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Linear stability analysis of stationary Euler flows for passive scalars and inhomogeneous fluids

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ABSTRACT

This thesis is concerned with the study of linear stability properties of some particular fluid flows, within the framework of quantitative hydrodynamic stability. In the last decade, the problem has received a lot of attention thanks to the introduction of new analytical techniques particularly useful to tackle classical problems that date back to the end of the nineteen century.

The purpose of this thesis is threefold. We first investigate the asymptotic decay properties of a passive scalar driven by a vortex with a power-law velocity field in the whole plane. In particular, we quantify the *enhanced dissipation* mechanism caused by fluid mixing, namely, the transfer of energy from large to small scales due to transport. We provide sharp (up to a logarithmic correction) bounds on the dissipation time-scales, which are faster than the standard diffusive one.

The rest of the thesis is dedicated to the study of the stability of parallel flows in 2D inhomogeneous fluids. The second problem we deal with concerns a linear stability analysis for the Couette flow with constant density in an isentropic compressible fluid. We consider both the inviscid and the viscous case. In the inviscid case, we give the first rigorous mathematical justification to a Lyapunov instability mechanism previously addressed in the physics literature. More precisely, we show that the L^2 norms of the density and the irrotational component of the velocity field grow as $t^{1/2}$. Instead, the solenoidal component of the velocity strongly converges to zero in L^2, a mechanism known as *inviscid damping*. In the viscous case, we present the first enhanced dissipation result for an inhomogeneous fluid. The exponential decay become effective on a time-scale $O(\nu^{-1/3})$, with ν being proportional to the inverse of the Reynolds number, and there is a large transient growth of order $O(\nu^{-1/6})$ caused by the inviscid instability. The estimates are valid also in a large Mach number regime.

We finally study the stability of a particular class of stratified shear flows. We consider an inhomogeneous, inviscid and incompressible fluid under the action of gravity, near shear flows close to Couette with an exponentially stratified density profile. Under the Miles-Howard criterion, we prove the inviscid damping for the velocity field and the (scaled) density. The decay rates are slower with respect to the classical homogeneous case without gravity. This is due to a Lyapunov instability mechanism for the vorticity that we characterize in the Couette case.

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Contents

CHAPTER 1

Introduction

The Navier-Stokes equations for an isentropic compressible fluid in a domain $D \subseteq \mathbb{R}^2$ are given by

$$\partial_t\widetilde{\rho} + \operatorname{div}(\widetilde{\rho}\boldsymbol{u}) = 0, \quad \text{in } D,\ t \geq 0 \tag{1.1a}$$

$$\widetilde{\rho}(\partial_t\boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}) + \frac{1}{M^2}\nabla p(\widetilde{\rho}) = \nu\Delta\boldsymbol{u} + \lambda\nabla\operatorname{div}(\boldsymbol{u}) + \rho\mathbf{F}, \tag{1.1b}$$

$$\boldsymbol{u}|_{t=0} = \boldsymbol{u}^{in}, \quad \widetilde{\rho}|_{t=0} = \widetilde{\rho}^{in}, \tag{1.1c}$$

where $\widetilde{\rho}$ is the density of the fluid, $\boldsymbol{u} = (u^x, u^y)$ the velocity field, $p(\widetilde{\rho}) = \widetilde{\rho}^\gamma/\gamma$ with $\gamma \geq 1$ is the pressure, M the Mach number, $\mathbf{F}$ is an external potential force, ν and λ are the shear and bulk viscosities coefficients respectively.[1] The shear viscosity ν is proportional to the inverse of the Reynolds number. We will always consider domains D without boundaries. In the absence of viscosities we are reduced to the Euler equations. When the velocity field is divergence free we have an incompressible fluid and if the density is constant we talk about a homogeneous incompressible fluid.

One of the simplest examples of non-constant solution to the previous equations is given by *shear flows*, namely

$$\rho_E = 1, \qquad \boldsymbol{u}_E = \begin{pmatrix} U(y) \\ 0 \end{pmatrix}, \tag{1.3}$$

[1]For a Newtonian fluid in dimension n the viscous stress tensor σ has the following structure

$$\sigma_{ij} = -P_{th}\delta_{ij} + 2\nu e_{ij} + \zeta e_{kk}\delta_{ij}, \qquad i,j = 1,\dots,n,$$

where P_{th} is the thermodynamic pressure, $e_{ij} = \frac{1}{2}(\partial^i u_j + \partial^j u_i)$ and ζ is the second viscosity coefficient. By decomposing e_{ij} as the sum of its isotropic and deviatoric part, namely $\sigma_{ij} = \frac{1}{n}\sigma_{kk}\delta_{ij} + (\sigma_{ij} - \frac{1}{n}\sigma_{kk}\delta_{ij})$, we rewrite the previous equation as

$$\sigma_{ij} = -P_{th}\delta_{ij} + 2\nu(e_{ij} - \frac{1}{n}\delta_{ij}e_{kk}) + \lambda\delta_{ij}e_{kk}, \qquad i,j = 1,\dots,n, \tag{1.2}$$

where $\lambda = \zeta + \frac{2\nu}{n} \geq 0$ is the *bulk viscosity* coefficient. It has to be nonnegative since the entropy cannot decrease, see [109, Ch. 49]. The *Stokes hypothesis* says that the mechanical pressure, namely $P_{me} = \operatorname{Tr}(\sigma)/n$ satisfies $P_{me} = -P_{th}$, which implies that $\lambda = 0$. However, the validity of the latter hypothesis is still debated [31, 84], since for a large class of fluids the bulk viscosity is not zero and in some case may also be very large. The first two terms on the right-hand side of (1.1b) follows from the last two terms in (1.2) when $n = 2$.

which are stationary solutions in the Euler regime. We have normalized the equilibrium density to the value one for convenience of notation. Euler himself, when introducing the equations for an incompressible non-viscous fluid, proposed a shear flow as an example of a particular solution, see [69, Par. XLVIII] and the recent translation [74, Par. 48]. Actually, in [69, Par.XXX], see also [74, Par.30], Euler showed that *circular flows*, which are given by

$$\rho_E = 1, \qquad \boldsymbol{u}_E(x,y) = U(\sqrt{x^2+y^2})\begin{pmatrix} -y \\ x \end{pmatrix}, \tag{1.4}$$

are stationary solutions to the non-viscous incompressible equations. When the density is not constant, we have also that *stratified shear flows*, given by

$$\rho_E = \bar{\rho}(y), \qquad\qquad \boldsymbol{u}_E = \begin{pmatrix} U(y) \\ 0 \end{pmatrix}, \tag{1.5}$$

are stationary solutions for the Euler equations with an external potential force (usually the gravity).

Having at hand some particular solutions, what does it mean for them to be stable? A first intuitive notion of stability can be formulated as follows: "*A given equilibrium is stable if an infinitely small variation at the initial time leads to infinitely small changes in future times. Otherwise it is said to be unstable*". This was essentially the classical notion of stability well known from the physical point of view, however, a precise mathematical definition of this concept has not been available for many years, for instance the paper of Lyapunov [120] dates back to 1892. In a more general framework, the understanding of stability properties of particular solutions of the Navier-Stokes equations enters in the field of *hydrodynamic stability theory* [8, 39, 63, 137, 156], which has its roots in the origins of the study of fluid dynamic problems. Starting from the second half of the 19th century theoretical contributions were given by Stokes, Helmholtz, Kelvin, Rayleigh and G. I. Taylor among many others.

The famous experiment of Reynolds [133] in 1883, showing the transition from laminar to turbulent flow in a circular pipe, drew attention towards the study of the stability of parallel flows. Indeed, a first mathematical approximation of this experiment is to assume radial symmetry, so that instead of a pipe one has an infinite channel. Then, the laminar equilibrium is given by a planar shear flow, see (1.3), whose velocity profile is $U(y) = L - y^2$, where $2L$ is the width of the channel. This particular shear flow was previously considered by Poiseuille in 1838 [142] and by Stokes in 1845 [140] and it is known as the *Poiseuille flow*.

Another crucial experiment was done by Mallock [122] and Couette [50] around 1888 with the aim of computing the value of the viscosity of water. In particular, they both set up an experimental apparatus consisting of a fluid confined between two rotating cylinders and some instabilities were observed depending on the angular velocities of the cylinders. Later, G. I. Taylor [144] in 1923 studied the problem from the theoretical point of view and the experimental one. In particular, in his experiments, he shows the transition between the laminar regime and turbulent one, with the formation of the so called *Taylor vortices*. Also in this case, a planar shear flow may be considered as a first mathematical approximation. More precisely, if the difference between the inner and the outer radius of the cylinder is very small compared to their length, neglecting changes along the vertical direction, the laminar regime may be described by a shear flow with a linear velocity profile $U(y) = Sy$ in an infinite channel, where S and the width of the channel are determined by the angular velocities of the cylinders. Nowadays we refer to this shear flow as the *Couette flow*. For more historical details, we refer to the nice paper of Donnely [60].

Regarding the study of stratified shear flows (1.5), in the essay [145] G. I. Taylor writes that: "It is well known that when the wind near the ground drops at night owing to the cooling of the ground, the wind at a higher level frequently remains unchanged so that the effect of a decrease in density with height is to enable a large velocity gradient to be maintained. This implies that the turbulence is suppressed or at any rate much reduced by the density gradient. To the mathematician this at once presents the problem of the stability of a fluid in which the density and velocity vary with height above the ground, regarded as a horizontal plane.". He was indeed one of the first who got interested in studying the stability of stratified shear flows, which became a relevant problem in oceanography and meteorology [159]. In particular, G.I. Taylor conjectured the so-called *Miles-Howard criterion* [128].

Organization of the chapter

In Section 1.1 we discuss different notions of stability and several classical results concerning the stability of shear flows. Linear stability properties of the Couette flow for a homogeneous incompressible fluid are studied in detail in Section 1.2, where we review also the most recent results available for shear flows in the context of incompressible fluids. In Section 1.3 we present the main results which we are going to discuss in this thesis.

1.1 Notions of stability and classical results

In order to investigate the stability of a given equilibrium it is natural to study the behaviour of a perturbation around it. Hence, in the system (1.1) one considers

$$\widetilde{\rho} = \rho_E + \rho, \qquad \boldsymbol{u} = \boldsymbol{u}_E + \boldsymbol{v},$$

and then has to infer properties of $\rho, \boldsymbol{v}$ by investigating the related system of *nonlinear* PDE's. The study of this system usually presents several mathematical difficulties, which were out of reach at the end of the 19th century and are still a challenging problem nowadays. As a first mathematical approximation towards the understanding of stability properties, it is natural to consider the *linearized* system. The linear stability analysis in the hydrodynamic context, in general, does not imply any stability result for the full nonlinear problem. However, it may be extremely useful to understand some of the underlying mechanisms in the dynamics of the perturbation and, in some cases, instability at the linear level may be transferred to the nonlinear one [6, 73, 87, 115].

One of the first problems under consideration in hydrodynamic stability theory was the case of shear flows for an inviscid and homogeneous incompressible fluid. In 1880 Rayleigh [132] proved his famous *inflection point theorem*, which says the following:

> Given a velocity profile $U(y)$, a necessary condition for instability is that $U''(y)$ changes sign.

To prove this, we consider the linearized equations in vorticity formulation, namely

$$\begin{aligned} &\partial_t\omega + U(y)\partial_x\omega - U''(y)\partial_x\psi = 0, && (1.6a)\\ &v = \nabla^\perp\psi, \quad \Delta\psi = \omega, \end{aligned}$$

where $\nabla^\perp = (-\partial_y, \partial_x)$, $\omega = \nabla^\perp \cdot v$ is the vorticity and ψ is the associated stream function. Equivalently, we have that the stream function satisfies

$$(\partial_t + U(y)\partial_x)\Delta\psi - U''(y)\partial_x\psi = 0. \tag{1.7}$$

Considering a perturbation of the form

$$\psi(t,x,y) = \varphi(y)e^{\lambda t + ikx}, \quad \text{for } k \in \mathbb{R},\ \lambda \in \mathbb{C}\setminus\{0\},$$

i.e. through a *normal-mode* analysis, we get the so called *Rayleigh's equation*

$$(\lambda + ikU(y))(\varphi'' - k^2\varphi) - ikU''(y)\varphi = 0. \tag{1.8}$$

It corresponds to a Fourier transform in x and a Laplace transform in t in (1.7) and λ is an eigenvalue of the linear operator associated to (1.7). The stability of the perturbation is then determined by the sign of $\mathrm{Re}(\lambda)$. Since we are looking for instabilities, one can assume that $\mathrm{Re}(\lambda) > 0$, hence we can divide (1.8) by $\lambda + ikU(y)$. It is then sufficient to multiply by the complex conjugate of φ, integrate by parts and consider the imaginary part of the resulting identity, given by

$$\mathrm{Re}(\lambda)\int \frac{U''(y)}{\lambda^2 + U^2(y)}|\varphi|^2(y)dy = 0.$$

The last identity cannot hold if U'' does not change sign, hence proving the Rayleigh's inflection point theorem. Fjørtoft [72] refined this proof to obtain a stronger result.

This particular case provides an example of *spectral* instability, namely the existence of an eigenvalue with positive real part. Such instabilities can be localized via the Howard's *semicirle theorem* [97] in the region

$$\left(\mathrm{Re}(\lambda) - \frac{1}{2}(U_{min} + U_{max})\right)^2 + (\mathrm{Im}(\lambda))^2 \leq \frac{1}{4}\left(U_{max} - U_{min}\right)^2,$$

where U_{min}, U_{max} are respectively the minimum and the maximum of the shear flow under consideration. Following a similar approach, a stability criterion can be obtained also for stratified shear flows, as in (1.5), the proof of which is due to Miles [128] and Howard [97]. It says the following:

> A stratified shear flow is spectrally stable if the *Richardson number*[2] is greater or equal than $1/4$.

[2]The Richardson number is defined as $\mathrm{Ri}(y) = \mathfrak{g}\left(\frac{-\bar\rho'}{\bar\rho(U')^2}\right)(y)$ where $\mathfrak{g}$ is the gravity and $\bar\rho, U$ are defined in (1.5).

As previously mentioned, this criterion was conjectured by Taylor [145] and prior relevant studies for stratified shear flows have been done also by Goldstein [83] and Synge [143] around the '30s. The *Miles-Howard criterion* will be relevant to our analysis in Chapter 4, where we present a detailed proof of it.

Spectral analysis is not the only available technique to tackle the stability problem. Kelvin in 1887 [107] was interested in studying the stability of the Couette flow, i.e. $U(y) = y$, for a viscous and homogeneous incompressible fluid. The particular structure of the linear equation, in this case, allowed Kelvin to find an exact solution to the problem and prove its stability at any Reynolds number. The problem in the inviscid case was also considered by Orr in 1907 [131], showing again the stability of the Couette flow and also highlighting other important mechanisms, see Section 1.2.

The studies of Kelvin and Orr entail *Lyapunov stability* for the Couette flow. More precisely:

> Given an equilibrium u_E and two Banach spaces X and Y, we say that u_E is *Lyapunov stable* if for all $\epsilon > 0$ there exists a $\delta > 0$ such that
>
> $$\text{if } \left\|\boldsymbol{v}^{in}\right\|_X \leq \delta, \text{ then } \|\boldsymbol{v}(t)\|_Y \leq \epsilon \text{ for any } t \geq 0.$$
>
> The equilibrium u_E is *Lyapunov unstable* if it is not stable.

For the Couette flow, the linear Lyapunov stability follows by studying the explicit solution. However, beyond this very basic example, Lyapunov stability for other shear flows was not considered for many years. Indeed, even at a linear level, the problem becomes much more subtle. In 1965 Arnold [3], by means of an elegant variational approach, proved that shear flows with $0 < c < U/U'' < C$ are *nonlinearly* Lyapunov stable. This was one of the first nonlinear results in hydrodynamic stability theory. Even with this very strong result at hand, many interesting cases remained open. For example, the Couette flow does not belong to the class considered by Arnold.

In some physically relevant cases, including the Poiseuille and the Couette flows, the available linear results do not agree with experimental and numerical observations [34, 112, 130, 137, 147]. One of the reasons behind this, is the possibility of a large *transient growth* of the perturbation. This means that, even if the flow is linearly stable, the disturbance can grow so much as to invalidate the linear regime. A Lyapunov stability analysis can detect such mechanism, as can be seen for example in the Couette case (Section 1.2), while a pure eigenvalue analysis cannot. Indeed, a transient growth phenomenon can occur if the operators involved are *non-normal*. In this case, we know that the eigenfunctions are not necessarily orthogonal. Hence, we may have an amplification even if all the eigenvalues have negative real parts. This problem was essentially already known to Orr in 1907 [131]. In general, Lyapunov and spectral stability are neither equivalent nor one is stronger than the other.

To elucidate the problem of transient growth, let us consider the following toy example

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} -\delta & 1 \\ 0 & -\delta \end{pmatrix}\begin{pmatrix} f \\ g \end{pmatrix} = A\begin{pmatrix} f \\ g \end{pmatrix}$$

for some $\delta > 0$. Here, the matrix A is non-normal, i.e. $AA^T \neq A^T A$, but clearly the system is spectrally stable. However, by solving explicitly we find that

$$f(t) = e^{-\delta t} f^{in} + t e^{-\delta t} g^{in}, \quad g(t) = e^{-\delta t} g^{in},$$

hence, by considering $0 < f^{in} = g^{in}$, for $t = \delta^{-1}$ one has

$$f(\delta^{-1}) = (1 + \delta^{-1})e^{-1} f^{in},$$

leading to an extreme transient growth if $\delta \ll 1$. Therefore, a pure eigenvalue analysis may be misleading when the involved operators are non-normal, which is a common feature in shear flows problems. Consequently, one should study deeper spectral properties of the linearized operator. This became evident with the paper of Trefethen et. al [148], where, by considering as an example the linear operator associated to the Couette and the Poiseuille flows, they investigated *pseudospectral properties*, meaning a characterization of the set of $\lambda \in \mathbb{C}$ where $\|(A - \lambda I)^{-1}\| \geq \epsilon^{-1}$, with A being the associated linear operator. Here, a careful analysis of the resolvent is required. In particular, "*if the pseudospectra protrude far into the upper half-plane, then substantial transient growth must be possible*"-[148]. If the operator is normal it is possible to show that the pseudospectrum consists of a set of balls of radius ϵ around the eigenvalues, while for a non-normal operator it is not possible to have a similar a priori geometrical description of this set.

1.2 Linear stability analysis for the incompressible Couette flow

We now study in detail the linear stability of the Couette flow $U(y) = y$ for a homogeneous and incompressible fluid and we review the most recent results related to shear flows problems.

The linearized system, in vorticity formulation, is

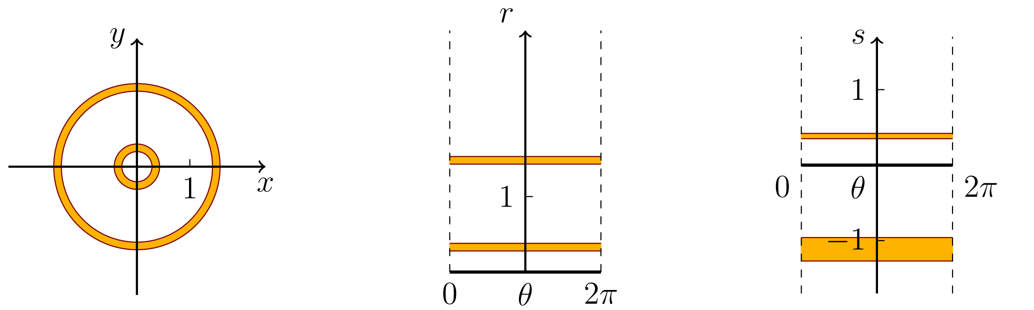
$$\partial_t \omega + y \partial_x \omega = \nu \Delta \omega, \quad \text{in } \mathbb{T} \times \mathbb{R},\ t \geq 0 \tag{1.9a}$$

$$v = \nabla^{\perp} \psi, \quad \Delta \psi = \omega, \tag{1.9b}$$

which, for $\nu = 0$, is (1.6) when $U(y) = y$. The domain is the infinite periodic strip $\mathbb{T} \times \mathbb{R}$, where $\mathbb{T}$ is the one dimensional torus (which we identify as $[0, 2\pi]$ with periodic boundary conditions). The choice of this domain is convenient in order to avoid boundary effects and contributions coming from low wavenumbers in x. The study of the Couette flow in $\mathbb{T} \times \mathbb{R}$ can be considered as a first step towards the understanding of the physically relevant case of perturbations around a radially symmetric vortex in the whole plane. Indeed, $\mathbb{T} \times \mathbb{R}$ is the domain that one would obtain by considering $\mathbb{R}^2$ in log-polar coordinates, see Figure 1.1. For more discussion about the relevance of this particular domain we refer to [16].

The explicit solution

When dealing with shear flows, in general one has a decoupling of the x-modes, see (1.8). In this case, we can perform the analysis at any fixed frequency. We denote the



(a) Cartesian coordinates (b) Polar coordinates (c) Log-polar coordinates

Figure 1.1: (a) Standard Cartesian coordinates $(x, y) \in \mathbb{R}^2$. (b) Polar coordinates defined as $x = r\cos(\theta), y = r\sin(\theta)$ with $(\theta, r) \in \mathbb{T} \times \mathbb{R}_+$. (c) Log-polar coordinates given by $x = e^s \cos(\theta), y = e^s \sin(\theta)$ with $(\theta, s) \in \mathbb{T} \times \mathbb{R}$.

