# Ergodicity, Differentiability, & Continuity

Hyunwoo Gu

# 1 Ergodicity

## 1.1 Notion of ergodicity

***Law of Large Numbers***. *Let $\xi_1, \cdots$ IID such that $\mathbb{E}\xi_1 < \infty$. Then*

$$\frac{1}{N}\sum_{n=1}^{N}\xi_n \xrightarrow{P} \mathbb{E}(\xi_1)$$

Let $X_T$ a stochastic process,

$$\frac{1}{N}\sum_{t=1}^{T}X_t \xrightarrow{P} c$$

$$\xi_n \xrightarrow{as} \xi \Leftrightarrow P\left\{w : \xi_n(w) \to \xi(w)\right\} = 1$$

$$\xi_n \xrightarrow{L^2} \xi \Leftrightarrow \mathbb{E}(\xi_n - \xi)^2 \to 0$$

$$\xi_n \xrightarrow{P} \xi \Leftrightarrow \forall\epsilon > 0 P(|\xi_n - \xi| > \epsilon) \to 0$$

$$\xi_n \xrightarrow{d} \xi \Leftrightarrow P(\xi_n \leq x) \to P(\xi \leq x), \forall x \in \mathbb{R}$$

## 1.2 Ergodicity of wide-sense stationary processes

***Proposition.***

*Let $X_t$ with $|K(s,t)| \leq \alpha$ such that $\exists \alpha$.*

$$C(T) := cov(X_T, M_T)$$

$$M_T := \frac{1}{T} \sum_{t=1}^{T} X_t$$

$$Var(M_T) \overset{T \to \infty}{\to} 0 \Leftrightarrow C(T) \overset{T \to \infty}{\to} 0$$

***Corollary.*** *Let $X_t$: weakly stationary, and $\gamma(\cdot)$ : autocovariance function. Then*

- *(i) $\frac{1}{T} \sum_{r=0}^{T-1} \gamma(r) \to 0$ when $T \to \infty$*
- *(ii) $\gamma(r) \to 0$ when $r \to \infty$*

Note that clearly $\mathbb{E}(X_t) = const$ then $cov(X_t, X_s) = \gamma(t-s)$.

**Proof**. I.

$$\mathbb{E}(X_t) = c \Leftrightarrow \mathbb{E}(M_T) = c$$

$$\mathrm{Var}(M_T) = \mathbb{E}\left[(M_T - c)^2\right] \overset{*}{\to} 0$$

$$\Rightarrow M_T \overset{L^2}{\to} c \Rightarrow M_T \overset{P}{\to} c$$

then $X_t$ : ergodic.

For $(*)$ part, check

$$C(T) = cov(X_T, \frac{1}{T}\sum_{t=1}^{T} X_T) = \frac{1}{T}\sum_{t=1}^{T}\gamma(T-t) = \frac{1}{T}\sum_{r=0}^{T-1}\gamma(r) \to 0$$

II. (**Stolz-Cesaro theorem**)

Let $a_b, b_n$ where $b_n$ : strictly increasing & unbounded. Then

$$\lim_{n\to\infty}\frac{a_n - a_{n-1}}{b_n - b_{n-1}} = q \Rightarrow \frac{a_n}{b_n} \overset{n\to\infty}{\to} q$$

Take

$$a_n := \sum_{r=0}^{n-1}\gamma(r)$$

$$b_n = n$$

Note that

$$\frac{a_n - a_{n-1}}{b_n - b_{n-1}} = \gamma(n-1) \to 0 = q$$

For example, $N_t$ : Poisson process with $\lambda$.

For $p > 0$, $X_t := N_{t+p} - N_t$.

$$\mathbb{E}(X_t) = \lambda(t+p) - \lambda t = \lambda p$$

$$K(t,s) = \gamma(t-s)$$

$$\gamma(r) = \begin{cases} \lambda(p - |r|) & |r| \le p \\ 0 & |r| > p \end{cases}$$

$$\gamma(r) \to 0$$

For example, $X_t := Acos(wt) + Bsin(wt)$, where $A, B$ random with $cov(A, B) = 0$

with $w := \pi/20$.

$$\mathbb{E}(A) = \mathbb{E}(B) = 0$$

$$Var(A) = Var(B) = 1$$

$$\mathbb{E}(X_t) = 0$$

$$K(t, s) = cos\left(w(t - s)\right)$$

$$\gamma(r) = cos(wr)$$

Note that it is **NOT true** that a stochastic process is stationary if and only if it is ergodic.

Assume that $X_t$ is weakly statinary and

$$\frac{1}{T}\sum_{r=0}^{T-1}\gamma(r) \to 0$$

but $\gamma(r) \to 0$ when $r \to \infty$. Then $X_t$ is ergodic.

