Continuous Probability Distribution

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Objectives

- Identify the properties of the normal distribution.
- Find the area under the standard normal distribution given various *z* values.
- Find probabilities for a normally distributed variable by transforming it into a standard normal variable.

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Objectives

- Find specific data values for given percentages using the standard normal distribution.
- Use the Central Limit Theorem to solve problems involving sample means.
- Solve problems involving exponential distribution.

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Mathematical Equation for the Normal Distribution

The mathematical equation for the normal distribution:

$$y = \frac{e^{-(x-\mu)^2/2\sigma^2}}{\sigma\sqrt{2\pi}}$$

where

$e \approx 2.718$

$\pi \approx 3.14$

μ = *population mean*

σ = *population standard deviation*

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Properties of the Theoretical Normal Distribution

- The normal distribution curve is **bell-shaped**.
- The mean, median, and mode are equal and located at the **center** of the distribution.
- The normal distribution curve is **unimodal** (single mode).

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Properties of the Theoretical Normal Distribution

- The curve is symmetrical about the mean.
- The curve is continuous.
- The curve never touches the *x*-axis.
- The total area under the normal distribution curve is equal to 1.

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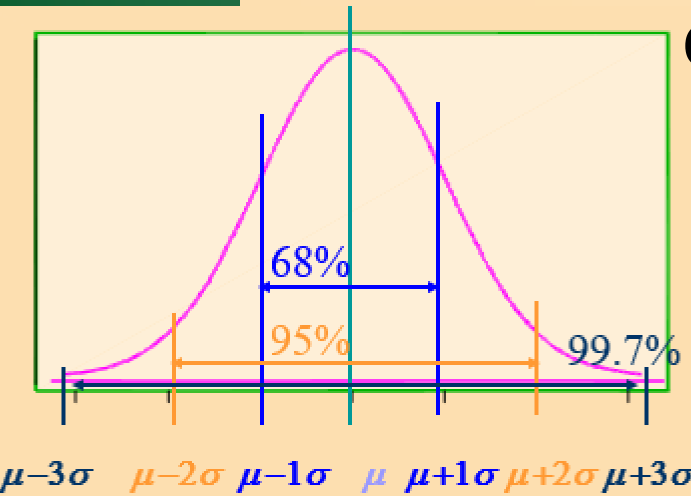
Properties of the Theoretical Normal Distribution



- The area under the normal curve that lies within
  - ✓ one standard deviation of the mean is approximately 0.68 (68%).
  - ✓ two standard deviations of the mean is approximately 0.95 (95%).
  - ✓ three standard deviations of the mean is approximately 0.997 (99.7%).

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The Standard Normal Distribution

$$z = \frac{value - mean}{standard\ deviation}$$

or

$$z = \frac{X - \mu}{\sigma}$$

- The **standard normal distribution** is a normal distribution with a mean of 0 and a standard deviation of 1.
- All normally distributed variables can be transformed into the standard normally distributed variable by using the formula for the standard score.

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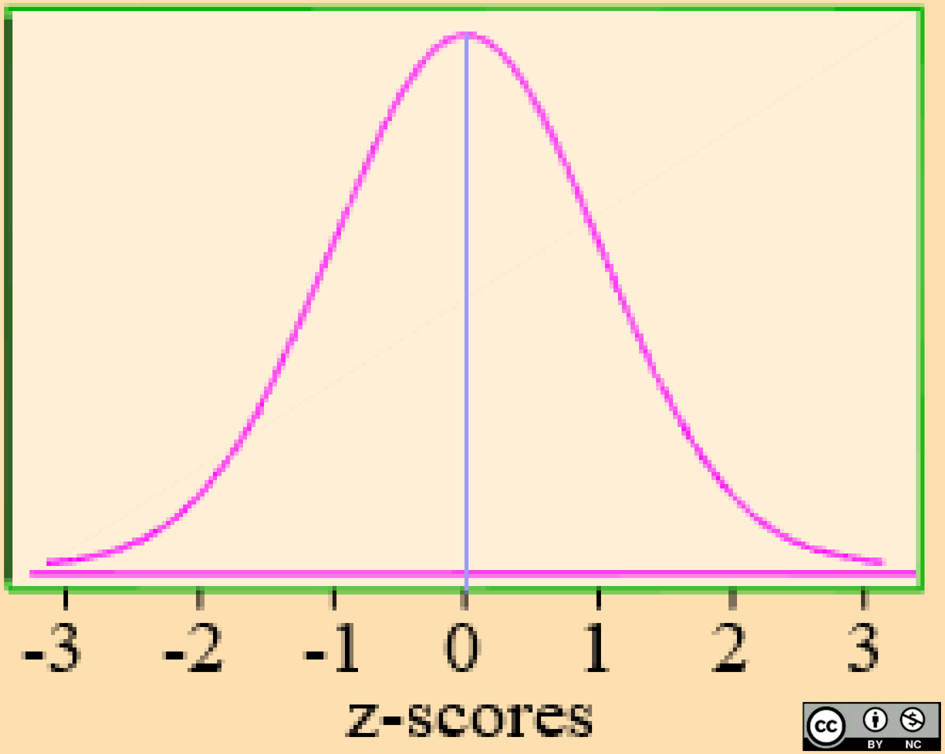
Area Under the Standard Normal Curve - Example

Example: Find the area under the standard normal curve between $z = 0$ and $z = 2.34$

