

Ergodicity, Differentiability, & Continuity

Hyunwoo Gu

1 Ergodicity

We have mentioned that **ergodicity** means **aperiodicity** and **recurrence** in discrete-time Markov chains. We now turn to general stochastic processes.

1.1 Notion of ergodicity

***Law of Large Numbers.** Let ξ_1, \dots IID such that $\mathbb{E}\xi_1 < \infty$. Then*

$$\frac{1}{N} \sum_{n=1}^N \xi_n \xrightarrow{P} \mathbb{E}(\xi_1)$$

Let X_T a stochastic process,

$$\frac{1}{N} \sum_{t=1}^T X_t \xrightarrow{P} c$$

$$\xi_n \xrightarrow{as} \xi \Leftrightarrow P\{w : \xi_n(w) \rightarrow \xi(w)\} = 1$$

$$\xi_n \xrightarrow{L^2} \xi \Leftrightarrow \mathbb{E}(\xi_n - \xi)^2 \rightarrow 0$$

$$\xi_n \xrightarrow{P} \xi \Leftrightarrow \forall \epsilon > 0 P(|\xi_n - \xi| > \epsilon) \rightarrow 0$$

$$\xi_n \xrightarrow{d} \xi \Leftrightarrow P(\xi_n \leq x) \rightarrow P(\xi \leq x), \forall x \in \mathbb{R}$$

1.2 Ergodicity of wide-sense stationary processes

Proposition.

Let X_t with $|K(s, t)| \leq \alpha$ such that $\exists \alpha$.

$$C(T) := \text{cov}(X_T, M_T)$$

$$M_T := \frac{1}{T} \sum_{t=1}^T X_t$$

$$\text{Var}(M_T) \xrightarrow{T \rightarrow \infty} 0 \Leftrightarrow C(T) \xrightarrow{T \rightarrow \infty} 0$$

Corollary. Let X_t : weakly stationary, and $\gamma(\cdot)$: autocovariance function. Then

- (i) $\frac{1}{T} \sum_{r=0}^{T-1} \gamma(r) \rightarrow 0$ when $T \rightarrow \infty$
- (ii) $\gamma(r) \rightarrow 0$ when $r \rightarrow \infty$

Note that clearly $\mathbb{E}(X_t) = \text{const}$ then $\text{cov}(X_t, X_s) = \gamma(t - s)$.

Proof. I.

$$\mathbb{E}(X_t) = c \Leftrightarrow \mathbb{E}(M_T) = c$$

$$\text{Var}(M_T) = \mathbb{E}[(M_T - c)^2] \xrightarrow{*} 0$$

$$\Rightarrow M_T \xrightarrow{L^2} c \Rightarrow M_T \xrightarrow{P} c$$

then X_t : ergodic.

For (*) part, check

$$C(T) = \text{cov}(X_T, \frac{1}{T} \sum_{t=1}^T X_t) = \frac{1}{T} \sum_{t=1}^T \gamma(T-t) = \frac{1}{T} \sum_{r=0}^{T-1} \gamma(r) \rightarrow 0$$

II. (Stolz-Cesaro theorem)

Let a_n, b_n where b_n : strictly increasing & unbounded. Then

$$\lim_{n \rightarrow \infty} \frac{a_n - a_{n-1}}{b_n - b_{n-1}} = q \Rightarrow \frac{a_n}{b_n} \xrightarrow{n \rightarrow \infty} q$$

Take

$$a_n := \sum_{r=0}^{n-1} \gamma(r)$$

$$b_n = n$$

Note that

$$\frac{a_n - a_{n-1}}{b_n - b_{n-1}} = \gamma(n-1) \rightarrow 0 = q$$

For example, N_t : Poisson process with λ .

For $p > 0$, $X_t := N_{t+p} - N_t$.

$$\mathbb{E}(X_t) = \lambda(t+p) - \lambda t = \lambda p$$

$$K(t, s) = \gamma(t - s)$$

$$\gamma(r) = \begin{cases} \lambda(p - |r|) & |r| \leq p \\ 0 & |r| > p \end{cases}$$

$$\gamma(r) \rightarrow 0$$

For example, $X_t := A \cos(wt) + B \sin(wt)$, where A, B random with $\text{cov}(A, B) = 0$

with $w := \pi/20$.

$$\mathbb{E}(A) = \mathbb{E}(B) = 0$$

$$\text{Var}(A) = \text{Var}(B) = 1$$

$$\mathbb{E}(X_t) = 0$$

$$K(t, s) = \cos(w(t - s))$$

$$\gamma(r) = \cos(wr)$$

Note that it is **NOT true** that a stochastic process is stationary if and only if it is ergodic.

Assume that X_t is weakly stationary and

$$\frac{1}{T} \sum_{r=0}^{T-1} \gamma(r) \rightarrow 0$$

but $\gamma(r) \rightarrow 0$ when $r \rightarrow \infty$. Then X_t is ergodic.

