Limit theorems for Markov Chains

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1 Discrete Renewal Equation

Theorem 1.1.

$$u_n - \sum_{k=-\infty}^{\infty} a_{n-k} u_k = b_n$$

for $n \in \mathbb{Z}$,

Proof(long).

Next, we have the following ergodic theorem for this case.

Theorem 1.2(The basic limit theorem of Markov chains).

*(a) Consider a **irreducible**, **ergodic**(recurrent and aperiodic) Markov chain. Let P_{ii}^n the probability of entering state i at the nth transition, $n = 0, 1, 2, \cdots$, given that $X(O) = i$ (the initial state is i) . By our earlier convention $P_{ii}^0 = 1$. Let f_{ii}^n the probability of first returning to state i at the nth transition, $n = 0, 1, 2, \cdots$ where $f_{ii}^0 = 0$. Thus*

$$P_{ii}^n - \sum_{k=0}^{n} f_{ii}^{n-k} P_{ii}^k = \begin{cases} 1 & n = 0 \\ 0 & n > 0 \end{cases}$$

Then,

$$\lim_{n\to\infty} P_{ii}^n = \frac{1}{\sum_{n=0}^{\infty} n f_{ii}^n}$$

(b) Under the same conditions as in (a),

$$\lim_{n\to\infty} P_{ji}^n = \lim_{n\to\infty} P_{ii}^n.$$

Proof.

Under these circumstances we prove that $\lim_{n\to\infty} \gamma_n = c$. In fact,

$$\begin{aligned} y_n - c &= \sum_{k=0}^{n} a_{n-k} x_k - c \sum_{m=0}^{\infty} a_m \\ &= \sum_{k=0}^{n} a_{n-k}(x_k - c) - c \sum_{m=n+1}^{\infty} a_m \end{aligned}$$

For $\epsilon > 0$ prescribed we determine $K(\epsilon)$ so that $|x_k - c| < \epsilon/3$ for all $k \geq K(\epsilon)$.

$$y_n - c = \sum_{k=0}^{K(\epsilon)} a_{n-k}(x_k - c) + \sum_{k=K(\epsilon)+1}^{n} a_{n-k}(x_k - c) - c \sum_{m=n+1}^{\infty} a_m$$

and so

$$|y_n - c| \leq \sum_{k=0}^{K(\epsilon)} a_{n-k}(x_k - c) + \sum_{k=K(\epsilon)+1}^{n} a_{n-k}(x_k - c) - c \sum_{m=n+1}^{\infty} a_m$$

where

$$M = \max_{k \geq 0} |x_k - c|$$

We choose $N(\epsilon)$ so that $|c| \sum_{m=n+1}^{\infty} a_m < \epsilon/3$ and

$$\sum_{k=0}^{K(t)} a_{n-k} \equiv \sum_{m=n-K(t)}^{n} a_m < \frac{\epsilon}{3M} \quad n \geq N(\epsilon)$$

Then

Remark 1.3. Let C be a recurrent class. Then $P_{ij}^n = 0$, for $i \in C, j \notin C$, and every n. Hence, once in C it i snot possible to leave C.

for one i in an aperiodic recurrent class, then $\pi_j > 0$ for all j in the class of i.

is recurrent we speak of the class as **null recurrent** or **weakly ergodic**.

Theorem 1.3. *In a* ***positive recurrent aperiodic*** *class with states* $j = 0, 1, 2, \cdots,$

$$\lim_{n \to \infty} P_{jj}^n = \pi_j = \sum_{i=0}^{\infty} \pi_i P_{ij}, \quad \sum_{i=0}^{\infty} \pi_i = 1$$

and the π*'s are uniquely determined by the set of equations*

1.1 Absorption Probabilities

Remark 3.1. If there are only a finite number of states, M, then there are no null states and not all states can be transient. In fact, since $\sum_{j=0}^{M-1} P_{ij}^n = 1$ for all n, it cannot happen that $\lim_{n\to\infty} P_{ij}^n = 0$ for all j.