Fourier transform as

$$\widehat{f}(k, \eta) = \frac{1}{2\pi} \iint_{\mathbb{T}\times\mathbb{R}} e^{-i(kx+\eta y)} f(x, y) dx dy,$$
$$f(x, y) = \frac{1}{2\pi} \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} e^{i(kx+\eta y)} \widehat{f}(k, \eta) d\eta,$$

and the x-average of the solution and the fluctuations around it as

$$f_0(y) = \frac{1}{2\pi} \int_{\mathbb{T}} f(x, y) dx, \qquad f_{\neq} = f - f_0.$$

From (1.9a), the x-average ω_0 of the solution satisfies the 1-D heat equation

$$\partial_t \omega_0 = \nu \partial_{yy} \omega_0, \quad \text{in } \mathbb{R},\ t \geq 0.$$

In particular, when $\nu = 0$, ω_0 is a conserved quantity. In order to solve (1.9a), we introduce the change of coordinates dictated by the background shear flow, i.e.

$$X = x - yt, \qquad Y = y.$$

We denote the functions in the new reference frame as $\Omega(t, X, Y) = \omega(t, X + tY, Y)$, $V(t, X, Y) = v(t, X + tY, Y)$. Then, from (1.9a) we deduce that

$$\partial_t \Omega = \nu(\partial_{XX} + (\partial_Y - t\partial_X)^2)\Omega.$$

Taking the Fourier transform in both variables we have

$$\partial_t \widehat{\Omega} = -\nu(k^2 + (\eta - kt)^2)\widehat{\Omega},$$

whose explicit solution is given by

$$\widehat{\Omega}(t, k, \eta) = e^{-\nu t\left(\frac{1}{3}k^2t^2 + (k^2+\eta^2) - \eta k t\right)} \widehat{\Omega}^{in}(k, \eta). \tag{1.10}$$

Thanks to the Biot-Savart's law, see (1.9b), we also get

$$\widehat{\boldsymbol{V}}(t, k, \eta) = \begin{pmatrix} i(\eta - kt) \\ -ik \end{pmatrix} \frac{e^{-\nu t\left(\frac{1}{3}k^2t^2 + (k^2+\eta^2) - \eta k t\right)}}{k^2 + (\eta - kt)^2} \widehat{\Omega}^{in}(k, \eta). \tag{1.11}$$

The two formulas above were already known by Kelvin in 1887, see [107, eq. (34)-(35)]. Let us now discuss separately the inviscid and the viscous case.

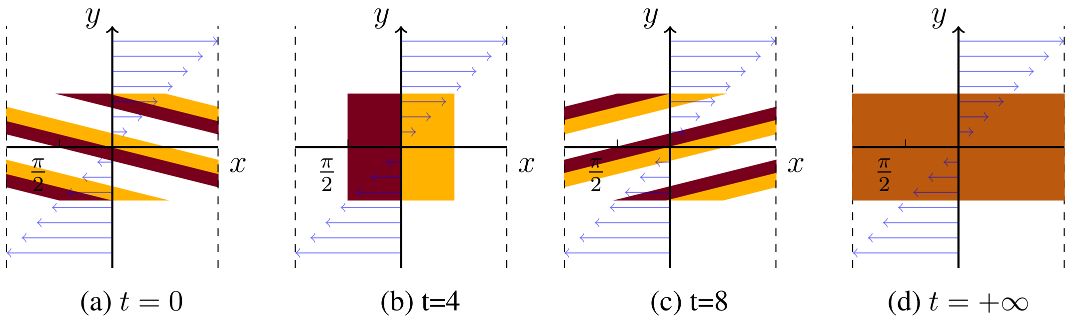


Figure 1.2: Evolution of the vorticity for $\nu = 0$ assuming constant and different values in the red and yellow regions. The blue arrows represent the background shear flow. In this case, the transient growth for the velocity field is caused by the un-mixing of the vorticity up to the critical time $t = 4$. Afterwards, the vorticity is being mixed along the streamlines of the Couette flow.

Mixing and inviscid damping

In the inviscid case, (1.9a) reduce to a simple transport equation for the vorticity. In particular, for $\nu = 0$, by (1.10) we infer that the vorticity is conserved along the streamlines of the Couette flow. Going back to the original variables we have

$$\widehat{\omega}(t,k,\eta) = \widehat{\omega}^{in}(k,\eta+kt), \quad \omega(t,x,y) = \omega^{in}(x-yt,y)$$

The first identity above is telling us that, for all the modes $k \neq 0$, there is a transfer of information to high frequencies. From the second one, due to the periodicity in x, we see that the vorticity is *mixed*, meaning that it is sent into smaller spatial scales as time goes on, see Figure 1.2.

A possible quantification of mixing is the decay of negative Sobolev norms [146]. In this particular case, one can easily infer explicit decay rates. Indeed, defining $\langle a,b\rangle^2 = 1 + a^2 + b^2$, for any $s \geq 0$ we get

$$\|\omega_{\neq}(t)\|^2_{H^{-s}} = \sum_{k\in\mathbb{Z}\setminus\{0\}} \int_{\mathbb{R}} \frac{\langle k,\eta\rangle^{2s}}{\langle k,\eta-kt\rangle^{2s}\langle k,\eta\rangle^{2s}} |\widehat{\omega}^{in}_{\neq}|^2(k,\eta)d\eta \lesssim \frac{1}{\langle t\rangle^{2s}} \left\|\omega^{in}_{\neq}\right\|^2_{H^s},$$

where in the last line we have used that $\langle a-b\rangle\langle a\rangle \gtrsim \langle b\rangle$. This implies Lyapunov stability from $H^s \to H^{-s}$ for any $s \geq 0$. Instead, one can deduce the Lyapunov instability from $L^2 \to H^s$ for any $s > 0$, since essentially any derivative in y result in a factor t.

Thanks to (1.11), we can deduce an interesting stabilizing effect for the velocity field, which was first observed by Orr in 1907 [131] and rigorously proved by Case in 1960 [36]. The denominator in the formula (1.11) reaches its maximum at $t = \eta/k$, also known as the *Orr's critical time*. This implies that the velocity experiences a *transient growth* up to the critical time after which, for any fixed frequency, it starts decaying. More precisely, from (1.11) we have

$$|\widehat{V}^x_{\neq}|(t,k,\eta) \leq \frac{1}{\sqrt{k^2+(\eta-kt)^2}}|\widehat{\Omega}^{in}_{\neq}|(k,\eta) \lesssim \frac{1}{|k|\langle t\rangle}\langle \eta/k\rangle |\widehat{\Omega}^{in}_{\neq}|(k,\eta).$$

Analogously

$$|\widehat{V}^y_{\neq}|(t,k,\eta) \lesssim \frac{1}{|k|\langle t\rangle^2}\langle \eta/k\rangle^2 |\widehat{\Omega}^{in}_{\neq}|(k,\eta).$$

Therefore, if we assume $\Omega^{in} \in H^2(\mathbb{T}\times\mathbb{R})$ one has the following decay estimates

$$\left\|V^x_{\neq}(t)\right\|_{L^2} \lesssim \frac{1}{\langle t\rangle}\left\|\Omega^{in}\right\|_{H^1}, \qquad \left\|V^y_{\neq}(t)\right\|_{L^2} \lesssim \frac{1}{\langle t\rangle^2}\left\|\Omega^{in}\right\|_{H^2}. \tag{1.12}$$

In addition, in view of the incompressibility and the conservation of the zero mode, we know that $\boldsymbol{v}_0(t,x,y) = (v_0^{in,x}(y),0)$. Hence, the bounds above give us the strong convergence of the velocity field towards the shear flow $(y+v_0^{in,x}(y),0)$. The linear Couette case highlights also another important feature. From (1.12) we see that the decay can be obtained only if the initial perturbation is regular enough. Even though we were not sharp with regularity assumptions to obtain (1.12), it is possible to show that if $\omega^{in} \in L^2$ then one still have that $\|\boldsymbol{v}\|_{L^2} \to 0$ strongly [116, 163]. However, explicit decay rates are not available.

The strong convergence to zero of the velocity field is now often called as *inviscid damping*, since it shares several analogies with the *Landau damping* observed in plasma physics. This connection was already drawn by Landau and Lifshitz [109]. For more details about the Landau damping we refer to the work of Caglioti and Maffei [32] and to the Mouhot and Villani's seminal paper [129], see also the more recent papers [18, 86].

The Couette flow is probably the simplest shear flow one can think of, however, the linear analysis just performed suggests many interesting mechanisms. Despite the simplicity of this flow, studying the full nonlinear inviscid dynamics gives rise to challenging mathematical problems, where most of the results have been obtained only in the last decade. In [116] Lin and Zeng proved the existence of non-parallel stationary states in any H^s neighbourhood of the Couette flow, with $s < 3/2$ for the vorticity and in the domain $\mathbb{T}\times[-1,1]$. This in particular implies that nonlinear inviscid damping, hence convergence towards a shear flow, cannot hold in this class. Then, a major breakthrough came in Bedrossian and Masmoudi's work [17], where it is shown nonlinear inviscid damping in $\mathbb{T}\times\mathbb{R}$ for perturbations in Gevrey-$1/s$ class[3] with $s > 1/2$. The nonlinear inviscid damping was obtained as a consequence of the asymptotic stability of the vorticity in this high-regularity space. The regularity requirements turned out to be sharp for the stability of the vorticity, as proved by Deng and Masmoudi in [55]. The nonlinear inviscid damping has been proved also in the channel $\mathbb{T}\times[0,1]$ by Ionescu and Jia [101], for perturbations supported away from the boundary. In principle, the inviscid damping may be valid also in the presence of instabilities for the vorticity, see for instance a related linear toy model proposed by Deng and Zillinger [56].

[3]A function f belongs to the Gevrey-$1/s$ class in $\mathbb{T}\times\mathbb{R}$ if

$$\|f\|^2_{\mathcal{G}^{\lambda,s}} = \sum_{k\in\mathbb{Z}}\int_{\mathbb{R}} e^{2\lambda(|k|+|\eta|)^s}|\widehat{f}|^2(k,\eta)d\eta < \infty.$$

When $s = 0$ we have a C^∞ function and for $s = 1$ we have an analytic function with radius of analyticity λ.

Similar behaviour to that of the Couette flow was expected also for more general shear flows, see [27] and references therein. However, the proof of similar results for more general shear flows is far from being trivial even at a linear level. In this direction, the linear inviscid damping has been proved by Zillinger [162, 163] for shear flows which are, in some sense, close to Couette. The result has been extended to a more general class of monotone shear flows, with completely different techniques, by Wei, Zhang and Zhao [153, 154]. Other available results concerning linear inviscid damping can be found in [57, 87, 104, 155, 164]. Ionescu and Jia [102] and Masmoudi and Zhao [125] proved, independently, the nonlinear inviscid damping for a class of monotone shear flows. Recently, Coti Zelati, Elgindi and Widmayer [48] proved the existence of (nonlinear) stationary structures near the Kolmogorov and the Poiseuille flows. A strongly related problem is the stability of vortices, considered by Bedrossian, Coti Zelati and Vicol [12] in the linear case. The nonlinear inviscid damping is valid also for perturbations around a point vortex, concentrated away from it, as proved by Ionescu and Jia [100].

Enhanced dissipation

We now turn our attention to the analysis of the viscous case. The equation (1.9a) is a particular case of advection-diffusion, where the vorticity can be considered as a passive scalar. When $\nu > 0$, we immediately have a stabilizing mechanism due to the dissipation. In particular, in absence of the advective term given by the background shear, one has decay on the standard diffusive time-scale $O(\nu^{-1})$. It is well-known that the presence of advection can enhance the dissipation, a mechanism also called *relaxation enhancement* or *shear-diffuse* extensively studied in the physics literature [4, 21, 64, 110, 119, 134]. An intuitive explanation can be given as follows: the advection transfer information to high frequencies (as we have seen previously) where dissipation is more efficient. In the Couette case, this mechanism can be fully characterized thanks to the explicit solution.

We first observe that for ω_0 there is no advection, hence it decays on the standard diffusive time-scale $O(\nu^{-1})$. More precisely, one has that

$$\|\omega_0(t)\|_{L^\infty} \lesssim \frac{1}{\sqrt{\nu t}} \|\omega_{0,in}\|_{L^1}, \quad \|\partial_y \omega_0(t)\|_{L^2} \lesssim \frac{1}{\sqrt{\nu t}} \|\omega_{0,in}\|_{L^2}, \tag{1.13}$$

see for example [79]. Instead, for all the other modes, first of all observe

$$\frac{1}{3}k^2t^2 + k^2 + \eta^2 - \eta k t = \frac{1}{3}k^2t^2 + k^2 + \left(\frac{1}{2}kt - \eta\right)^2 - \frac{1}{4}k^2t^2 \geq \frac{1}{12}k^2t^2.$$

Then, by combining Plancherel's Theorem with (1.10) we deduce

$$\|\Omega_{\neq}(t)\|_{L^2} \leq e^{-\frac{1}{12}\nu t^3} \left\|\Omega^{in}\right\|_{L^2}. \tag{1.14}$$

Consequently, from (1.11) we also get

$$\|\boldsymbol{V}_{\neq}(t)\|_{L^2} \leq e^{-\frac{1}{12}\nu t^3} \left\|\Omega^{in}\right\|_{L^2}.$$

Assuming regularity on the initial data, one can combine the exponential decay with the algebraic one obtained in (1.12) to have

$$\left\|V^x_{\neq}(t)\right\|_{L^2} \lesssim \frac{e^{-\frac{1}{12}\nu t^3}}{\langle t\rangle} \left\|\Omega^{in}\right\|_{H^1}, \qquad \left\|V^y_{\neq}(t)\right\|_{L^2} \lesssim \frac{e^{-\frac{1}{12}\nu t^3}}{\langle t\rangle^2} \left\|\Omega^{in}\right\|_{H^2}. \quad (1.15)$$

The estimates (1.14)-(1.15) give a dissipation time-scale of order $O(\nu^{-\frac{1}{3}})$, which is much faster with respect to the diffusive one for $\nu \ll 1$. Before times $O(\nu^{-\frac{1}{3}})$ the dissipation is not efficient and the inviscid dynamics dominate, meaning that we will see the transient growth and, having enough regularity, inviscid damping (1.15) for the velocity field. Hence, we will see a separation of time-scales between two phenomena: perturbations around Couette goes first towards the shear flow $(y + v^x_0(t, y), 0)$ on a time-scale $O(\nu^{-\frac{1}{3}})$. This is due to the inviscid mechanism of un-mixing/mixing of the vorticity, see Figure 1.2. Then, given the decay of the zero mode (1.13), the shear flow converges back to the Couette flow on a time-scale $O(\nu^{-1})$. This dynamics was essentially confirmed also at the nonlinear level by Bedrossian, Masmoudi and Vicol [19], under some high-regularity assumptions similar to the inviscid case.

In general, a precise quantification of the enhanced dissipation mechanism is much harder to obtain, even for a passive scalar. Mathematically rigorous results in this direction have started to appear only recently, probably one of the first one was given by Constantin et.al. in [43]. We discuss more about this mechanism for passive scalars in Chapter 2, where we study the relaxation-enhancement generated by circular flows in $\mathbb{R}^2$.

For other shear flows, at the linear level the presence of a nonlocal term (the last one in the left-hand side of (1.6a)) lead to challenging mathematical problems [42, 87, 155].

In the nonlinear framework, one of the first stability results was proved by Romanov [135] in 1973 for the Couette flow in an infinite 3-D pipe. Then, a question of great interest for the nonlinear stability of shear flows is the quantification of *transition thresholds*, which can be formulated as follows:

> Given two Banach spaces X, Y, find the smallest $\gamma = \gamma(X) \geq 0$ such that if
>
> $$\left\|\boldsymbol{v}^{in}\right\|_X \ll \nu^\gamma \quad \text{then} \quad \begin{cases} \|\boldsymbol{v}(t)\|_Y \ll 1 & \text{for all } t > 0, \\ \|\boldsymbol{v}(t)\|_Y \to 0 & \text{for } t \to \infty. \end{cases}$$
>
> For $\|\boldsymbol{v}^{in}\|_X \gtrsim \nu^\gamma$ instabilities are possible.

This problem, not in those mathematical terms, was essentially raised up right after the experiment of Reynolds [133], see also [88, 95], and has been extensively studied numerically [108, 137, 148]. From the mathematical point of view, we remark that γ crucially depends on the choice of the norms of the initial data. The enhanced dissipation and the inviscid damping mechanisms play a crucial role in determining γ (possibly in an optimal way). Recently, many results estimated the values of γ in different cases:

- 2-D Couette flow: in $\mathbb{T} \times \mathbb{R}$, Gevrey regularity $\gamma = 0$ [19], Sobolev regularity $\gamma \leq 1/3$ [123] and *critical* regularity $\gamma \leq 1/2$ [124]. In $\mathbb{T} \times [-1, 1]$, $X = H^2$ and $\gamma \leq 1/2$ [40].

- 2-D shear flows close to Couette: in $\mathbb{T} \times \mathbb{R}$ and Sobolev regularity $\gamma \leq 1/2$ [20].

- 2-D Poiseuille flow: in $\mathbb{T} \times \mathbb{R}$, X is a weighted L^2 space and $\gamma < 3/4$ [47].

- 2-D Couette flow in the Boussinesq approximation: in $\mathbb{T} \times \mathbb{R}$, Sobolev regularity [54, 166]. The values of γ depends on the anisotropy of the viscosity coefficients and the presence [166] or not [54] of a stably stratified temperature.

- 3-D Couette flow: in $\mathbb{T} \times \mathbb{R} \times \mathbb{T}$, Gevrey regularity $\gamma \leq 1$ [13]. In the same setting, for $\gamma = 2/3^+$ characterization of instabilities above the stability threshold [14]. Sobolev regularity $\gamma \leq 3/2$ [15] and recently improved to $\gamma \leq 1$ in H^2 [151]. In $\mathbb{T} \times [-1, 1] \times \mathbb{T}$, $X = H^2$ and $\gamma \leq 1$ [41].

- 3-D Couette flow in a uniform magnetic field (incompressible MHD equations): in $\mathbb{T} \times \mathbb{R} \times \mathbb{T}$, Sobolev regularity $\gamma \leq 1$ [117].

A more extensive discussion of results until 2018 is given in Bedrossian, Germain and Masmoudi's review paper [16].

Regarding the stability of vortices, the nonlinear enhanced dissipation for the Lamb-Oseen vortex was proved by Gallay [76], based on a crucial estimate given by Li, Wei and Zhang [113]. For a complete literature review of stability of vortices we refer to the papers of Gallay [77, 78].

1.3 Statement of the main results

We now introduce the main results obtained in this thesis. We begin our analysis with the study of decay properties of a passive scalar satisfying a linear advection-diffusion equation in the whole plane. The drift is given by a vortex with a power-law velocity field and we aim at studying mixing properties and the enhanced dissipation mechanism.

Then, we turn our attention to the linear stability analysis of shear flows for heterogeneous fluids. The mathematical literature, in this case, is substantially less developed with respect to the one in the homogeneous case.

For an isentropic compressible fluid, in the absence of external forces, we investigate in detail the stability properties of the homogeneous Couette flow in the Euler and Navier-Stokes case. In the inviscid case, we confirm an instability mechanism predicted in the physics literature. For a viscous compressible fluid, we prove that there is enhanced dissipation on a time-scale $O(\nu^{-\frac{1}{3}})$.

We finally consider a class of stratified shear flows, as in (1.5), for an incompressible and inviscid fluid under the action of gravity. In particular, we consider exponentially stratified densities and shear flows which are *close* to Couette. In this case, under the Miles-Howard criterion, we prove inviscid damping for the velocity field and the density.

Let us now present the main theorems obtained for three problems outlined above.

Mixing and enhanced dissipation for circular flows

In Chapter 2, we consider the linear advection-diffusion equation in the whole plane (2.1). The advection is driven by the radially symmetric velocity field

$$\boldsymbol{u}_q(x,y) = \big(x^2+y^2\big)^{q/2}\begin{pmatrix}-y\\ x\end{pmatrix}, \quad q \geq 1,$$

which is regular and divergence-free. As observed in (1.4), we also know that $\boldsymbol{u}_q$ is a stationary solution of the incompressible Euler equations, namely a vortex with a power-law velocity field. To state our main result, we need to introduce the following weighted L^2-based norm

$$\|g\|_X^2 := \int_0^\infty \int_{\mathbb{T}} |g(r,\theta)|^2 r\mathrm{d}r\mathrm{d}\theta + \int_0^\infty \int_{\mathbb{T}} r^{2(q-1)}|g(r,\theta)|^2 r\mathrm{d}r\mathrm{d}\theta,$$

where $(r,\theta)\in\mathbb{R}_+\times\mathbb{T}$ are the standard polar coordinates. For the sake of simplicity, we also assume to have an initial datum whose average over circles is zero, since their dynamics is not influenced by $\boldsymbol{u}_q$ and one has decay on the standard diffusive time-scale. In Chapter 2 we consider the general case. We denote with $\nu \geq 0$ the diffusivity coefficient. We then have the following.

Theorem 1.3.1. *Let $q\geq 1$ and $\nu\geq 0$. Assume that $\int_{\mathbb{T}} f^{in}(r,\theta)d\theta = 0$ for any $r\geq 0$.*
If $\nu = 0$ and $f^{in}\in H^1(\mathbb{R}^2)$ then for all $t\geq 0$

$$\|f(t)\|_{H^{-1}} \lesssim \frac{1}{(1+t)^{p_q}}\left\|f^{in}\right\|_{H^1}, \qquad p_q = \frac{2}{\max\{q,2\}}. \tag{1.16}$$

If $\|f^{in}\|_X < +\infty$ then there exists $\varepsilon_0\in(0,1)$ such that for every $\nu\in(0,1]$ and $t\geq 0$, there hold

$$\|f(t)\|_X \lesssim e^{-\varepsilon_0\lambda_\nu t}\left\|f^{in}\right\|_X, \qquad \lambda_\nu = \frac{\nu^{\frac{q}{q+2}}}{1+\frac{2(q-1)}{q+2}|\ln\nu|}. \tag{1.17}$$

The estimate (1.16) quantifies the mixing properties of the flow $\boldsymbol{u}_q$, namely how fast is the tracer stirred along the streamlines of the flow in the absence of diffusivity. The bound (1.17) entails enhanced dissipation, where the decay-rates depend on the "flatness" of $\boldsymbol{u}_q$ near the origin. We remark that having an exponential decay in the whole plane is not a priori trivial since we do not have the Poincaré inequality. A precise quantification of the time-scales of dissipation in the case of the radial flows considered here are of interest in the physics literature [134]. This result was obtained in collaboration with M. Coti Zelati [45].

