Markov Chains

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1 Markov Chains

A discrete time Markov chain $\{X_n\}$ is a Markov stochastic process whose state space is a countable or finite set.

> ***Definition***. *The **Markov chain** $\{X_n\}$ is a stochastic process such that*
>
> $$P(X_n = j | X_{n-1} = i_{n-1}, \cdots, X_0 = i_0) = P(X_n = j | X_{n-1} = i_{n-1})$$
>
> *for any $i_0, \cdots, i_{n-1}, j \in \mathcal{S}$, the state space.*

We have

$$\begin{aligned} P\left[X_0 = i_0, \cdots, X_n = i_n\right] &= P[X_n = i_n | X_0 = i_0, \cdots, X_{n-1} = i_{n-1}] \\ &\cdot P(X_0 = i_0, \cdots, X_{n-1} = i_{n-1}) \\ &= P(X_n = i_n | X_{n-1} = i_{n-1}) \cdots P(X_1 = i_1 | X_0 = i_0) P(X_0 = i_0) \end{aligned}$$

Chapman-Kolmogorov Equation

Definition. *A stochastic process $\{X_t\}_a^b$ is said to satisfy the Markov property if for any $a \leq t_1 < \cdots < t_n < t \leq b$, the equality*

$$P(X_t \leq x | X_{t_1}, \cdots, X_{t_n}) = P(X_t \leq x | X_{t_n})$$

holds for any $x \in \mathbb{R}$, or equivalently, the equality

$$P(X_t \leq x | X_{t_i} = y_i, i = 1, \cdots, n) = P(X_t \leq x | X_{t_n} = y_n)$$

holds for any $y_i \in \mathbb{R}$.

Lemma. *Suppose a stochastic process X_t, $a \leq t \leq b$ is adapted to a filtration $\{\mathcal{F}_t : a \leq t \leq b\}$ and satisfies the condition*

$$P(X_t \leq x | \mathcal{F}_s) = P(X_t \leq x | X_s), \forall s < t, x \in \mathbb{R}$$

then X_t is a Markov process

Proof. Let $t_1 < t_2 < \cdots < t_n < t$ and $x \in \mathbb{R}$. Then

$$\begin{aligned} P(X_t \leq x | X_{t_1}, \cdots, X_{t_n}) &= \mathbb{E}\left[P(X_t \leq x | \mathcal{F}_{t_n}) | X_{t_1}, \cdots, X_{t_n}\right] \\ &= \mathbb{E}\left[P(X_t \leq x | X_{t_n}) | X_{t_1}, \cdots, X_{t_n}\right] \quad = P(X_t \leq x | X_{t_n}) \end{aligned}$$

Define the conditional probability $P_{s,x}(t, dy) := P(X_t \in dy | X_s = x)$ a **transition probability** of a Markov process X_t.

Definition. *The equality is called* ***Chapman-Kolmogorov equation***:

$$P_{s,x}(t, A) = \int_{-\infty}^{\infty} P_{u,z}(t, A) P_{s,x}(u, dz)$$

for all $s < u < t$, $x \in \mathbb{R}$, *and* $A \in \mathcal{B}(\mathbb{R})$, *the Borel field of* $\mathbb{R}$.

This is due to

$$P(X_{t_1} \leq c_1, \cdots, X_{t_n} \leq c_n) = \int_{-\infty}^{c_1} \cdots \int_{-\infty}^{c_n} P_{t_{n-1}, x_{n-1}}(t_n, dx_n)$$
$$\times \cdots P_{t_1, x_1}(t_2, dx_2) \nu(dx_1)$$

1.1 Examples of Markov chains

Spatially homogeneous Markov chains

Let $\xi_1, \xi_2, \cdots$ IID samples such that $P(\xi = i) = a_i$. Let $\eta_n := \sum_{i=1}^{n} \xi_i$

$$P = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & \cdots \\ 0 & a_0 & a_1 & a_2 & \cdots \\ 0 & 0 & a_0 & a_1 & \cdots \\ \vdots & & & & \end{bmatrix}$$

Note that

$$P(X_{n+1} = j | X_n \ i) = \begin{cases} a_{j-i} & j \geq i \\ 0 & j < i \end{cases}$$

One-dimensional random walks

$$P = \begin{bmatrix} r_0 & p_0 & 0 & 0 & \cdots \\ q_1 & r_1 & p_1 & 0 & \cdots \\ 0 & q_2 & r_2 & p_2 & \cdots \\ \vdots & & & & \end{bmatrix}$$

Note that $P(X_{n+1} = i+1|X_n = i) = p_i$, $P(X_{n+1} = i-1|X_n = i) = q_i$, $P(X_{n+1} = i|X_n = i) = r_i$

