Fourier Transform(sup)

Hyunwoo Gu

Ideas of Derivation

Suppose $f(x)$: a periodic of 2π

$$\hat{f}(n) = \int_{-\infty}^{\infty} f(y) exp(-iny)$$

Properties

Prancherel's theorem.

$$\int_{-\infty}^{\infty} |f(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat{f}(\omega)|^2 d\omega$$

Proof. Note that

$$=$$

Theorem. *For f: 2π-periodic with $f^{(r)}$ being absolutely continuous,*

$$|\hat{f}(k)| = \mathcal{O}\left(\frac{1}{|k|^{r+1}}\right)$$

Proof. Note that

$$=$$

Examples

Let $f(t) := e^{-at}\mathbf{1}_{t>0}$ for $a > 0$. Then

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-ax} e^{-i\omega t} dt$$

since $e^{-at-i\omega t} = e^{-at}\left(cos(-\omega t) + isin(-\omega t)\right)$

DFT and FFT

We have the following of DFT.

Circular shift property. *For some $0 \leq m \leq N-1$, define the shift by m as follows:*

$$y_m[n] = \begin{cases} x[n-m+N] & 0 \leq n < m \\ x[n-m] & m \leq n \leq N-1 \end{cases}$$

Then we have the DFT of the circular shift $y_m[n]$ of $x[n]$ by m is given by

$$Y_m[k] = X[k]e^{-i\frac{2\pi km}{N}}$$

Theorem. *Assume that $N = N_1N_2$ and the data $x[n], n = 0, \cdots, N-1$, are of N_1-periodic. Suppose that $Y[l], l = 0, \cdots, N_1 - 1$ are the DFTs of N_1 data $x[n], n = 0, \cdots, N_1 - 1$. Then the DFT of $x[n]$ is given by*

$$X[k] = \begin{cases} N_2Y[l] & k = 1N_2, l = 0, \cdots, N_1 - 1 \\ 0 & o.w. \end{cases}$$

Assume that $N = N_1N_2$ and the data $x[n], n = 0, 1, \cdots, N-1$ are of N_1-periodic. We then have the following decomposition of DFT $X[k], k = 0, \cdots, N$

$$\begin{aligned} X[k] &= \sum_{n=0}^{N-1} x[n]e^{-i\frac{2\pi kn}{N}} \\ &= \sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2-1} x[n_2N_1 + n_1]e^{-i\frac{2\pi k(n_2N_1+n)1}{N}} \\ &= \end{aligned}$$

where for each n_1,

$$\sum_{n_2=0}^{N_2-1} x[n_2N_1 + n_1]e^{-i\frac{2\pi kn_2}{N_2}}$$

is a Fourier transform of N_2 data

$$x[n_2N_1 + n_1], n_2 = 0, \cdots, N_2 - 1$$

Since $x[n]$ is N_1-periodic, we have

$$x[n_2 N_1 + n_1] = x[]$$

thus,

$$\begin{aligned}
X[k] &= N_2 \sum_{n_1=0}^{N_1-1} x[n_1] e^{-i\frac{2\pi k n_1}{N}} \\
&= N_2 \sum_{n_1=0}^{N_1-1} x[n_1] e^{-i\frac{2\pi k n_1}{N_1}} \\
&= N_2 Y[l]
\end{aligned}$$

where $Y[l]$ is the DFT of N_1 data $x[n], n = 0, \cdots, N_1 - 1$.

Note that from $\beta_j := \frac{1}{N} \sum_{k=0}^{N-1}$

$$\begin{aligned}
\beta_j &= \frac{1}{N} \left(f_0 + f_1 e^{-ijx_1} + \cdots + f_{N-1} e^{-i(N-1)x_1} \right) \\
&=
\end{aligned}$$

where

Start with $m = 1$. Let $M := 2^{m-1}$, $R := 2^{n-1}$, for $m = 1, 2, \cdots$. From $2R(m) = 2^n$ phase polynomial $P_r^{(0)}(x)$, which are constant functions for $m = 1, r = 0, 1, \cdots, 2R-1$, find $R(m)$ phase polynomials $P_r^{(1)}, r = 0, 1, 2, \cdots, R-1$ by

$$P_r^{(1)} = \beta_{r,0}^{(1)} + \beta_{r,1}^{(1)} e^{ix}$$

using the fact that

$$P_r^{(1)}$$

From the $2R(m) = 2^{n-1}$ phase polynomials $P_r^{(1)}(x)$ for $r = 0, 1, 2, \cdots, 2R-1$, find $R(m) = 2^{n-1}$ phase

As the following algorithm:

Examples

The Fourier series of $f(x) := sin^5(x)$ is as follows :