The Couette flow in an isentropic compressible fluid

In Chapter 3, we deal with perturbations around the homogeneous Couette flow, i.e. (1.3) with $U(y) = y$, in an isentropic compressible fluid, see (1.1). We neglect external forces and we consider the domain $\mathbb{T} \times \mathbb{R}$. The linearized system is given in (3.1). We define the Helmholtz projection operators in the usual way, namely

$$\boldsymbol{v} = \nabla\Delta^{-1}(\mathrm{div}(\boldsymbol{v})) + \nabla^{\perp}\Delta^{-1}\omega := Q[\boldsymbol{v}] + P[\boldsymbol{v}],$$

where we recall $\nabla^{\perp} = (-\partial_y, \partial_x)$ and the vorticity is given by $\omega = \nabla^{\perp} \cdot \boldsymbol{v}$.

To state the main result in a simpler way, we assume that $\rho_0^{in} = \boldsymbol{v}_0^{in} = 0$ and we denote with $C_{in} = C_{in}(\rho^{in}, \alpha^{in}, \omega^{in})$ a suitable combination of Sobolev norms of the initial data. We remark that in this case the dynamics of the zero x-mode is not completely trivial and it is studied in Section (3.1). We have the following.

Theorem 1.3.2. *Let $\nu, \lambda \geq 0$ and $M > 0$ be such that $\nu + \lambda \leq 1/2$ and $M \leq \min\{(\nu+\lambda)^{-\frac{1}{2}}, \nu^{-\frac{1}{3}}\}$. Let $\rho^{in}, \boldsymbol{v}^{in} \in H^7(\mathbb{T} \times \mathbb{R})$ be such that $\rho_0^{in} = \boldsymbol{v}_0^{in} = 0$. Then, the following inequalities holds:*

$$\|Q[\boldsymbol{v}](t)\|_{L^2} + \frac{1}{M}\|\rho(t)\|_{L^2} \leq \langle t\rangle^{\frac{1}{2}} e^{-\frac{1}{32}\nu^{\frac{1}{3}}t} C_{in}, \tag{1.18}$$

$$\|P[\boldsymbol{v}]^x(t)\|_{L^2} \leq M\frac{e^{-\frac{1}{64}\nu^{\frac{1}{3}}t}}{\langle t\rangle^{\frac{1}{2}}} C_{in} + \frac{e^{-\frac{1}{12}\nu^{\frac{1}{3}}t}}{\langle t\rangle}\left\|\omega^{in}\right\|_{H^1}, \tag{1.19}$$

$$\|P[\boldsymbol{v}]^y(t)\|_{L^2} \leq M\frac{e^{-\frac{1}{64}\nu^{\frac{1}{3}}t}}{\langle t\rangle^{\frac{3}{2}}} C_{in} + \frac{e^{-\frac{1}{12}\nu^{\frac{1}{3}}t}}{\langle t\rangle^2}\left\|\omega^{in}\right\|_{H^2}. \tag{1.20}$$

Let $\nu = \lambda = 0$ and $s \geq 0$. Up to a nowhere dense set of initial data $\rho^{in}, \alpha^{in}, \omega^{in} \in H^s(\mathbb{T} \times \mathbb{R})$ one has

$$\|Q[\boldsymbol{v}](t)\|_{L^2} + \frac{1}{M}\|\rho(t)\|_{L^2} \geq \langle t\rangle^{\frac{1}{2}} C_{in}. \tag{1.21}$$

Notice that when $M = 0$ (and $\mathrm{div}(\boldsymbol{v}^{in}) = \rho^{in} = 0$), formally the estimates (1.19)-(1.20) give the same result as in the incompressible case considered in the previous section. In the inviscid case, the estimate (1.21) implies a Lyapunov instability for a generic class of initial data. This gives the first rigorous mathematical proof of an instability mechanism formally derived in the physics literature [5, 37, 38, 81, 82, 92], whereas the bounds (1.19)-(1.20) show the inviscid damping for the solenoidal component of the velocity field. In the viscous case, we obtain the first enhanced dissipation estimate for a heterogeneous fluid. The time-scale of dissipation is $O(\nu^{-\frac{1}{3}})$, same as in the incompressible case. However, from (1.18) we see the possibility of a large transient growth of order $O(\nu^{-\frac{1}{6}})$ due to the inviscid instability. This result is in agreement with the (linear) numerical simulation considered by Farrell and Ioannou [70], where, however, the order of magnitude with respect to ν were not explicitly observed. The results in the theorem above are obtained in collaboration with P. Antonelli and P. Marcati, in particular, the inviscid case was treated in [2].

Linear inviscid damping for exponentially stratified shear flows near Couette

In Chapter 4, we study linear stability properties for stratified shear flows, see (1.5), in a heterogeneous incompressible fluid under the action of gravity. The domain is $\mathbb{T} \times \mathbb{R}$. In particular, we consider an exponentially stratified density, i.e. $\bar{\rho} = e^{-\beta y}$ with $\beta \geq 0$, and shear flows close to Couette, meaning that $U' \approx 1$ and $U'' \approx 0$. The linearization around this equilibrium can be found in (4.3).

We need to introduce the parameter $R = \beta \mathfrak{g}$, where $\mathfrak{g}$ is gravity, which is also called the Brunt-Väisälä frequency and it is equal to the Richardson number multiplied by $(U')^2$. In view of the analogies between stratified fluids and the Boussinesq approximation, explained in Section 4.0.1, it is convenient to think of R and $\beta \geq 0$ as two *independent parameters*, see Remark 4.0.3. The result is stated in terms of a scaled density which, in the case $\beta > 0$, is given by $q = \rho/(\beta\bar{\rho})$. We again assume that $q_0^{in} = \omega_0^{in} = 0$ since in this case the x-averages are conserved quantities. We then have the following.

Theorem 1.3.3. *Let $R > 1/4$ and $\beta \geq 0$ be arbitrarily fixed. Let $\omega^{in} \in H^1(\mathbb{T} \times \mathbb{R})$ and $q^{in} \in H^2(\mathbb{T} \times \mathbb{R})$ be such that $q_0^{in} = \omega_0^{in} = 0$. There exists a small constant $\varepsilon_0 = \varepsilon_0(\beta, R) \in (0,1)$ with the following property. If $\varepsilon \in (0, \varepsilon_0]$ and*

$$\|U' - 1\|_{H^6} + \|U''\|_{H^5} \leq \varepsilon, \tag{1.22}$$

then for every $t \geq 0$ we have

$$\|q(t)\|_{L^2} + \|v^x(t)\|_{L^2} \lesssim \frac{1}{\langle t \rangle^{\frac{1}{2}-\delta_\varepsilon}} \left(\left\|\omega^{in}\right\|_{L^2} + \left\|q^{in}\right\|_{H^1} \right), \tag{1.23}$$

$$\|v^y(t)\|_{L^2} \lesssim \frac{1}{\langle t \rangle^{\frac{3}{2}-\delta_\varepsilon}} \left(\left\|\omega^{in}\right\|_{H^1} + \left\|q^{in}\right\|_{H^2} \right), \tag{1.24}$$

where $\delta_\varepsilon = 2\sqrt{\varepsilon}$.

The hypothesis (1.22) quantifies how close the background shear is to the Couette one. Since $U' \approx 1$ and we assume $R > 1/4$, the stability result holds essentially under the Miles-Howard criterion. The estimates (1.23)-(1.24) give us inviscid damping for the (scaled) density and the velocity field. The time-rates are slower with respect to the incompressible and homogeneous case. In the Couette case, it can be proved that this loss is compensated by a Lyapunov instability for the vorticity, as we show in Corollary 4.0.5. The inviscid damping for the Couette case was previously studied, with completely different techniques, in [91, 157]. The result for more general shear flows is new. This problem was done in collaboration with R. Bianchini and M. Coti Zelati [22].

CHAPTER 2

Mixing and enhanced dissipation for a passive scalar advected by circular flows in the whole plane

In this chapter, which is based on a joint work with M. Coti Zelati [45], we aim at studying decay properties of the solution to the following linear advection-diffusion equation

$$\begin{aligned} &\partial_t f + \boldsymbol{u}_q \cdot \nabla f = \nu \Delta f, \quad \text{in } \mathbb{R}^2,\ t \geq 0, \\ &f|_{t=0} = f^{in}, \end{aligned} \tag{2.1}$$

where f is a scalar function, $\nu \geq 0$ denotes the diffusivity coefficient, f^{in} is an assigned mean-free initial datum and $\boldsymbol{u}_q : \mathbb{R}^2 \to \mathbb{R}^2$ is a regular, radially symmetric and divergence-free velocity field given by

$$\boldsymbol{u}_q(x, y) = \left(x^2 + y^2\right)^{q/2} \begin{pmatrix} -y \\ x \end{pmatrix},$$

with $q \geq 1$ an arbitrary fixed exponent. In this way, the background velocity field generates a counter-clockwise rotating motion, which is also a stationary solution to the 2D incompressible Euler equations, see (1.4). In view of the radial symmetry of $\boldsymbol{u}_q$, it is convenient to rewrite (2.1) in polar coordinates $(r, \theta) \in [0, \infty) \times \mathbb{T}$, namely

$$\begin{aligned} &\partial_t f + r^q \partial_\theta f = \nu \Delta f, \quad \text{in } (r, \theta) \in [0, \infty) \times \mathbb{T},\ t \geq 0, \\ &f|_{t=0} = f^{in}, \end{aligned} \tag{2.2}$$

where here and in what follows, Δ denotes the Laplace operator in polar coordinates

$$\Delta = \partial_{rr} + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_{\theta\theta}.$$

At a first glance, similarities of the equation (2.2) with the linearization around the 2D Couette flow presented in Section 1.2 are evident. Indeed, at least heuristically, one expect that f is mixed along the streamlines of the background velocity field, which means an energy transfer towards small spatial scales, and, if diffusivity is present, this mechanism should enhance the dissipation being the viscosity more efficient at high frequencies, see Section 1.2. However, a rigorous justification of this scenario will be more involved with respect to the Couette case, where it was possible to have an explicit solution at hand.

Before stating the main results, we notice that the average of circles, defined for every $r \geq 0$ by

$$\langle f\rangle_\theta(t,r) = \frac{1}{2\pi}\int_{\mathbb{T}} f(t,r,\theta)\mathrm{d}\theta,$$

satisfies the two-dimensional radially symmetric heat equation

$$\begin{aligned} &\partial_t\langle f\rangle_\theta = \nu\big(\partial_{rr} + \frac{1}{r}\partial_r\big)\langle f\rangle_\theta, \quad \text{in } r\in[0,\infty),\ t\geq 0,\\ &\langle f\rangle_\theta|_{t=0} = \langle f^{in}\rangle_\theta, \end{aligned} \tag{2.3}$$

which is exactly the analogue of the zero x-mode in the Couette case. Hence, if $\nu = 0$ then $\langle f\rangle_\theta$ is conserved along the dynamics, otherwise $\langle f\rangle_\theta$ decay on the time-scale dictated by the classical diffusion process, namely proportional to $1/\nu$. Therefore, we are interested in studying decay properties of $f - \langle f\rangle_\theta$, which are going to depend on the exponent q. In the following we state the main results of this chapter and then we comment about them.

When the diffusivity is not present, i.e. $\nu = 0$, we adapt the result obtain by Coti Zelati, Delgadino and Elgindi [44] in the unit disk to the domain $\mathbb{R}^2$.

Theorem 2.0.1. *Let $q \geq 1$, $\nu = 0$ and $f^{in} \in H^1(\mathbb{R}^2)$ be the initial datum of* (2.2). *Then*

$$\left\|f(t) - \langle f^{in}\rangle_\theta\right\|_{H^{-1}} \leq \frac{C_0}{(1+t)^{p_q}}\left\|f^{in} - \langle f^{in}\rangle_\theta\right\|_{H^1}, \tag{2.4}$$

where

$$p_q = \frac{2}{\max\{q,2\}},$$

and C_0 is a constant depending only on q.

Then, for $\nu > 0$ the main original contribution of this chapter is a precise quantification of the dissipation enhancement effect in $\mathbb{R}^2$. Since we are working on the whole plane, we need to introduce a weighted L^2-based norm defined by

$$\|g\|_X^2 := \int_0^\infty\int_{\mathbb{T}} |g(r,\theta)|^2 r\mathrm{d}r\mathrm{d}\theta + \int_0^\infty\int_{\mathbb{T}} r^{2(q-1)}|g(r,\theta)|^2 r\mathrm{d}r\mathrm{d}\theta. \tag{2.5}$$

We then have the following.

Theorem 2.0.2. *Let $q \geq 1$ and f^{in} be such that $\|f^{in}\|_X < \infty$. There exist constants $\varepsilon_0 \in (0,1)$ and $C_0 \geq 1$ (explicitly computable and depending only on q) such that the following holds: for every $\nu \in (0,1]$ there hold the decay estimates*

$$\|\langle f(t)\rangle_\theta\|_{L^\infty} \leq \frac{C_0}{\sqrt{\nu t}}\|\langle f^{in}\rangle_\theta\|_{L^2}, \qquad \forall t\geq 0, \tag{2.6}$$

and

$$\|f(t) - \langle f(t)\rangle_\theta\|_X \leq C_0 \mathrm{e}^{-\varepsilon_0 \lambda_\nu t} \left\|f^{in} - \langle f^{in}\rangle_\theta\right\|_X, \qquad \forall t \geq 0, \tag{2.7}$$

where

$$\lambda_\nu = \frac{\nu^{\frac{q}{q+2}}}{1 + \frac{2(q-1)}{q+2}|\ln \nu|} \tag{2.8}$$

is the decay rate.

Theorem 2.0.1 describe the mixing properties of the radial flow under consideration. More precisely, the H^{-1} norm of the advected scalar quantity is one of the possible quantification of mixing [146], called *functional mixing scale*. The study of mixing is a problem of great interest in physics and engineering, with studies that dates back to the '50s [52, 66]. In recent times, the problem received an enormous interest also from the mathematical point of view, where an highly incomplete list of papers includes [1, 29, 160, 68, 103, 118, 138, 158, 150, 165]. For the specific case of radial flows under consideration, when the evolution is confined in a unit disk, the case $q = 2$ was studied in [51], where the authors focus on obtaining sharp estimates for the *geometric mixing scale*[1]. A general power-law in the unit disk was considered in [44], where the same decay time-rates of Theorem 2.0.1 were obtained. The proof of Theorem 2.0.1 is analogous to the one given in [44] by means of a stationary phase argument. For the sake of completeness we include a detailed proof here.

Theorem 2.0.2 instead describes the *shear-diffuse mechanism* that has been studied extensively in the physics literature [4, 64, 134, 110]. Rigorous mathematical results have started to appear only in recent times [30, 150, 53, 15, 44, 47, 43, 11, 152, 10, 19, 20, 13, 113, 9, 76, 99, 155, 14, 151, 114, 87], and the field has quickly attracted enormous interest. The main difficulty here is to quantify the separation of time-scales that happens between the evolution of θ-independent modes and the others, which are first advected towards small scales and then diffusion takes over.

In the radial case studied in this paper, the picture is quite intuitive and simple to describe: if $\nu \ll 1$, mixing *along* streamlines is most efficient on a time-scale between $1/\lambda_\nu$ and the classical diffusive one proportional to $1/\nu$. At these times, (2.7) becomes arbitrarily small as $\nu \to 0$ and the passive scalar tends to relax to a θ-independent state, represented by the average $\langle f\rangle_\theta$, which remains order 1 by (2.6). Then, diffusion takes over and dissipation happens mainly *across* streamlines (see Figure 2.1 below). It is thus evident the analogy with the Couette flow discussed in Section 1.2. From a quantitative standpoint, the enhanced dissipation mechanism is most efficient if the background flow is not "too flat" at the origin. In fact, this is the same that happens in the case of shear flows, in which the decay rate is determined exclusively by the flatness of the critical points (see [10]). The presence of the log-correction in (2.8) is probably an artifact of the proof (notice that this contribution is no present at $q = 1$), but the decay rates are otherwise sharp, as recently proved by Drivas and Coti Zelati [46]. The sharpness was also suggested by numerical evidences [127].

The main source of difficulties to prove Theorem 2.0.2 is the fact that we are working in the whole space $\mathbb{R}^2$. Indeed, on the one hand, we do not have a Poincaré inequality, so that exponential decay estimates are, generically speaking, far from being

[1]A quantification of mixing introduced by Bressan in [29].

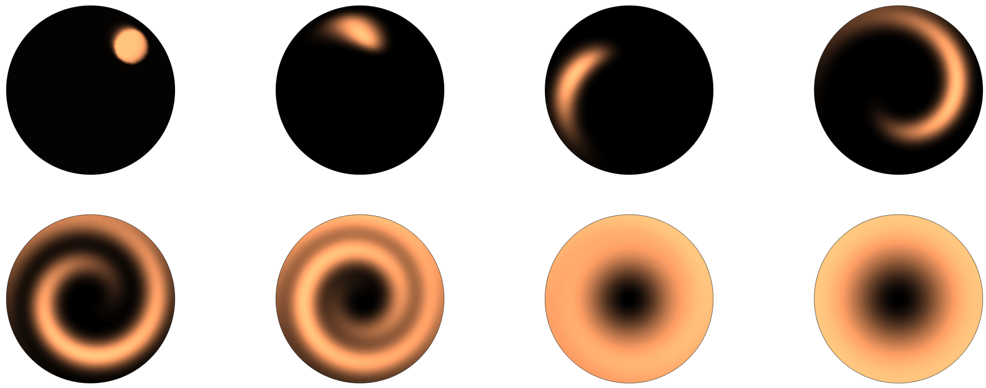


Figure 2.1: The evolution of an initial concentration subject radial stirring and small diffusion (think of milk stirred in a coffee mug!). At the beginning, advection is the main effect, while at the end the solution has reached radial symmetry and only undergoes diffusion.

trivial. On the other hand, the possibility of growth of the solution at infinity forces us to work in weighted spaces (hence the X-norm defined in (2.5)), adding a few technicalities in closing the estimates. To the best of our knowledge, the only other available result of enhanced dissipation in $\mathbb{R}^2$ was previously given by Gallay in [76] when considering the nonlinear stability analysis of the Lamb-Oseen vortex.

The proof of Theorem 2.0.2 relies on ideas originated in kinetic theory to study the long-time behavior of collisional models [59, 58, 94, 93], and is based on a technique known as *hypocoercivity* [149].

Combining the ideas introduced in this case and the those of [10], it is possible to treat the case of more general radial flows, where r^q in (2.2) is replaced by an arbitrary smooth function $u(r)$, and the case of a bounded domain (a disk), by imposing suitable no-flux boundary conditions on f. In this case, the weights in the X-norm in (2.5) become redundant, but no substantial change in the rate (2.8) is expected, except possibly in the case $q = 1$ (see [10]).

Outline of the chapter

The next Section 2.1 is devoted to the study of the inviscid problem and we present the proof of Theorem 2.0.1. In Section 2.2 we consider the $\nu > 0$ case and we aim at proving Theorem 2.0.2. The proof is divided in several steps. In Section 2.2.1 we first derive the energy balances that will be crucial in order to set up the hypocoercivity scheme. We also show two fundamental lemmas, that hold in general for radial functions, that we need to use in Section 2.2.2, where a Fourier-localized version of Theorem 2.0.2 is proven.

Notations and conventions

In what follows, we will use $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ for the standard real L^2 norm and scalar product, respectively defined as

$$\langle g,\tilde g\rangle=\int_0^\infty\int_{\mathbb{T}} g(r,\theta)\tilde g(r,\theta)r\mathrm{d}r\mathrm{d}\theta,\qquad \|g\|^2=\int_0^\infty\int_{\mathbb{T}}|g(r,\theta)|^2 r\mathrm{d}r\mathrm{d}\theta.$$

Given $g\in L^2$, we can expand it in Fourier series in the angular θ variable, namely

$$g(r,\theta)=\sum_{\ell\in\mathbb{Z}}\mathfrak{g}_\ell(r)\mathrm{e}^{i\ell\theta},\qquad \mathfrak{g}_\ell(r)=\frac{1}{2\pi}\int_0^{2\pi}g(r,\theta)\mathrm{e}^{-i\ell\theta}\mathrm{d}\theta.$$

It is worth noticing here that if a function g is smooth, then

$$\mathfrak{g}_\ell(r)\sim r^{|\ell|}(a_0+a_1r^2+a_2r^4+\ldots),\tag{2.9}$$

as it can be seen by simply applying the operator Δ to $\mathfrak{g}_\ell$ (see [7]). This is useful when performing integration by parts and make sure that no term at $r=0$ arises. We will often make use of functions that are localized on a single band in θ-frequency. Thus, for $k\in\mathbb{N}_0$ we set

$$g_k(r,\theta):=\sum_{|\ell|=k}\mathfrak{g}_\ell(r)\mathrm{e}^{i\ell\theta}.\tag{2.10}$$

This way we may write

$$g(r,\theta)=\sum_{k\in\mathbb{N}_0}g_k(r,\theta),$$

as a sum of *real-valued* functions g_k that are localized in θ-frequency on a single band $\pm k$, $k\in\mathbb{N}_0$. In particular, for the θ-average of a function g, we have $\langle g\rangle_\theta=g_0=\mathfrak{g}_0$.

We will not distinguish between the two dimensional $L^2(r\mathrm{d}r\mathrm{d}\theta)$ and the one dimensional $L^2(r\mathrm{d}r)$ spaces, as no dimensional property will be used. Notice that for θ-independent functions, the norms only differ by a constant.

