Limit Theorems for Renewal Processes

Hyunwoo Gu

1 The discrete-time Renewal Equation

1.1 The Renewal Theorem

Recall that

$$M(t) = \mathbb{E}[N(t)] = \sum_{j=1}^{\infty} F_j(t)$$

where $F_j(t) = Pr[S_j \le t]$ for $t \ge 0$. We call M(t) a **renewal function**.

Let us now show that the renewal function M(t) satisfies the equation

$$M(t) = F(t) + \int_0^t M(t - y)dF(y), \quad t \ge 0$$

$$\Leftrightarrow M(t) = F(t) + F * M(t), \quad t \ge 0$$

Renewal Equations

An integral equation of the following form is called a **renewal equation**.

$$A(t) = a(t) + \int_0^t A(t-x)dF(x), \quad t \ge 0$$

Theorem 4.1. Suppose a is a bounded function. Then there uniquely exists A bounded on finite intervals satisfying

$$A(t) = a(t) + \int_0^t A(t-x)dF(y)$$

Namely,

$$A(t) = a(t) + \int_0^t a(t-x)dM(x)$$

where $M(t) := \sum_{k=1}^{\infty} F_k(t)$: the renewal function.

Proof. We verify first that A specified fulfilles the requisite boundedness properties and solves

$$A = a + F * a + F_2 * (a + F * A)$$

$$= a + F * a = F_2 * a + F_3 * A$$

$$= a + \left(\sum_{k=1}^{n-1} F_k\right) * a + F_n * A$$

Next observe that

Theorem 4.2. Let $\{X_t\}$ be a renewal process with $\mu = \mathbb{E}X_1 < \infty$. Then

$$\lim_{t \to \infty} \frac{1}{t} M(t) = \frac{1}{\mu}$$

Proof. Note that $t < S_{N(t)+1}$. Thus

$$t < \mathbb{E}\left[S_{N(t)+1}\right] = \mu \left[1 + M(t)\right]$$

and therefore

$$\frac{1}{t}M(t) > \frac{1}{\mu} - \frac{1}{t}$$

It follows that

$$\lim_{t\to\infty} \frac{1}{t} M(t) \ge \frac{1}{\mu}$$

To establish the opposite inequality, let c > 0 be arbitrary, and set

$$X_i^c = \begin{cases} X_i & X_i \le c \\ c & X_i > c \end{cases}$$

and consider the renwal process having lifetimes $\{X_i^c\}$. Let $\{S_n^c\}$ and $\{N^c(t)\}$ denote the waiting times and counting process, respectively, for this **truncated renewal process** generated by $\{X_i^c\}$. Since the random variables X_i^c are uniformly bounded by c, it is clear that $t + c \geq S_{N^c(t)+1}^c$, and therefore

$$t + c \ge \mathbb{E}\left[S_{N^c(t)+1}^c\right] = \mu^c \left[1 + M^c(t)\right]$$

where

From $N^c(t) \ge N(t)$ from $X_i^c \le X_i$, and $M^c(t) \ge M(t)$, we have

$$t + c \ge \mu^c \left[1 + M(t) \right]$$

$$\therefore \frac{1}{t} M(t) \le \frac{1}{\mu_c} + \frac{1}{t} \left(\frac{c}{\mu^c} - 1 \right)$$

Hence

$$\limsup_{t\to\infty} \frac{1}{t}M(t) \le \frac{1}{\mu^c}, \quad \forall c > 0$$

since

$$\lim_{c \to \infty} \mu^c = \lim_{c \to \infty} \int_0^c [1 - F(x)] dx$$
$$= \int_0^\infty [1 - F(x)] dx = \mu$$

while the left-hand side is fixed, we deduce that

$$\lim_{t\to\infty} \frac{1}{t} M(t) \le \lim_{c\to\infty} \frac{1}{\mu^c} = \frac{1}{\mu}$$

2 The Continuous-time Renewal Theorem

$$M(t+h) - M(t) \to \frac{h}{\mu}, \quad t \to \infty$$

In words, the expected number of renewals in an interval with length h is sapproximately h/μ , provided the process has been in operation for a long duration.