$\Rightarrow P(0 < z < 2.34)$

Ans: 0.4904

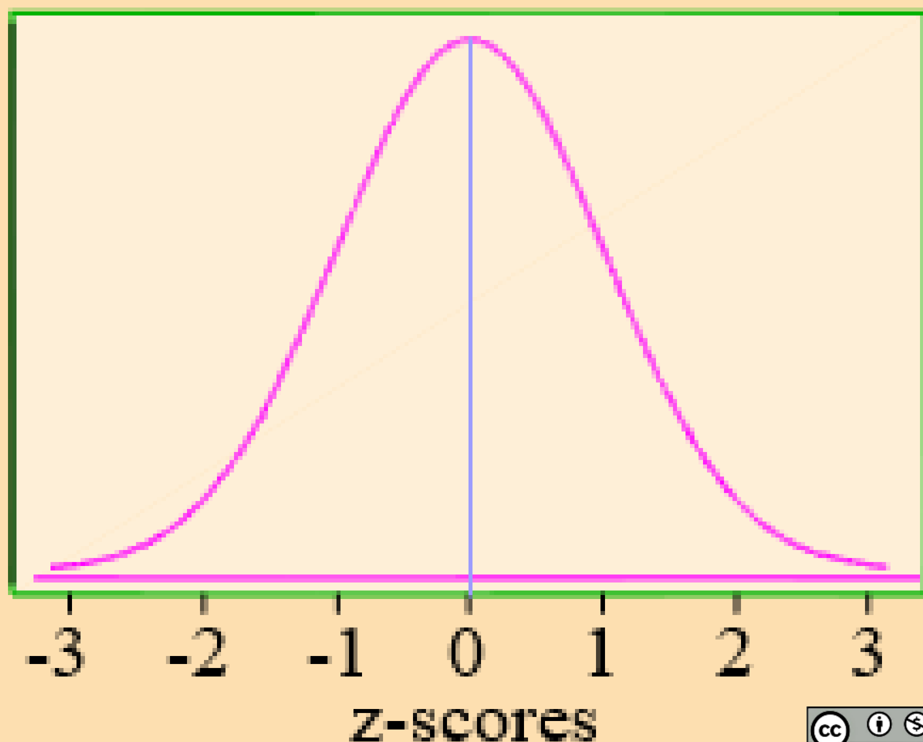


Area Under the Standard Normal Curve - Example

Example: **Find the area under the standard normal curve between $z = 0$ and $z = -1.75$**

$\Rightarrow P(-1.75 < z < 0)$

Ans. **0.4599**

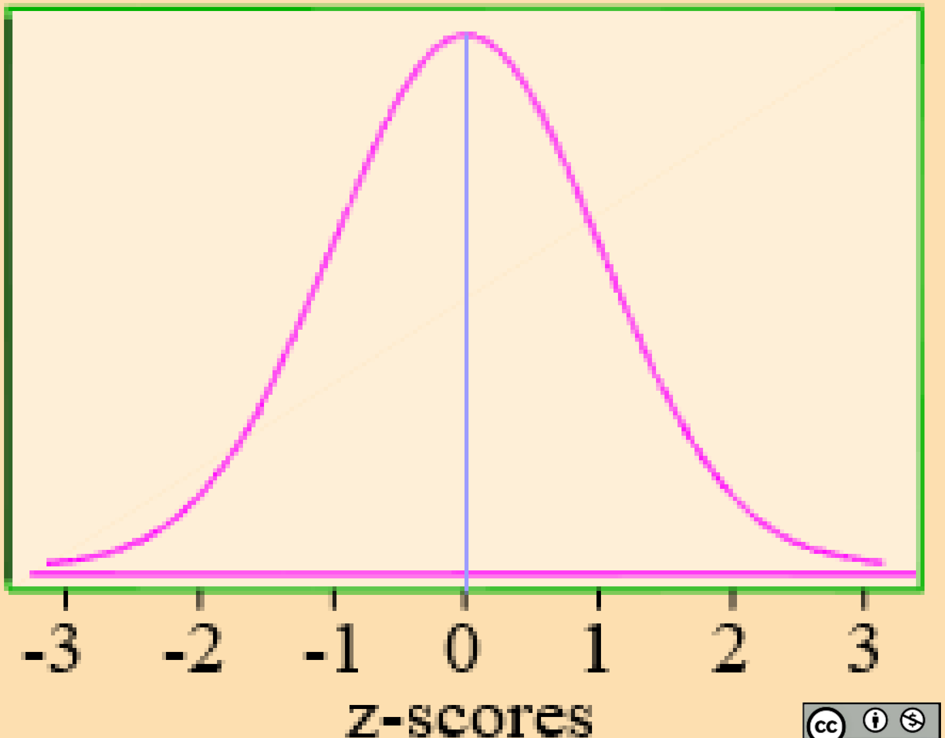


Area Under the Standard Normal Curve - Example

Example: Find the area to the right of $z = 1.11$

$\Rightarrow P(z > 1.11)$

Ans. 0.1335



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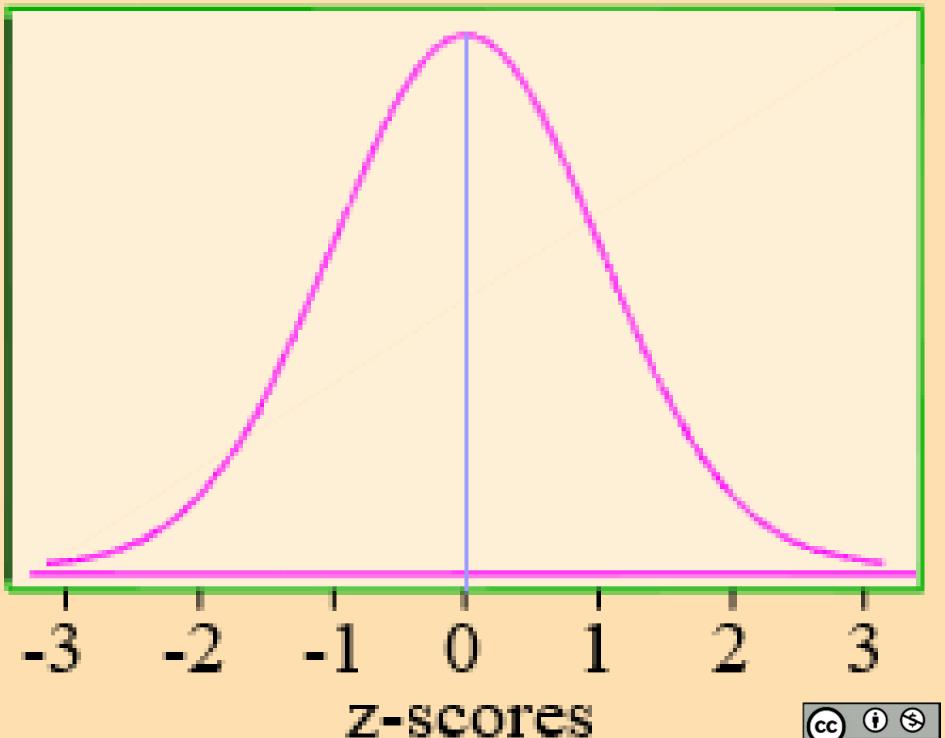
Area Under the Standard Normal Curve - Example

Example: Find the area between $z = 2$ and $z = 2.47$ $\Rightarrow P(2 < z < 2.47)$

