Levy processes

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1 Introduction to the theory of Levy processes

Here we will be able to:

- understand the main properties of Levy processes
- construct ta Levy process from an infinitely

1.1 Definition of a Levy process

Levy processes are widely used for modelings of jump dynamics.

N_t	W_t	Levy
$N_0 = 0$ inpt. incr. sta. incr. $N_t - N_s \sim Pois(\lambda(t-s))$	$W_0 = 0$ inpt. incr. sta. incr. $W_t - W_s \sim N(0, t-s)$	$L_0 = 0$ inpt. incr. sta. incr. $L_t - L_s \sim P(t-s)$

where $P(\cdot)$: **infinite divisible** distribution.

1.2 Stochastic continuity and cadlag paths

L_t is **stochastically continuous** if $L_{t+h} \overset{P,h\to 0}{\to} L_t$, or $\forall \epsilon > 0$

$$P[|L_{t+h} - L_t] > \epsilon] \overset{h\to 0}{\to} 0$$

The sample path of **Levy process** follows a cadlag path.

1.3 Examples of Levy processes

The class of **infinitely divisible distributions**.

> ***Definition***. *An RV ξ is a **infinitely divisible distribution**, if*
>
> $$\xi \overset{d}{\equiv} Y_1 \oplus \cdots \oplus Y_n$$

It is not true that the distribution is stable iff it is infinitely divisible.

Example 1.

Example 2.

1.4 Characteristic Exponent

> ***Proposition***. *For $\forall$ Levy process L_t*
>
> $\exists \psi : \mathbb{R} \to \mathbb{C}$ *such that*

$$\phi_{L_t}(u) = \mathbb{E}\left[e^{iuL_t}\right] = e^{t\psi(u)}$$

1.5 Properties of Levy Characteristic Exponent

Levy measure.

Definition.

$$\nu(B) = \mathbb{E}[\#t \in [0,1] : \Delta X_t \in B]$$

for any $B \subset \mathbb{R} - \{0\}$

1.6 Levy-Khintchine theorem

$$\phi_{X_t}(u) = exp\left[t(iu\mu - \sigma^2 u^2/2) + \int_{\mathbb{R}} \left(e^{iux} + 1 - iux\mathbf{1}_{|x|<1}\right)\nu(dx)\right]$$

Example 1. For X_t : of bounded variation,

$$\sum_{k=1}^{n} |X_{t_k} - X_{t_{k-1}}| \overset{max|t_i - t_{i-1}| \to 0}{\longrightarrow} < \infty$$

Note that the Brownian motion is not a process of bounded variation. X_t is of bounded variation iff

$$\sigma = 0, \int_{|x|<1} x\nu(dx) < \infty$$

By letting $\sigma \equiv 0$, we have

$$\phi_{X_t}(u) = exp\left\{t\left(iu\tilde{\mu} + \int(e^{iux} - 1)\nu(dx)\right)\right\}$$

where $\tilde{\mu} := \mu - \int_{|x|<1} xd\nu(dx)$.

Example 2. For X_t : **compound Poisson process**,

$$\sigma = 0, \int_{\mathbb{R}} \nu(dx) = \nu(\mathbb{R}) < \infty$$

Example 3. For X_t : **Subordinators**,

$$X_t \geq 0 \quad a.s. \Leftrightarrow X_t \geq X_s \quad a.s.$$

Properties.

We have the following property:

$$\int_{|x|<1} x^2\nu(dx) < \infty, \int_{|x|>1} \nu(dx) < \infty$$

$$J := \int_{|x|<1} \left(e^{iux} - 1 - 0\right)$$

1.7 Modeling of jump-type dynamics

How to estimate that the Levy measure of a Levy process from the following data?

$$X_t : X_\Delta$$

CIR model for the stochastic volatility is

$$c\sqrt{V_t}dW_t$$

Levy process can be used to model the properties of jobs.

Let X_t be of bounded variation, and

$$\phi_{X_\Delta}(u) := exp\left\{\Delta\left(iu\mu + \int_{\mathbb{R}}(e^{iux}-1)S(x)dx\right)\right\}$$

Note that

$$\phi_{X_\Delta}(u) := exp\left\{\Delta\left(iu\mu + \int_{\mathbb{R}}(e^{iux}-1)S(x)dx\right)\right\}$$

Let X_t be a Levy process of bounded variation with Levy triplet (μ, σ^2, ν) and Levy density $s(x)$. The correct form of the characteristic exponent is given as

$$\psi(u) = iu\left(\mu - \int_{|x|<1} xs(x)dx\right) + \int_{\mathbb{R}}(e^{iux}-1)s(x)dx$$

with μ probably equal to zero. Note that it is also true that nonzero μ can be of bounded variation.

1.8 Time-changed stochastic processes. Monroe theorem

The following are the **stylized facts** of financial data.

1. Stochastic time change

For $\{X_t\}$: Levy process, and a subordinator $T(s)$, and thus $X_{T(s)}$. The **Monroe's theorem** states the equivalence of the set of $W_{T(s)}$ types and the set of submartingales, where W, T are possibly dependent.

Assume that $W \perp T$. Note that

$$\phi_{X_{T(s)}}(u) = \mathcal{L}_{T(s)}\left(-\psi(u)\right)$$

where $\psi(u)$ is the characteristic exponent of process X.

2. Stochastic volatility

For **Black-Scholes**,

$$d(lnS_t) = \left(\mu - \sigma^2/2\right) dt + \sigma dW_t$$

where the volatility parameter $\sigma \to V_t \geq 0$. For example, **Cox-Ingersoll-Ross** is defined as the SDE

$$dV_t = (a - bV_t)dt + c\sqrt{V_t}dW_t$$

Quizzes

(Quiz 1-3). $X_t := bt + \sigma W_t + cN_t$, where W_t: Brownian motion, N_t: a Poisson process with λ, and W_t, N_t independent, $b, c \in \mathbb{R}$, $\sigma \geq 0$. Denote the Levy measure of this process bt ν.

(Quiz 1). Find the characteristic function of this process.

(Answer). $exp\{iubt + \lambda t(e^{icu} - 1) - \frac{t(\sigma u)^2}{2}\}$

(Quiz 2). What is measure ν of a Borel set B?

(Answer). $\nu(B) = \lambda$, if $1 \in B$ and 0 otherwise.

Considering that the jump part of the Levy process discussed is

- c: real number
- N_t : Poisson process with λ

Thus $\nu(B) = \lambda \mathbf{1}_{c \in B}$. Computing the integral in the characteristic exponent, returns a pointwise evaluation in the point c, $\lambda(e^{iuc} - 1)$.

(Quiz 3). Let X_t be a Levy process. What is the correct expression for $\text{Var}(X_t)$ in terms of characteristic exponent ψ?

(Answer). $\text{Var}(X_t) = -t\psi''(0)$

(Quiz 4). Let X_t be a Levy process, assuming $X_1 \sim N(0, 1)$. Find the mean and the variance of X_t

(Answer). It is just the Brownian motion. Therefore, $\mathbb{E}[X_t] = 0, \text{Var}(X_t) = t$.

(Quiz 5). Let $X_t = bt + N_t$, where N_t is a Poisson process with λ and $b \in \mathbb{R}$. Find the Levy triplet of this process.

(Answer). $(b+\lambda, 0, \nu)$, where $\nu(B) = \lambda \mathbf{1}_{1 \in B}$ for any Borel set B.

The first term of a Levy triplet should correspond to the coefficient of the drift term, i.e. b, if X is of the form

$$X_t = bt + \sigma W_t + cN_t$$