Definition 5.1. A point α of a distribution function F is called a **point** of increase if for a positive ϵ

$$F(\alpha + \epsilon) - F(\alpha - \epsilon) > 0$$

A distribution function is **arithmetic** if there exists $\lambda > 0$ such that F exhibits points of increase exclusively among $0, \pm \lambda, \pm 2\lambda, \cdots$. The largest such λ is called the **span** of F.

Note that F that has a continuous part is **not arithmetic**. The distribution function of a discrete RVs having positible values $0, 1, 2, \cdots$ is arithmetic with span 1.

Definition 5.2. A point α of a distribution function F is called a **point** of increase if for a positive ϵ

$$F(\alpha + \epsilon) - F(\alpha - \epsilon) > 0$$

Every monotonic function g which is absolutely integrable in the sense that

$$\int_0^\infty |g(t)|dt < \infty$$

is directly Riemann integrable. Manifestly, all finite linear combinations of monotone functions satisfying the above are also directly Riemann integrable.

Theorem 5.1. (The Basic Renewal Theorem). Let F be the distribution function of a positive random variable with mean μ . Suppose that a is directly Riemann integrable and that A is the solution of the renewal equation

$$A(t) = a(t) + \int_0^t A(t-x)dF(x)$$

(i) If F is not arithmetic, then

$$\lim_{t \to \infty} A(t) = \begin{cases} \frac{1}{\mu} \int_0^\infty a(x) dx & \mu < \infty \\ 0 & \mu = \infty \end{cases}$$

(ii) If F is arithmetic with span λ , then $\forall c > 0$,

$$\lim_{t \to \infty} A(t) = \begin{cases} \frac{\lambda}{\mu} \sum_{n=0}^{\infty} a(c+n\lambda) & \mu < \infty \\ 0 & \mu = \infty \end{cases}$$

In a simpler form, we say that

Combining

An obvious generalization of integer-valued inter-renewal intervals is that of interrenewals that occur only at integer multiples of some real number $\lambda > 0$. Such a distribution is called an **arithmetic distribution**. The **span** of an arithmetic distribution is the largest number λ s.t. this property holds.

3 Applications of the Renewal Theorem

3.1 Limiting Distribution of the Excess Life

3.2 Asymptotic Expansion of the Renewal Function

Suppose F is a nonarithmetic distribution with a finite variance σ^2 . Under these assumptions we will determine the second term in the asymptotic expansion of M(t) by proving

$$\lim_{t \to \infty} \{ M(t) - \mu^{-1} t \} = \frac{\sigma^2 - \mu^2}{2\mu^2}$$

4 Applications of the Renewal Theorem

4.1 Delayed Renewal Processes

$$M_D(t) = \int_0^\infty \mathbb{E}[N(t)|X_t = x]dG(x)$$
$$= \int_0^t \{1 + M(t - x)\}dG(x)$$
$$= G(t) + \int_0^t M(t - x)dG(x)$$
$$= G(t) + \int_0^t G(t - x)dM(x)$$

4.2 Stationary Renewal Processes

A delayed renewal process for which the first life has the distribution function

$$G(x) = \mu^{-1} \int_0^x \{1 - F(y)\} dy$$

4.3 Alternating and Markov Processes

An alternating renewal process is a sequence Y_1, Y_2, \cdots of independent RVs, where

$$Y_1, Y_{r+1}, Y_{2r+1}, \dots \sim F_1$$

 $Y_2, Y_{r+2}, Y_{2r+2}, \dots \sim F_2$
 $\dots \vdots \dots$
 $Y_r, Y_{2r}, Y_{3r}, \dots \sim F_r$

4.4 Central Limit Theorem for Renewals

4.5 Characterization of the Poisson Process

Poisson process is a special type of a renewal process. Let $\{X_k\}$ be a renewal process with $\mathbb{E}[X_k] := \mu < \infty$ and $F(x) = P\{X_k \leq x\}$. Assuming F(0) = 0, define

$$F_t(x) = \begin{cases} F(x) & 0 \le x < t \\ 1 & t \le x \end{cases}$$

which is basically the distribution function for $min\{X_k, t\}$

Theorem 8.1.. (a) If there exists a sequence $\{t_j\}$, where $t_j \to \infty$ as $j \to \infty$, and for which the current life δ_t satisfies

$$F_{t_j}(x) = Pr[\delta_{t_j} \le x], \forall x$$

then F is an exponential distribution.