# 2 Stochastic Differentiation

## 2.1 Definition of a stochastic derivative

$X_t$ is differentiable at $t = t_0$, if

$$\frac{X_{t_0+h} - X_{t_0}}{h} \xrightarrow{L^2} \eta := X^1_{t_0}$$

for $h \to 0$.

$$\mathbb{E}\left(\frac{X_{t_0+h} - X_{t_0}}{h} - \eta\right)^2 h \overset{\to}{\to} 00$$

***Proposition***. *Let* $X_t : \mathbb{E}(X_t^2) < \infty$ *Then*

$$X_t : \quad diff \quad t = t_0 \Leftrightarrow \begin{cases} m(t) = \mathbb{E}(X_t) & diff \quad t = t_0 \\ \frac{\partial^2}{\partial t \partial s} K(t,s) & \exists (t_0, t_0) \end{cases}$$

**Ex1**. Let $X_t$ is weakly stationary, i.e., $m(t) = const$, and $K(t,s) = \gamma(t - s)$.

$$\frac{\partial^2 K}{\partial t \partial s}|_{(t_0,t_0)} = -\gamma''(0)$$

For example, $\gamma(r) = e^{-L|r|}$, then $X_t$ is not differentiable.

For another example, $\gamma(r) = cos(\alpha r)$, then $X_t$ is differentiable.

**Ex2**. Brownian motion is not diff at any $t = t_0$.

$$K(t,s) = min(t,s)$$

$$\begin{aligned} K(t_0+h,t_0) - K(t_0,t_0) &= \frac{min(t_0,t_0+h)-t_0}{h} \\ &= \begin{cases} 0 & h>0 \\ 1 & h<0 \end{cases} \end{aligned}$$

thus there does not exist a result $h \to 0$.

**Ex3**. Let $X_t$ : independent increments with $X_0 = 0$. Then

$$K(t,s) = cov(X_t, X_s) \overset{t \geq s}{=} cov(X_t - X_s, X_s) + Cov(X_s, X_s) = var(X_{min(t,s)})$$

## 2.2 Continuity in the mean-squared sense

Let

$$X_t \overset{L^2}{\to} X_{t_0} \Leftrightarrow \mathbb{E}(X_t - X_{t_0})^2 t \overset{\to}{\to} t_0 0$$

such that $\mathbb{E}(X_t) = 0$.

***Proposition.***

- *$K(t,s)$ is const $(t_0,t_0)$, then $X_t$ is const in the mean squared sense $t = t_0$*
- *$X_t$ : const in the mean squared sense $t = t_0, s_0$, then $K(t,s)$ is const at $(t_0, s_0)$.*

**Proof**. (i)

$$\mathbb{E}(X_t - X_{t_0})^2 = K(t,t) - 2K(t,t_0) + K(t_0,t_0) \overset{t \to t_0}{\to} 0$$

(ii)

$$K(t,s) \pm K(t_0,s) - K(t_0,s_0) = K(t,s) - K(t_0,s) + K(t_0,s) - K(t_0,s_0)$$

$$\begin{aligned}|K(t,s) - K(t_0,s)| &= \mathbb{E}\left[(X_t - X_{t_0})X_s\right] \\ &\leq \sqrt{\mathbb{E}(X_t - X_{t_0})^2} \cdot \sqrt{\mathbb{E}X_s^2} \to 0\end{aligned}$$

> ***Corollary***.
>
> $K(t,s)$ *is const at* $t_0, s_0 \Leftrightarrow K(t,s)$ *is const at* $(t_0,t_0)$

**Proof**. Let $K(t,s)$ is const at diagonal, i.e. $(t_0,t_0)$. Then by **(i)**, we have $X_t$ : const at $t = t_0$.

Then by **(i)**, we have $K(t,s)$ is const at $(t_0,s_0)$.

THe following are the correct statements:

- If $K(t,s)$ is continuous at any $(t_0,s_0) \in \mathbb{R}^2$, then $X_t$ is continuous in MSS at $\forall t$.
- If $X_t$ is continuous in MSS at $t_0, s_0$ then $K(t,s)$ is continuous at $(t_0,s_0)$ and $(s_0,t_0)$.
- If $K(t,s)$ is continuous at the diagonal, it is also continuous at any $(t_0,s_0) \in \mathbb{R}^2$

## Quizzes

**(Quiz 1)**. Let $X_t := cos(wt + \theta)$ be a stochastic process and $\theta \sim Unif(0,2\pi)$, with $w = \pi/10$. Classify this process.

**(Answer) Ergodic** and **weak stationary**.

**(Quiz 2)**. Let $X_t := \epsilon_t + \xi cos(\pi t/12)$, $t = 1, 2, \cdots$, where $\xi, \epsilon_1, \epsilon_2, \cdots$ are IID standard normal random variables.

**(Answer)** Not **weak stationry**, but **ergodic**.

**(Quiz 3)**. Assume that for a process $X_t$ it is known that $\mathbb{E}(X_t) = \alpha + \beta t$, $cov(X_t, X_{t+h} = e^{-h\lambda}$ for all $h \geq 0, t > 0$, and some constants $\lambda > 0, \alpha, \beta$. Classify the process $Y_t := X_{t+1} - X_t$.

**(Answer)** $Y_t$ is weakly stationary and ergodic.

**(Quiz 4)**. Let $X_t := \sigma W_t + ct$, where $W_t$ is Brownian motion, $\sigma, c > 0$. Choose the correct statements about this process.

**(Answer)** $X_t$ has **continuous trajectories**.

**(Quiz 5)**. Let $X_t$ have an autocovariance function $\gamma(r) := e^{-\alpha|r|}$. Is $Y_t := X_t + w$ an ergodic process?

**(Answer)** Yes, if $w$ is a constant.