The same argument restricted to recurrent classes shows that there are no null states. Let $C, C_1, C_2, \cdots$ denote recurrent classes. We define $\pi_i(C)$ as the probability that the process will be ultimately absorbed into the recurrent class C if the initial state is the transient state i.

Theorem 3.1. *Let $j \in C$, where C: an aperiodic recurrent class. Then for $i \in T$, we have*

$$\lim_{n\to\infty} P_{ij}^n = \pi_i(C) \lim_{n\to\infty} P_{jj}^n = \pi_i(C)\pi_j$$

Proof. Clearly $\pi_i^n(C) = \sum_{k\in C} \pi_{ik}^n(C)$ where $\pi_{ik}^n(C)$ represents the probability starting from state i of being absorbed at the nth transition into class C at state k. We have

$$\pi_i(C) = \sum_{\nu=1}$$

Therefore $\forall \epsilon > 0$, $\exists$ a finite number of states $C' \subset C$ and an integer $N(\epsilon) = N$ such that

$$|\pi_i(C) - \sum_{\nu=1}^{n} \sum_{k\in C'} \pi_{ik}^{\nu}(C)| < \epsilon$$

i.e.

Combining these relations, we have

Example (The gambler's ruin on $n+1$ states).

$$P = \begin{bmatrix} 1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

1.2 Criteria for Recurrence

Let us prove theorems which will be useful in determining whether a given Markov chain is recurrent or transient and then we apply them to several examples

> ***Theorem 4.1.*** *Let $\mathcal{B}$ be an irreducible Markov chain whose state space is labeled by the nonnegative integers. Then a necessary and sufficient condition that $\mathcal{B}$ be transient (i.e. each state is a transient state) is taht the system of equations*
>
> $$\sum_{j=0}^{\infty} P_{ij}\gamma_j = \gamma_i, \quad i \neq 0$$
>
> *have a bounded nonconstant solution.*

Proof. For the sufficiency, let the transition matrix for $\mathcal{B}$ be

$$P = \|P_{ij}\| =] \; [$$

where the zero state has been converted into an absorbing barrier while the transition probabilities governing the motion among the other states are unchanged. Let us denote

If the chain is recurrent, then

$$\lim_{n\to\infty}\tilde{P}^n_{i0} = 1$$

and

where

> ***Theorem 4.2.*** *In an irreducible Markov chain a sufficient condition for recurrence is that there exists a sequence* $\{\gamma_i\}$ *such that*
>
> $$\sum_{j=0}^{\infty} P_{ij}\gamma_j \le \gamma_i \quad i \ne 0, \gamma_i \to \infty$$

Proof. Using the same notation as in the previous theorem, we have

$$\sum_{j=0}^{\infty} \hat{P}_{ij}\gamma_j \le \gamma_i, \quad \forall i$$

Since

Given ϵ, we choose $M(\epsilon)$ such that $1/\gamma_i \le \epsilon$ for $i \ge M(\epsilon)$. Now

$$\sum_{j=0}^{M-1} \tilde{P}^m_{ij}\gamma_j + \sum_{j=M}^{\infty} \tilde{P}^m_{ij}\gamma_j \le \gamma_i$$

and so

$$\sum_{j=0}^{M-1} \tilde{P}^m_{ij}\gamma_j + \min_{r\ge M}\{\gamma_r\} \sum_{j=M}^{\infty} \tilde{P}^m_{ij} \le \gamma_i$$

Since

$$\sum_{j=0}^{\infty} \tilde{P}_{ij}^{m} = 1$$

we have

$$\sum_{j=0}^{M-1} \tilde{P}_{ij}^{m}\gamma_j + \min_{r\geq M}\{\gamma_r\}\left(1 - \sum_{j=0}^{M-1} \tilde{P}_{ij}^{m}\right) \leq \gamma_i$$

As observed in the proof of the preceding theorem,

$$\lim_{n\to\infty}\tilde{P}_{ij}^{n} = 0, \quad j \neq 0$$

Thus passing to the limit as $m \to \infty$, we obtain for each i,

$$\tilde{\pi}_i(C_0)\gamma_0 + \min_{r\geq M}\{\gamma_r\}(1 - \tilde{\pi}_i(C_0)) \leq \gamma_i$$