Stationarity & Linear Filters

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1 Spectral density of a wide-sense stationary process

Bocher

$\phi(u) : \mathbb{R} \to \mathbb{C}, \phi(u) = \mathbb{E}[e^{iu\xi}]$: characteristic function, iff

- 1) ϕ is constant
- 2) ϕ is positive semidefinite, $\sum_{j=1}^{n} z_j \bar{z}_k \phi(u_j - u_k) \geq 0$, $\forall (z_1, \cdots, z_n) \in \mathbb{C}^n, \forall (u_1, \cdots, u_n) \in \mathbb{R}^n$.
- 3) $\phi(0) = 1$

If 1), 2) are met, we have

$$\exists \mu : \phi(\mu) = \int e^{iux} \mu(dx)$$

X_t : weakly stationary.

$$\sigma : K(t, s) = \sigma(t - s)$$

if σ is constant, and $\int |\sigma(u)|du < \infty$

$$\mathcal{F}$$

For example, there does not exist a stochastic process with the covariance $K(t, s) := sin(\lambda(t - s))$, since it is not positive semi-definite.

$$g(x) :=$$

Example 1) $WN(0, \sigma^2)$

Note that

$$\gamma(u) = \sigma^2 1_{\{u=0\}}$$

$$g(x) = \frac{\sigma^2}{2\pi}$$

Example 2) $MA(1)$

> ***Proposition.***
>
> $$\exists \lim_{n\to\infty} P_{ij}(n) = \pi_j^* > 0$$
>
> $$\sum_{j=1}^{M} \pi_j^* = 1$$

2 Stochastic integration

Quizzes

(Quiz 1). Let Y_n be a stochastic process, such that

$$Y_{n+1} := \alpha Y_n + X_n$$

for $n = 0, 1, \cdots$. Assume $Y_0 := 0, |\alpha| < 1$ and X_n : a sequence of IID standard normal RVs. Determine whether Y_n is stationary and find its mean and variance.

(Answer) $\forall t > s$

$$\begin{aligned} K(t,s) &= Cov(\alpha^{t-1}X_0 + \cdots \alpha^0 X_{t-1}, \alpha^{s-1}X_0 + \cdots + \alpha^0 X_{s-1}) \\ &= \alpha^{t-1}\alpha^{s-1} + \alpha^{t-2}\alpha^{s-2} + \cdots + \alpha^{t-s} \\ &= \alpha^{t-s}\frac{1-\alpha^{2s}}{1-\alpha^2} \end{aligned}$$

where $K(t,s)$ also depends on s, not only on $t - s$.

(Quiz 2). Let W_t be a Brownian motion and define $X_t := (1-t)W_{t/(1-t)}$, for $t \in (0,1)$. Is X_t stationary?

(Answer) No,

$$\begin{aligned} K(t,s) &= (1-t)(1-s)Cov(W_{t/(1-t)}, W_{s/(1-s)}) \\ &= (1-t)(1-s)Cov(W_{t/(1-t)} - W_{s/(1-s)} + W_{s/(1-s)}, W_{s/(1-s)} - W_0) \\ &= (1-t)(1-s)Var(W_{s/(1-s)}) = s(1-t) \end{aligned}$$

which is not weakly stationary.

(Quiz 3). Let X_t be a process with independent and stationry increments and $\exists h > 0$. Moreover, $\mathbb{E}(X_t) = 0, \mathbb{E}|X_t|^2 < \infty$. Is $Y_t = X_{t+h} - X_t$ a wide-sense stationary process?

(Answer) Yes,

Note that if increments of a process is stationry, then the process is stationary.

(Quiz 4). Let the autocovariance function of some stochastic process X_t be $\gamma_X(u) := \begin{cases} 3 & u = 0 \\ 1 & u = \pm 2 \\ 0 & o.w. \end{cases}$ Find the **spectral density** of $Y_t := 3X_t + 2X_{t-1} + X_{t-2}$.

(Answer) Note that

$$
\begin{aligned}
g_X(u) &= \frac{1}{2\pi}(3 + e^{-2iu} + e^{2iu}) \\
&= \frac{1}{2\pi}(3 + 2cos(2u)) \\
g_Y(u) &= g_X(u)|\mathcal{F}[\rho](u)|^2
\end{aligned}
$$

where $\rho(h) =:= \begin{cases} 3 & h = 0 \\ 2 & h = 1 \\ 1 & h = 2 \\ 0 & o.w. \end{cases}$. Therefore

$$
\begin{aligned}
\mathcal{F}[\rho](u) &= e^{2iu} + 2e^{iu} + 3 \\
[\mathcal{F}[\rho](u)]^2 &= \mathcal{F} \times \bar{\mathcal{F}} \\
&= (e^{2iu} + 2e^{iu} + 3) \times (e^{-2iu} + 2e^{-iu} + 3) \\
&= 9 + 4 + 1 + 8(e^{iu} + e^{-iu}) + 3(e^{2iu} + e^{-2iu}) \\
&= 14 + 3 \cdot 2cos(2u) + 8 \cdot 2cos(u) \cdot g_Y(u) \\
&= \frac{1}{2\pi}(3 + 2cos(2u))(14 + 3 \cdot 2cos(2u) + 8 \cdot 2cos(u))
\end{aligned}
$$

(Quiz 4). Let the autocovariance function of some stochastic process X_t be $\gamma_X(u) := \begin{cases} 3 & u = 0 \\ 1 & u = \pm 2 \\ 0 & o.w. \end{cases}$ Find the **spectral density** of $Y_t := 3X_t + 2X_{t-1} + X_{t-2}$.

(Answer) Note that for $t > s + h$, we have

$$K(t, s) = Cov(W_{t+h} - W_t, W_{s+h} - W_s) = 0$$

For $t \leq s + h$, we have

$$
\begin{aligned}
K(t, s) &= Cov(W_{t+h} - W_t, W_{s+h} - W_s) \\
&= Cov(W_{t+h} - W_{s+h} + W_{s+h} - W_t, W_{s+h} - W_s) \\
&= Cov(W_{s+h} - W_t, W_{s+h} - W_t + W_t - W_s) \\
&= Var(W_{s+h} - W_t) = h - |t - s| \\
g_Y(u) &= g_X(u)|\mathcal{F}[\rho](u)|^2
\end{aligned}
$$