In polar coordinates, the gradient and Laplace operators become

$$\nabla=\begin{pmatrix}\partial_r\\ \frac{1}{r}\partial_\theta\end{pmatrix},\qquad \Delta=\partial_{rr}+\frac{1}{r}\partial_r+\frac{1}{r^2}\partial_{\theta\theta},$$

respectively. Whenever we apply them to a function localized in one frequency band $k\in\mathbb{N}_0$ (see (2.10)), we will often tacitly use that

$$\|\nabla g_k\|^2=\|\partial_r g_k\|^2+k^2\left\|\frac{g_k}{r}\right\|^2.\tag{2.11}$$

Accordingly, we say that $g\in H^1(\mathbb{R}^2)$ if

$$\|g\|^2_{H^1}=\sum_{k\in\mathbb{N}_0}\|g_k\|^2+\|\partial_r g_k\|^2+k^2\left\|\frac{g_k}{r}\right\|^2<+\infty.$$

The corresponding $H^{-1}(\mathbb{R}^2)$ norm is then characterized by duality as follows

$$\|g\|_{H^{-1}}=\sup_{\|\varphi\|_{H^1}=1}\left|\sum_{k\in\mathbb{N}_0}\int_0^\infty g_k(r)\varphi_k(r)rdr\right|.\tag{2.12}$$

2.1 Mixing properties for $\nu = 0$

This section is devoted to the proof of Theorem 2.0.1. First of all, by setting $\nu = 0$ and taking the Fourier transform in the angular variable in (2.2), we get

$$\begin{aligned} &\partial_t f_k + ikr^q f_k = 0, \quad \text{for } k \in \mathbb{Z}\,, r \in [0,+\infty) \text{ and } t \geq 0, \\ &f_k|_{t=0} = f_k^{in}. \end{aligned} \tag{2.13}$$

Since $f_0 = \langle f(t) \rangle_\theta$ is conserved along the dynamics, namely $f_0 = f_0^{in}$, being the equation linear it is sufficient to prove the bound (2.4) when $f_0^{in} = 0$. It is then possible to solve explicitly the equation (2.13) and obtain that

$$f_k(t,r) = e^{-ikr^q t} f_k^{in}(r).$$

In view of the explicit formula, for $t \leq 1$ there is nothing to prove, hence we assume that $t \geq 1$. Then, we are going to proceed via a standard stationary phase argument, as done in [44] and in [10] in a related setting.

Proof of Theorem 2.0.1. The proof will be done for $f^{in} \in C_0^\infty(\mathbb{R}^2)$. The original statement can be recovered by passing to the limit in the norms (since $C_0^\infty(\mathbb{R}^2)$ is dense in $H^1(\mathbb{R}^2)$). In order to prove (2.4), in view of (2.12), let $\varphi \in H^1(\mathbb{R}^2)$ be such that $\|\varphi\|_{H^1} = 1$. We then need to control

$$\int_0^\infty f_k(r,t)\varphi_k(r) r dr = \int_0^\infty e^{-ikr^q t} f_k^{in}(r)\varphi_k(r) r dr,$$

which will be done by considering separately the case $1 \leq q < 2$ and $q \geq 2$.

Case $1 \leq q < 2$

Integrating by parts we deduce that

$$\begin{aligned} \int_0^\infty e^{-ikr^q t} f_k^{in}(r)\varphi_k(r) r dr &= -\frac{1}{iqkt}\int_0^\infty r^{1-q}\left(\frac{\mathrm{d}}{\mathrm{d}r} e^{-ikr^q t}\right) f_k^{in}(r)\varphi_k(r) r dr \\ &= \frac{1}{iqkt}\int_0^\infty e^{-ikr^q t}\frac{\mathrm{d}}{\mathrm{d}r}\left(f_k^{in}(r)\varphi_k(r) r^{2-q}\right) dr \end{aligned}$$

where the boundary term vanish since $1 \leq q < 2$. We now claim

$$\frac{1}{|k|qt}\left|\int_0^\infty e^{-ikr^q t}\frac{\mathrm{d}}{\mathrm{d}r}\left(f_k^{in}(r)\varphi_k(r) r^{2-q}\right) dr\right| \leq \frac{C_0}{|k|t}\left(\|f_k^{in}\| + \|\nabla f_k^{in}\|\right), \tag{2.14}$$

where C_0 depends only on q. Having the inequality above at hand, the proof of (2.4) follows by summing up in k.

To prove the last claim, since $1 \leq q < 2$ observe that

$$\begin{aligned} \int_0^\infty |(\partial_r f_k^{in})\varphi_k r^{2-q}| dr &\leq \int_0^1 |(\partial_r f_k^{in})\varphi_k r^{1-q}| r dr + \int_1^\infty |(\partial_r f_k^{in})\varphi_k| r dr \\ &\leq \int_0^1 |(\partial_r f_k^{in})\frac{1}{r}\varphi_k| r dr + \|\partial_r f_k^{in}\|\,\|\varphi_k\| \\ &\leq \|\partial_r f_k^{in}\|\left(\left\|\frac{\varphi_k}{r}\right\| + \|\varphi_k\|\right). \end{aligned}$$

Hence, in account of (2.11) and the fact that $\|\varphi\|_{H^1} = 1$ we conclude

$$\int_0^\infty |(\partial_r f_k^{in})\varphi_k r^{2-q}| dr \leq \left\|\partial_r f_k^{in}\right\|.$$

Analogously we have

$$\begin{aligned}\int_0^\infty |f_k^{in}(\partial_r\varphi)_k r^{2-q}| dr &\leq \|\partial_r\varphi_k\| \left(\left\|\frac{f_k^{in}}{r}\right\| + \left\|f_k^{in}\right\|\right)\\ &\leq \left\|\nabla f_k^{in}\right\| + \left\|f_k^{in}\right\|.\end{aligned}$$

Finally, observe that

$$\begin{aligned}\int_0^\infty |f_k^{in}\varphi_k r^{1-q}| dr &\leq \int_0^1 \left|\frac{f_k^{in}}{r}\frac{\varphi_k}{r}\right| r dr + \int_1^\infty |f_k^{in}\varphi_k| r dr\\ &\leq \left\|\frac{f_k^{in}}{r}\right\| \left\|\frac{\varphi_k}{r}\right\| + \left\|f_k^{in}\right\| \|\varphi_k\|\\ &\leq \left\|\nabla f_k^{in}\right\| + \left\|f_k^{in}\right\|,\end{aligned}$$

hence proving the claim (2.14) and concluding the proof of (2.4) in the case $1 \leq q < 2$.

Case $q \geq 2$

Let $R > 0$ to be chosen later and split the integral as follows

$$\int_0^\infty e^{-ikr^q t} f_k^{in}\varphi_k r dr = \int_0^R e^{-ikr^q t} f_k^{in}\varphi_k r dr + \int_R^\infty e^{-ikr^q t} f_k^{in}\varphi_k r dr.$$

Then notice that

$$\int_0^R |f_k^{in}||\varphi_k| r dr = \int_0^R r^2 \left|\frac{f_k^{in}}{r}\right| \left|\frac{\varphi_k}{r}\right| r dr \leq R^2 \left\|\frac{f_k^{in}}{r}\right\| \left\|\frac{\varphi_k}{r}\right\|.$$

For the second integral, integrating by parts we get

$$\begin{aligned}\int_R^\infty e^{-ikr^q t} f_k^{in}(r)\varphi_k(r) r dr =& -\frac{1}{ikqt}\int_R^\infty r^{1-q}\left(\frac{\mathrm{d}}{\mathrm{d}r}e^{-ikr^q t}\right) f_k^{in}(r)\varphi_k(r) r dr\\ =&\frac{1}{iqkt}\int_R^\infty e^{-ikr^q t}\frac{\mathrm{d}}{\mathrm{d}r}\left(f_k^{in}(r)\varphi_k(r) r^{2-q}\right) dr\\ &- \left[\frac{1}{ikt} r^{2-q} e^{-ikr^q t} f_k^{in}(r)\varphi_k(r)\right]_{r=R}^{r=+\infty},\end{aligned}$$

and, since $q \geq 2$, the only contribution coming from the boundary term is when $r = R$. In order to control this part, first notice that for any function ϕ in H^1 such that $\phi_0 = 0$ one has

$$\frac{1}{2}|\phi_k|^2(r) = \int_0^r \phi_k\partial_r\phi_k dr = \int_0^r \frac{\phi_k}{r}\partial_r\phi_k r dr \leq \left\|\frac{\phi_k}{r}\right\| \|\partial_r\phi_k\|,$$

meaning that $\|\phi_k\|_{L^\infty} \leq \|\nabla \phi_k\|$. Therefore we get

$$|R^{2-q} f_k^{in}(R) \varphi_k(R)| \leq R^{2-q} \left\|\nabla f_k^{in}\right\|,$$

where we have also used that $\|\varphi\|_{H^1} = 1$. To control the remaining integral, since $q > 2$ we directly have that

$$\int_R^\infty r^{2-q} \left(\left| (\partial_r f_k^{in}) \frac{\varphi_k}{r} \right| + \left| \frac{f_k^{in}}{r} (\partial_r \varphi_k) \right| + \left| \frac{f_k^{in}}{r} \frac{\varphi_k}{r} \right| \right) r dr \leq R^{2-q} \left\|\nabla f_k^{in}\right\|.$$

Hence, by combining the estimates above we get

$$\left| \int_0^\infty e^{-ikr^q t} f_k^{in} \varphi_k r dr \right| \leq \left(R^2 + R^{2-q} \frac{C_0}{|k|t} \right) \left\|\nabla f_k^{in}\right\|.$$

Choosing $R = \left(\frac{|k|t}{C_0} \right)^{-\frac{1}{q}}$ we conclude the proof of the Theorem 2.0.1. $\square$

2.2 Enhanced dissipation when $\nu > 0$

The main purpose of this section is to prove Theorem 2.0.2. As said, the proof is based on the hypocoercivity method whose steps, roughly speaking, can be summarized as follows. First look for the available energy balances, where a key point is the anti-symmetry of the transport operator and its commutation properties with differential operators. The second step is to set up an energy functional by suitably combining the available energy identities with the aim of finding a Grönwall's inequality. In particular, the last step require some non-trivial technical tools. We hence organize this section as follows.

In Section 2.2.1 we first derive the energy balances. Then, we present two useful inequalities that plays a central role to close the hypocoercivity scheme. The first one, given in Lemma 2.2.4, is essentially the spectral gap estimate given in [10, Proposition 2.7] and originally inspired by [75]. However, we will prove the inequality via a straightforward splitting and optimization. The second key inequality is presented in Lemma 2.2.5 and is reminiscent of an Hardy-type inequality.

In Section 2.2.2 we complete the set up of the hypocoercivity scheme and we prove Theorem 2.0.2.

2.2.1 Energy balances and two useful inequalities

In the following proposition we derive several energy balances from (2.2).

Proposition 2.2.1. *Let f be a smooth solution of* (2.2). *Then there hold the energy balances*

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|f\|^2+\nu\|\nabla f\|^2=0, \tag{2.15}$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla f\|^2+\nu\|\Delta f\|^2=-q\left\langle r^{q-1}\partial_\theta f,\partial_r f\right\rangle, \tag{2.16}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\langle r^{q-1}\partial_\theta f,\partial_r f\right\rangle+q\left\|r^{q-1}\partial_\theta f\right\|^2=-2\nu\left\langle r^{q-1}\partial_r\partial_\theta f,\Delta f\right\rangle-\nu q\left\langle r^{q-2}\partial_\theta f,\Delta f\right\rangle, \tag{2.17}$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|r^{q-1}\partial_\theta f\right\|^2+\nu\left\|r^{q-1}\partial_\theta\nabla f\right\|^2=2\nu(q-1)^2\left\|r^{q-2}\partial_\theta f\right\|^2. \tag{2.18}$$

Remark 2.2.2. By projecting (2.2) onto the ℓ-th Fourier mode, namely perform the Fourier transform in θ, we see that the dynamics of each mode is decoupled. Therefore, in the proposition above we can replace f by f_ℓ, ∂_θ by $i\ell$ and consider only the real part of the scalar products involved. In the next section we discuss more about this point.

Remark 2.2.3. Notice that the most singular term in the origin, for $q\in(1,2)$, is the one on the right-hand side of (2.18), namely $|k|\,\|r^{q-2}f_k\|$ (when localized in a frequency band k). In view of (2.9), if f is smooth we have that

$$f_k(r,\theta)\sim r^k,\qquad \text{as } r\to 0,$$

making the term finite for $k\geq 1$.

Proof of Proposition 2.2.1. In the proof, we will use several times the antisymmetry of the the transport operator, namely that

$$\langle r^q\partial_\theta f,g\rangle=-\langle f,r^q\partial_\theta g\rangle, \tag{2.19}$$

for every f,g sufficiently regular.

The equality (2.15) follows by multiplying (2.2) by f, integrating by parts and using (2.19). To prove (2.16), multiply (2.2) by $r\Delta f$ and integrate to get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\langle f,\Delta f\rangle+\langle r^q\partial_\theta f,\Delta f\rangle=\nu\|\Delta f\|^2,$$

so that integrating by parts we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla f\|^2+\left\langle\nabla\left(r^q\partial_\theta f\right),\nabla f\right\rangle+\nu\|\Delta f\|^2=0.$$

Then just observe that by (2.19) we have

$$\left\langle\nabla\left(r^q\partial_\theta f\right),\nabla f\right\rangle=\langle r^q\partial_\theta\nabla f,\nabla f\rangle+q\left\langle r^{q-1}\partial_\theta f,\partial_r f\right\rangle=q\left\langle r^{q-1}\partial_\theta f,\partial_r f\right\rangle,$$

so that (2.16) follows.

We now turn to (2.17). Using (2.2) we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left\langle \partial_\theta f, r^{q-1}\partial_r f\right\rangle =&\nu\left[\left\langle \partial_\theta \Delta f, r^{q-1}\partial_r f\right\rangle + \left\langle \partial_\theta f, r^{q-1}\partial_r \Delta f\right\rangle\right]\\ &-\left\langle r^q\partial_{\theta\theta} f, r^{q-1}\partial_r f\right\rangle\\ &-\left\langle \partial_\theta f, r^{q-1}r^q\partial_r\partial_\theta f\right\rangle - q\left\langle r^{q-1}\partial_\theta f, r^{q-1}\partial_\theta f\right\rangle.\end{aligned}$$

Integrating by parts the third term in the right-hand side above, we have

$$-\left\langle r^q\partial_{\theta\theta} f, r^{q-1}\partial_r f\right\rangle = \left\langle r^q\partial_\theta f, r^{q-1}\partial_\theta\partial_r f\right\rangle,$$

which cancels with the fourth term. Moreover,

$$\begin{aligned}\left\langle \partial_\theta f, r^{q-1}\partial_r\Delta f\right\rangle &= \left\langle r^q\partial_\theta f, \frac{1}{r}\partial_r \Delta f\right\rangle = -\left\langle \frac{1}{r}\partial_r\left(r^q\partial_\theta f\right), \Delta f\right\rangle\\ &= -\left\langle r^{q-1}\partial_r\partial_\theta f, \Delta f\right\rangle - q\left\langle r^{q-2}\partial_\theta f, \Delta f\right\rangle.\end{aligned}$$

So thanks to these computations we get that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\langle \partial_\theta f, r^{q-1}\partial_r f\right\rangle + q\left\|r^{q-1}\partial_\theta f\right\|^2 = -2\nu\left\langle r^{q-1}\partial_r\partial_\theta f, \Delta f\right\rangle - \nu q\left\langle r^{q-2}\partial_\theta f, \Delta f\right\rangle,$$

hence proving (2.17).

Finally, to prove (2.18), we use (2.19) and compute the following

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|r^{q-1}\partial_\theta f\right\|^2 &= \nu\left\langle r^{q-1}\partial_\theta f, r^{q-1}\partial_\theta\Delta f\right\rangle - \left\langle r^{q-1}\partial_\theta f, r^{q-1}r^q\partial_{\theta\theta} f\right\rangle\\ &= \nu\left\langle r^{2(q-1)}\partial_\theta f, \partial_\theta\Delta f\right\rangle = -\nu\left\langle \nabla\left(r^{2(q-1)}\partial_\theta f\right), \partial_\theta\nabla f\right\rangle.\end{aligned}$$

Then computing the last term and using once more (2.19), we have

$$\begin{aligned}-\nu\left\langle \nabla\left(r^{2(q-1)}\partial_\theta f\right), \partial_\theta\nabla f\right\rangle =& -\nu\left\|r^{q-1}\partial_\theta\nabla f\right\|^2\\ &- 2\nu(q-1)\left\langle r^{2(q-1)}\partial_\theta f, \frac{1}{r}\partial_r\partial_\theta f\right\rangle.\end{aligned}$$

Rewrite the last scalar product of the previous equality as

$$\begin{aligned}\left\langle r^{2(q-1)}\partial_\theta f, \frac{1}{r}\partial_r\partial_\theta f\right\rangle &= -\left\langle \frac{1}{r}\partial_r\left(r^{2(q-1)}\partial_\theta f\right), \partial_\theta f\right\rangle\\ &= -\left\langle \frac{1}{r}\partial_r\partial_\theta f, r^{2(q-1)}\partial_\theta f\right\rangle\\ &\quad - 2(q-1)\left\langle \frac{1}{r}r^{2(q-1)-1}\partial_\theta f, \partial_\theta f\right\rangle,\end{aligned}$$

or equivalently we have that

$$\left\langle r^{2(q-1)}\partial_\theta f, \frac{1}{r}\partial_r\partial_\theta f\right\rangle = -(q-1)\left\|r^{q-2}\partial_\theta f\right\|^2.$$

So putting everything together we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|r^{q-1}\partial_\theta f\right\|^2 = -\nu\left\|r^{q-1}\partial_\theta\nabla f\right\|^2 + 2\nu(q-1)^2\left\|r^{q-2}\partial_\theta f\right\|^2,$$

proving (2.18). The proof is over. □

Now we present the two key inequalities, whose proof is postponed at the end of this subsection. One of the main point in the proof of Theorem 2.0.2 is to set up a proper energy functional and derive a Grönwall type estimate. In view of Proposition 2.2.1, we immediately see that $\|f\|$ does not appear on the left-hand side of (2.15)-(2.18), therefore we need to recover the L^2 norm from some term already present on the left-hand side of (2.15)-(2.18). In particular, we have the following.

Lemma 2.2.4. *Let $q \geq 1$, and let $g \in H^1(\mathbb{R}^2)$ be such that $r^{q-1}g \in L^2(\mathbb{R}^2)$. Assume $k \in \mathbb{N}$, and let g_k be defined as in* (2.10)*. Then, for any $\sigma > 0$ the following inequality holds*

$$\sigma^{\frac{q-1}{q}} \|g_k\|^2 \leq \sigma \left\|\frac{g_k}{r}\right\|^2 + \left\|r^{q-1}g_k\right\|^2 \leq \sigma \|\nabla g_k\|^2 + \left\|r^{q-1}g_k\right\|^2. \tag{2.20}$$

Another crucial point is the control of the term on the right-hand side of (2.18). In view of (2.11), one may think to exploit the fact that

$$2\nu(q-1)^2 \left\|r^{q-2}(\partial_\theta f)_k\right\|^2 \leq \frac{2\nu(q-1)^2}{|k|^2} \left\|r^{q-1}(\partial_\theta \nabla f)_k\right\|^2 \tag{2.21}$$

in order to absorb this term on the left-hand side of (2.18) (clearly for $k \neq 0$). However, this would possible only if $(q-1) < |k|/\sqrt{2}$. To overcome this difficulty we have the following.

Lemma 2.2.5. *Let $q \geq 1$ and $g \in H^1(\mathbb{R}^2)$ be such that $r^{q-1}g \in L^2(\mathbb{R}^2)$. Assume $k \in \mathbb{N}$, and let g_k be defined as in* (2.10)*. There exists a constant $c_q \geq 1$ depending only on q such that*

$$\frac{1}{c_q}\sigma^{\frac{1}{q}} \left\|r^{q-2}g_k\right\|^2 \leq \sigma \|\nabla g_k\|^2 + \left\|r^{q-1}g_k\right\|^2, \tag{2.22}$$

for any $\sigma > 0$.

Remark 2.2.6. The exponents of σ given in Lemma 2.2.4 and 2.2.5 are the natural ones obtained by a simple scaling argument.

Remark 2.2.7. The constant c_q in Lemma 2.2.5 is explicitly given, indeed, for $q \geq 2$ is $c_q = 2$, for $q = 1$ is $c_q = 1$ and for $1 < q < 2$ is $c_q = q^{-1}(1-q)^{-\frac{1}{q}}$, hence going to infinity for $q \to 1$. However, in account of (2.21) we do not need to use Lemma 2.2.5 for, let's say, $1 < q < 1+1/(2\sqrt{2})$, meaning that we will always have a uniform bound on c_q.

We conclude this subsection with the proof of the two lemmas above.

Proof of Lemma 2.2.4. The proof will be performed for a smooth function g. It is clear that the original statement holds upon passing to the limit in the various norms. Let $R > 0$ to be chosen, then

$$\begin{aligned}
\sigma^{\frac{q-1}{q}} \|g_k\|^2 &= \sigma^{\frac{q-1}{q}} \iint_{r\leq R} r^2 \frac{|g_k|^2}{r^2} r\mathrm{d}r\mathrm{d}\theta + \sigma^{\frac{q-1}{q}} \iint_{r>R} \frac{1}{r^{2(q-1)}} r^{2(q-1)}|g_k|^2 r\mathrm{d}r\mathrm{d}\theta \\
&\leq \sigma^{\frac{q-1}{q}} R^2 \left\|\frac{g_k}{r}\right\|^2 + \sigma^{\frac{q-1}{q}} R^{-2(q-1)} \left\|r^{q-1}g_k\right\|^2,
\end{aligned}$$

where in the last line we use the fact that $q \geq 1$. Hence choosing

$$R = \sigma^{\frac{1}{2q}},$$

we infer that

$$\sigma^{\frac{q-1}{q}} \|g_k\|^2 \leq \sigma \left\| \frac{g_k}{r} \right\|^2 + \left\| r^{q-1} g_k \right\|^2.$$

The second inequality simply follows by (2.11). Hence we have proved the lemma. □

Proof of Lemma 2.2.5. We preliminary note that the case $q = 1$ is trivial, since $\|g_k/r\| \leq \|\nabla g_k\|$ so (2.22) holds with $c_q = 1$. As for the rest, we divide the proof in two cases, namely $1 < q < 2$ and $q \geq 2$.