Gambler's ruin

Let

- $N - i$: Casino's initial wealth
- i : Gambler's initial wealth

For $S := \{0, 1, \cdots, N\}$

$$P_{i,i+1} = p, P_{i,i-1} = 1 - p$$

$$P_{0,0} = 1, P_{N,N} = 1$$

We have the state spaces

$$S = \{0\} \cup \{1, \cdots, N-1\} \cup \{N\}$$

Two-dimensional random walks

Three-dimensional random walks

Success runs

Branching processes

1.2 Classifications of states of a Markov chains

Accessibility

State j is **accessible** from state i if $\exists n \geq 0$ such that $P_{ij}^n > 0$. Two states i and j, each accessible to the other, are saide to **communicate**. If i and j do not communicate, then either

$$P_{ij}^n = 0, \forall n \geq 0 \quad or \quad P_{ji}^n = 0, \forall n \geq 0$$

The properties of **communicaiton** as an **equivalence relation**:

- (**Reflexivity**) : a consequence from the definition of $P_{ij}^0 = \delta_{ij}$
- (**Symmetry**) : a consequence from the definition
- (**Transitivity**)

(**Proof of transitivity**). $i \leftrightarrow j$ and $j \leftrightarrow k$ imply that there exist $n, m \in \mathbb{N}$ such that $P_{ij}^n > 0$ and $P_{jk}^m > 0$. Thus

$$P_{ij}^{n+m} = \sum_{r=0}^{\infty} P_{ir}^n P_{rk}^m \geq P_{ij}^n P_{jk}^m > 0$$

And the similar argument shows the opposite way.

We can say all the states are equivalent if there is no inner loop comming back to the state with probability 1.

Periodicity

Period of state i, $d(i)$ is the greatest common divisor(GCD) of all integers $n \geq 1$ where $P_{ii}^n > 0$, i.e.

$$d(i) \equiv GCD\{n : P_{ii}^n > 0\}$$

If $d(i) = 1$, then the state i is called **aperiodic**.

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

Theorem 4.1. *Periodicity is a **class property**, i.e, if $i \leftrightarrow j$ then*

$$d(i) = d(j)$$

Theorem 4.2.. *If state i has period $d(i)$ then there exists an integer N depending on i such that $\forall n \geq N$,*

$$P_{ii}^{nd(i)} > 0$$

which asserts that a return to state i can occur at all sufficiently large multiples of the period $d(i)$.

Recurrence

Let us define

$$f_{ii}^n := P(X_n = i, X_{n-1} \neq i, \cdots, X_1 \neq 1 | X_0 = i)$$

Note that

$$P_{ii}^n := \sum_{k=0}^{n} f_{ii}^k P_{ii}^{n-k}$$

where $f_{ii}^0 := 0$.

***Definition**. The **generating function** $P_{ij}(s)$ of the sequence $\{P_{ij}^n\}$ is*

$$P_{ij}(s) = \sum_{n=0}^{\infty} P_{ij}^n s^n, \forall |s| < 1$$

In a similar manner, the generating function of the sequence $\{f_{ij}^n\}$ is

$$F_{ij}(s) = \sum_{n=0}^{\infty} f_{ij}^n s^n, \forall |s|$$

We say a state i is **recurrent** if and only if $\sum_{n=1}^{\infty} f_{ii}^n = 1$. i.e. it has **finite first return time**, a.s., where the first return time can be defined

$$\tau_{ii} := \begin{cases} min(n \geq 1 : X_n = i | X_0 = i) & \\ \infty & P(X_n = i | X_0 = i) = 0 \end{cases}$$

where $\sum_{n=1}^{\infty} f_{ii}^n = Pr(\tau_{ii} < \infty)$.

Theorem 5.1.. *A state i is recurrent if and only if*

$$\sum_{n=1}^{\infty} P_{ii}^n = \infty$$

Proof Assume i is recurrent, that is, $\sum_{n=1}^{\infty} f_{ii}^n = 1$. Then by Lemma 5.1,

$$\lim_{s \to 1-} \sum_{n=0}^{\infty} f_{ii}^n s^n = \lim_{s \to 1-} F_{ii}(s) = 1$$

Thus using the fact that

$$\lim_{s \to 1-} P_{ii}(s) = \lim_{s \to 1-} \sum_{n=0}^{\infty} P_{ii}^n s^n = \infty$$

Corollary 5.1.. *If $i \leftrightarrow j$ and if i is recurrent then j is recurrent.*

i.e. **recurrence is the class property**.

Fact 1. *In irreducible MCs, all states are either* ***recurrent*** *or* ***transient***.