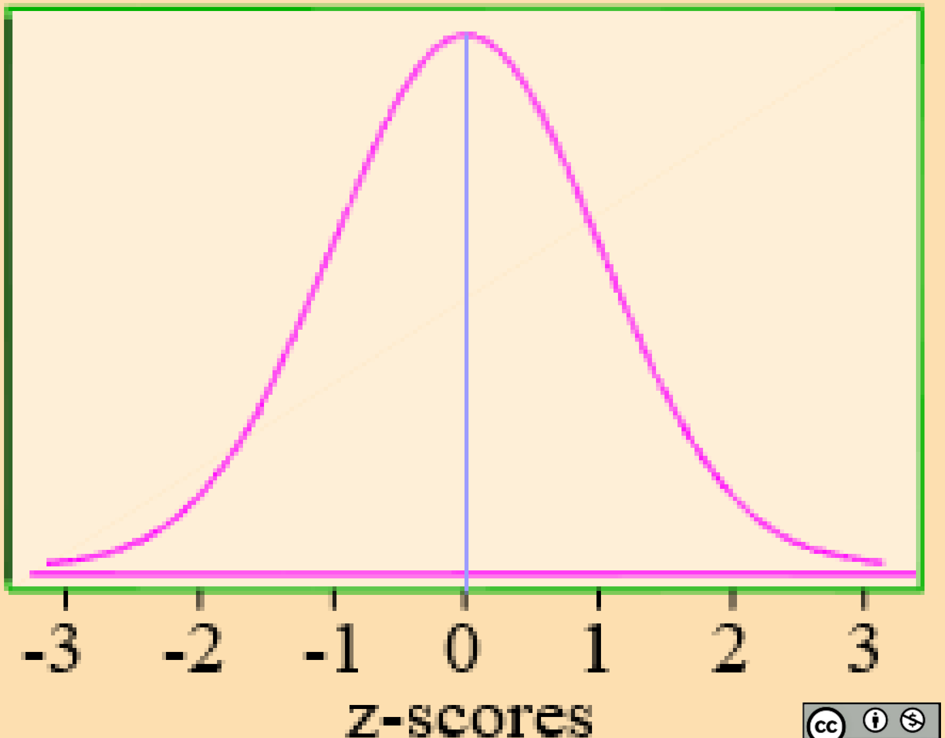
Ans. 0.0160



Area Under the Standard Normal Curve - Example

Example: Find the area between $z = 1.68$ and $z = -1.37$ $\Rightarrow P(-1.37 < z < 1.68)$

Ans. 0.8682

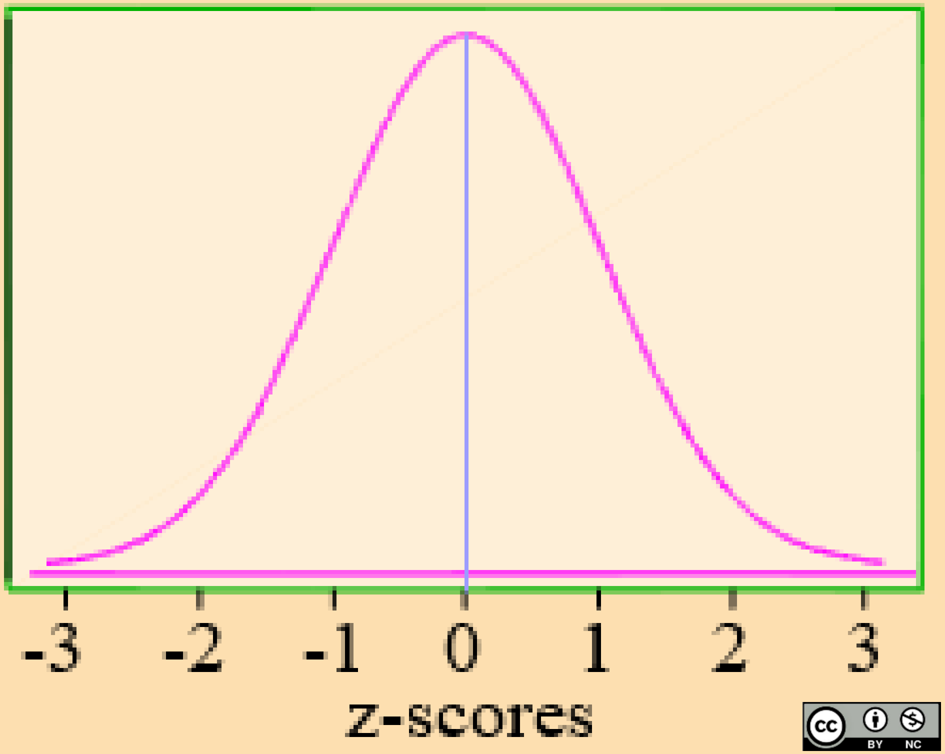


Area Under the Standard Normal Curve - Example

Example: **Find the area to the right of $z = -1.16$**

$\Rightarrow P(z > -1.16)$

Ans. **0.8770**



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Applications of Standard Normal Distribution

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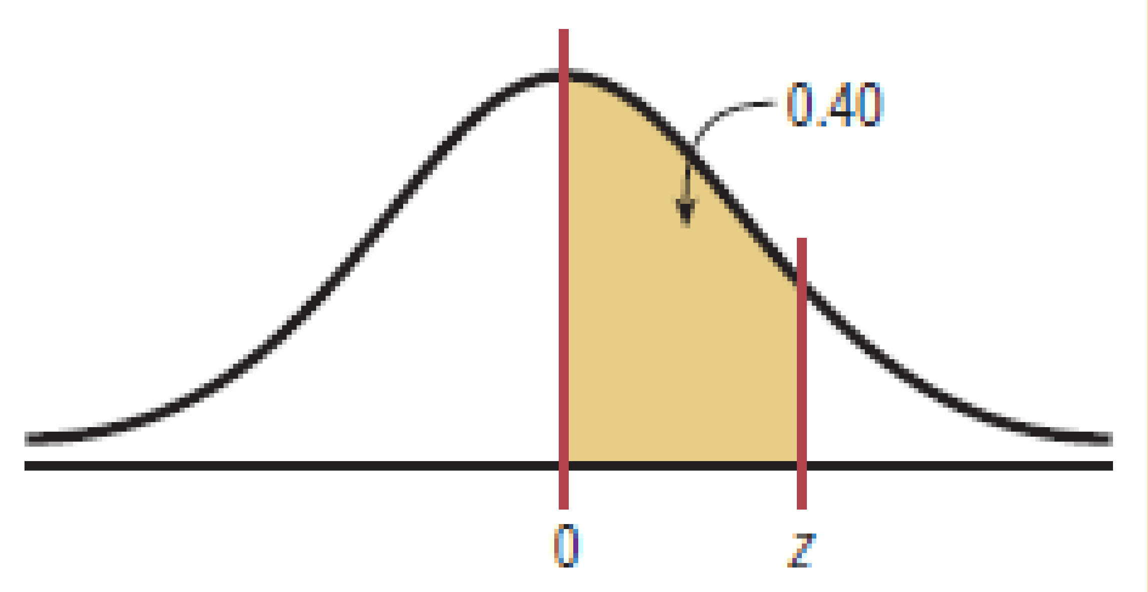
The Standard Normal Distribution

Find the z value that corresponds to the given area.