Case $q \geq 2$

In this case, let $R > 0$ to be chosen later. Then we get that

$$\begin{aligned}
\sigma^{\frac{1}{q}} \left\| r^{q-2} g_k \right\|^2 &= \sigma^{\frac{1}{q}} \iint_{r \leq R} r^{2(q-2)} |g_k|^2 r \mathrm{d}r \mathrm{d}\theta + \sigma^{\frac{1}{q}} \iint_{r > R} \frac{1}{r^2} r^{2(q-1)} |g_k|^2 r \mathrm{d}r \mathrm{d}\theta \\
&\leq \sigma^{\frac{1}{q}} R^{2(q-2)} \|g_k\|^2 + \sigma^{\frac{1}{q}} R^{-2} \left\| r^{q-1} g_k \right\|^2,
\end{aligned}$$

where in the last line, for the first term, we have used the fact that $q \geq 2$. Now we take advantage of Lemma 2.2.4, so choose

$$R = \sigma^{\frac{1}{2q}},$$

to get that

$$\begin{aligned}
\sigma^{\frac{1}{q}} \left\| r^{q-2} g_k \right\|^2 &\leq \sigma^{\frac{q-1}{q}} \|g_k\|^2 + \left\| r^{q-1} g_k \right\|^2 \\
&\leq \sigma \|\nabla g_k\|^2 + 2 \left\| r^{q-1} g_k \right\|^2,
\end{aligned}$$

and the proof is completed with $c_q = 2$.

Case $1 < q < 2$

Writing down explicitly, we have that

$$\begin{aligned}
\left\| r^{q-2} g_k \right\|^2 &= \iint r^{2(q-2)} |g_k|^2 r \mathrm{d}r \mathrm{d}\theta = \frac{1}{2(q-2)+2} \iint \partial_r (r^{2(q-2)+2}) |g_k|^2 \mathrm{d}r \mathrm{d}\theta \\
&= -\frac{1}{q-1} \iint r^{2(q-2)+1} g_k \partial_r g_k r \mathrm{d}r \mathrm{d}\theta, \qquad (2.23)
\end{aligned}$$

where the boundary term disappear since $1 < q < 2$. Now let $a > 0$ to be chosen later and rewrite the previous integral, without constants in front, as follows

$$\begin{aligned}
&\left| \iint (\partial_r g_k) g_k r^{2(q-2)+1} r \mathrm{d}r \mathrm{d}\theta \right| \\
&\quad \leq \iint |\partial_r g_k| \frac{|g_k|^a}{r^a} r^{2(q-2)+1+a} |g_k|^{1-a} r \mathrm{d}r \mathrm{d}\theta \\
&\quad \leq \|\partial_r g_k\| \left(\iint \frac{|g_k|^{ap}}{r^{ap}} r \mathrm{d}r \mathrm{d}\theta \right)^{\frac{1}{p}} \left(\iint r^{(2(q-2)+1+a)p'} |g_k|^{(1-a)p'} r \mathrm{d}r \mathrm{d}\theta \right)^{\frac{1}{p'}}, \qquad (2.24)
\end{aligned}$$

where we have used Hölder's inequality, in particular we need $\frac{1}{p} + \frac{1}{p'} = \frac{1}{2}$. Now we impose that

$$ap = 2, \qquad (2(q-2) + 1 + a)p' = 2(q-1).$$

Solving the previous linear system, combined with the Hölder constrains, gives that

$$a = \frac{2-q}{q}, \qquad p = \frac{2q}{2-q}, \qquad p' = \frac{q}{q-1}.$$

Then by (2.23), (2.24) and the choice of the exponents, we have that

$$(q-1)\left\|r^{q-2}g_k\right\|^2 \leq \|\partial_r g_k\| \left\|\frac{g_k}{r}\right\|^{\frac{2-q}{q}} \left\|r^{q-1}g_k\right\|^{\frac{2(q-1)}{q}}. \tag{2.25}$$

Then multiply (2.25) by $k^{\frac{2-q}{q}}$, to get that

$$\begin{aligned} k^{\frac{2-q}{q}}(q-1)\left\|r^{q-2}g_k\right\|^2 &\leq \|\partial_r g_k\| \left\|\frac{(\partial_\theta g)_k}{r}\right\|^{\frac{2-q}{q}} \left\|r^{q-1}g_k\right\|^{\frac{2(q-1)}{q}} \\ &\leq \|\nabla g_k\|^{\frac{2}{q}} \left\|r^{q-1}g_k\right\|^{\frac{2(q-1)}{q}}, \end{aligned}$$

where in the last line it is crucial that $1 < q < 2$. Then since $1/q + (q-1)/q = 1$, we apply Young's inequality, with a free parameter λ, to get that

$$k^{\frac{2-q}{q}}(q-1)\left\|r^{q-2}g_k\right\|^2 \leq \frac{1}{q}\lambda^q \|\nabla g_k\|^2 + \frac{q-1}{q}\lambda^{-\frac{q}{q-1}} \left\|r^{q-1}g_k\right\|^2,$$

or equivalently

$$k^{\frac{2-q}{q}} q\lambda^{\frac{q}{q-1}} \left\|r^{q-2}g_k\right\|^2 \leq \frac{1}{q-1}\lambda^{\frac{q^2}{q-1}} \|\nabla g_k\|^2 + \left\|r^{q-1}g_k\right\|^2.$$

Then let $\sigma = (q-1)^{-1}\lambda^{q^2/(q-1)}$ and we get that

$$k^{\frac{2-q}{q}} q(q-1)^{\frac{1}{q}}\sigma^{\frac{1}{q}} \left\|r^{q-2}g_k\right\|^2 \leq \sigma \|\nabla g_k\|^2 + \left\|r^{q-1}g_k\right\|^2.$$

Since $k \geq 1$, this concludes the proof of Lemma 2.2.5 by choosing $c_q = q^{-1}(q-1)^{-\frac{1}{q}}$. $\square$

2.2.2 Hypocoercivity setting

In this subsection we prove Theorem 2.0.2 by proceeding in several steps. Firstly, we decouple equation (2.2) in the various Fourier modes in θ. Then we set up a proper energy functional that satisfies a suitable inequality that allows to prove an enhanced decay at the rate (2.8) *without* the log-correction. As this functional involves norms of higher derivatives, we then show how to pass to the weighted L^2 estimate (2.7) by giving up a log-correction in the rate. All the estimates that we perform in this section are done in a smooth setting. Then one recover the general case by a standard approximation argument and passing to the limit.

The equation mode-by-mode

By taking the Fourier transform in θ of (2.2), we deduce that for each $\ell \in \mathbb{Z}$ we have

$$\begin{aligned} &\partial_t \mathfrak{f}_\ell + i\ell r^q \mathfrak{f}_\ell = \nu \left(\partial_{rr} + \frac{1}{r}\partial_r - \frac{\ell^2}{r^2} \right) \mathfrak{f}_\ell, \quad \text{in } r \in [0,\infty),\ t \geq 0, \\ &\mathfrak{f}_\ell|_{t=0} = \mathfrak{f}_\ell^{in}. \end{aligned} \tag{2.26}$$

It is apparent that the equation decouples in the Fourier modes, and one can treat each equation separately. However, this implies a great deal of notation due to the fact that the various $\mathfrak{f}_\ell$'s are complex functions. Instead, we prefer to study (2.2) with an initial datum concentrated on a Fourier band $|\ell| = k \in \mathbb{N}_0$, since none of the estimates involved are the same for the mode k and $-k$. Hence, we will study, as explained in (2.10), the evolution of

$$f_k(t, r, \theta) := \sum_{|\ell|=k} \mathfrak{f}_\ell(t, r) \mathrm{e}^{i\ell\theta},$$

which satisfies (2.2) but also has nice property with respect to norms, such as (2.11). With this convention, observe that Proposition 2.2.1 holds true with f replaced by f_k.

Remark 2.2.8 (The zeroth mode)**.** As mentioned in the introduction, f_0 satisfies the diffusion equation (2.3), which is precisely the two-dimensional heat equation under radial symmetry assumptions. Estimate (2.6) is therefore a classical one, which can be proven via the Green's function of the heat operator (see e.g. [79]).

The modified energy functional

From now on we will always assume that $k \geq 1$. The main result is the following theorem which provide us with the Grönwall estimate for a suitable energy functional.

Theorem 2.2.9. *Let $q \geq 1$. There exists a constant $\varepsilon_0 \in (0,1)$ such that the following holds: there exist positive numbers $\alpha_0, \beta_0, \gamma_0$ only depending on ε_0 such that for each integer $k \geq 1$ and $\nu > 0$ with $\nu k^{-1} \leq 1$ the energy functional*

$$\begin{aligned} \Phi_k = \frac{1}{2} \Bigg[&\|f_k\|^2 + \alpha_0 \frac{\nu^{\frac{2}{q+2}}}{k^{\frac{2}{q+2}}} \|\nabla f_k\|^2 + 2q\beta_0 \frac{\nu^{\frac{2-q}{q+2}}}{k^{\frac{4}{q+2}}} \left\langle r^{q-1}\partial_\theta f_k, \partial_r f_k \right\rangle \\ &+ \gamma_0 \frac{\nu^{\frac{-2(q-1)}{q+2}}}{k^{\frac{6}{q+2}}} \left\| r^{q-1} \partial_\theta f_k \right\|^2 \Bigg] \end{aligned} \tag{2.27}$$

satisfies the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t} \Phi_k + 2\varepsilon_0 \nu^{\frac{q}{q+2}} k^{\frac{2}{q+2}} \Phi_k \leq 0, \tag{2.28}$$

for all $t \geq 0$.

Notice that the previous theorem already encode an enhanced dissipation estimate for the functional Φ_k, which however is not sufficient to directly prove Theorem 2.0.2 since Φ_k require a control on the H^1 norm, whereas Theorem 2.0.2 is stated only in terms of the weighted L^2 norm. Nevertheless, the main Theorem 2.0.2 follows as a consequence of Theorem 2.2.9 by essentially exploiting smoothing properties of the heat kernel. Since the proof is not straightforward and requires some additional care to close the estimate in the space X, we postpone the proof of Theorem 2.0.2 to the end of this chapter.

We now proceed with the proof Theorem 2.2.9.

Proof of Theorem 2.2.9. For simplicity of notation we will always omit the subscript k throughout all the proof. Also, in the course of this proof, c_1, c_2 and c_3 will denote specific constants, depending on q and independent of $\nu, k, \alpha_0, \beta_0, \gamma_0$. Define the following energy functional

$$\Phi = \frac{1}{2}\left[\|f\|^2 + \alpha\|\nabla f\|^2 + 2q\beta\left\langle r^{q-1}\partial_\theta f, \partial_r f\right\rangle + \gamma\left\|r^{q-1}\partial_\theta f\right\|^2\right],$$

where α, β, γ are chosen in accordance with (2.27) as

$$\alpha = \alpha_0\frac{\nu^{\frac{2}{q+2}}}{k^{\frac{2}{q+2}}}, \qquad \beta = \beta_0\frac{\nu^{\frac{2-q}{q+2}}}{k^{\frac{4}{q+2}}}, \qquad \gamma = \gamma_0\frac{\nu^{\frac{-2(q-1)}{q+2}}}{k^{\frac{6}{q+2}}}. \tag{2.29}$$

As we go along the proof, we will identify several constraints on such coefficients and show at the end that they can be satisfied. In addition, also the powers of ν, k appearing in (2.29) are dictated by those constrains, as we explain in Remark 2.2.10.

First of all, observe that

$$2q\beta|\left\langle r^{q-1}\partial_\theta f, \partial_r f\right\rangle| \leq 2q\beta\left\|r^{q-1}\partial_\theta f\right\|\,\|\partial_r f\| \leq \frac{\alpha}{2}\|\nabla f\|^2 + \frac{2q^2\beta^2}{\alpha}\left\|r^{q-1}\partial_\theta f\right\|^2.$$

As a first condition on the coefficients α, β, γ, we require that

$$\frac{\beta^2}{\alpha\gamma} \leq \frac{1}{4q^2}. \tag{2.30}$$

In this way, (2.30) guarantees that

$$\begin{aligned}\Phi &\geq \frac{1}{4}\left(2\|f\|^2 + \alpha\|\nabla f\|^2 + \gamma\left\|r^{q-1}\partial_\theta f\right\|^2\right)\\ \Phi &\leq \frac{1}{4}\left(2\|f\|^2 + 3\alpha\|\nabla f\|^2 + 3\gamma\left\|r^{q-1}\partial_\theta f\right\|^2\right).\end{aligned} \tag{2.31}$$

Then, thanks to the energy balances given in Proposition 2.2.1, we have that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Phi + &\nu\|\nabla f\|^2 + \alpha\nu\|\Delta f\|^2 + \beta q^2\left\|r^{q-1}\partial_\theta f\right\|^2 + \gamma\nu\left\|r^{q-1}\partial_\theta\nabla f\right\|^2\\ &= -\alpha q\left\langle r^{q-1}\partial_\theta f, \partial_r f\right\rangle - 2\beta q\nu\left\langle r^{q-1}\partial_r\partial_\theta f, \Delta f\right\rangle\\ &\quad - \beta q^2\nu\left\langle r^{q-2}\partial_\theta f, \Delta f\right\rangle + 2\gamma\nu(q-1)^2\left\|r^{q-2}\partial_\theta f\right\|^2.\end{aligned} \tag{2.32}$$

Now we need to estimate the terms on the right-hand side of (2.32). We control the scalar product terms just by Cauchy–Schwarz inequality. In particular, we have that

$$\alpha q\left\langle r^{q-1}\partial_\theta f, \partial_r f\right\rangle \leq \frac{\nu}{4}\|\nabla f\|^2 + \frac{\alpha^2 q^2}{\nu}\left\|r^{q-1}\partial_\theta f\right\|^2, \tag{2.33}$$

and

$$2\beta q\nu \left\langle r^{q-1}\partial_r\partial_\theta f, \Delta f\right\rangle \leq \frac{\alpha\nu}{4}\left\|\Delta f\right\|^2 + \frac{4\beta^2 q^2\nu}{\alpha}\left\|r^{q-1}\partial_\theta \nabla f\right\|^2. \tag{2.34}$$

Moreover,

$$\begin{aligned}\beta q^2\nu \left\langle r^{q-2}\partial_\theta f, \Delta f\right\rangle &\leq \frac{\alpha\nu}{4}\left\|\Delta f\right\|^2 + \frac{\beta^2 q^4\nu}{\alpha}\left\|r^{q-2}\partial_\theta f\right\|^2 \\ &\leq \frac{\alpha\nu}{4}\left\|\Delta f\right\|^2 + \frac{\beta^2 q^4\nu}{\alpha}\left\|r^{q-1}\nabla f\right\|^2,\end{aligned} \tag{2.35}$$

where the last inequality follows since $\nabla = (\partial_r, \partial_\theta/r)$. Now assume that

$$\frac{\alpha^2}{\beta} \leq \frac{\nu}{4}, \tag{2.36}$$

in order to absorb the last term of (2.33) on the left-hand side of (2.32). So thanks to (2.33), (2.34) and (2.35), we obtain that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Phi+&\frac{3\nu}{4}\left\|\nabla f\right\|^2 + \frac{\alpha\nu}{2}\left\|\Delta f\right\|^2 + \frac{3\beta q^2}{4}\left\|r^{q-1}\partial_\theta f\right\|^2 + \nu\gamma\left\|r^{q-1}\partial_\theta\nabla f\right\|^2 \\ &\leq 2\gamma\nu(q-1)^2\left\|r^{q-2}\partial_\theta f\right\|^2 + \frac{4\beta^2q^2\nu}{\alpha}\left\|r^{q-1}\partial_\theta\nabla f\right\|^2 + \frac{\beta^2 q^4\nu}{\alpha}\left\|r^{q-1}\nabla f\right\|^2 \\ &= 2\gamma\nu(q-1)^2\left\|r^{q-2}\partial_\theta f\right\|^2 + \frac{2\beta^2q^2\nu}{\alpha}\left(2+\frac{q^2}{2k^2}\right)\left\|r^{q-1}\partial_\theta\nabla f\right\|^2,\end{aligned} \tag{2.37}$$

where the last one follows since we are localized at frequencies k. Now further restrict (2.30) as follows:

$$\frac{\beta^2}{\alpha\gamma} \leq \frac{1}{4q^2(2+q^2/2)} =: \frac{1}{c_1}. \tag{2.38}$$

Thanks to (2.38) and the fact that $k \geq 1$, we can absorb the last term on the right-hand side of (2.37) in the left-hand side to infer that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Phi + \frac{3\nu}{4}\left\|\nabla f\right\|^2 + \frac{\alpha\nu}{2}\left\|\Delta f\right\|^2 + \frac{3\beta q^2}{4}\left\|r^{q-1}\partial_\theta f\right\|^2 \\ + \frac{\gamma\nu}{2}\left\|r^{q-1}\partial_\theta\nabla f\right\|^2 \leq 2\gamma\nu(q-1)^2\left\|r^{q-2}\partial_\theta f\right\|^2.\end{aligned} \tag{2.39}$$

It remains to estimate the last term above, which is not present for $q = 1$ (see Remark 2.2.11 below). Then, as observed in (2.21) if

$$q \leq 1 + \frac{1}{2\sqrt{2}},$$

we directly control it with the last term on the left-hand side of (2.39). All the remaining cases will be done via the estimate (2.22), by choosing

$$\sigma = \frac{\nu}{2\beta q^2 k^2}. \tag{2.40}$$

Since $k \geq 1$, we find that there exists some constant $c_2 \geq 1$ such that

$$\frac{1}{c_2}\left(\frac{\nu}{k^2}\right)^{\frac{1}{q}} \beta^{\frac{q-1}{q}} \left\|r^{q-2}\partial_\theta f\right\|^2 \leq \frac{\nu}{8}\left\|\nabla f\right\|^2 + \frac{\beta q^2}{4}\left\|r^{q-1}\partial_\theta f\right\|^2, \tag{2.41}$$

where, for c_q as in Lemma 2.2.5, we can define

$$\frac{1}{c_2} := \frac{1}{c_q} q^{\frac{2(q-1)}{q}} 2^{-(2+\frac{1}{q})},$$

and notice that, since we have $q > 1 + 1/(2\sqrt{2})$, c_q is uniformly bounded above and below by a universal constant, see also Remark 2.2.7. Let us impose for the moment that

$$2\gamma\nu(q-1)^2 \leq \frac{1}{c_2}\left(\frac{\nu}{k^2}\right)^{\frac{1}{q}} \beta^{\frac{q-1}{q}}. \tag{2.42}$$

In this way, from (2.39) and (2.41) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi + \frac{5\nu}{8}\left\|\nabla f\right\|^2 + \frac{\beta q^2}{2}\left\|r^{q-1}\partial_\theta f\right\|^2 \leq 0. \tag{2.43}$$

All that is missing now is the norm of f in (2.43). With this in mind, we apply (2.20) with the same choice of σ as in (2.40) and deduce that

$$\frac{1}{c_3}\nu^{\frac{q-1}{q}}\left(k^2\beta\right)^{\frac{1}{q}}\left\|f\right\|^2 \leq \frac{\nu}{8}\left\|\nabla f\right\|^2 + \frac{\beta q^2}{4}\left\|r^{q-1}\partial_\theta f\right\|^2, \qquad c_3 := 2^{\frac{3q-1}{q}} q^{-\frac{2}{q}}.$$

Using this in (2.43), we find

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi + \frac{1}{c_3}\nu^{\frac{q-1}{q}}\left(k^2\beta\right)^{\frac{1}{q}}\left\|f\right\|^2 + \frac{\nu}{2}\left\|\nabla f\right\|^2 + \frac{\beta q^2}{4}\left\|r^{q-1}\partial_\theta f\right\|^2 \leq 0.$$

In view of (2.29), this becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi + \frac{1}{c_3}\nu^{\frac{q}{q+2}} k^{\frac{2}{q+2}} \beta_0^{\frac{1}{q}}\left\|f\right\|^2 + \frac{\nu}{2}\left\|\nabla f\right\|^2 + \frac{\beta q^2}{4}\left\|r^{q-1}\partial_\theta f\right\|^2 \leq 0,$$

or, equivalently,

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi + \frac{1}{c_3}\nu^{\frac{q}{q+2}} k^{\frac{2}{q+2}} \beta_0^{\frac{1}{q}}\left[2\left\|f\right\|^2 + \frac{c_3}{\alpha_0\beta_0^{\frac{1}{q}}}\alpha\left\|\nabla f\right\|^2 + c_3\frac{\beta_0^{\frac{q-1}{q}}}{\gamma_0}\gamma\left\|r^{q-1}\partial_\theta f\right\|^2\right] \leq 0.$$

Here it is crucial that in front of α and γ in the brackets, we have something independent of ν and k. In fact we want to use equivalence (2.31), so assuming

$$\frac{c_3}{\alpha_0\beta_0^{\frac{1}{q}}} \geq 3, \tag{2.44}$$

and

$$c_3\frac{\beta_0^{\frac{q-1}{q}}}{\gamma_0} \geq 3, \tag{2.45}$$

we conclude that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi + 2\varepsilon_0 \nu^{\frac{q}{q+2}} k^{\frac{2}{q+2}} \Phi \le 0, \tag{2.46}$$

where ε_0 is determined as follows: let

$$\delta = \min\left\{\frac{1}{2c_2(q-1)^2}, \frac{c_3}{3}\right\},$$

and choose $\alpha_0, \beta_0, \gamma_0$ as

$$\alpha_0 = \frac{c_1}{\delta}\beta_0^{\frac{q+1}{q}}, \qquad \beta_0 = \min\left\{\frac{\delta^2}{4c_1^2}, \frac{c_3\delta}{3c_1}\right\}^{\frac{q}{q+2}}, \qquad \gamma_0 = \delta\beta_0^{\frac{q-1}{q}},$$

and it is straightforward to verify that $\alpha_0, \beta_0, \gamma_0$ satisfies (2.36), (2.38), (2.42), (2.44) and (2.45). Then ε_0 could be chosen as

$$\varepsilon_0 = \frac{\beta_0^{\frac{1}{q}}}{2c_3},$$

hence proving (2.46). The proof is concluded.

□

Remark 2.2.10. To see why α, β and γ have exactly the form (2.29), let us look for example at the scaling in ν. Assume that $\alpha = \nu^i$, $\beta = \nu^j$ and $\gamma = \nu^n$. Then by (2.36), (2.38) and (2.42), one needs to impose

$$2i = 1 + j, \qquad 2j = n + i, \qquad 1 + n = \frac{1}{q} + j\left(\frac{q-1}{q}\right).$$

Then solving explicitly the system we infer (2.29). One argues analogously also for the scaling in k.

Remark 2.2.11 (The case $q = 1$)**.** As noted in the proof, the case $q = 1$ does not require estimating the most dangerous error term in (2.39). In fact, one does not even need to include the γ-term in the functional, since from (2.27) it is apparent that the γ-term is of the same order as $\|f_k\|$. This is also the reason why Lemma 2.2.5 is not needed in this case.

