Limit Theorems for Renewal Processes

Hyunwoo Gu

1 Definitions and Related Concepts

$$\mathbb{E}N(t) = M(t)$$

where $M(t)$ is called the **renewal function**.

2 Examples

2.1 Poisson Processes

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3.1 Poisson Processes

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Renewal Equations

An integral equation of the following form is called a **renewal equation**.

$$A(t) = a(t) + \int_0^t A(t-x)dF(x), \quad t \geq 0$$

Theorem 4.1. *Suppose a is a bounded function. Then there uniquely exists A bounded on finite intervals satisfying*

$$A(t) = a(t) + \int_0^t A(t-x)dF(y)$$

Namely,

$$A(t) = a(t) + \int_0^t a(t-x)dM(x)$$

where $M(t) := \sum_{k=1}^{\infty} F_k(t)$: the renewal function.

Proof. We verify first that A specified fulfilles the requisite boundedness properties and solves

$$\begin{aligned} A &= a + F * a + F_2 * (a + F * A) \\ &= a + F * a = F_2 * a + F_3 * A \\ &= a + \left(\sum_{k=1}^{n-1} F_k \right) * a + F_n * A \end{aligned}$$

Next observe that

Theorem 4.2. *Let $\{X_t\}$ be a renewal process with $\mu = \mathbb{E}X_1 < \infty$. Then*

$$\lim_{t\to\infty} \frac{1}{t} M(t) = \frac{1}{\mu}$$

Proof. Note that $t < S_{N(t)+1}$. Thus

$$t < \mathbb{E}\left[S_{N(t)+1}\right] = \mu\left[1 + M(t)\right]$$

and therefore

$$\frac{1}{t}M(t) > \frac{1}{\mu} - \frac{1}{t}$$

It follows that

$$\text{liminf}_{t\to\infty}\frac{1}{t}M(t) \geq \frac{1}{\mu}$$

To establish the opposite inequality, let $c > 0$ be arbitrary, and set

$$X_i^c = \begin{cases} X_i & X_i \leq c \\ c & X_i > c \end{cases}$$

and consider the renwal process having lifetimes $\{X_i^c\}$. Let $\{S_n^c\}$ and $\{N^c(t)\}$ denote the waiting times and counting process, respectively, for this **truncated renewal process** generated by $\{X_i^c\}$. Since the random variables X_i^c are uniformly bounded by c, it is clear that $t + c \geq S^c_{N^c(t)+1}$, and therefore

$$t + c \geq \mathbb{E}\left[S^c_{N^c(t)+1}\right] = \mu^c\left[1 + M^c(t)\right]$$

where

From $N^c(t) \geq N(t)$ from $X_i^c \leq X_i$, and $M^c(t) \geq M(t)$, we have

$$t + c \geq \mu^c \left[1 + M(t)\right]$$

$$\therefore \quad \frac{1}{t} M(t) \leq \frac{1}{\mu_c} + \frac{1}{t}\left(\frac{c}{\mu^c} - 1\right)$$

Hence

$$\text{limsup}_{t\to\infty} \frac{1}{t} M(t) \leq \frac{1}{\mu^c}, \quad \forall c > 0$$

since

$$\begin{aligned} \text{lim}_{c\to\infty} \mu^c &= \text{lim}_{c\to\infty} \int_0^c \left[1 - F(x)\right] dx \\ &= \int_0^\infty [1 - F(x)] dx = \mu \end{aligned}$$

while the left-hand side is fixed, we deduce that

$$\text{limsup}_{t\to\infty} \frac{1}{t} M(t) \leq \text{lim}_{c\to\infty} \frac{1}{\mu^c} = \frac{1}{\mu}$$

5 The Renewal Theorem

Recall that

$$M(t) = \mathbb{E}[N(t)] = \sum_{j=1}^{\infty} F_j(t)$$

where $F_j(t) = Pr[S_j \leq t]$ for $t \geq 0$. We call $M(t)$ a **renewal function**.

Let us now show that the renewal function $M(t)$ satisfies the equation

$$M(t) = F(t) + \int_0^t M(t-y)dF(y), \quad t \geq 0$$

$$\Leftrightarrow M(t) = F(t) + F * M(t), \quad t \geq 0$$

$$M(t+h) - M(t) \to \frac{h}{\mu}, \quad t \to \infty$$

In words, the expected number of renewals in an interval with length h i sapproximately h/μ, provided the process has been in operation for a long duration.

Definition 5.1. *A point α of a distribution function F is called a* ***point of increase*** *if for a positive ϵ*

$$F(\alpha + \epsilon) - F(\alpha - \epsilon) > 0$$

A distribution function is **arithmetic** if there exists $\lambda > 0$ such that F exhibits points of increase exclusively among $0, \pm\lambda, \pm 2\lambda, \cdots$. The largest such λ is called the **span** of F.

