Stochastic Integration
& Ito Formula

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1 Stochastic Integration

1.1 Different types of stochastic integrals

We consider the following types of **stochastic integrals**:

- $\int_a^b X_t dt$
- $\int_a^b f(t) dW_t$
- $\int_a^b X_t dW_t$
- $\int_a^b X_t dH_t$

1.2 Integrals of the type $\int X_t dt$

Consider the first case. For $X_t : \Omega \times \mathbb{R}_+ \to \mathbb{R}$, we fix some omega, and for any omega the integral is a **Riemann integral**.

$$\int_a^b X_t(\omega) dt = \lim \sum_{k=1}^{n} X_{t_{k-1}}(\omega)(t_k - t_{k-1})$$

for $a = t_0 < t_1 < \cdots < t_n = b$. Note that

$$\mathbb{E}\left(\sum_{k=1}^{n} X_{t_{k-1}}(\omega)(t_k - t_{k-1}) - \int_a^b X_t dt\right)^2 \to 0$$

> ***Theorem***. *Let $m(t), K(t,s)$ be continuous. Then* ***there exists*** $\int_a^b X_t dt$

$$\mathbb{E}\int_a^b X_t dt = \int_a^b \mathbb{E}X_t dt$$

$$\operatorname{Var}\int_a^b X_t dt = \int_a^b \int_a^b K(t,s)dtds$$

1.3 Integrals of the type $\int f(t)dW_t$

Now we consider

$$\int_a^b f(t)dW_t$$

which is known as **Wiener integral**. Let us constrain our attention to f such that

$$f \in \mathcal{L}^2([a,b])$$

Recalling that the inner product of two functions is defined $\langle f, g\rangle := \int_a^b fgdx$, we define

$$f_n \overset{\mathcal{L}^2}{\to} f \Leftrightarrow \langle f_n - f, f_n - f\rangle \to 0$$

Stage 1. Consider a **step function**

$$f(x) = \sum_{i=1}^{n} \alpha_i \mathbf{1}_{t_{i-1} \le x < t_i}$$

with $a = t_0 < t_1 < \cdots < t_n = b$, and $\alpha_i \in \mathbb{R}$. Then

$$\int_a^b f(t) dW_t = \sum_{i=1}^{n} \alpha_i \left(W_{t_i} - W_{t_{i-1}} \right)$$

> ***Theorem***. *Denote $\int_a^b f(t)dW_t = I(f)$. If f : step function, then*
>
> $$I(f) \sim N\left(0, \int_a^b f^2(x)dx\right)$$

Proof. Find $\mathbb{E}\left[\sum_{i=1}^{n} \alpha_i \left(W_{t_i} - W_{t_{i-1}}\right)\right]$ and

$$\text{Var}\left[\sum_{i=1}^{n} \alpha_i \left(W_{t_i} - W_{t_{i-1}}\right)\right] = \sum_{i=1}^{n} \alpha_i^2 \text{Var}\left(W_{t_i} - W_{t_{i-1}}\right) = \sum_{i=1}^{n} \alpha_i^2 (t_i - t_{i-1}) = \int f^2(x)dx$$

Stage 2. Consider a sequence of **step functions**, $\{f_n\}$, such that $f_n \overset{\mathcal{L}^2}{\to} f$, i.e. $\int_a^b ()^2 \, dt \overset{n \to \infty}{\to} 0$. Then

$$I(f) := \text{lim}_{n\to\infty} I(f_n) = \text{lim}_{n\to\infty} \int_a^b f_n(t) dW_t$$

where the limit is understood in a mean squared sense, i.e.

$$\mathbb{E}\left[(I(f_n) - I(f))^2\right] \overset{n \to \infty}{\to} 0$$

The above implies that the limit of $I(f_n)$ does not depend on the choice of f_n.

Theorem. *Let* $\{f_n\}, \{\hat{f}_n\}$ *be sequences of step functions such that* $f_n \overset{\mathcal{L}^2}{\to} f, \tilde{f}_n \to f$*, then*

$$\lim_{n\to\infty} I(f_n) = \lim_{n\to\infty} I(\tilde{f}_n)$$

Proof. Note that $I(f_n) - I(\tilde{f}_n) = I(f_n - \tilde{f}_n)$, where $f_n - \tilde{f}_n$: step function, $\forall n$, and it follows $N(0, L)$, where L : L2 norm between f_n, $\tilde{f}_n$. Thus,

$$\mathbb{E}\left((I(f_n) - I(f))^2\right) = \int_a^b \left(f_n(x) - \tilde{f}_n(d)\right)^2 dx \overset{n\to\infty}{\to} 0$$

Theorem. $\forall f \in \mathcal{L}^2(a, b)$

$$I(f) \sim N\left(0, \int_a^b f^2(x)dx\right)$$

Proof. Since f can be represented $\lim f_n$,

$$I(f) = \lim_{n\to\infty} I(f_n)$$

where $I(f_n) \sim N\left(0, \int_a^b f_n^2(x)dx\right)$, and using the ergodicity of Gaussian process, i.e. for all invariant event E, after time shift, either $P(E) = 1$ or $P(E) = 0$.

