Intro & Renewal Process

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1 Introduction to Stochastic Processes

1.1 Introduction

σ**-algebra** on a set Ω is a collection F of subsets of Ω :

- **Includes Ω itself**,
- Closed under **complement**
- Closed under **countable unions**

where the pair (Ω, F) is called a **measurable space** or **Borel space**. More rigorously, letting X be some set, and $\mathcal{P}(X)$ be its power set. Then $\Sigma \subseteq \mathcal{P}(X)$ is called σ**-algebra** on X if

- $X \in \Sigma$
- If $A \in \Sigma$, then $X - A \in \Sigma$
- If $A_1, A_2, \cdots \in \Sigma$, then $A_1 \cup A_2 \cup \cdots \in \Sigma$

where from De Morgan's laws Σ is also closed under countable intersections. For example, $\{\Omega, \emptyset\}$ is the smallest possible σ-algebra on Ω, whereas the largest possible σ-algebra on Ω is $2^\Omega := \mathcal{P}(\Omega)$. Elements of the σ-algebra, i.e. an ordered pair (Ω, F) is called a **measurable set**.

For example, let $(\Omega, \mathcal{F}, P)$ be a probability space in Bernoulli scheme, where $(a_1, \cdots, a_n), a_i \in \{0, 1\}$.

- Ω, **sample space**: $\{0,1\}^n$, $|\Omega| = 2^n$
- $\mathcal{F}$, **filtration**: $|\mathcal{F}| = 2^{2^n}$, since it is the power set.
- P, the **probability** measure

1.2 Stochastic functions

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a **probability space**(or probability triple). **Random variable** is a function $\xi : \Omega \to \mathbb{R}$ such that $\xi^{-1}(B) \in \mathcal{F}, \forall B \in \mathcal{B}(\mathbb{R})$

For time T, $X : T \times \Omega \to \mathbb{R}$ is **random function** if $X(t, \cdot)$ is a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ for all t.

$T = \mathbb{R}_+$ case is called **random process**, whereas $T = \mathbb{R}_+^n$ case is called **random field**, where

- Discrete time random process, $T = \mathbb{N}$ or $\mathbb{Z}$
- Continuous time random process, $T = \mathbb{R}_+$ or $\mathbb{R}$

Note that **any stochastic process at any fixed time** is **a random variable**.

Let $X : T \times \Omega \to \mathbb{R}, T = \mathbb{R}_+$. Let a **finite-dimensional distribution** $(X_{t_1}, X_{t_2}, \cdots, X_{t_n})$ for $t_1, \cdots, t_n \in \mathbb{R}$ be given. A **trajectory**(path) is $T \to \mathbb{R}$ for a fixed ω, $(X_{t_1}(\omega), X_{t_2}(\omega), \cdots, X_{t_n}(\omega))$.

For example, $X_t = \xi t$, for $P(\xi = 1) = P(\xi = 2) = 1/2$ has only two possible trajectories, since the only source of randomness if ξ. Note that for t_1, t_2,

$$P(X_{t_1} \leq x_1, X_{t_2} \leq x_2) = \begin{cases} 0 & min(x_1/t_1, x_2/t_2) < 1 \\ 1/2 & min(x_1/t_1, x_2/t_2) \in [1, 2) \\ 1 & min(x_1/t_1, x_2/t_2) \in [2, \infty) \end{cases}$$

1.3 Renewal processes

Let $S_0 = 0$, $S_n = S_{n-1} + \xi_n$, where $\xi_i > 0$ IID. Letting $N_t := \text{argmax}_k\{S_k \leq t\}$,

$$F \to \mathbb{E}N_t$$

For $X \perp Y$, we have the **convolution**

$$F_{X+Y}(x) = F_X * F_Y := \int_{\mathbb{R}} F_X(x-y)dF(y)$$

Theorem. *For $S_n = S_{n-1} + \xi_n$, ξ_i IID,*

- $u(t) = \sum_{n=1}^{\infty} F^{n*}(t) < \infty$
- $u(t) = \mathbb{E}N_t$

1.4 Laplace transform

For $f : \mathbb{R}_+ \to \mathbb{R}$, **Laplace transform** is defined as

$$\mathcal{L}_f(s) = \int_0^\infty e^{-sx} f(x) dx$$

- f : density of ξ, then $\mathcal{L}_f(s) = m(s) = \mathbb{E}[e^{-s\xi}]$
- $\mathcal{L}_{f_1 * f_2}(s) = \mathcal{L}_{f_1}(s)\mathcal{L}_{f_2}(s)$
- F : $\mathcal{L}_F(s) = \mathcal{L}_p(s)/s$