Fact 2. *In* ***finite***, *irreducible MCs, all states are either* ***recurrent***.

For example,

- An asymmetric 1D random walk is **transient**, since there is a probability of absorption
- A symmetric 1D random walk is **aperiodic**, since it has period 2.
- In **gambler's ruin**, the states are **irreducible**

1.3 Ergodic theorem

Ergodic theorem. *Let X_t irreducible, ergodic(recurrent and aperiodic) Markov chain. Then*

$$\exists \lim_{n\to\infty} P_{ij}(n) = \pi_j^* > 0$$

$$\sum_{j=1}^{M} \pi_j^* = 1$$

Corollary1. $\pi P = \pi$

Corollary2. $\lim_{n\to\infty} P(X_n = j) = \pi_j$

where **corollary2** is equivalent to

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} I(X_k = j | X_0 = j) = \pi_j$$

We have the following additional theorem:

***Theorem 1.1.*.** *Let $\{a_k\}$, $\{u_k\}$, $\{b_k\}$ be sequences indexed by $k = 0, \pm 1, \pm 2, \cdots$ that the GCD of the integer k for which $a_k > 0$ is 1. If the renewal equation, for $n = 0, \pm 1, \pm 2, \cdots$*

$$u_n - \sum_{k=-\infty}^{\infty} a_{n-k} u_k = b_n$$

is satisfied by a bounded sequence $\{u_n\}$ *of real numbers, then*

$$\exists \lim_{n\to\infty} u_n, \exists \lim_{n\to-\infty} u_n$$

. Furthermore, if

$$\lim_{n\to-\infty} u_n = 0$$

then

$$\lim_{n\to\infty} u_n$$

Quizzes

(Quiz 1). Find the stationary distribution of the following P:

$$P = \begin{bmatrix} 0 & 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 1 & 0 & 0 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 0 & 1/2 & 0 & 0 & 1/2 \\ 0 & 1/2 & 1/2 & 0 & 0 \end{bmatrix}$$

(Answer) The left-eigenvector corresponding to eigenvalue 1 is as follows:

$$(1/12, 3/12, 5/12, 1/12, 2)$$

(Quiz 2). Jane and Peter are playing chess. For Jane, the probabilities of wining, drawing, and losing a game number t are (w, d, l). Peter is slightly more emotional.

- If he wins in the previous game, then $(w + \epsilon, d, l - \epsilon)$
- If he draws in the previous game, then (w, d, l)
- If he loses in the previous game, then $(w - \epsilon, d, l + \epsilon)$

Find the condition which guarantees that the probability of wining in stationry distribution for Peter is larger than that for Jane.

(Answer) Note that

$$P_{Jane} = \begin{bmatrix} w & d & l \\ w & d & l \\ w & d & l \end{bmatrix} P_{Peter} = \begin{bmatrix} w+\epsilon & d & l-\epsilon \\ w & d & l \\ w-\epsilon & d & l+\epsilon \end{bmatrix}$$

It is direct that $Rank(P^T_{Jane}) = 1$, thus the multiplicity of $\lambda = 0$ is at least 2. Note that $\lambda = 1$ is also an eigenvalue, where the corresponding eigenvector is (w, d, l).

For P^T_{Peter}, we need some algebra to get

$$x_1 = \frac{w(1-\epsilon) - l\epsilon}{1 - 2\epsilon}$$

then by $w < x_1$, we have

$$l < w$$

(Quiz 3).

Is the following P **ergodic**?

$$P = \begin{bmatrix} 0 & 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(Answer) Note that it is **irreducible** and **recurrent**, but not **aperiodic**.

(Quiz 4).

Tell the number of equivalence classes, and all the periodic states of the following transition matrix:

$$P = \begin{bmatrix} 1/3 & 1/3 & 1/3 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 1/4 & 1/4 & 0 & 1/2 \\ 0 & 1/2 & 0 & 1/2 \end{bmatrix}$$

(Answer) Note that it is **irreducible** and **recurrent**. Note that $P_{11} > 0$, thus all the states are aperiodic.

(Quiz 5).

Assume that there is a series of integer numbers, in which numbers $0, 1, \cdots, 9$ appear randomly and independently of each other with equal probabilities. Let x_n be a quantity of different numbers in n first elements of the series. Find a stationary distribution of this chain.

(Answer)

$$P = \begin{bmatrix} 1/9 & 8/9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2/9 & 7/9 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3/9 & 6/9 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4/9 & 5/9 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5/9 & 4/9 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6/9 & 3/9 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 7/9 & 2/9 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8/9 & 1/9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Clearly the left eigenvector corresponding to $\lambda := 1$ is $(0, \cdots, 1)$.