1.4 Integrals of the type $\int X_t dW_t$

Recall that **filtration** is a sequence of σ-algebras $\mathcal{F}_t$ on $(\Omega, \mathcal{F}, \mathbb{P})$: $\mathcal{F}_t \subset \mathcal{F}_s, \forall t \leq s$.

Given the filtration, we can define the space of the processes X_t, $\mathcal{L}_{ad}([a, b], \Omega)$ with the following defining properties:

- 1. X_t : $\mathcal{F}_t$-adapted where X_t: $\mathcal{F}_t$-measurable for all t

- 2. $\int_a^b \mathbb{E}X_t^2 dt < \infty$

Note that X_t: F_t-measurable for all t means

$$\{X_t \in B\} \subset \mathcal{F}_t, \forall t, \forall B \in \mathcal{B}(\mathbb{R})$$

Additionally, W_t is defined $\mathcal{F}_t$-Brownian motion if

- 1) W_t : $\mathcal{F}_t$-adapted
- 2) $W_t - W_s \perp \mathcal{F}_s$, $\forall t > s$

First, we will define the integral for **step processes**:

$$\sum \xi_{i-1} \mathbf{t_{i-1}} \leq \mathbf{t} < \mathbf{t_i}$$

Second, $X_t \in \mathcal{L}^2_{ad}$

For example, assume W_t is a Brownian motion. The interval from 0 to T is divided into n parts by points $t_1, \cdots, t_{n-1}$ and $t_0 = 0, t_n = T$. We have

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left(W_{t_i} - W_{t_{i-1}}\right)^2 = t$$

Let us **formally define**

$$\int_a^b X_t dW_t$$

where $X_t \in \mathcal{L}^2$ and W_t: $\mathcal{F}_t$-Brownian motion. Note that the definition heavily depends on the notion of **filtration**.

Stage1. If

$$X_t = \sum_{i=1}^{n} \xi_{i-1} \mathbf{1}_{t_{i-1} \leq t < t_i}$$

then

$$I(X_t) := \sum_{i=1}^{n} \xi_{i-1} \left(W_{t_i} - W_{t_{i-1}} \right)$$

Stage2. For $X_t \in \mathcal{L}^2_{ad}$ and X^n_t : step processes, and

$$\int_a^b \mathbb{E}(X^n_t - X_t)^2 dt \to 0$$

then

$$I(X_t) := \lim_{n\to\infty} I(X^n_t)$$

in the sense that $\mathbb{E}(X^n_t - X_t)^2 \to 0$.

Now let us focus on how to construct such a step sequence

Theorem. *Let $m(t), K(t,s)$ be continuous. Then*

$$X^n_t := \sum_{i=1}^{n} X_{t_{i-1}} \mathbf{t_{i-1} \leq t < t_i}$$

$$X^n_t \to X_t$$

in the sense that $\int_a^b \mathbb{E}(X_t^n - X_t)^2 \to 0.$

Proof

Note that

$$\begin{aligned}\mathbb{E}(X_t - X_s)^2 &= \mathbb{E}X_t^2 - 2\mathbb{E}X_tX_s + \mathbb{E}X_s^2 \\ &= \big(K(t,t) + m^2(t)\big) - 2\left(K(t,s) + m(t)m(s)\right) + \big(K(s,s) + m^2(s)\big) \to 0\end{aligned}$$

as $s \to t$. Thus the process is **continuous in the mean squared sense**, i.e. $X_t^n \overset{n\to\infty}{\to} X_t$. Note that the legality of the change of limit and integral is guaranteed by **Leibniz theorem**. Note that

$$\int =$$

$$\phi_\xi(u) =$$

This theorem basically means that one can exactly take the sequence X_{t_n} as a sequence of step processes to construct sequence on the stage 2.

