

# The Queueing Theory

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## 1 Definitions

The **first** character describes the **arrival process** to the queue.

- **Memorylessness**: Poisson arrival process
- **Deterministic**: interarrival interval is nonrandom
- **General**: general interarrival distribution

The **second** character describes the **service process**, with  $M$ : exponentially distributed service times.

The **third** character describes the **number of servers**.

Sometimes a fourth character is added which gives the **number of customers** that can be saved in the queue plus service facility.

It is often assumed that the service times are IID, independent of the arrival epochs, and independent of which server is used.

**M/M/1 queue**(Gallager). A queueing system with a Poisson arrival system with rate  $\lambda$  and a single server that serves arriving customers in order with a service time distribution  $F(y) = 1 - \exp(-\mu y)$ .

The service times are IID and independent of the interarrival intervals. During any period when the server is busy, customers leave the system according to a Poisson process

**M/G/ $\infty$  queue**(Gallager). A queueing system with Poisson arrivals, a general service distribution, and an infinite number of servers. From the infinite number of servers, no arriving customers are ever queued. The service time  $Y_i$  of customer  $i$  is IID over  $i$  with CDF  $G(y)$ .

## 2 Examples

### 2.1 Example 1

Let us consider the following queueing model, where the transition matrix is given as

$$P_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

If  $\sum k a_k \leq 1$ , the process is recurrent.

In order to ascertain whether the process  $\mathcal{B}$  is null recurrent or positive recurrent we first deal with the following auxiliary problem of some independent interest.

We set

$$G(s) = \frac{b_{-1}}{s} + b_0 + b_1 s + b_2 s^2 + \dots$$

Our is to determine  $U(s)$  in terms of  $G(s)$ . To this end we write the usual renewal relations

$$\gamma_1 = b_{-1}, \quad \gamma_k = \sum_{j=0}^{\infty} b_j \gamma_{j-1}^{(j+1)}, \quad k \geq 2$$

This decomposition, in conjunction with the Markov property, yields the following expression for the average recurrence time:

**Case 1.**

Referring again to the queueing process., we identify the  $b$ 's and the  $a$ 's by  $a_k = bk_1$  and let  $Z_{ij}$  = length of time (number of transitions) required., starting in state  $i$ , to reach state  $j < i$  for the first time.

is precisely the random variable  $Z$  examined above whose generating function  $U(s)$  was determined. Since

## 2.2 Example 2

The state of the process is the length of the waiting line where, in each unit of time, one person arrives and  $k$  persons are served with probability  $a_k > 0$ ,  $k = 0, 1, \dots$ .

if there are at least transition probability matrix may be easily evaluated as

Therefore, if  $Lkak < 1$  the system admits no bounded solution and therefore, in particular, no stationary distribution exists so that the process is either null recurrent or transient.

We now prove that the system of equation

To sum up, if

- $\sum ka_k < 1$ , the process is **transient**
- $\sum ka_k = 1$ , the process is **null recurrent**

- $\sum ka_k > 1$ , the process is **positive recurrent**

$$\sum ka_k < 1 = 1,$$