Ans. $z=1.28$

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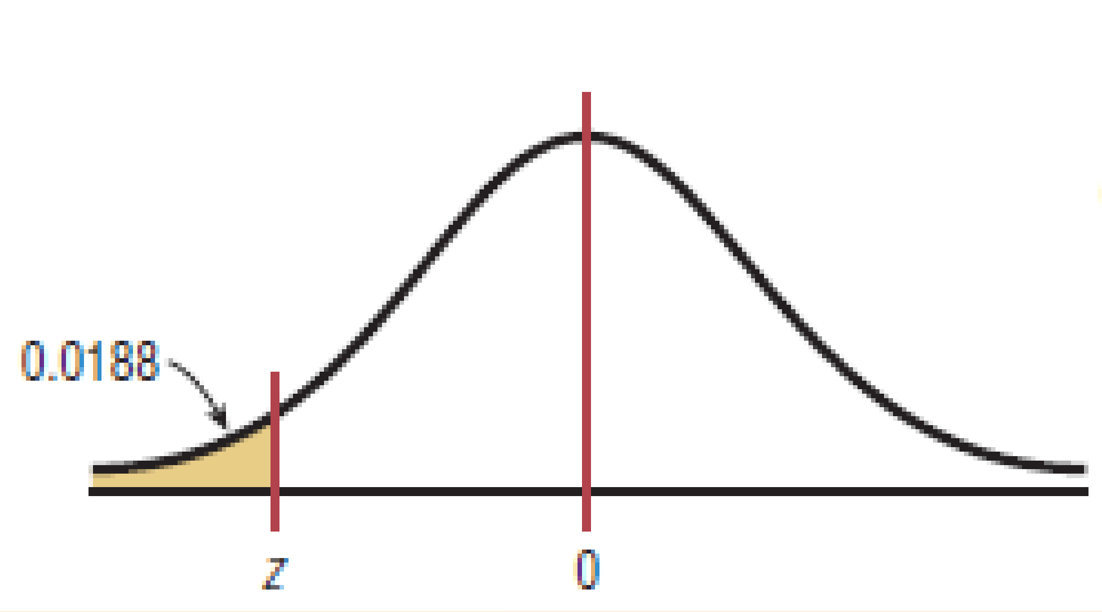
The Standard Normal Distribution

Find the z value that corresponds to the given area.



Ans. $z=-2.08$

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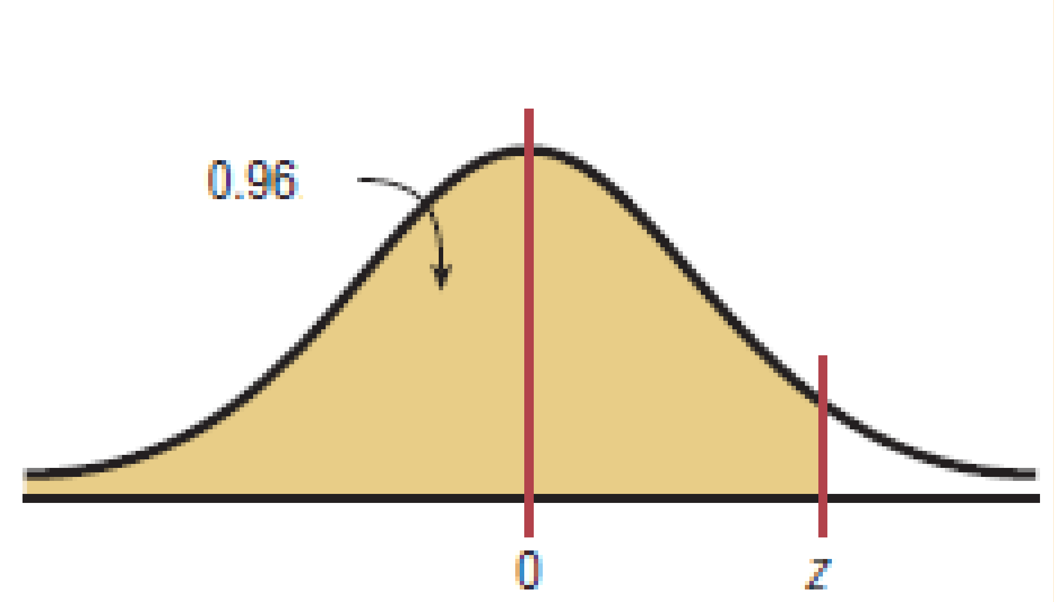
The Standard Normal Distribution

Find the z value that corresponds to the given area.



Ans. z=1.75

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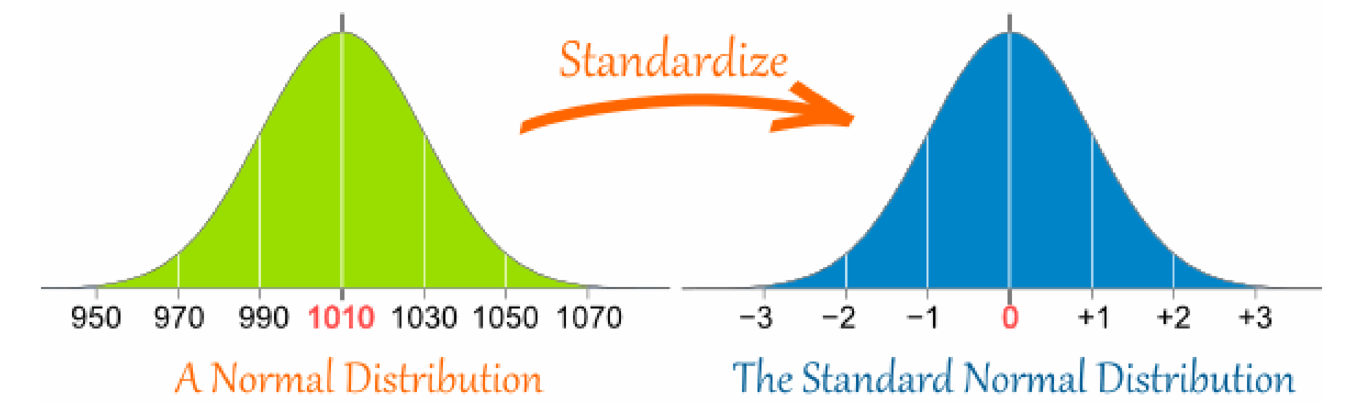
The Standard Normal Distribution

$$z = \frac{X - \mu}{\sigma}$$

$$X = z\sigma + \mu$$



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Applications of the Normal Distribution - Example

Each month, an American household generates an average of 28 pounds of newspaper for garbage or recycling. Assume the standard deviation is 2 pounds. Assume the amount generated is normally distributed. If a household is selected at random, find the probability of its generating:

(a) more than 30.2 pounds per month.