Note that F that has a continuous part is **not arithmetic**. The distribution function of a discrete RVs having posiible values $0, 1, 2, \cdots$ is arithmetic with span 1.

Definition 5.2. *A point α of a distribution function F is called a* ***point of increase*** *if for a positive ϵ*

$$F(\alpha + \epsilon) - F(\alpha - \epsilon) > 0$$

Every monotonic function g which is absolutely integrable in the sense that

$$\int_0^\infty |g(t)|dt < \infty$$

is **directly Riemann integrable**. Manifestly, all finite linear combinations of monotone functions satisfying the above are also directly Riemann integrable.

Theorem 5.1. (The Basic Renewal Theorem). *Let F be the distribution function of a positive random variable with mean μ. Suppose that a is directly Riemann integrable and that A is the solution of the renewal equation*

$$A(t) = a(t) + \int_0^t A(t-x)dF(x)$$

(i) If F is not arithmetic, then

$$\lim_{t\to\infty} A(t) = \begin{cases} \frac{1}{\mu}\int_0^\infty a(x)dx & \mu < \infty \\ 0 & \mu = \infty \end{cases}$$

(ii) If F is arithmetic with span λ, then $\forall c > 0$,

$$\lim_{t\to\infty} A(t) = \begin{cases} \frac{\lambda}{\mu}\sum_{n=0}^\infty a(c+n\lambda) & \mu < \infty \\ 0 & \mu = \infty \end{cases}$$

In a simpler form, we say that

6 Generalizations and Variations of Renewal Theorem

6.1 Stationary renewal processes

A delayed renewal process for which the first life has the following distribution function is called a **stationary renewal process**.

$$G(x) = \mu^{-1} \int_0^x [1 - F(x)] dy$$

6.2 Cumulative and Related Processes

Suppose

6.2.1 Renewal Processes Involving Two Components to Each Renewal Interval

Interpreting Y_i as a cost or value associated with the ith renewal cycle.

Terminating renewal processes

Suppose we allow the possibility of infinite interoccurence

Thus we have

$$\begin{aligned} \lim_{t\to\infty} \hat{A}(t) &= \lim_{t\to\infty} e^{\lambda t}[M(\infty) - M(t)] \\ &= \left\{ \lambda \int_0^\infty x e^{\lambda x} dF(x) \right\}^{-1} \end{aligned}$$

Combining

An obvious generalization of integer-valued inter-renewal intervals is that of inter-renewals that occur only at integer multiples of some real number $\lambda > 0$. Such a distribution is called an **arithmetic distribution**. The **span** of an arithmetic distribution is the largest number λ s.t. this property holds.

7 Applications of the Renewal Theorem

7.1 Limiting Distribution of the Excess Life

7.2 Asymptotic Expansion of the Renewal Function

Suppose F is a nonarithmetic distribution with a finite variance σ^2. Under these assumptions we will determine the second term in the asymptotic expansion of $M(t)$ by proving

$$\lim_{t\to\infty}\{M(t) - \mu^{-1}t\} = \frac{\sigma^2 - \mu^2}{2\mu^2}$$

7.3 Delayed Renewal Processes

$$\begin{aligned}
M_D(t) &= \int_0^\infty \mathbb{E}[N(t)|X_t = x]dG(x) \\
&= \int_0^t \{1 + M(t-x)\}dG(x) \\
&= G(t) + \int_0^t M(t-x)dG(x) \\
&= G(t) + \int_0^t G(t-x)dM(x)
\end{aligned}$$

7.4 Stationary Renewal Processes

A delayed renewal process for which the first life has the distribution function

$$G(x) = \mu^{-1} \int_0^x \{1 - F(y)\}dy$$

7.5 Alternating and Markov Processes

An **alternating renewal process** is a sequence $Y_1, Y_2, \cdots$ of independent RVs, where

$$\begin{aligned}
Y_1, Y_{r+1}, Y_{2r+1}, \cdots &\sim F_1 \\
Y_2, Y_{r+2}, Y_{2r+2}, \cdots &\sim F_2 \\
&\cdots \vdots \cdots \\
Y_r, Y_{2r}, Y_{3r}, \cdots &\sim F_r
\end{aligned}$$