Reconstruction of the X-norm

Theorem 2.2.9 does not directly imply the decay in the X-norm as stated in Theorem 2.0.2. In the following we prove Theorem 2.0.2 in the case $q \ge 1$, and for initial data localized at band $k \in \mathbb{N}$. It is convenient to define the functional

$$W_k(t) = \frac{1}{2}\|f_k(t)\|^2 + \frac{\gamma_0}{4}\left\|r^{q-1} f_k(t)\right\|^2,$$

which is equivalent to the X-norm. Here we prove the following version of Theorem 2.0.2, localized at each Fourier frequency in θ.

Proposition 2.2.12. *Let $q \geq 1$ and $\nu k^{-1} \in (0,1]$. Then*

$$W_k(t) \leq C_0 W_k(0) \exp\left(-\frac{2\varepsilon_0 \nu^{\frac{q}{q+2}} k^{\frac{2}{q+2}}}{1+\frac{2(q-1)}{q+2}(|\ln\nu|+\ln k)} t\right), \tag{2.47}$$

for all $t \geq 0$, and where $C_0 \geq 1$ is a constant independent of ν, k.

Notice that Proposition 2.2.12 implies a linear semigroup estimate for (2.26) in a weighted L^2-space. Theorem 2.0.2, and in particular (2.7), simply follows by summing over $k \in \mathbb{N}_0$ in (2.47).

Proof of Proposition 2.2.12. First of all, recall the definition of γ given in (2.29). Then observe that for $\nu k^{-1} \in (0,1]$ we get

$$\frac{\gamma}{\gamma_0} k^2 = \left(\frac{k}{\nu}\right)^{\frac{2(q-1)}{q+2}} \geq 1, \tag{2.48}$$

since $q \geq 1$. So, in view of the equivalence (2.31) for the functional Φ_k, we infer that

$$W_k(t) \leq \frac{1}{2}\|f_k(t)\|^2 + \frac{\gamma}{4}\left\|r^{q-1}\partial_\theta f_k(t)\right\|^2 \leq \Phi_k(t). \tag{2.49}$$

Now we introduce the following two fixed times

$$T_{\nu,k} := \frac{1}{2\varepsilon_0 \nu^{\frac{q}{q+2}} k^{\frac{2}{q+2}}}, \qquad T_{\nu,k,\ln} := \frac{1+\frac{2(q-1)}{q+2}(|\ln\nu|+\ln k)}{2\varepsilon_0 \nu^{\frac{q}{q+2}} k^{\frac{2}{q+2}}},$$

where $T_{\nu,k,\ln}$ is for technical convenience and $T_{\nu,k}$ is the expected time scale for the enhanced dissipation mechanism. We divide the proof in two cases.

Case $0 < t < T_{\nu,k,\ln}$

In this case, Theorem 2.0.2 is proved if we are able to show that

$$W_k(t) \leq C_1 W_k(0). \tag{2.50}$$

Notice that for $q = 1$, this is trivial. In all other cases, by (2.15), we have that

$$\|f_k(T)\|^2 + 2\nu\int_0^T \|\nabla f_k(\tau)\|^2 \mathrm{d}\tau = \left\|f_k^{in}\right\|^2, \qquad \forall T \geq 0. \tag{2.51}$$

In particular, we know that $\|f_k(\cdot)\|^2$ is monotonically decreasing, and

$$\|f_k(t)\|^2 \leq \left\|f_k^{in}\right\|^2. \tag{2.52}$$

Now we need to infer some property on $\|r^{q-1}f_k(t)\|^2$. In view of (2.21), for $1 \leq q \leq 1+1/(2\sqrt{2})$ by (2.18) we have that $\|r^{q-1}f_k(t)\|^2$ is monotonically decreasing, hence concluding the proof for this range of q. In other cases, we cannot hope in general for some monotonicity. Then we proceed as follows: consider the equality (2.18) and define $C_q = 4(q-1)^2$. By Lemma 2.2.5 we have that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|r^{q-1}f_k\right\|^2 \leq \nu C_q^q \left\|\nabla f_k\right\|^2 + \nu\left\|r^{q-1}f_k\right\|^2.$$

Then for any $t \in (0, T_{\nu,k,\ln})$ we infer that

$$\begin{aligned}\left\|r^{q-1}f_k(t)\right\|^2 &\leq \left(\left\|r^{q-1}f_k^{in}\right\|^2 + \nu C_q^q \int_0^t \mathrm{e}^{-\nu s}\left\|\nabla f_k(s)\right\|^2 \mathrm{d}s\right)\mathrm{e}^{\nu t}\\ &\leq \left(\left\|r^{q-1}f_k^{in}\right\|^2 + \frac{C_q^q}{2}\left\|f_k^{in}\right\|^2\right)\mathrm{e}^{\nu t},\end{aligned} \tag{2.53}$$

where in the last line we have bounded $\mathrm{e}^{-\nu s}$ by 1 and used the energy equality (2.51). Finally notice that

$$\nu T_{\nu,k,\ln} = \left(\frac{\nu}{k}\right)^{2/(q+2)} \frac{1 + \frac{2(q-1)}{q+2}\left(|\ln\nu| + \ln k\right)}{2\varepsilon_0} \leq \widetilde{C}_1,$$

for some $\widetilde{C}_1$ which depends only on q and ε_0. The last inequality follows simply because for any $\beta > 0$ we have $\lim_{x\to 0} x^\beta \ln x = 0$ and we are assuming that $\nu k^{-1} \in (0,1]$. So from (2.53) we get that

$$\left\|r^{q-1}f_k(t)\right\|^2 \leq \left(\left\|r^{q-1}f_k^{in}\right\|^2 + \frac{C_q^q}{2}\left\|f_k^{in}\right\|^2\right)\mathrm{e}^{\widetilde{C}_1}. \tag{2.54}$$

By combining (2.52) with (2.54) we prove (2.50) for a proper C_1, hence proving Proposition 2.2.12 for all $t \leq T_{\nu,k,\ln}$. □

Case $t \geq T_{\nu,k,\ln}$

First of all, consider the energy equality (2.51) for $T = T_{\nu,k}$. Invoking the mean value theorem, there exists

$$t^* \in (0, T_{\nu,k}),$$

such that

$$2\nu T_{\nu,k}\left\|\nabla f_k(t^*)\right\|^2 \leq \left\|f_k^{in}\right\|^2,$$

but recalling the definition of α, see (2.29), we infer that

$$\frac{\alpha}{\alpha_0}\left\|\nabla f_k(t^*)\right\|^2 \leq \varepsilon_0\left\|f_k^{in}\right\|^2. \tag{2.55}$$

Using (2.55) in the equivalence (2.31) for Φ_k, we get

$$\Phi_k(t^*) \leq \frac{1}{2}\left\|f_k(t^*)\right\|^2 + 3\alpha_0\varepsilon_0\left\|f_k^{in}\right\|^2 + \frac{3}{4}\gamma k^2\left\|r^{q-1}f_k(t^*)\right\|^2. \tag{2.56}$$

Since $t^* \leq T_{\nu,k} \leq T_{\nu,k,\ln}$, by combining (2.52) and (2.54) with (2.56), we conclude that

$$\begin{aligned}\Phi_k(t^*) &\leq 3\frac{\gamma k^2}{\gamma_0}\left[\left(\frac{\gamma_0}{6\gamma k^2} + \frac{\alpha_0\varepsilon_0\gamma_0}{\gamma k^2}\right)\left\|f_k^{in}\right\|^2 + \frac{\gamma_0}{4}\left\|r^{q-1}f_k^{in}\right\|^2\right]\\ &\leq \widetilde{C}_2\gamma k^2 W_k(0),\end{aligned} \tag{2.57}$$

where the last one follows by (2.48) and $\widetilde{C}_2$ is a constant that does not depend on ν, k. Then by (2.49), (2.57) and applying the differential inequality (2.28) starting from t^*, we have that

$$\begin{aligned} W_k(t) \leq \Phi_k(t) &\leq \mathrm{e}^{-2\varepsilon_0 \nu^{\frac{q}{q+2}} k^{\frac{2}{q+2}} (t-t^*)} \Phi_k(t^*) \\ &\leq \widetilde{C}_2 \mathrm{e}^{2\varepsilon_0 \nu^{\frac{q}{q+2}} k^{\frac{2}{q+2}} t^*} \gamma k^2 \mathrm{e}^{-2\varepsilon_0 \nu^{\frac{q}{q+2}} k^{\frac{2}{q+2}} t} W_k(0) \\ &\leq \widetilde{C}_2 \gamma_0 \mathrm{e}^{\frac{\gamma k^2}{\gamma_0}} \mathrm{e}^{-2\varepsilon_0 \nu^{\frac{q}{q+2}} k^{\frac{2}{q+2}} t} W_k(0), \end{aligned}$$

where in the last inequality we have used the fact that $t^* \leq T_{\nu,k}$. For $q = 1$, we are done, since $\gamma k^2 \sim 1$. For $q > 1$, notice that

$$\mathrm{e}^{\frac{\gamma k^2}{\gamma_0}} \mathrm{e}^{-2\varepsilon_0 \nu^{\frac{q}{q+2}} k^{\frac{2}{q+2}} t} = \mathrm{e} \left(\frac{k}{\nu} \right)^{\frac{2(q-1)}{q+2}} \mathrm{e}^{-2\varepsilon_0 \nu^{\frac{q}{q+2}} k^{\frac{2}{q+2}} t} = \mathrm{e}^{1 + \frac{2(q-1)}{q+2} (|\ln \nu| + \ln k) - 2\varepsilon_0 \nu^{\frac{q}{q+2}} k^{\frac{2}{q+2}} t}.$$

Now let $a = 1 + \frac{2(q-1)}{q+2} (|\ln \nu| + \ln k)$ and $b = 2\varepsilon_0 \nu^{\frac{q}{q+2}} k^{\frac{2}{q+2}}$. By the definition of $T_{\nu,k,\ln}$, we are interested in times $t \geq a/b$. Then observe the following basic inequality

$$a - bt \leq 1 - \frac{b}{a} t, \qquad \text{for } t \geq \frac{a}{b},$$

which is true for any $a > 1$ and $b > 0$. So finally we have that

$$W_k(t) \leq C_2 W_k(0) \exp\left(-\frac{2\varepsilon_0 \nu^{\frac{q}{q+2}} k^{\frac{2}{q+2}}}{1 + \frac{2(q-1)}{q+2} (|\ln \nu| + \ln k)} t \right),$$

where $C_2 = \mathrm{e} \widetilde{C}_2 \gamma_0$. Then by choosing $C_0 = \max\{C_1, C_2\}$ we conclude the proof of Proposition 2.2.12.

Remark 2.2.13 (No log-correction for $q = 1$)**.** We stress once more that for $q = 1$, there is no logarithmic correction for $W_k(t)$ in (2.47). This is essentially due to the fact that the γ-term is of the same order as $\|f_k\|$.

CHAPTER 3

Linear stability analysis for the 2D isentropic compressible Couette flow

In this chapter, we study the linear stability properties of the Couette flow when the fluid under consideration satisfy the 2D isentropic compressible Navier-Stokes system. The results that we present in the following are obtained in collaboration with P. Antonelli and P. Marcati, the inviscid case appeared in [2].

We consider a perturbation around the homogeneous Couette flow, i.e. $\boldsymbol{u}_E = (y, 0)$ and $\rho_E = 1$, given by

$$\widetilde{\rho} = \rho + \rho_E, \qquad \boldsymbol{u} = \boldsymbol{u}_E + \boldsymbol{v},$$

for $\tilde{\rho}, \boldsymbol{u}$ satisfying (1.1), in the domain $\mathbb{T} \times \mathbb{R}$ and in absence of external forces. The linearized system read as follows

$$\partial_t \rho + y\partial_x \rho + \operatorname{div}(\boldsymbol{v}) = 0, \quad \text{in } \mathbb{T} \times \mathbb{R},\ t \geq 0, \tag{3.1a}$$

$$\partial_t \boldsymbol{v} + y\partial_x \boldsymbol{v} + \begin{pmatrix} v^y \\ 0 \end{pmatrix} + \frac{1}{M^2}\nabla\rho = \nu\Delta\boldsymbol{v} + \lambda\nabla\operatorname{div}(\boldsymbol{v}), \tag{3.1b}$$

where we set $p'(1) = 1$.

Before stating our main results in this case, let us first review the literature about compressible shear flows, which is less developed with respect to the incompressible case.

In the Euler regime, the Rayleigh inflection point Theorem and Howard's semicircle Theorem has been extended to the compressible case respectively by Lees and Lin in 1946 [111] and Eckart in 1963 [67]. Both of them were rederived with a simpler proof by Blumen [23]. Other spectral stability conditions can be found in [24, 61, 141]. In the review paper [96] there is the extension of Arnold's method to prove Lyapunov type stability properties in the case of strictly concave shear flows for a 2D isentropic compressible fluid. The linearization around the Couette flow in the 2D isentropic compressible Euler dynamics was considered in the physics literature, both from the numerical point of view and from the theoretical one, a highly incomplete list of papers includes [5, 37, 38, 92, 90] and references therein. By adding the effect of a Coriolis forcing term, the problem has been considered as a first model to understand the formation of spiral arms in a rotating disk galaxy by Goldreich and Lynden-Bell [81, 82]. In particular, in [82, Sec. 5-6] they directly consider the linearized initial value problem and they perform the Fourier analysis after a change of coordinates done in order to follow the background shear, closely related to what can be done in the incompressible case, see Section 1.2. Then, they derive a second order ODE satisfied by the density in the Fourier space, from which, appealing to some formal approximation, they deduce an instability phenomenon that appears specifically due to the compressibility of the flow. More precisely, they argue that the square of the density, in the frequency space, can grow linearly in time. The problem without Coriolis force, which is the one studied in this chapter, with analogous computations was considered by Chagelishvili et al. in [37, 38], where a linear growth of $|\rho|^2 + |v|^2$ is observed. In [25] the analysis of Goldreich and Lynden-Bell was revisited, supplemented with numerical simulations and are also highlighted similarities to the case studied in [38]. Recently, more refined numerical simulations and analysis can be found in [5, 92], where the results are justified with some formal asymptotic expansion.

For the viscous compressible plane Couette flow, Glatzel in 1988 [80] has investigated linear stability properties via a normal mode analysis, see also [65, 98]. Hanifi et al. in [90] have numerically investigated a transient growth mechanism in the non-isothermal case, showing that the maximum transient growth scales as $O(\nu^{-2})$ and increases with increasing Mach number, see also the more recent result [121]. Then, Farrell and Ioannou in [70] considered the linear problem (3.1a)-(3.1b) and showed a rapid transient energy growth, that at large Mach numbers greatly exceed the expected one in the incompressible case, which is then damped due to the effect of viscosity. By some heuristic argument, the authors have also observed that the transient growth is due to purely inviscid and compressible effects in agreement with [37, 38]. It is interesting to notice that in the numerical simulations shown in [70, Fig. 1], although not explicitly observed, it is evident that the energy starts being damped on a time scale $O(\nu^{-\frac{1}{3}})$, meaning that *enhanced dissipation* is possible, see Section 1.2 and Chapter 2. On the mathematical literature, we mention the more recent result obtained by Kagei [105], where he considered the problem in an infinite channel in dimension n, and proved the nonlinear asymptotic stability for small disturbances to the Couette flow in L^2 for sufficiently small Mach and Reynolds numbers, showing decay rates as the n-1 dimensional heat equation. See also [106] for the extension of this result to more general planar shear flows.

Statement of the results

In this chapter, we will confirm and make more precise the linear inviscid instability phenomena found in the above mentioned literature. In particular, we are able to prove that the solenoidal component of the velocity field experience inviscid damping whereas the irrotational component and the density have a linear Lyapunov instability for a *generic* class of initial data. Then, in the viscous case, we confirm the observations made in [70] by showing that dynamics is qualitatively the same to the inviscid case on a time scale $O(\nu^{-\frac{1}{3}})$, after which viscosity become effective and the perturbations decay exponentially fast, meaning that there is an enhanced dissipation mechanism. This is not a priori trivial in view of the absence of viscosity in the continuity equation. We remark that the estimates obtained here are in agreement with the incompressible case in the formal limit $M \to 0$.

Before stating our main result, for any velocity field $\boldsymbol{v}$, we denote

$$\alpha = \mathrm{div}(\boldsymbol{v}), \qquad \omega = \nabla^{\perp} \cdot \boldsymbol{v}.$$

The Helmholtz projection operators are defined in the usual way, namely

$$\boldsymbol{v} = (v^x, v^y)^T = \nabla \Delta^{-1} \alpha + \nabla^{\perp} \Delta^{-1} \omega := Q[\boldsymbol{v}] + P[\boldsymbol{v}], \tag{3.2}$$

where $\nabla^{\perp} = (-\partial_y, \partial_x)^T$. The system (3.1a)-(3.1b), in terms of (ρ, α, ω) read as

$$\partial_t \rho + y \partial_x \rho + \alpha = 0, \qquad \text{in } \mathbb{T} \times \mathbb{R}, t \geq 0, \tag{3.3}$$

$$\partial_t \alpha + y \partial_x \alpha + 2 \partial_x v^y + \frac{1}{M^2} \Delta \rho = (\nu + \lambda) \Delta \alpha, \tag{3.4}$$

$$\partial_t \omega + y \partial_x \omega - \alpha = \nu \Delta \omega. \tag{3.5}$$

The second component of the velocity v^y can be recovered by means of the Helmholtz decomposition

$$v^y = \partial_y (\Delta^{-1}) \alpha + \partial_x (\Delta^{-1}) \omega, \tag{3.6}$$

hence (3.3)-(3.5) is a closed system in terms of the variables (ρ, α, ω).

In the following, we are going to denote

$$f_0(y) = \frac{1}{2\pi} \int_{\mathbb{T}} f(x, y) dx. \tag{3.7}$$

For the sake of brevity, we do not explicit the dependence of the bounds with respect to the Sobolev norms of the initial data. In particular, we simply write $C^j_{in} = C^j_{in}(\rho^{in}, \alpha^{in}, \omega^{in})$, with $j = 1, 2, \ldots$, to indicate a suitable combination of Sobolev norms of the initial data. Those constants may also depend on quantities increasing with respect to the Mach number, namely $(1 + M)^{\beta}$ or $\exp(M^{\beta})$ for some $\beta \geq 1$. A more precise statement of the theorem below will be given in Sections 3.2 and 3.3.

Theorem 3.0.1. *Let $\nu, \lambda \geq 0$ and $M > 0$ be such that $\nu + \lambda \leq 1/2$ and $M \leq \min\{(\nu + \lambda)^{-\frac{1}{2}}, \nu^{-\frac{1}{3}}\}$. Let $\rho^{in} \in H^7(\mathbb{T} \times \mathbb{R})$ and $\alpha^{in}, \omega^{in} \in H^6(\mathbb{T} \times \mathbb{R})$. Then, the x-average of the solution satisfy*

$$\|\alpha_0(t)\|_{L^2} + \frac{1}{M} \|\partial_y \rho_0(t)\|_{L^2} \leq \frac{C^1_{in}}{(1 + \nu t)^{\frac{1}{2}}}, \tag{3.8}$$

$$\|v^y_0(t)\|_{L^2} + \frac{1}{M} \|\rho_0(t)\|_{L^2} + \|\omega_0(t)\|_{L^2} \leq C^2_{in},$$

whereas for the fluctuations around it the following inequalities holds:

$$\|(Q[\boldsymbol{v}] - Q[\boldsymbol{v}]_0)(t)\|_{L^2} + \frac{1}{M} \|(\rho - \rho_0)(t)\|_{L^2} \leq \langle t \rangle^{\frac{1}{2}} e^{-\frac{1}{32} \nu^{\frac{1}{3}} t} C^3_{in}, \tag{3.9}$$

$$\|(P[\boldsymbol{v}]^x - P[\boldsymbol{v}]^x_0)(t)\|_{L^2} \leq M \frac{e^{-\frac{1}{64} \nu^{\frac{1}{3}} t}}{\langle t \rangle^{\frac{1}{2}}} C^4_{in} + \frac{e^{-\frac{1}{12} \nu^{\frac{1}{3}} t}}{\langle t \rangle} (\|\omega^{in} + \rho^{in}\|_{H^1}), \tag{3.10}$$

$$\|P[\boldsymbol{v}]^y(t)\|_{L^2} \leq M \frac{e^{-\frac{1}{64} \nu^{\frac{1}{3}} t}}{\langle t \rangle^{\frac{3}{2}}} C^5_{in} + \frac{e^{-\frac{1}{12} \nu^{\frac{1}{3}} t}}{\langle t \rangle^2} (\|\omega^{in} + \rho^{in}\|_{H^2}). \tag{3.11}$$

Let $\nu = \lambda = 0$ and $s \geq 0$. Up to a nowhere dense set of initial data $\rho^{in}, \alpha^{in}, \omega^{in} \in H^s(\mathbb{T} \times \mathbb{R})$ one has

$$\|(Q[\boldsymbol{v}] - Q[\boldsymbol{v}]_0)(t)\|_{L^2} + \frac{1}{M}\|(\rho - \rho_0)(t)\|_{L^2} \geq \langle t \rangle^{\frac{1}{2}} C^6_{in}. \tag{3.12}$$

Notice that when $M = 0$ (and $\alpha^{in} = \rho^{in} = 0$), formally the estimates (3.10)-(3.11) give the same result as in the incompressible case, see Section 1.2.

We remark that the dynamics of the x-averages decouples with respect to fluctuations around it, as we will show in Section 3.1 where we comment more about the evolution of the zero x-mode.

In the following, we discuss the results given in the theorem above and we outline the strategy of proof by considering separately the inviscid and the viscous case, which we investigate in Section 3.2 and Section 3.3 respectively.

Inviscid case

For $\nu = \lambda = 0$, the estimates (3.9) and (3.12) give the first rigorous justification to the growth predicted in [5, 37, 38, 92], where, in order to implement a WKB asymptotic analysis, the authors had to restrict themselves to a small Mach number regime. We emphasize that our result stated in Theorem 3.0.1 is actually more general since it removes the smallness assumption on the Mach number. We also see that only the density and the irrotational part of the velocity field are growing, whereas in (3.10)-(3.11) we show an inviscid damping result for the solenoidal component of the velocity, with slower decay with respect to the incompressible case (Section 1.2). Indeed, this slow down of the inviscid damping is exactly compensated by the time growth of the compressible part of the fluid.