For the last property,

$$LHS = -\int_{\mathbb{R}_+} F(x) \frac{d(e^{-sx})}{s} = -\frac{F(x)e^{-sx}}{s} \|_0^\infty + \frac{1}{s} \int_{\mathbb{R}_+} p(x)e^{-sx} dx = \frac{1}{s} \int_{\mathbb{R}_+} p(x)e^{-sx} dx$$

Consider

$$F \to \mathbb{E}N_t^-$$

where

$$\mathbb{E}N_t = u(t) = \sum_{n=1}^\infty F^{n*}(t)$$

$$= F(t) + \sum_{n=1}^\infty F^{n*}(t) * F(t)$$

$$u = F + u * F = F + u * p$$

where $\int_{\mathbb{R}} u(x-y) dF(y) = \int_{\mathbb{R}} u(x-y) p(y) dy$

Note that

$$\begin{aligned}\mathcal{L}_u(s) &= \mathcal{L}_F(s) + \mathcal{L}_u(s) \cdot \mathcal{L}_p(s) \\ &= \frac{\mathcal{L}_p(s)}{1 - \mathcal{L}_p(s)}\end{aligned}$$

So we can follow

- $F \to \mathcal{L}_p$
- $\mathcal{L}_p \to \mathcal{L}_u$
- $\mathcal{L}_u \to u$

where the inverse Laplace transform can be obtained using Bromwich integral

For example, let $\{S_n\}_{n=1}^{\infty}$ to be

$$S_n := S_{n-1} + \xi_n$$

where $\xi_i \sim p(x) = e^{-x}/2 + e^{-2x}, x > 0$. Then $\mathbb{E}(N_t^-)$?

$p \to \mathcal{L}(p)$.

$$\mathcal{L}(p) = \frac{1}{2(s+1)} + \frac{1}{s+2} = \frac{3s+4}{2(s+1)(s+2)}$$

1.5 Limit theorems for renewal processes

Consider a stochastic process $\{S_n\}_{n=1}^{\infty}$,

$$S_n := S_{n-1} + \xi_n$$

for IID $\xi_i > 0$.

Theorem 1. *Assume* $\mu := \mathbb{E}(\xi_1) < \infty$. *Then*

$$N_t/t \overset{t\to\infty}{\to} 1/\mu, a.s.$$

This is analogous to ***SLLN***, *where*

$$(\xi_1 + \cdots + \xi_n)/n \overset{t\to\infty}{\to} \mu, a.s.$$

Proof

$$S_{N_t} \le t \le S_{N_{t+1}}$$

$$\frac{N_t}{S_{N_{t+1}}} \le \frac{N_t}{t} \le \frac{N_t}{S_{N_t}}$$

where we have

$$\lim_{t\to\infty} \frac{N_t}{S_{N_t}} = \lim_{n\to\infty} \frac{n}{S_n} = 1/\mu$$

$$\lim_{t\to\infty} \frac{N_t}{S_{N_{t+1}}} = \lim_{t\to\infty} \frac{N_t}{N_{t+1}} \frac{N_{t+1}}{S_{N_{t+1}}} = 1/\mu$$

Theorem 2. *Assume* $\sigma^2 = \mathrm{Var}(\xi_1) < \infty$. *Then*

$$Z_t := \frac{N_t - t/\mu}{\sigma\sqrt{t}/\mu^{3/2}} \xrightarrow{d} N(0,1)$$

This is analogous to ***CLT***, *where*

$$\frac{\xi_1 + \cdots + \xi_n - n\mu}{\sigma\sqrt{n}} \xrightarrow{d} N(0,1)$$

Proof

Note that

$$P\left(\frac{S_n - n\mu}{\sigma\sqrt{n}} \leq x\right) \to P(x)$$

Karlin: Renewal processes

1 Renewal Process: Definition and Concepts

A **renewal (counting) process** $\{N(t) : t \geq 0\}$ is a nonnegative integer-valued stochastic process that registers the successive

cf. A **counting process** is a process $\{N(t) : t \geq 0\}$ such that

- $N(t) \geq 0$
- $N(t)$ is an integer
- $s \leq t$, then $N(s) \leq N(t)$

2 Examples

2.1 Poisson Processes

A Poisson process $\{N(t) : t \geq 0\}$ with λ is a renewal counting process having the exponential interoccurrence distribution

$$F(x) = 1 - e^{-\lambda x}, x \geq 0$$

Note that

$$P(W_r > t) = P(N_t \leq r - 1)$$

where W_r is the time taken for rth event, and N_t is the number of events that cumulated until time t. By integral by parts, we can obtain

$$\int_t^\infty \frac{\lambda^r y^{r-1} exp(-\lambda y)}{\Gamma(r)} dy = \sum_{k=0}^{r-1} \frac{exp(-\lambda t(\lambda t)^k)}{k!}$$