$\sum_{j=0}^{\infty} \hat{P}_{ij}\gamma_j \leq \gamma_i, \quad \forall i$$

Since

Given ϵ, we choose $M(\epsilon)$ such that $1/\gamma_i \leq \epsilon$ for $i \geq M(\epsilon)$. Now

$$\sum_{j=0}^{M-1} \tilde{P}_{ij}^{m} \gamma_j + \sum_{j=M}^{\infty} \tilde{P}_{ij}^{m} \gamma_j \leq \gamma_i$$

and so

$$\sum_{j=0}^{M-1} \tilde{P}_{ij}^{m} \gamma_j + \min_{r \geq M}\{\gamma_r\} \sum_{j=M}^{\infty} \tilde{P}_{ij}^{m} \leq \gamma_i$$

Since

$$\sum_{j=0}^{\infty} \tilde{P}_{ij}^{m} = 1$$

we have

$$\sum_{j=0}^{M-1} \tilde{P}_{ij}^{m} \gamma_j + \min_{r \geq M}\{\gamma_r\} \left(1 - \sum_{j=0}^{M-1} \tilde{P}_{ij}^{m}\right) \leq \gamma_i$$

As observed in the proof of the preceding theorem,

$$\lim_{n \to \infty} \tilde{P}_{ij}^{n} = 0, \quad j \neq 0$$

Thus passing to the limit as $m \to \infty$, we obtain for each i,

$$\tilde{\pi}_i(C_0)\gamma_0 + \min_{r \geq M}\{\gamma_r\}(1 - \tilde{\pi}_i(C_0)) \leq \gamma_i$$

0.3 A Queueing Example

If $\sum ka_k \leq 1$, the process is recurrent.

In order to ascertain whether the process $\mathcal{B}$ is null recurrent or positive recurrent we first deal with the following auxiliary problem of some independent interest.

We set

$$G(s) = \frac{b_{-1}}{s} + b_0 + b_1 s + b_2 s^2 + \cdots$$

Our is to determine $U(s)$ in terms of $G(s)$. To this end we write the usual renewal relations

$$\gamma_1 = b_{-1}, \quad \gamma_k = \sum_{j=0}^{\infty} b_j \gamma_{j-1}^{(j+1)}, \quad k \geq 2$$