(b) between 27 and 31 pounds per month

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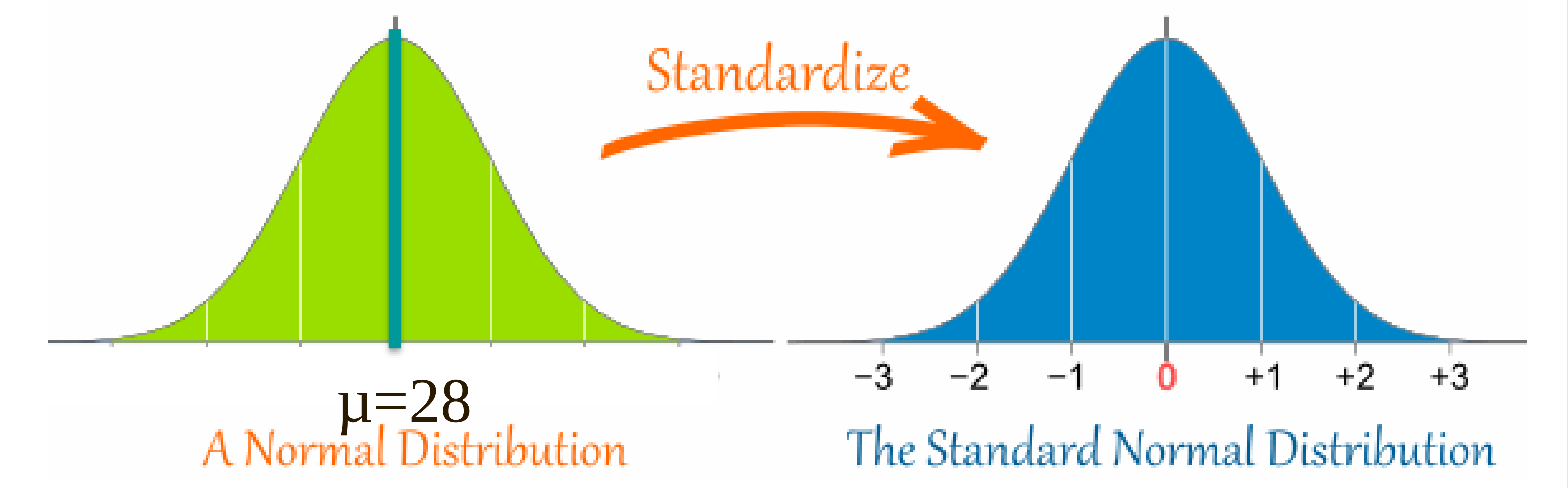
Applications of the Normal Distribution - Example



Given: μ=28 and σ=2

a.) Find: P(x > 30.2)

$$z = \frac{x-\mu}{\sigma} = \frac{30.2-28}{2} = 1.1$$

Answer: P(x > 30.2)=P(z > 1.1)=0.1357 or 13.57%

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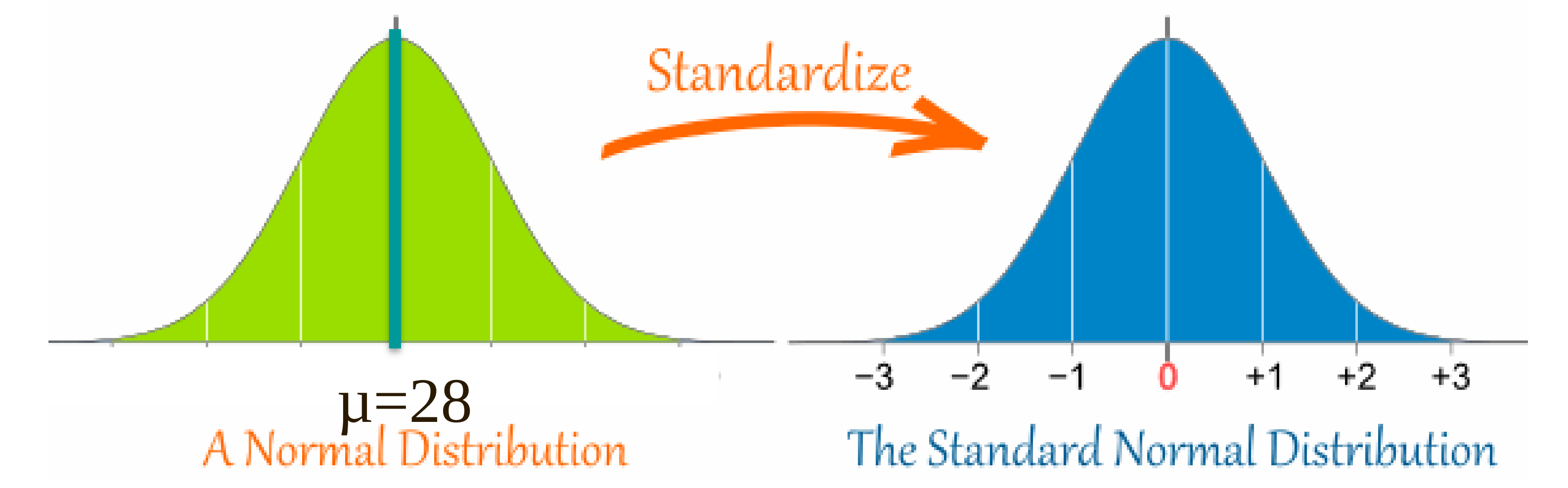
Applications of the Normal Distribution - Example



Given: μ=28 and σ=2

b.) Find: P(27< x < 31)

$$z_1 = \frac{x_1 - \mu}{\sigma} = \frac{27-28}{2} = -0.5; \quad z_2 = \frac{x_2 - \mu}{\sigma} = \frac{31-28}{2} = 1.5$$

Answer: P(27< x <31) = P(-0.5< z <1.5)=0.6247

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Applications of the Normal Distribution - Example

An exclusive college desires to accept only the top 10% of all graduating seniors based on the results of a national placement test. This test has a mean of 500 and a standard deviation of 100. Find the cutoff score for the exam. Assume the variable is normally distributed.