In [38] the authors says that there is an "*emergence of acoustic waves from vortices*". Theorem 3.0.1 implies that the density and the irrotational part of the velocity exhibit a growth in time even when the initial perturbation satisfies $\rho^{in} = \alpha^{in} = 0$. This can be seen from the linearized equations (3.3)-(3.5), where the identity (3.6) for v^y yields a source term, depending on the vorticity, in the equation for the divergence (3.4). See Figure 3.1 for some numerical simulations.

Remark 3.0.2. For some particular initial data the lower bound in (3.12) may not be valid. However, as we shall see in Proposition 3.2.8, we are able to explicitly construct an arbitrary small perturbation of the initial data, at any fixed frequency k, η, for which the lower bound holds true. The perturbation is chosen such that the set is nowhere dense in any Sobolev space in which the initial data is taken, which implies a *generic* Lyapunov type instability.

Remark 3.0.3. When $\nu = \lambda = 0$, by adding (3.3) to (3.5), it is immediate to see that $\rho + \omega$ is conserved along the characteristics. This can be seen as the linear analogue of the potential vorticity $\widetilde{\omega}/\widetilde{\rho}$ being transported along the flow, where $\widetilde{\omega} = \nabla^\perp \cdot \boldsymbol{u}$ for $\widetilde{\rho}, \boldsymbol{u}$ satisfying (1.1). Indeed, at least formally, a direct computation shows that

$$\partial_t \left(\frac{\widetilde{\omega}}{\widetilde{\rho}}\right) + \boldsymbol{u} \cdot \nabla \left(\frac{\widetilde{\omega}}{\widetilde{\rho}}\right) = 0.$$

Then, since we are considering perturbations around the Couette flow we have $\widetilde{\omega} = -1+\omega$ and $\widetilde{\rho} = 1+\rho$, meaning that $(-1+\omega)/(1+\rho) = (-1+\omega^{in})/(1+\rho^{in})$ along the flow generated by $\boldsymbol{u}$. More precisely, in Lagrangian coordinates we deduce

$$\omega(t, \mathbf{X}(\mathbf{x}_{in}, t)) + \frac{1-\omega^{in}(\mathbf{x}_{in})}{1+\rho^{in}(\mathbf{x}_{in})}\rho(t, \mathbf{X}(\mathbf{x}_{in}, t)) = \frac{\rho^{in}(\mathbf{x}_{in}) + \omega^{in}(\mathbf{x}_{in})}{1+\rho^{in}(\mathbf{x}_{in})},$$

where $\mathbf{X}$ is the flow associated to u, namely $\partial_t \mathbf{X}(t, \mathbf{x}_{in}) = \boldsymbol{u}(\mathbf{X}(t, \mathbf{X}_{in}))$, $\mathbf{X}(0, \mathbf{x}_{in}) = \mathbf{x}_{in}$. Assuming that $|\omega^{in}| \ll 1$ and $|\rho^{in}| \ll 1$ one has $(1-\omega^{in})/(1+\rho^{in}) \approx 1$. Hence, by the previous heuristic argument at the nonlinear level, we see why, at least formally, the conservation of $\rho+\omega$ can be considered as a linear approximation of the conservation of the potential vorticity for perturbations around the Couette flow with constant density.

The exact conservation along the Couette flow of $\rho+\omega$ plays a central role in our analysis in the inviscid case. In addition, this conservation law connects compressible and incompressible phenomena, namely an increase of the vorticity need to be compensated by a decrease for the density and the other way around. This interplay between density and vorticity is also the cause of the slow-down of the inviscid damping for the solenoidal component of the velocity.

Let us now briefly discuss the strategy of proof for the Theorem 3.0.1 when $\nu = \lambda = 0$. First of all, we remove the transport terms by defining the change of coordinates dictated by the background shear, as done also in the incompressible case. Then, on this reference frame we have the exact conservation of the quantity $\rho+\omega$, so that we are able to reduce the degrees of freedom for the system (3.3)-(3.5) and write a 2×2 system only involving the density and the divergence in the moving frame. Taking its Fourier transform in all the space variables, it can be studied as a 2×2 non-autonomous dynamical system at any fixed frequency k, η. In particular, performing a suitable symmetrization via time dependent Fourier multipliers, we can infer an energy estimate useful to deduce some property of the associated semigroup. Once the dynamics at any fixed frequency is understood, Theorem 3.0.1 follows as a consequence and can be proved by going back to the original variables.

We point out that, from the 2×2 non-autonomous system in the Fourier space, one may also consider the second order equation satisfied by the density and study a single scalar equation, as done for example in [5, 37, 38, 82, 92]. However, it does not seem immediate to infer properties of this equation without the aid of some formal approximation, see Remark 3.2.2.

We present a more precise statement of Theorem 3.0.1 in Theorems 3.2.5 and 3.2.7, where we consider separately the upper and lower bounds respectively. In Figures 3.1 and 3.3 we show some numerical simulations of the system at fixed frequencies k, η for different values of the Mach number.

Viscous case

Theorem 3.0.1 for $\nu > 0$ gives a rigorous mathematical justification for the observations made in [70]. At least to our knowledge, it appears to be the first enhanced dissipation estimate in the compressible case. In the bound (3.9) we see the possibility of a large transient growth of order $O(\nu^{-\frac{1}{6}})$ on a time scale $O(\nu^{-\frac{1}{3}})$. This growth is due to the instability mechanism found in the inviscid case. Instead, the bounds (3.10)-(3.11) combines inviscid damping and enhanced dissipation for the solenoidal component of the velocity. See Figure 3.5 for a numerical simulation.

The numerical observations made in [70, 90, 121] shows that the transient growth increases at increasing Mach number, see for example [90, Fig. 9]. In Theorem 3.0.1 we have not an explicit dependence since, as previously mentioned, we are neglecting constants which can grow exponentially fast with respect to the Mach number. However, as we explain in Remark 3.3.6, by refining the energy estimate done to prove Theorem 3.0.1, it should be possible to improve the constants up to $O(M^\beta)$ for some $\beta \geq 1$. This would imply that the density may experience a transient growth of order $O(M^{\beta+1}\nu^{-\frac{1}{6}})$. It may be of interest to estimate the optimal β.

Remark 3.0.4 (Restrictions on the Mach number)**.** In Theorem 3.0.1 we have to restrict our analysis to the case of Mach numbers which satisfy $M \leq \min\{(\nu+\lambda)^{-\frac{1}{2}}, \nu^{-\frac{1}{3}}\}$. Clearly, for $\nu, \lambda \ll 1$ the last assumption is not really restrictive since in most physical applications $M \leq 1$ and in the astrophysical context $M \sim 10-50$ [25, 26, 136]. However, as we explain in Remark 3.3.6, the condition can be easily relaxed to

$$M \leq \min\{(\nu+\lambda)^{-\frac{1}{2}}, \delta^{-1}\nu^{-\frac{1}{3}}\}$$

for any $0 < \delta \leq 1$, at the price of deteriorating the decay rates by a factor δ^{-1}, namely instead of $e^{-c\nu^{\frac{1}{3}}t}$ one has $e^{-\delta c\nu^{\frac{1}{3}}t}$. The hypothesis $M \leq (\nu+\lambda)^{-\frac{1}{2}}$ is required also to obtain decay for the x-average, see (3.8) and Theorem 3.1.1, and it seems to be a more rigid condition. Heuristically, the larger the Mach number the weaker is the coupling among ρ and α. Hence, it is more difficult to exploit the structure of the equation in order to get dissipation for the density.

Remark 3.0.5 (Absence of shear viscosity)**.** If we set $\nu = 0$ and $\lambda > 0$ the dissipation is present only in the equation for the divergence. This particular case is not immediately covered by Theorem 3.0.1 since one may infer more properties. For example, for $\nu = 0$ we have the conservation of $\rho + \omega$ along the characteristics. Therefore, one can study a 2×2 system as done in the inviscid case with the additional dissipation for the divergence. One may prove that if $\rho^{in} + \omega^{in} = 0$ then there is enhanced dissipation on a time scale $O(\lambda^{-\frac{1}{3}})$ whereas if $\rho^{in} + \omega^{in} \neq 0$ there is a constant forcing term which prevents the convergence towards zero. We do not discuss this case in more details.

Remark 3.0.6 (Regularity of the initial data)**.** In Theorem 3.0.1, since we want to combine the inviscid and viscous dynamics, we are not interested in providing sharp regularity assumptions on the initial data. It is indeed natural to pay regularity in order to take out time rates, as also observed in the incompressible case, see Section 1.2.

In view of the previous remark, we stress that in the viscous case it is not necessary to lose regularity and we are able to infer the following.

Theorem 3.0.7. *Let $\nu > 0$, $\lambda \geq 0$ and $M > 0$ be such that $\nu + \lambda \leq 1/2$ and $M \leq \min\{(\nu+\lambda)^{-\frac{1}{2}}, \nu^{-\frac{1}{3}}\}$. Assume that $\rho^{in} \in H^1(\mathbb{T}\times\mathbb{R})$, $\alpha^{in}, \omega^{in} \in L^2(\mathbb{T}\times\mathbb{R})$.*

Then

$$\begin{aligned}\|(\alpha-\alpha_0)(t)\|_{L^2}+&\frac{1}{M}\|\nabla(\rho-\rho_0)(t)\|_{L^2}+\|(\omega-\omega_0)(t)\|_{L^2}\\&\lesssim\nu^{-\frac{1}{2}}e^{-\frac{\nu^{\frac{1}{3}}}{64}t}\left(\left\|\alpha^{in}\right\|_{L^2}+\frac{1}{M}\left\|\nabla\rho^{in}\right\|_{L^2}+\left\|\omega^{in}\right\|_{L^2}\right).\end{aligned}\tag{3.13}$$

In addition, the following inequality holds

$$\begin{aligned}\|(\boldsymbol{v}-\boldsymbol{v}_0)(t)\|_{L^2}+&\frac{1}{M}\|(\rho-\rho_0)(t)\|_{L^2}\\&\lesssim\nu^{-\frac{1}{6}}e^{-\frac{\nu^{\frac{1}{3}}}{64}t}\left(\left\|\alpha^{in}\right\|_{L^2}+\frac{1}{M}\left\|\nabla\rho^{in}\right\|_{L^2}+\left\|\omega^{in}\right\|_{L^2}\right)\end{aligned}\tag{3.14}$$

In the bound (3.13), at the price of having worst estimates with respect to the one in Theorem 3.0.1, we see that we do not lose derivatives to get the exponential decay for the quantities on the left-hand side. Then, the bound (3.14) does not straightforwardly follow by (3.13), it is indeed a consequence of a careful choice of some Fourier multipliers used to prove (3.13). In addition, since $\langle t\rangle^{\frac{1}{2}}\lesssim\nu^{-\frac{1}{6}}\exp((\nu^{\frac{1}{3}}t)/64)$, the bound (3.14) agrees with (3.9) in terms of order of magnitude of the maximal possible growth. The estimates and the method of proof of Theorem 3.0.7 can be extremely useful in order to extend this linear result to prove a transition threshold in Sobolev spaces for the fully nonlinear case, which we aim at studying.

We now comment about the strategy of proof of Theorem 3.0.1 and Theorem 3.0.7. Being similar, we outline here the main ideas in both cases.

When viscosity is present, we have to overcome two main difficulties. First of all, as can be seen by summing up (3.3) and (3.5), the conservation of $\rho+\omega$ along the Couette flow no longer holds, which is a crucial point in the inviscid case. Therefore, we cannot reduce the analysis to the study of a 2×2 system. The second point is that since we do not have a dissipative term in (3.3), it is not a priori trivial to have decay for the density. However, we will be able to recover the exponential decay via a weighted energy estimate where it is crucial to exploit the coupling between ρ and α,

More precisely, as done in the inviscid case, we first remove the transport terms via the standard change of coordinates and we perform the Fourier transform in both space variables, leading us to the study of a 3×3 system in the Fourier space. It is then crucial to replace the vorticity with another auxiliary quantity, i.e. $\rho+\omega-\nu M^2\alpha$, which satisfy a more complicated equation with respect to ω but has a better structure to make use of this variable in energy estimates. Then, we are able to define a weighted energy functional in terms of $(\rho,\alpha,\rho+\omega-\nu M^2\alpha)$ for which we can infer a Grönwall's type estimate. The weights are suitable time-dependent Fourier multipliers. The main difference between the proof of Theorem 3.0.1 and Theorem 3.0.7 is the choice of the weights.

Outline of the chapter

We begin our analysis with the study of the dynamics of the x-averages in Section 3.1. In Section 3.2 we consider the inviscid problem in order to prove Theorem 3.0.1 when

$\nu = \lambda = 0$. In Section 3.3 we turn our attention to the viscous case. Here, we first prove Theorem 3.0.1 in Subsection 3.3.1. Then, in Subsection 3.3.2 we present the proof of Theorem 3.0.7.

Notations

In this chapter, when using the symbol $\lesssim$ we are neglecting constants which do not depend on ν but may depend on $(1+M)^\beta$ or $\exp(M^\beta)$ for some $\beta \geq 1$. However, we keep track of constants which goes to zero as $M \to 0$.

When it will be clear from the context whether we are working in the physical space or in the frequency space, by an abuse of notation, we will not distinguish between pseudo-differential operators and their own symbols.

We say that $f \in H^{s_1}_x H^{s_2}_y$ whenever

$$\|f\|^2_{H^{s_1}_x H^{s_2}_y} = \sum_k \int \langle k\rangle^{2s_1}\langle \eta\rangle^{2s_2}|\hat{f}|^2(k,\eta)d\eta < +\infty,$$

whereas we denote the norm in the usual $H^s(\mathbb{T}\times\mathbb{R})$ space as

$$\|f\|^2_{H^s} = \sum_k \int \langle k,\eta\rangle^{2s}|\hat{f}|^2(k,\eta)d\eta.$$

Let $Z(t) = (Z_1(t), Z_2(t))^T : [t_0,+\infty)\times\mathbb{C}^2 \to \mathbb{C}^2$ and $\mathcal{L}(t) : [t_0,+\infty)\times\mathbb{C}^{2\times 2} \to \mathbb{C}^{2\times 2}$. Given the following 2D non-autonomous dynamical system

$$\frac{\mathrm{d}}{\mathrm{d}t}Z = \mathcal{L}(t)Z,$$

we define the standard Picard iteration

$$\begin{aligned} &\Phi_{\mathcal{L}}(t,t_0) = \mathbb{1} + \sum_{n=1}^{\infty}\mathcal{I}_n(t,t_0),\\ &\mathcal{I}_{n+1}(t,t_0) = \int_{t_0}^{t}\mathcal{L}(\tau)\mathcal{I}_n(\tau,t_0)d\tau, \qquad \mathcal{I}_1(t,t_0) = \int_{t_0}^{t}\mathcal{L}(\tau)d\tau. \end{aligned} \tag{3.15}$$

$\Phi_{\mathcal{L}}$ is the evolution operator associated to $\mathcal{L}$. In particular it satisfies the group property, $\Phi_{\mathcal{L}}(t,t_0) = \Phi_{\mathcal{L}}(t,s)\Phi_{\mathcal{L}}(s,t_0)$ for any $t, s \geq 0$.

In order to remove the transport term from the equations, we will always make the following change of coordinates

$$X = x - yt, \quad Y = y. \tag{3.16}$$

In particular, the differential operators change as follows

$$\begin{aligned} &\partial_x = \partial_X,\\ &\partial_y = \partial_Y - t\partial_X,\\ &\Delta = \Delta_L := \partial_{XX} + (\partial_Y - t\partial_X)^2. \end{aligned} \tag{3.17}$$

In the new reference frame, which we shall often refer to as the *moving frame*, we also define the functions

$$\begin{aligned} R(t,X,Y) &= \rho(t,X+tY,Y), \\ A(t,X,Y) &= \alpha(t,X+tY,Y), \\ \Omega(t,X,Y) &= \omega(t,X+tY,Y). \end{aligned} \tag{3.18}$$

We denote the symbol associated to $-\Delta_L$ as

$$p(t,k,\eta) = k^2 + (\eta - kt)^2. \tag{3.19}$$

Moreover

$$(\partial_t p)(t,k,\eta) = -2k(\eta - kt) \tag{3.20}$$

is the symbol associated to the operator $2\partial_X(\partial_Y - t\partial_X)$.

3.1 Dynamics of the $k = 0$ modes

In this section, we investigate in detail the dynamics of the x-averages of the perturbations. Due to the structure of the shear flow and the fact that the equations are linear, it is clear that the zero mode in x has an independent dynamics with respect to other modes. Consequently, in our analysis we can decouple the evolution of the $k = 0$ mode from the rest of the perturbation. From a mathematical point of view, the filtering of the x-average out of the dynamics is a necessary condition in order to have a good definition for the inverse of some differential operators, naturally appearing in the equations, see (3.6).

The system (3.3)-(3.4) when projected onto the $k = 0$ frequency, recalling (3.7), read as follows

$$\partial_t \rho_0 = -\alpha_0, \tag{3.21}$$

$$\partial_t \alpha_0 = (\nu + \lambda)\partial_{yy}\alpha_0 - \frac{1}{M^2}\partial_{yy}\rho_0,$$

$$\partial_t \omega_0 = \alpha_0 + \nu\partial_{yy}\omega_0. \tag{3.22}$$

Since v_0^y satisfy the same equation of $\alpha_0 = \partial_y v_0^y$, we will identify $\partial_y^{-1}\alpha_0 = v_0^y$.

When $\nu = \lambda = 0$ the dynamics of (ρ_0, α_0) is given by a standard 1-D wave equation, namely

$$\partial_{tt}\rho_0 - M^{-2}\partial_{yy}\rho_0 = 0, \qquad \text{in } \mathbb{R}, \tag{3.23}$$

and by adding (3.21) to (3.22) we get

$$\partial_t(\rho_0 + \omega_0) = 0,$$

hence $\omega_0 = \rho_0^{in} + \omega_0^{in} - \rho_0$. Therefore, when $\nu = \lambda = 0$ the dynamics of the $k = 0$ mode is completely determined by solving (3.23). By the explicit representation formula for (3.23), we know that there is not any decay for the zero modes.

When $\nu > 0$, one has an explicit representation of the solution in the Fourier space. However, from this formula it is not immediate to infer decay properties of solutions to (3.21)-(3.22). We are then going to derive decay properties of the $k = 0$ by using an energy method in a similar way to what was done by Guo and Wang in [89]. In order to perform the energy estimate, it will be convenient to replace the equation (3.22) with

$$\partial_t(\rho_0 + \omega_0 - \nu M^2\alpha_0) = \nu\partial_{yy}(\rho_0 + \omega_0 - \nu M^2\alpha_0) - \lambda\nu M^2\partial_{yy}\alpha_0.$$

In particular, we have the following.

Theorem 3.1.1. *Let $\nu, \lambda \geq 0$ and $\rho^{in}, \alpha^{in}, \omega^{in}$ be the initial data of* (3.3)-(3.5)*. Then, the solution (ρ, α, ω) can be decomposed as $\rho = \rho_0 + \rho_{\neq}$, $\alpha = \alpha_0 + \alpha_{\neq}$, $\omega = \omega_0 + \omega_{\neq}$ where $(\rho_{\neq}, \alpha_{\neq}, \omega_{\neq})$ satisfy* (3.3)-(3.5) *and $(\rho_0, \alpha_0, \omega_0)$ satisfy* (3.21)-(3.22)*. For the $k = 0$ mode we have the following: for any $\ell \geq 0$, let*

$$\begin{aligned}\mathcal{E}^\ell(t) = &\left\|\partial_y^\ell\alpha_0(t)\right\|_{L^2}^2 + \left\|\partial_y^{\ell-1}\alpha_0(t)\right\|_{L^2}^2 + \left\|\partial_y^\ell(\omega_0 + \rho_0 - \nu M^2\alpha_0)\right\|_{L^2}^2 \\ &+ \frac{1}{M^2}(\left\|\partial_y^{\ell+1}\rho_0(t)\right\|_{L^2}^2 + \left\|\partial_y^\ell\rho_0(t)\right\|_{L^2}^2),\end{aligned}$$

where $\partial_y^{-1}\alpha_0 = v_0^y$. If $M \leq (\nu + \lambda)^{-\frac{1}{2}}$ and $\mathcal{E}_{in}^\ell, \mathcal{E}_{in}^0 < +\infty$ then

$$\mathcal{E}^\ell(t) \leq \frac{4\mathcal{E}_{in}^\ell}{(\nu C_{in}^\ell t + 1)^\ell}, \tag{3.24}$$

where $C_{in}^0 = 0$ and $C_{in}^\ell = C\max\{1, (\mathcal{E}_{in}^\ell/\mathcal{E}_{in}^0)^{\frac{1}{\ell}}\}$ for $\ell \geq 1$ and some constant C which does not depend on ℓ, ν, λ. In addition we have that

$$\rho_0^{in} = \alpha_0^{in} = \omega_0^{in} = 0 \Longrightarrow \rho_0(t) = \alpha_0(t) = \omega_0(t) = 0. \tag{3.25}$$

Remark 3.1.2. In view of the theorem above, it is equivalent to study the dynamics of $(\rho - \rho_0, \alpha - \alpha_0, \omega - \omega_0)$ or (ρ, α, ω) assuming that $\rho_0^{in} = \alpha_0^{in} = \omega_0^{in} = 0$. In the rest of this chapter, for simplicity of notation, we will always consider the second case.

For any $N \geq 0$, from the previous theorem we infer that

$$\begin{aligned}\|\alpha_0(t)\|_{H^N} + \frac{1}{M}\|\partial_y\rho_0(t)\|_{H^N} &\lesssim \frac{\sqrt{\mathcal{E}_{in}^N}}{(1 + \nu t)^{\frac{1}{2}}}, \\ \|\omega_0(t)\|_{H^N} &\lesssim \sqrt{\mathcal{E}_{in}^N}.\end{aligned}$$

Hence, α_0 and $\partial_y\rho_0$ have the same decay as if ρ_0, v_0^y had satisfied the standard 1-D heat equation, see for instance [89, Theorem 1.1].