2 Stochastic Differentiation

2.1 Definition of a stochastic derivative

X_t is differentiable at $t = t_0$, if

$$\frac{X_{t_0+h} - X_{t_0}}{h} \xrightarrow{L^2} \eta := X_{t_0}^1$$

for $h \rightarrow 0$.

$$\mathbb{E} \left(\frac{X_{t_0+h} - X_{t_0}}{h} - \eta \right)^2 \xrightarrow{h \rightarrow 0} 0$$

Proposition. Let $X_t : \mathbb{E}(X_t^2) < \infty$ Then

$$X_t : \text{diff at } t = t_0 \Leftrightarrow \begin{cases} m(t) = \mathbb{E}(X_t) & \text{diff at } t = t_0 \\ \frac{\partial^2}{\partial t \partial s} K(t, s) & \exists(t_0, t_0) \end{cases}$$

Ex1. Let X_t is weakly stationary, i.e., $m(t) = \text{const}$, and $K(t, s) = \gamma(t - s)$.

$$\frac{\partial^2 K}{\partial t \partial s} \Big|_{(t_0, t_0)} = -\gamma''(0)$$

For example, $\gamma(r) = e^{-L|r|}$, then X_t is not differentiable.

For another example, $\gamma(r) = \cos(\alpha r)$, then X_t is differentiable.

Ex2. Brownian motion is not diff at any $t = t_0$.

$$K(t, s) = \min(t, s)$$

$$\begin{aligned} K(t_0 + h, t_0) - K(t_0, t_0) &= \frac{\min(t_0, t_0 + h) - t_0}{h} \\ &= \begin{cases} 0 & h > 0 \\ 1 & h < 0 \end{cases} \end{aligned}$$

thus there does not exist a result $h \rightarrow 0$.

Ex3. Let X_t : independent increments with $X_0 = 0$. Then

$$K(t, s) = \text{cov}(X_t, X_s) \stackrel{t \geq s}{=} \text{cov}(X_t - X_s, X_s) + \text{Cov}(X_s, X_s) = \text{var}(X_{\min(t, s)})$$

2.2 Continuity in the mean-squared sense

Let

$$X_t \xrightarrow{L^2} X_{t_0} \Leftrightarrow \mathbb{E}(X_t - X_{t_0})^2 \xrightarrow{t \rightarrow t_0} 0$$

such that $\mathbb{E}(X_t) = 0$.

Proposition.

- $K(t, s)$ is const (t_0, t_0) , then X_t is const in the mean squared sense $t = t_0$
- X_t : const in the mean squared sense $t = t_0, s_0$, then $K(t, s)$ is const at (t_0, s_0) .

Proof. (i)

$$\mathbb{E}(X_t - X_{t_0})^2 = K(t, t) - 2K(t, t_0) + K(t_0, t_0) \xrightarrow{t \rightarrow t_0} 0$$

(ii)

$$K(t, s) \pm K(t_0, s) - K(t_0, s_0) = K(t, s) - K(t_0, s) + K(t_0, s) - K(t_0, s_0)$$

$$|K(t, s) - K(t_0, s)| = \mathbb{E}[(X_t - X_{t_0})X_s]$$

$$\leq \sqrt{\mathbb{E}(X_t - X_{t_0})^2} \cdot \sqrt{\mathbb{E}X_s^2} \rightarrow 0$$

Corollary.

$$K(t, s) \text{ is const at } t_0, s_0 \Leftrightarrow K(t, s) \text{ is const at } (t_0, t_0)$$

Proof. Let $K(t, s)$ is const at diagonal, i.e. (t_0, t_0) . Then by (i), we have X_t : const at $t = t_0$.

Then by (i), we have $K(t, s)$ is const at (t_0, s_0) .

The following are the correct statements:

- If $K(t, s)$ is continuous at any $(t_0, s_0) \in \mathbb{R}^2$, then X_t is continuous in MSS at $\forall t$.
- If X_t is continuous in MSS at t_0, s_0 then $K(t, s)$ is continuous at (t_0, s_0) and (s_0, t_0) .
- If $K(t, s)$ is continuous at the diagonal, it is also continuous at any $(t_0, s_0) \in \mathbb{R}^2$

Quizzes

(Quiz 1). Let $X_t := \cos(wt + \theta)$ be a stochastic process and $\theta \sim \text{Unif}(0, 2\pi)$, with $w = \pi/10$. Classify this process.

(Answer) Ergodic and weak stationary.

(Quiz 2). Let $X_t := \epsilon_t + \xi \cos(\pi t/12)$, $t = 1, 2, \dots$, where $\xi, \epsilon_1, \epsilon_2, \dots$ are IID standard normal random variables.

(Answer) Not weak stationary, but ergodic.

(Quiz 3). Assume that for a process X_t it is known that $\mathbb{E}(X_t) = \alpha + \beta t$, $\text{cov}(X_t, X_{t+h}) = e^{-h\lambda}$ for all $h \geq 0, t > 0$, and some constants $\lambda > 0, \alpha, \beta$. Classify the process $Y_t := X_{t+1} - X_t$.

(Answer) Y_t is weakly stationary and ergodic.

(Quiz 4). Let $X_t := \sigma W_t + ct$, where W_t is Brownian motion, $\sigma, c > 0$. Choose the correct statements about this process.

(Answer) X_t has continuous trajectories.

(Quiz 5). Let X_t have an autocovariance function $\gamma(r) := e^{-\alpha|r|}$. Is $Y_t := X_t + w$ an ergodic process?

(Answer) Yes, if w is a constant.