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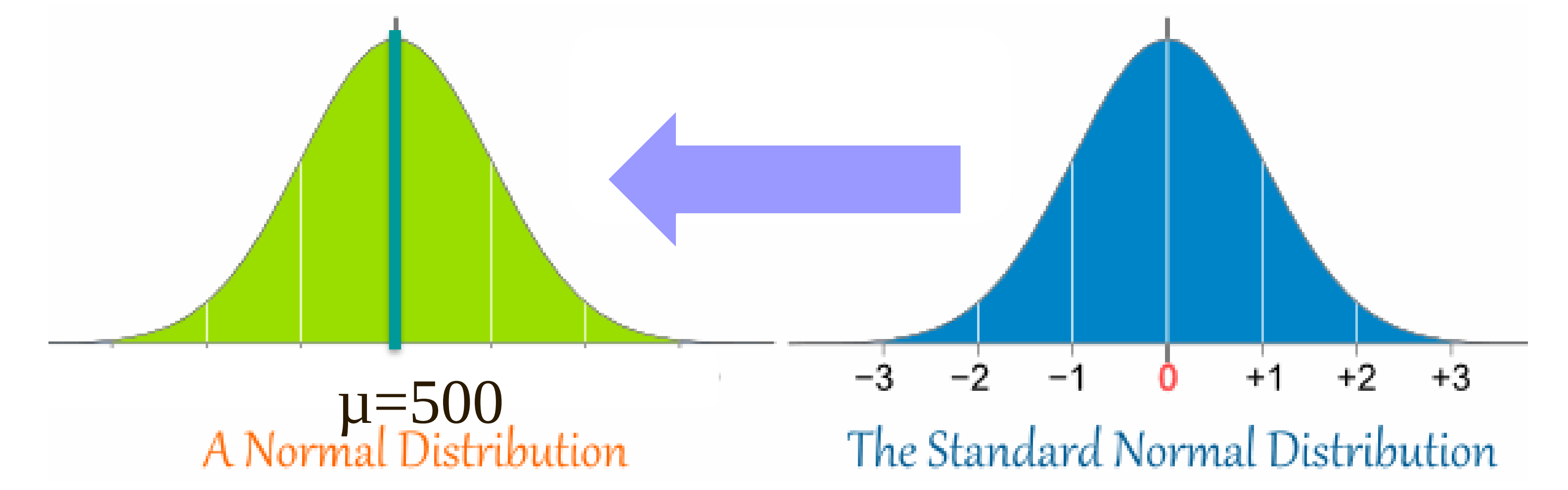
Applications of the Normal Distribution - Example



Given: μ=500 and σ=100 and area of top 10%

Find: cut-off score (x)

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Applications of the Normal Distribution - Example

Values Needed:
μ=500
σ=100
z=1.28

$$z = \frac{x - \mu}{\sigma}$$

$$1.28 = \frac{x - 500}{100}$$

$$1.28(100) = x - 500$$

$$1.28(100) + 500 = x$$

$x = 628$ – cut-off score

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Applications of the Normal Distribution - Example

For a medical study, a researcher wishes to select people in the middle 60% of the population based on blood pressure. If the mean systolic blood pressure is 120 and the standard deviation is 8, find the upper and lower readings that would qualify people to participate in the study.

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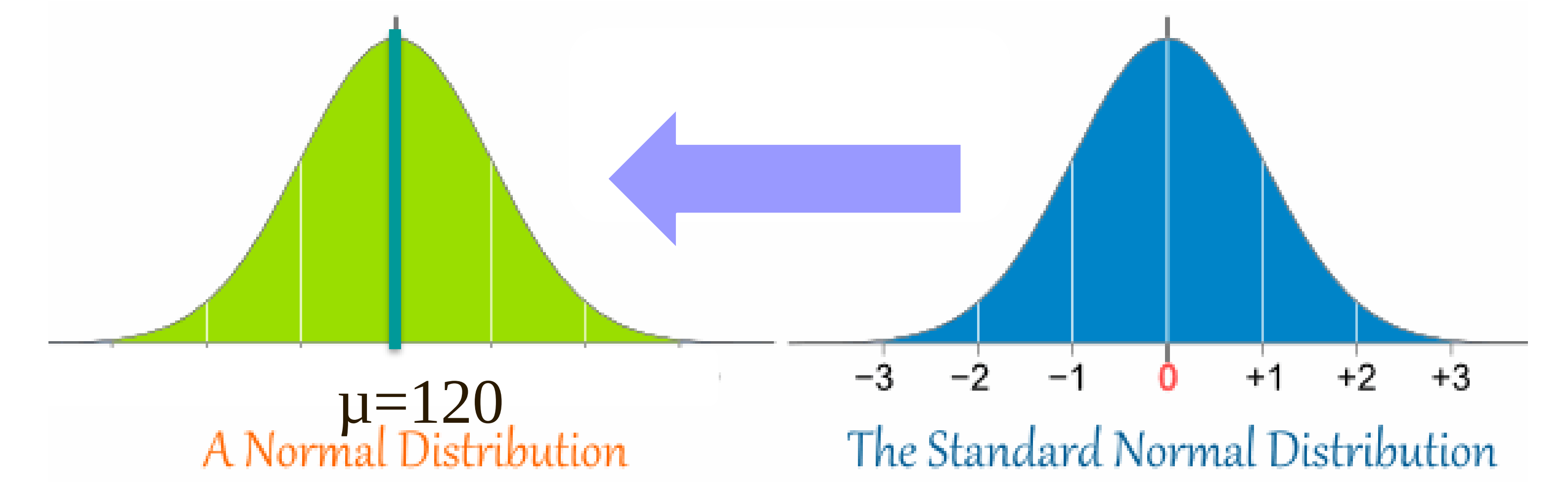
Applications of the Normal Distribution - Example



Given: $\mu=120$ and $\sigma=8$ and

area of the middle 60%

Find: upper and lower reading (x_1 and x_2)

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Applications of the Normal Distribution - Example

Values Needed:
μ=120
σ=8
z_1=-0.84
z_2=0.84

$$z = \frac{x-\mu}{\sigma}$$

$$-0.84 = \frac{x-120}{8}$$

$$-0.84(8) = x - 120$$

$$-0.84(8) + 120 = x$$

$$x = 113.28$$

(lower reading)

$$z = \frac{x-\mu}{\sigma}$$

$$0.84 = \frac{x-120}{8}$$

$$0.84(8) = x - 120$$

$$0.84(8) + 120 = x$$

$$x = 126.72$$

(upper reading)

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Sampling Distribution and the Central Limit Theorem

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Distribution of Sample Means

- **Distribution of Sample means:** A sampling distribution of sample means is a distribution obtained by using the means computed from random samples of a specific size taken from a population.

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Distribution of Sample Means

- **Sampling error** is the difference between the sample measure and the corresponding population measure due to the fact that the sample is not a perfect representation of the population.

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Properties of the Distribution of Sample Means

- The mean of the sample means $(\mu_{\bar{x}})$ will be the same as the population mean (μ).
- The standard deviation of the sample means (also called the **standard error of the mean**) $(\sigma_{\bar{x}})$ will be smaller than the standard deviation of the population, and it will be equal to the population standard deviation (σ) divided by the square root of the sample size $(\sqrt{n})$.

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Properties of the Distribution of Sample Means - Example

Suppose a professor gave an 8-point quiz to a small class of four students. The results of the quiz were 2, 6, 4, and 8. Assume the four students constitute the population.