7.6 Central Limit Theorem for Renewals

7.7 Characterization of the Poisson Process

Poisson process is a special type of a renewal process. Let $\{X_k\}$ be a renewal process with $\mathbb{E}[X_k] := \mu < \infty$ and $F(x) = P\{X_k \leq x\}$. Assuming $F(0) = 0$, define

$$F_t(x) = \begin{cases} F(x) & 0 \leq x < t \\ 1 & t \leq x \end{cases}$$

which is basically the distribution function for $min\{X_k, t\}$

***Theorem 8.1.**. (a) If there exists a sequence $\{t_j\}$, where $t_j \to \infty$ as $j \to \infty$, and for which the current life δ_t satisfies*

$$F_{t_j}(x) = Pr[\delta_{t_j} \leq x], \forall x$$

*then F is an **exponential distribution**.*

(b) If there exists a sequence $\{t_j\}$, where $t_j \to \infty$ as $j \to \infty$, and for which

$$F(x) = Pr[\gamma_{t_j} \leq x], \forall x$$

*then F is an **exponential distribution**.*

Proof. By the result of (6.5), i.e.

the limiting distribution of the current life δ_t i s

$$\lim_{t\to\infty} Pr\{\delta_t > y\} = \mu^{-1} \int_y^\infty [1 - F(z)]dz$$

Letting t increase along t_j with due account of the hypothesis of the theorem, we derive the functional equation

$$1 - F(y) = \mu^{-1} \int_y^\infty [1 - F(z)]dz$$

The right-hand side is clearly differentiable in y, yielding the elementary first-order DE

$$\frac{d}{dy}[1 - F(y)] = -\frac{1}{\mu}[1 - F(y)]$$

whose solution, subject to $F(0) = 0$, is

$$1 - F(y) = e^{-\lambda y}, \lambda := 1/\mu$$

8 Superposition of Renewal Processes

Definition 9.1. *The triangular array $\{N_{ni}(t)\}$ is called* ***infinitesimal*** *if for every $t \geq 0$,*

$$\lim_{n\to\infty} \max_{1\leq i\leq k_n} F_{ni}(t) = 0$$

Theorem 9.1. *Let $\{N_{ni}(t)\}$ be an infinitesimal array of renewal processes with superposition $N_n(t)$. Then*

$$\lim_{n\to\infty} Pr\{N_n(t) = j\} = \frac{e^{-\lambda t(\lambda t)^j}}{j!}, \quad j = 0, 1, 2, \cdots$$

if and only if

$$\lim_{n\to\infty} \sum_{i=1}^{k_n} F_{ni}(t) = \lambda t$$

Proof.

(1) *Necessity.* For $j = 0$, we obtain

$$\lim_{n\to\infty} Pr\{N_t(t) = 0\} = e^{-\lambda t}$$

or, equivalently,

(1) *Sufficiency.* Let us follow an induction on m to show

$$\lim_{n\to\infty} Pr\{N_n(t) = m\} = \frac{e^{-\lambda t(\lambda t)^m}}{m!}, \quad m = 0, 1, 2, \cdots$$

For $m = 0$,

Example 1. Suppose $F(t)$ is a distribution function for which $F(0) = 0$, $F'(0) = \lambda > 0$. Let

$$F_{ni}(t) = F(t/n), \quad i = 1, \cdots, n$$

and, for all n, let $N_{ni}(t), i = 1, \cdots, n$ be independent renewal counting processes with interoccurence distribution F_{ni}. Then $N_{ni}(t)$ is a triangular array. Furthermore, since

Hence, the distribution of the superposition $N_n(t)$ converges to the Poisson process.

Theorem 7.1. *Let $\{X_n\}$ be a renewal process for which $\mu = \mathbb{E}[X_1] < \infty$, $\sigma^2 = \mathbb{E}[(X_1 - \mu)^2] < \infty$. Then*

$$\lim_{t\to\infty} Pr\left\{\frac{N(t) - t/\mu}{\sqrt{t\sigma^2/\mu^3}} < x\right\} = \Phi(x)$$

where $\Phi(\cdot)$: standard normal integral.

Theorem 9.2. *Let $N_1(t)$ and $N_2(t)$ be two independent renewal processes with the same interoccurrence distribution F having mean μ. Let $N(t) = N_1(t) + N_2(t)$. If $N(t)$ is also a renewal process, then $N_1(t), N_2(t), N(t)$ are all Poisson.*

Proof. Let H be the interoccurrence distribution for $N(t)$. Then

$$\begin{aligned} 1 - H(x) &= Pr[N(x) = 0] \\ &= Pr[N_1(x) = 0, N_2(x) = 0] \\ &= [1 - F(x)]^2 \end{aligned}$$

Let

$$\frac{1}{v}\int_x^\infty [1 - H(y)]dy = \frac{1}{\mu^2}\left\{\int_x^\infty [1 - F(y)]dy\right\}^2$$

where $v := \int_0^\infty [1 - H(y)]dy$. Both sides are differentiable with respect to x, and earlier we noted $1 - H(x) = [1 - F(x)]^2$.