Levy processes

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1 Introduction to the theory of Levy processes

Here we will be able to:

- understand the main properties of Levy processes
- construct ta Levy process from an infinitely

1.1 Different types of stochastic integrals

1.2 Examples of Levy processes

The class of **infinitely divisible distributions**.

Definition. *An RV ξ is a **infinitely divisible distribution**, if*

$$\xi \overset{d}{\equiv} Y_1 \oplus \cdots \oplus Y_n$$

Note that

$$\phi_\xi(u) =$$

It is not true that the distribution is stable iff it is infinitely divisible.

Definition. *An RV ξ is a* ***infinitely divisible distribution****, if*

$\exists \psi : \mathbb{R} \to \mathbb{C}$ *such that*

$$\phi_{L_t}(u) = \mathbb{E}\left[e^{iuL_t}\right] = e^{t\psi(u)}$$

Example 1.

Example 12.

1.3 Characteristic Exponent

1.4 Properties of Levy Characteristic Exponent

Levy measure.

1.5 Levy-Khintchine triplet

$$\phi_{X_t}(u) = exp\left[t(iu\mu - \sigma^2 u^2/2) + \int_{\mathbb{R}} \left(e^{iux} + 1 - iux\mathbf{1}_{|x|<1}\right)\right]$$

Example 1. For X_t : of bounded variation,

$$\sum_{k=1}^{n} |X_{t_k} - X_{t_{k-1}}| \overset{max|t_i - t_{i-1}| \to 0}{\longrightarrow}$$

$\sigma = 0$

Note that the Brownian motion is not a process of bounded variation.

1.6 Modeling of jump-type

How to estimate that the Levy measure of a Levy process from the following data?

$$X_t : X_\Delta$$

CIR model for the stochastic volatility is

$$c\sqrt{V_t}dW_t$$

Levy process can be used to model the properties of jobs.

Let X_t be of bounded variation, and

$$\phi_{X_\Delta}(u) := exp\left\{\Delta\left(iu\mu + \int_{\mathbb{R}}(e^{iux} - 1)S(x)dx\right)\right\}$$

Note that

$$\phi_{X_\Delta}(u) := exp\left\{\Delta\left(iu\mu + \int_{\mathbb{R}}(e^{iux} - 1)S(x)dx\right)\right\}$$

Let X_t be a Levy process of bounded variation with Levy triplet (μ, σ^2, ν) and Levy density $s(x)$. The correct form of the characteristic exponent is given as

$$\psi(u) = iu\left(\mu - \int_{|x|<1} xs(x)dx\right) + \int_{\mathbb{R}}(e^{iux} - 1)s(x)dx$$

with μ probably equal to zero. Note that it is also true that nonzero μ can be of bounded variation.

Quizzes

(Quiz 1-3). $X_t := bt + \sigma W_t + cN_t$, where W_t: Brownian motion, N_t: a Poisson process with λ, and W_t, N_t independent, $b, c \in \mathbb{R}$, $\sigma \geq 0$. Denote the Levy measure of this process bt ν.

(Quiz 1). Find the characteristic function of this process.

(Answer). $exp\{iubt + \lambda t(e^{icu} - 1) - \frac{t(\sigma u)^2}{2}\}$

(Quiz 2). What are the mean, variance, covariance function of this process?

(Answer). No answer.

(Quiz 3). What is measure ν of a Borel set B?

(Answer). $\nu(B) = \lambda$, if $1 \in B$ and 0 otherwise.

Considering that the jump part of the Levy process discussed is

- c: real number
- N_t : Poisson process with λ

Thus $\nu(B) = \lambda \mathbf{1}_{c \in B}$. Computing the integral in the characteristic exponent, returns a pointwise evaluation in the point c, $\lambda(e^{iuc} - 1)$.

(Quiz 4). Let X_t be a Levy process. What is the correct expression for $\mathrm{Var}(X_t)$ in terms of characteristic exponent ψ?

(Answer). $\mathrm{Var}(X_t) = -t\psi''(0)$

(Quiz 5). Let X_t be a Levy process, assuming $X_1 \sim N(0,1)$. Find the mean and the variance of X_t

(Answer). $\mathbb{E}[X_t] = 0, \mathrm{Var}(X_t) = t$.

(Quiz 6). Let $X_t = bt + N_t$, where N_t is a Poisson process with λ and $b \in \mathbb{R}$. Find the Levy triplet of this process.

(Answer). $(b + \lambda, 0, \nu$, where $\nu(B) = \lambda \mathbf{1}_{1 \in B}$ for any Borel set B.

The first term of a Levy triplet should correspond to the coefficient of the drift term, i.e. b, if X is of the form

$$X_t = bt + \sigma W_t + cN_t$$