- 1. Find the mean and standard deviation of the population.
- **Answer:** $\mu=5$; $\sigma=2.236067977$

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Properties of the Distribution of Sample Means - Example

Suppose a professor gave an 8-point quiz to a small class of four students. The results of the quiz were 2, 6, 4, and 8. Assume the four students constitute the population.

- 2. Consider all samples of size 2 taken with replacement. Then show that (i) $\mu_{\bar{x}} = \mu$ and (ii) $\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}}$
- (i) **Solution:** Samples of size 2={(2,2),(2,6),(2,4),(2,8),(6,2),(6,6),(6,4),(6,8),(4,2),(4,6),(4,4),(4,8),(8,2),(8,6),(8,4),(8,8)}

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Properties of the Distribution of Sample Means - Example

- 2. Consider all samples of size 2 taken with replacement. Then show that (i) $\mu_{\overline{x}} = \mu$ and (ii) $\sigma_{\overline{x}} = \frac{\sigma}{\sqrt{n}}$
- (i) **Solution:** Samples of size 2={(2,2),(2,6),(2,4),(2,8),(6,2),(6,6),(6,4),(6,8),(4,2),(4,6),(4,4),(4,8),(8,2),(8,6),(8,4),(8,8)}
- $\overline{x}$ ={2, 4, 3, 5, 4, 6, 5, 7, 3, 5, 4, 6, 5, 7, 6, 8}
- $\mu_{\overline{x}} = 5$
- $\mu = 5$ (see question 1)
- $\therefore \mu_{\overline{x}} = \mu$

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Properties of the Distribution of Sample Means - Example

- 2. Consider all samples of size 2 taken with replacement. Then show that (i) $\mu_{\overline{x}} = \mu$ and (ii) $\sigma_{\overline{x}} = \frac{\sigma}{\sqrt{n}}$
- (ii) **Solution:** Samples of size 2={(2,2),(2,6),(2,4),(2,8),(6,2),(6,6),(6,4),(6,8),(4,2),(4,6),(4,4),(4,8),(8,2),(8,6),(8,4),(8,8)}
- $\overline{x}$ ={2, 4, 3, 5, 4, 6, 5, 7, 3, 5, 4, 6, 5, 7, 6, 8}
- $\sigma_{\overline{x}} = 1.58113883$
- Sample size (n) = 2

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Properties of the Distribution of Sample Means - Example

- (ii) Solution: Samples of size 2={(2,2),(2,6),(2,4),(2,8),(6,2),(6,6),(6,4),(6,8),(4,2),(4,6),(4,4),(4,8),(8,2),(8,6),(8,4),(8,8)}
- $\overline{x}$ ={2, 4, 3, 5, 4, 6, 5, 7, 3, 5, 4, 6, 5, 7, 6, 8}
- $\boldsymbol{\sigma_{\overline{x}} = 1.58113883}$
- Sample size (n) = 2
- $\boldsymbol{\sigma}$ = 2.236067977 (see question 1)
- $\boldsymbol{\frac{\sigma}{\sqrt{n}} = \frac{2.236067977}{\sqrt{2}} = 1.58113883}$
- $\boldsymbol{\therefore \sigma_{\overline{x}} = \frac{\sigma}{\sqrt{n}}}$

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The Central Limit Theorem

- As the sample size ***n*** increases, the shape of the distribution of the sample means taken from a population with mean μ and standard deviation of σ will approach a normal distribution. As previously shown, this distribution will have a mean μ and standard deviation $\sigma / \sqrt{n}$.

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The Central Limit Theorem

The central limit theorem can be used to answer questions about sample means in the same manner that the normal distribution can be used to answer questions about individual values. The only difference is that a new formula must be used for the z-values. It is

$$z = \frac{\overline{X} - \mu}{\sigma / \sqrt{n}}.$$

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The Central Limit Theorem - Example

A.C. Neilsen reported that children between the ages of 2 and 5 watch an average of 25 hours of TV per week. Assume the variable is normally distributed and the standard deviation is 3 hours. If 20 children between the ages of 2 and 5 are randomly selected, find the probability that the mean of the number of hours they watch TV is greater than 26.3 hours.

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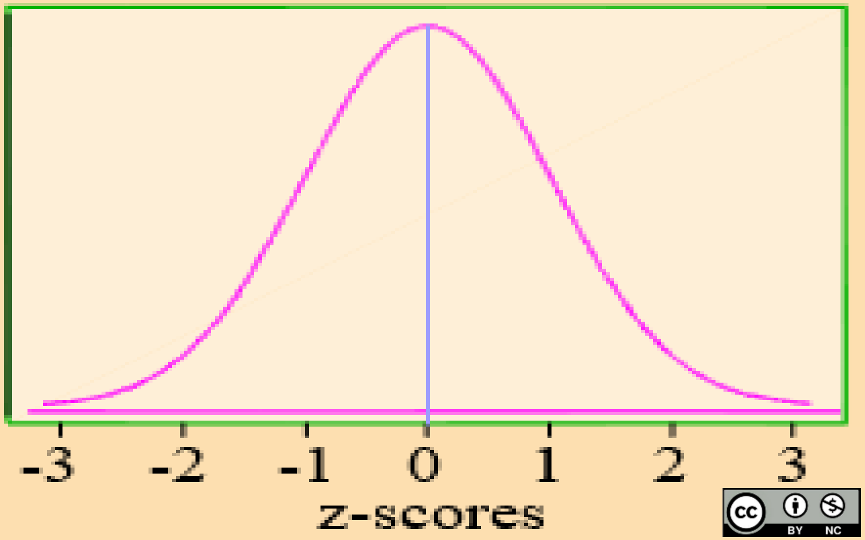
The Central Limit Theorem - Example

- **Given: μ=25, σ=3, n=20 and $\bar{x}$=26.3**
- **Find: P($\bar{x}$ > 26.3)**
- **Solution:** $z = \frac{\bar{x}-\mu}{\frac{\sigma}{\sqrt{n}}} = \frac{26.3-25}{\frac{3}{\sqrt{20}}} = 1.94$

P($\bar{x}$ > 26.3)

=P(z > 1.94)=0.0262



The Central Limit Theorem - Example

The average age of a vehicle registered in the United States is 8 years, or 96 months. Assume the standard deviation is 16 months. If a random sample of 36 cars is selected, find the probability that the mean of their age is between 90 and 100 months.