2 Ito Formula

2.1 Integrals of the type $\int X_t dH_t$

$$\int_a^b X_t dH_t$$

where H_t : Ito process, which can be represented as

$$H_t = H_0 + \int_0^t b_s ds + \int_0^t \sigma_s dW_s$$

We assume that there is some **filtration** behind this integral, $\mathcal{F}_t$. In other words, b_s, σ_s : processes adapted to $\mathcal{F}_t$, W_t is $\mathcal{F}_t$-Brownain motion, and H_0 us a RV measruable wrt $\mathcal{F}_0$.

Here, H_t is such a process, and X_t is some process adapted to the same filtration.

If

$$X_t : \int_a^b |X_s b_x| + X_s^2 b_x^2 dx < \infty$$

then

$$\int_a^b X_t dH_t = \int_a^b b_s X_s dx + \int_a^b \sigma_s X_s dW_s$$

Note that the first integral is the first type. The second integral is the third type of the integrals. We have the **equivalent representation**:

$$dH_t = b_t dt + \sigma_t dW_t$$

Theorem. *Let H_t be a Ito process, and $f(t,x)$: twice differentiable, continuous. Then*

$$f(t,H_t) = f(0,H_0)+\int_0^t f_1'(s,H_s)dx+\int_0^t f_2'(s,H_s)dH_s+\frac{1}{2}\int_0^t f_{22}''(s,H_s)\sigma_s^2 ds$$

2.2 Ito's formula

Let us show how the Ito formula can be used for calculating the stochastic integrals of the following type:

$$\int_0^t g(s,W_s)dW_s$$

where f: antiderivative of g wrt the second argument, i.e. $f_2' = g$. Note that $H_t = W_t$.

$$f(t,W_t) = f(0,0) + \int_0^t f_1'(s,W_s)ds + \int_0^t g(s,W_s)dW_s + \frac{1}{2}\int_0^t g_2'(s,W_s)ds$$

$$\int_0^t g(s,W_s)dW_s = f(t,W_t) - f(0,0) - \int_0^t f_1'(s,W_s) + \frac{1}{2}g_2'(s,W_s)ds$$

For instance, consider $\int_0^t W_s dW_s$.

2.3 Black-Scholes model

Black-Scholes formula is widely used for **modeling of stock prices**.

$$dX_t = X_t\mu dt + x_t\sigma dW_t, \quad \sigma > 0$$

Note that

$$X_t = X_0 + \mu \int_0^t X_s dx + \sigma \int_0^t X_s dW_s$$

where $f(t, x) := \ln(x)$.

2.4 Vasicek model

The **Vasicek model** is defined as the solution for the following stochastic DE:

$$dX_t = (a - bX_t)dt + cdW_t$$

where $b, c > 0$. This model is a so called **mirror model**. There is a sense considering the following relationship:

$$b\left(1/b - X_t\right)dt$$

thus b is called the speed of reversion.

$$\begin{aligned} d\left(X_t e^{bt}\right) &= bX_t e^{bt} dt + e^{bt}\left(\right) \\ &= ae^{bt}dt + ce^{bt}dW_t \end{aligned}$$

Therefore the solution of the above stochastic differential equation is the following:

$$X_t = e^{-bt}X_0 + \frac{a}{b}(1 - e^{-bt}) + c\int_0^t e^{b(s-t)}dW_s$$

e

2.5 Ornstein-Uhlenbeck process

$$mdV_t = dW_t - \lambda V_t dt$$

where W_t : Brownian motion, λ : friction coefficient. We use Ito formula with the following function f,

$$f(t,x) = xe^{\lambda t/m}$$

we have the solution

$$V_t = e^{\lambda t/m}\left(V_0 + \frac{1}{\mu}\int_0^t e^{\lambda s/m} dW_s\right)$$

If we assume that $V_0 \sim N\left(0, \frac{1}{2\lambda m}\right) \perp W_t$, then V_t is a Gaussian process such that

$$K(t,s) = \frac{m}{2\lambda} e^{-\frac{\lambda}{m}|t-s|}$$

which is **stationary** in both and weak and wide senses.

Quizzes

(Quiz 1). Let $I(f) = \int_0^1 t^2 dW_t$. Find the mean of $I(f)$.

(Answer)

Let $f(t,x) = xt^2$. Then $f_2' := \frac{\partial f}{\partial x} = t^2$. Thus

$$f(1,W_t)=W_1 1^2=0+\int_0^1 2tW_t dt+\int_0^1 t^2 dW_t+0$$

$$\int_0^1 t^2 dW_t=W_1-\int_0^1 2tW_t dt$$

$$\therefore\quad \mathbb{E}\left(\int_0^1 t^2 dW_t\right)=0-\int_0^1 2t\mathbb{E}W_t dt=0$$

(Quiz 2). Let $I(f)=\int_0^1 t^2 dW_t$. Find the variance of $I(f)$.