(b) If there exists a sequence $\{t_j\}$, where $t_j \to \infty$ as $j \to \infty$, and for which

$$F(x) = Pr[\gamma_{t_i} \le x], \forall x$$

then F is an exponential distribution.

Proof. By the result of (6.5), i.e.

the limiting distribution of the current life δ_t is

$$\lim_{t\to\infty} \Pr\{\delta_t > y\} = \mu^{-1} \int_y^\infty [1 - F(z)] dz$$

Letting t increase along t_j with due account of the hypothesis of the theorem, we derive the functional equation

$$1 - F(y) = \mu^{-1} \int_{y}^{\infty} [1 - F(z)] dz$$

The right-hand side is clearly differentiable in y, yielding the elementary first-order DE

$$\frac{d}{dy}[1 - F(y)] = -\frac{1}{\mu}[1 - F(y)]$$

whose solution, subject to F(0) = 0, is

$$1 - F(y) = e^{-\lambda y}, \lambda := 1/\mu$$

4.6 Superposition of Renewal Processes

Definition 9.1. The triangular array $\{N_{ni}(t)\}$ is called **infinitesimal** if for every $t \geq 0$,

$$\lim_{n \to \infty} \max_{1 \le i \le k_n} F_{ni}(t) = 0$$

Theorem 9.1. Let $\{N_{ni}(t)\}$ be an infinitesimal array of renewal processes with superposition $N_n(t)$. Then

$$\lim_{n\to\infty} Pr\{N_n(t) = j\} = \frac{e^{-\lambda t(\lambda t)^j}}{j!}, \quad j = 0, 1, 2, \dots$$

if and only if

$$\lim_{n\to\infty} \sum_{i=1}^{k_n} F_{ni}(t) = \lambda t$$

Proof.

(1) Necessity. For j = 0, we obtain

$$\lim_{n\to\infty} \Pr\{N_t(t)=0\} = e^{-\lambda t}$$

or, equivalently,

(1) Sufficiency. Let us follow an induction on m to show

$$\lim_{n\to\infty} Pr\{N_n(t) = m\} = \frac{e^{-\lambda t(\lambda t)^m}}{m!}, \quad m = 0, 1, 2, \dots$$

For m = 0,

Example 1. Suppose F(t) is a distribution function for which F(0) = 0, $F'(0) = \lambda > 0$. Let

$$F_{ni}(t) = F(t/n), \quad i = 1, \cdots, n$$

and, for all n, let $N_{ni}(t)$, $i = 1, \dots, n$ be independent renewal counting processes with interoccurrence distribution F_{ni} . Then $N_{ni}(t)$ is a triangular array. Furthermore, since

Hence, the distribution of the superposition $N_n(t)$ converges to the Poisson process.

Theorem 9.2. Let $N_1(t)$ and $N_2(t)$ be two independent renewal processes with the same interoccurrence distribution F having mean μ . Let $N(t) = N_1(t) + N_2(t)$. If N(t) is also a renewal process, then $N_1(t), N_2(t), N(t)$ are all Poisson.

Proof. Let H be the interoccurrence distribution for N(t). Then

$$1 - H(x) = Pr[N(x) = 0]$$
$$= Pr[N_1(x) = 0, N_2(x) = 0]$$
$$= [1 - F(x)]^2$$

Let

$$\frac{1}{v} \int_{x}^{\infty} [1 - H(y)] dy = \frac{1}{\mu^{2}} \left\{ \int_{x}^{\infty} [1 - F(y)] dy \right\}^{2}$$

where $v := \int_0^\infty [1 - H(y)] dy$. Both sides are differentiable with respect to x, and earlier we noted $1 - H(x) = [1 - F(x)]^2$.