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The Central Limit Theorem - Example

- **Given: μ=96, σ=16, n=36, $\overline{x_1} = 90$ and $\overline{x_2} = 100$**
- **Find: P(90 < $\bar{x}$ < 100)**

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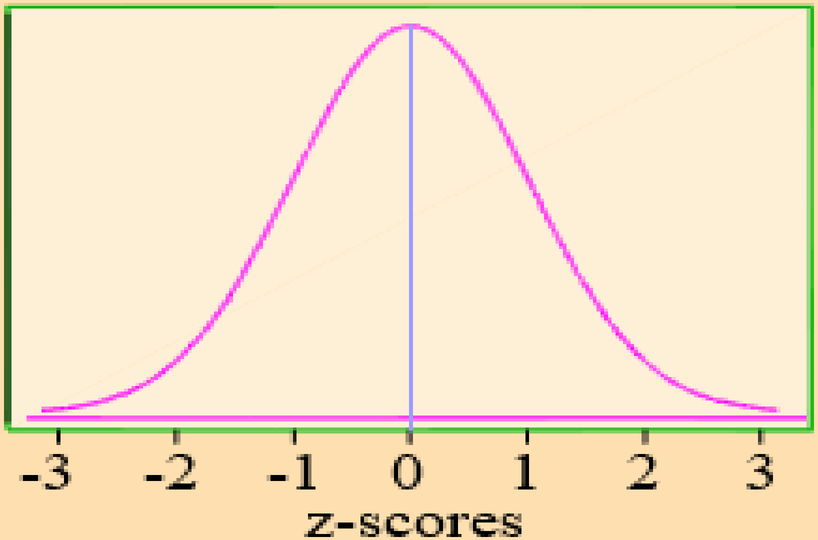
The Central Limit Theorem - Example

- **Solution**: $z_1 = \dfrac{\overline{x}_1 - \mu}{\frac{\sigma}{\sqrt{n}}} = \dfrac{90-96}{\frac{16}{\sqrt{36}}} = -2.25$

$$z_2 = \dfrac{\overline{x}_2 - \mu}{\frac{\sigma}{\sqrt{n}}} = \dfrac{100-96}{\frac{16}{\sqrt{36}}} = 1.5$$



P(90 < $\bar{x}$ < 100)

= P(-2.25 < z < 1.5) = 0.921

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Exponential Distribution

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Exponential Distribution

- The exponential distribution is often concerned with the amount of time until some specific event occurs.

Examples

- the amount of time (beginning now) until an earthquake occurs
- number of minutes we need to wait before a customer enters our shop
- the amount of time, in months, a car battery lasts.

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Exponential Distribution

- The equation for the **exponential distribution** is defined by

$$p(X \le x) = 1 - e^{-\lambda x}, \quad x \ge 0$$

where

$p(X \le x)$ - the probability that X will be less than or equal to some specific value

λ - the population mean number of occurrences per unit

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Exponential Distribution

Example:

- Suppose that customers arrive at a bank's ATM at an average rate of 20 per hour. If a customer has just arrived, what is the probability that the next customer will arrive within 6 minutes?
- Given: λ = 20 customers/hr, x=0.1hr
- $\boldsymbol{P(X \leq x) = 1 - e^{-\lambda x}}$

$$p(X \leq 0.1) = 1 - e^{-20(0.1)}$$

$$= 1 - e^{-2}$$

$$= 1 - 0.1353$$

$$= 0.8647, \quad \text{or almost a } 86.47\% \text{ chance}$$

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Exponential Distribution

Example:

- **Accidents occur with a Poisson distribution at an average of 4 per week. What is the probability that at least two weeks will elapse between accident?**
- **Given: λ = 4 accidents/week, x=2weeks**
- $\boldsymbol{P(X \geq x) = e^{-\lambda x}}$
- $\boldsymbol{P(X \geq 2) = e^{-(4)(2)} = 0.00034}$

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Reference:

Bluman, A. (2012). *Elementary Statistics: A Step by Step Approach*,8e. McGraw-Hill Higher Ed.

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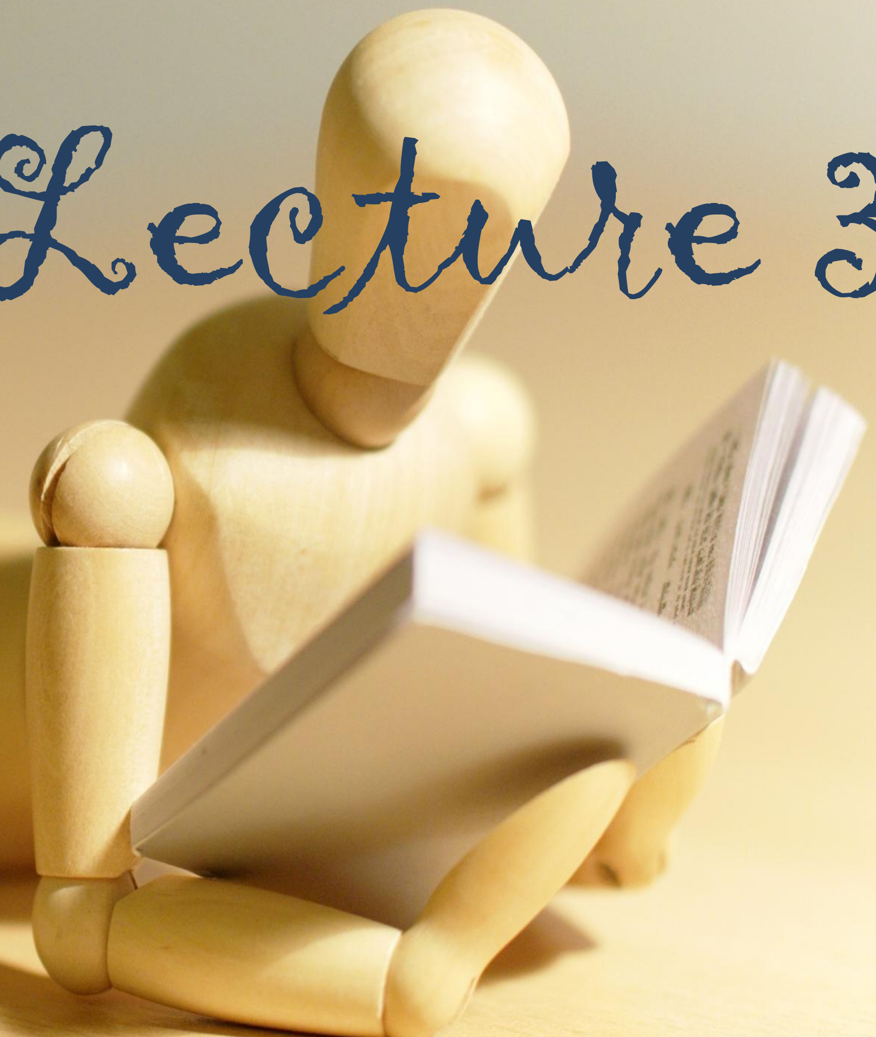

End of Lecture 3

Anthony S. Morfe_CAS_Math Cluster

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