(Answer)

$$\begin{aligned}\mathrm{Var}\left(\int_0^1 t^2 dW_t\right)&=\mathrm{Var}W_1+\mathrm{Var}\int_0^1 2tW_t dt-2\mathrm{cov}\left(W_1,\int_0^1 2tW_t dt\right)\\&=1+2\int_0^1\int_0^t cov(2tW_t,2sW_s)dsdt\end{aligned}$$

(Quiz 3). Let N_t be a Poisson process. Find the mean, covariance function and variance of $I(f):=\int_0^t N_s ds$, for $t>s\geq 0$.

(Answer)

Note that

$$\mathbb{E}[I(f)]=\int_0^t \mathbb{E}N_s dx=\lambda t^2/2$$

and

$$
\begin{aligned}
K(T,S) &= 2\int_0^T\int_0^S cov(N_t,N_s)dtds \\
&= 2\left\{\int_0^S\int_0^S cov(N_t,N_s)dtds + \int_0^S\int_S^T cov(N_t,N_s)dtds\right\}
\end{aligned}
$$

(Quiz 4). Let

$$
X_t := \begin{cases} \xi_1, & t\in[0,1) \\ \xi_2, & t\in[1,2) \\ \xi_3, & t\geq 2 \end{cases}
$$

, where ξ_1,ξ_2,ξ_3 are IID RVs having exponential distribution with parameter λ. Find the mean and the variance of $\int_0^T X_t dt$

(Answer)

$$
\begin{aligned}
\mathbb{E}\left[\int_0^T X_t dt\right] &= \int_0^T \mathbb{E}X_t dt \\
&= \int_0^T 1/\lambda dt \\
&= \frac{T}{\lambda}
\end{aligned}
$$

For $T<1$,

$$
\begin{aligned}
\text{Var}\left[\int_0^T X_t dt\right] &= 2\int_0^T\int_0^t \text{cov}(\xi_1,\xi_1)dsdt \\
&= \int_0^T 2t/\lambda^2 dt \\
&= \frac{T^2}{\lambda^2}
\end{aligned}
$$

For $1\leq T<2$,

$$
\begin{aligned}
\operatorname{Var}\left[\int_0^T X_t dt\right] &= \operatorname{Var}\left[\int_0^1 X_t dt + \int_1^T X_t dt\right] \\
&= \operatorname{Var}[\int_0^1 \xi_1 dt] + \operatorname{Var}[\int_1^T \xi_2 dt] \\
&= 2\int_0^1 \int_0^t cov()
\end{aligned}
$$

(Quiz 5). Find the equivalent expression for the stochastic integral $\int_0^T W_t^2 dW_t$, where W_t is a Brownian motion.

(Answer)

Let $f(t, W_t) = W_t^3/3$ then

$$
\begin{aligned}
f_1(t, W_t) &= 0 \\
f_2(t, W_t) &= W_t^2 \\
f_{22}(t, W_t) &= 2W_t
\end{aligned}
$$

Thus we have

$$
\frac{1}{3}W_T^3 = 0 + 0 + \int_0^T W_t^2 dW_t + \frac{1}{2}\int_0^T 2W_s \sigma_s^2 ds
$$

therefore,

$$
\int_0^T W_t^2 dW_t = \frac{1}{3}W_T^3 - \int_0^T W_s ds
$$

(Quiz 6). Compute the variance of the stochastic integral $\int_0^T W_t dW_t$, where W_t is a Brownian motion

(Answer)

Letting $f(t, W_t) = W_t^2/2$, we have

$$
\begin{aligned}
f_1(t, W_t) &= 0 \\
f_2(t, W_t) &= W_t \\
f_{22}(t, W_t) &= 1
\end{aligned}
$$

Thus

$$\frac{1}{2} W_T^2 = 0 + 0 + \int_0^T W_t dW_t + \int_0^T \sigma_s^2 ds$$

implying

$$\int_0^T W_t dW_t = \frac{1}{2} W_T^2 - T$$

thus

$$
\begin{aligned}
\mathrm{Var} \int_0^T W_t dW_t &= \mathrm{Var}\left(W_T^2/2\right) \\
&= \frac{1}{4} \left(N(0, T)\right)^4 = T^2/2
\end{aligned}
$$