We now present the proof of Theorem 3.1.1.

Proof. First of all, $(\rho_{\neq}, \alpha_{\neq}, \omega_{\neq})$ satisfy (3.3)-(3.5) since $\partial_x(f_0) = 0$. The proof of (3.25) follows by the linearity of the system (3.21)-(3.22). To prove (3.24), we define

$$E^\ell(t) = \frac{1}{2}\left(\mathcal{E}^\ell(t) - \frac{(\nu + \lambda)}{2}\left\langle\partial_y^\ell\rho_0(t), \partial_y^\ell\alpha_0(t)\right\rangle\right).$$

Since $M(\nu+\lambda)\leq 1$ we have

$$\frac{1}{4}\mathcal{E}^\ell(t)\leq E^\ell(t)\leq \mathcal{E}^\ell(t), \tag{3.26}$$

namely the functional E^ℓ is coercive. Then, by a direct computation we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E^\ell(t)&+(\nu+\lambda)\left(\left\|\partial_y^{\ell+1}\alpha_0\right\|_{L^2}^2+\left\|\partial_y^{\ell}\alpha_0\right\|_{L^2}^2+\frac{1}{4M^2}\left\|\partial_y^{\ell+1}\rho_0\right\|_{L^2}^2\right)\\&+\nu\left\|\partial_y^{\ell+1}(\rho_0+\omega_0-\nu M^2\alpha_0)\right\|_{L^2}^2\\=&\frac{\nu+\lambda}{4}\left\|\partial_y^{\ell}\alpha_0\right\|_{L^2}^2+\frac{(\nu+\lambda)^2}{4}\left\langle\partial_y^{\ell+1}\alpha_0,\partial_y^{\ell+1}\rho_0\right\rangle\\&+\lambda\nu M^2\left\langle\partial_y^{\ell+1}\alpha_0,\partial_y^{\ell+1}(\rho_0+\omega_0-\nu M^2\alpha_0)\right\rangle.\end{aligned}$$

Using again that $M^2(\nu+\lambda)\leq 1$, we have

$$\begin{aligned}\frac{(\nu+\lambda)^2}{4}\left|\left\langle\partial_y^{\ell+1}\alpha_0,\partial_y^{\ell+1}\rho_0\right\rangle\right|\leq&\frac{(\nu+\lambda)^2}{8}\left\|\partial_y^{\ell+1}\alpha_0\right\|_{L^2}^2+M^2(\nu+\lambda)\frac{(\nu+\lambda)}{8M^2}\left\|\partial_y^{\ell+1}\rho_0\right\|_{L^2}^2\\\leq&\frac{(\nu+\lambda)}{8}\left\|\partial_y^{\ell+1}\alpha_0\right\|_{L^2}^2+\frac{(\nu+\lambda)}{8M^2}\left\|\partial_y^{\ell+1}\rho_0\right\|_{L^2}^2,\end{aligned}$$

and

$$\begin{aligned}\lambda\nu M^2\left|\left\langle\partial_y^{\ell+1}\alpha_0,\partial_y^{\ell+1}(\rho_0+\omega_0-\nu M^2\alpha_0)\right\rangle\right|\leq&\frac{\nu}{2}\left\|\partial_y^{\ell+1}\alpha_0\right\|_{L^2}^2\\&+\frac{\nu}{2}\left\|\partial_y^{\ell+1}(\rho_0+\omega_0-\nu M^2\alpha_0)\right\|_{L^2}^2.\end{aligned}$$

Consequently we infer

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E^\ell(t)&+\frac{(\nu+\lambda)}{8}\left(\left\|\partial_y^{\ell+1}\alpha_0\right\|_{L^2}^2+\left\|\partial_y^{\ell}\alpha_0\right\|_{L^2}^2+\frac{1}{M^2}\left\|\partial_y^{\ell+1}\rho_0\right\|_{L^2}^2\right)\\&+\frac{\nu}{2}\left\|\partial_y^{\ell+1}(\rho_0+\omega_0-\nu M^2\alpha_0)\right\|_{L^2}^2\leq 0\end{aligned}. \tag{3.27}$$

Therefore, by combining (3.26) with (3.27) we prove that for any $\ell\geq 0$

$$\mathcal{E}^\ell(t)\lesssim\mathcal{E}^\ell_{in}. \tag{3.28}$$

To prove (3.24), we need to reconstruct some power of the energy functional by providing lower bounds for the positive terms appearing in (3.27). Hence, we first recall the following interpolation inequality, see also [89, Lemma A.4],

$$\left\|\partial_y^\ell f\right\|_{L^2}\leq\left\|\partial_y^{\ell+1}f\right\|_{L^2}^{\frac{\ell}{\ell+1}}\left\|f\right\|_{L^2}^{\frac{1}{\ell+1}}, \tag{3.29}$$

which can be proved directly via Plancherel's Theorem as follows

$$\left\|\partial_y^\ell f\right\|_{L^2}^2=\int_{\mathbb{R}}|\eta|^{2\ell}|\widehat{f}|^{\frac{2\ell}{\ell+1}}|\widehat{f}|^{\frac{2}{\ell+1}}d\eta\leq\left\||\eta|^{\ell+1}\widehat{f}\right\|_{L^2}^{\frac{2\ell}{\ell+1}}\left\|f\right\|_{L^2}^{\frac{2}{\ell+1}},$$

where in the last inequality we have applied the Hölder inequality with exponents $p = (\ell+1)/\ell$ and $p' = \ell + 1$. In addition, by (3.28) we know that

$$\|\alpha_0\|_{L^2}^2 + \left\|\partial_y^{-1}\alpha_0\right\|_{L^2}^2 + \frac{1}{M^2}\|\rho_0\|_{L^2}^2 + \left\|\rho_0 + \omega_0 - \nu M^2\alpha_0\right\|_{L^2}^2 \lesssim \mathcal{E}_{in}^0. \tag{3.30}$$

Therefore, for $\ell \geq 1$ from (3.29) and (3.30) we get

$$\begin{aligned}\left\|\partial_y^{\ell+1}\alpha_0\right\|_{L^2} + \frac{1}{M^2}\left\|\partial_y^{\ell+1}\rho_0\right\|_{L^2} \gtrsim& \left\|\partial_y^{\ell}\alpha_0\right\|_{L^2}^{1+\frac{1}{\ell}}\|\alpha_0\|_{L^2}^{-\frac{1}{\ell}} + \frac{1}{M^2}\left\|\partial_y^{\ell}\rho_0\right\|_{L^2}^{1+\frac{1}{\ell}}\|\rho_0\|_{L^2}^{-\frac{1}{\ell}}\\ \gtrsim&(\left\|\partial_y^{\ell}\alpha_0\right\|_{L^2}^{1+\frac{1}{\ell}} + (\frac{1}{M^2}\left\|\partial_y^{\ell}\rho_0\right\|_{L^2})^{1+\frac{1}{\ell}})(\mathcal{E}_{in}^0)^{-\frac{1}{2\ell}}.\end{aligned} \tag{3.31}$$

Similarly we have

$$\begin{aligned}&\left\|\partial_y^{\ell}\alpha_0\right\|_{L^2} = \left\|\partial_y^{\ell+1}(\partial_y^{-1}\alpha_0)\right\|_{L^2} \gtrsim \left\|\partial_y^{\ell-1}\alpha_0\right\|_{L^2}^{1+\frac{1}{\ell}}(\mathcal{E}_{in}^0)^{-\frac{1}{2\ell}},\\ &\left\|\partial_y^{\ell+1}(\rho_0+\omega_0-\nu M^2\alpha_0)\right\|_{L^2} \gtrsim \left\|\partial_y^{\ell}(\rho_0+\omega_0-\nu M^2\alpha_0)\right\|_{L^2}^{1+\frac{1}{\ell}}(\mathcal{E}_{in}^0)^{-\frac{1}{2\ell}}.\end{aligned}$$

In account of (3.28), we observe also that

$$\begin{aligned}\frac{1}{M^2}\left\|\partial_y^{\ell+1}\rho_0\right\|_{L^2} =&\frac{1}{M^2}\left\|\partial_y^{\ell+1}\rho_0\right\|_{L^2}^{1+\frac{1}{\ell}}\left\|\partial_y^{\ell+1}\rho_0\right\|_{L^2}^{-\frac{1}{\ell}}\\ \gtrsim&(\frac{1}{M^2}\left\|\partial_y^{\ell+1}\rho_0\right\|_{L^2})^{1+\frac{1}{\ell}}(\mathcal{E}_{in}^{\ell})^{-\frac{1}{2\ell}}.\end{aligned} \tag{3.32}$$

In particular, by combining the estimates (3.31)-(3.32) we infer

$$\begin{aligned}&\left\|\partial_y^{\ell+1}\alpha_0\right\|_{L^2}^2 + \left\|\partial_y^{\ell}\alpha_0\right\|_{L^2}^2 + \frac{1}{M^2}\left\|\partial_y^{\ell+1}\rho_0\right\|_{L^2}^2 + \left\|\partial_y^{\ell+1}(\rho_0+\omega_0-\nu M^2\alpha_0)\right\|_{L^2}^2\\ &\quad\gtrsim\left(\left\|\partial_y^{\ell}\alpha_0\right\|_{L^2}^2 + \left\|\partial_y^{\ell-1}\alpha_0\right\|_{L^2}^2 + \frac{1}{2M^2}\left\|\partial_y^{\ell}\rho_0\right\|_{L^2}^2\right)^{1+\frac{1}{\ell}}(\mathcal{E}_{in}^0)^{-\frac{1}{\ell}}\\ &\qquad+\left\|\partial_y^{\ell}(\rho_0+\omega_0-\nu M^2\alpha_0)\right\|_{L^2}^{1+\frac{1}{\ell}}(\mathcal{E}_{in}^0)^{-\frac{1}{2\ell}} + (\frac{1}{2M^2}\left\|\partial_y^{\ell+1}\rho_0\right\|_{L^2}^2)^{1+\frac{1}{\ell}}(\mathcal{E}_{in}^{\ell})^{-\frac{1}{\ell}}\\ &\quad\gtrsim(\mathcal{E}^{\ell}(t))^{1+\frac{1}{\ell}}(\max\{\mathcal{E}_{in}^{\ell},\mathcal{E}_{in}^0\})^{-\frac{1}{\ell}}.\end{aligned}$$

Consequently, appealing to (3.26), combining the bound above with (3.27) we have

$$\frac{\mathrm{d}}{\mathrm{d}t}E^{\ell}(t) + \nu C(\max\{\mathcal{E}_{in}^{\ell},\mathcal{E}_{in}^0\})^{-\frac{1}{\ell}}E^{\ell}(t)^{1+\frac{1}{\ell}} \leq 0,$$

where C is a constant independent of ℓ. Hence, from Grönwall's Lemma we get

$$E^{\ell}(t) \leq E_{in}^{\ell}(\nu\widetilde{C}_{in}^{\ell}t+1)^{-\ell},$$

where $\widetilde{C}_{in}^{\ell} = C(E_{in}^{\ell})^{\frac{1}{\ell}}(\max\{\mathcal{E}_{in}^{\ell},\mathcal{E}_{in}^0\})^{-\frac{1}{\ell}}$, whence proving (3.24) in view of (3.26). □

3.2 The inviscid case

In this section we investigate in detail the inviscid case. More precisely, we are going to prove the results stated in Theorem 3.0.1 when $\nu = \lambda = 0$, for which it is convenient to treat separately the analysis for the upper and lower bounds, respectively given in Theorem 3.2.5 and Theorem 3.2.7. As observed in Remark 3.1.2, we can remove the x-average from the dynamics, so we will prove the results only for initial perturbations without the $k = 0$ mode, namely $\rho_0^{in} = \alpha_0^{in} = \omega_0^{in} = 0$.

To proceed with the analysis of the system (3.3)-(3.5), in order to eliminate the transport term we make the change of coordinates (3.16) and we use the notation defined in (3.17)-(3.18).

By adding (3.3) to (3.5), we find out that $\rho + \omega$ is transported by the Couette flow. Hence, by defining

$$\Xi(t, X, Y) := R(t, X, Y) + \Omega(t, X, Y),$$

we have that $\partial_t \Xi = 0$. Consequently

$$\Omega(t, X, Y) = \Xi^{in}(X, Y) - R(t, X, Y), \tag{3.33}$$

where $\Xi^{in} = \omega^{in} + \rho^{in}$. In view of (3.6), we also have

$$\begin{aligned} V^y =&(\partial_Y - t\partial_X)\Delta_L^{-1} A + \partial_X \Delta_L^{-1}\Omega \\ =&(\partial_Y - t\partial_X)\Delta_L^{-1} A + \partial_X \Delta_L^{-1}\Xi^{in} - \partial_X \Delta_L^{-1} R. \end{aligned}$$

We can thus rewrite the system (3.3)-(3.5) in the moving frame only in terms of A and R as follows

$$\partial_t R = - A, \tag{3.34}$$

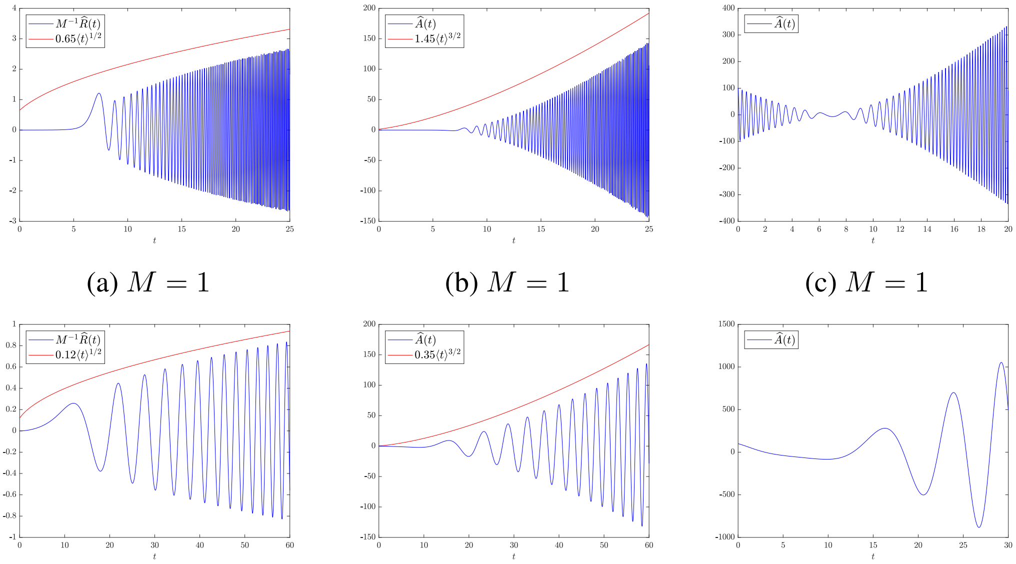
$$\begin{aligned} \partial_t A =& - 2\partial_X(\partial_Y - t\partial_X)(\Delta_L^{-1})A + \left(-\frac{1}{M^2}\Delta_L + 2\partial_{XX}(\Delta_L^{-1})\right) R \\ & - 2\partial_{XX}(\Delta_L^{-1})\Xi^{in}. \end{aligned} \tag{3.35}$$

We stress again the importance of the identity (3.33), which not only allow us to study the system only in terms of density and divergence but also relates compressible and incompressible effects, see Remark 3.0.3.

In view of the particular choice of the domain, it is now natural to perform the analysis in the Fourier space.

3.2.1 Analysis in the Fourier space

We first take the Fourier transform in both space variables of the system (3.34)-(3.35), which become a non-autonomous 2×2 dynamical system at each fixed frequency (k, η). Then, by properly weighting the density and the divergence we characterize the evolution semigroup of the associated homogeneous problem, i.e. $\Xi^{in} = 0$, which is a key point in order to prove Theorem 3.0.1.



(a) $M = 1$ (b) $M = 1$ (c) $M = 1$

(d) $M = 50$ (e) $M = 50$ (f) $M = 50$

Figure 3.1: Numerical simulations of the system (3.36)-(3.37) at fixed frequencies $k = 3, \eta = 21$ for different values of the Mach number. The red lines are the expected asymptotic behaviours. In the figures (a), (b), (d) and (e) we consider $\widehat{R}^{in} = \widehat{A}^{in} = 0$ and $\widehat{\Xi}^{in} = 5$. In the figures (c) and (f), we set $\widehat{R}^{in} = 20$, $\widehat{A}^{in} = 50$ and $\widehat{\Xi}^{in} = 5$. Notice the transient decay for the divergence, see Remark 3.2.1, up to times close to the critical one, namely $t = 7$.

By taking the Fourier transform of the system (3.34)-(3.35), recalling the notation introduced in (3.19)-(3.20), we get that

$$\partial_t \widehat{R} = -\widehat{A} \tag{3.36}$$

$$\partial_t \widehat{A} = \frac{\partial_t p}{p}\widehat{A} + \left(\frac{p}{M^2} + \frac{2k^2}{p}\right)\widehat{R} - \frac{2k^2}{p}\widehat{\Xi}^{in}. \tag{3.37}$$

Since in what follows we consider k, η as fixed parameters, we will omit their dependence for the quantities under study. In Figure 3.1 we show some numerical simulations of the system above.

Remark 3.2.1 (Transient decay)**.** From (3.37), since for $t < \eta/k$ one has $\partial_t p < 0$, the first term in the right-hand side of (3.37) acts as a damping term for $\widehat{A}$. Instead, $\partial_t p > 0$ for $t > \eta/k$, hence it induces a growth on $\widehat{A}$. In the incompressible case, see Section 1.2, the velocity may experience a transient growth, here, we see that the divergence may have a *transient decay*, see Figures 3.1c and 3.1f. To balance the growth generated by this term we need to properly weight $\widehat{R}$ and $\widehat{A}$.

Remark 3.2.2 (Wave equation for $\widehat{R}$)**.** Combining the equations (3.36)-(3.37) we have that

$$\partial_{tt}\widehat{R} - \frac{\partial_t p}{p}\partial_t \widehat{R} + \left(\frac{p}{M^2} + \frac{2k^2}{p}\right)\widehat{R} = \frac{2k^2}{p}\widehat{\Xi}^{in}.$$

In the physics literature the equation above is solved approximately for $M \ll 1$ [5, 37, 38, 82, 92], for example in [5] is used a WKB approximation. Indeed, for $\Xi^{in} = 0$ and assuming $M \ll 1$ one can say that the previous equation is approximated by

$$M^2 \partial_{tt} \widehat{R}_{app} = -p \widehat{R}_{app},$$

then, making a WKB ansatz, i.e. $\widehat{R}_{app}(t) = \exp(\delta^{-1} \sum_{n=0}^{+\infty} \delta^n S_n(t))$, a first order approximation satisfy

$$\widehat{R}_{app}(t) \approx S_1(t) = p^{\frac{1}{4}}(t) \left(c_1 e^{\frac{i}{M} \int_0^t \sqrt{p(\tau)} d\tau} + c_2 e^{-\frac{i}{M} \int_0^t \sqrt{p(\tau)} d\tau} \right). \tag{3.38}$$

In particular, recalling that $p = k^2 + (\eta - kt)^2$, the previous formal analysis suggest that $|\widehat{R}|^2$ should grow linearly in time. In the following, we essentially prove the validity of this asymptotic behaviour without the aid of any formal approximation.

The weighted variables

In order to study the system (3.36)-(3.37), let us first divide (3.37) by p (possible since $k \neq 0$) to have that

$$\begin{aligned} \partial_t \widehat{R} &= -\widehat{A} \\ \partial_t \left(\frac{\widehat{A}}{p} \right) &= \left(\frac{1}{M^2} + \frac{2k^2}{p^2} \right) \widehat{R} - \frac{2k^2}{p^2} \widehat{\Xi}^{in}. \end{aligned}$$

We want to look for a proper symmetrization of the system above. So we define

$$Z(t) = (Z_1(t), Z_2(t))^T = \left(\frac{\widehat{R}}{M p^{\frac{1}{4}}}(t), \ \frac{\widehat{A}}{p^{\frac{3}{4}}}(t) \right)^T. \tag{3.39}$$

Observe that if we are able to get a uniform bound on $|Z|$, in view of the weight on R, we will match the asymptotic behaviour predicted by (3.38). By a direct computation we find that $Z(t)$ satisfy

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} Z(t) &= L(t) Z(t) + F(t) \widehat{\Xi}^{in}, \\ Z(0) &= Z^{in} \end{aligned} \tag{3.40}$$

where

$$L(t) = \begin{bmatrix} -\dfrac{\partial_t p}{4p} & -\dfrac{\sqrt{p}}{M} \\ \dfrac{\sqrt{p}}{M} + \dfrac{2Mk^2}{p^{3/2}} & \dfrac{\partial_t p}{4p} \end{bmatrix}, \quad F(t) = \begin{pmatrix} 0 \\ -\dfrac{2k^2}{p^{7/4}} \end{pmatrix} \tag{3.41}$$

and

$$Z^{in} = \left(\frac{1}{M(k^2 + \eta^2)^{\frac{1}{4}}} \widehat{R}^{in}, \frac{1}{(k^2 + \eta^2)^{\frac{3}{4}}} \widehat{A}^{in} \right)^T \tag{3.42}$$

A key property coming from the choice of the weights on R, A given in the definition (3.39) is that the matrix $L(t)$ is trace-free.

We now have to deal with a non-autonomous 2D dynamical system. The solution of (3.40), given by Duhamel's formula, is

$$Z(t) = \Phi_L(t,0)\bigg(Z^{in} + \int_0^t \Phi_L(0,s)F(s)\widehat{\Xi}^{in}ds\bigg),$$

where Φ_L is the evolution operator defined in (3.15). Notice that $\Phi_L \neq \exp(L)$ since $L(t)L(s) \neq L(s)L(t)$. Therefore, everything is reduced in studying properties of the operator Φ_L, which is equivalent to the study of the homogeneous problem associated to (3.40).

Properties of Φ_L

In order to investigate properties of (3.40) when $\Xi^{in} = 0$, let us first consider the following toy example

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}Z(t) &= \begin{bmatrix} -a & -b \\ d & a \end{bmatrix} Z(t), \\ Z(0) &= Z^{in}, \end{aligned}$$

where $a, b, d \in \mathbb{R}$, $b, d \neq 0$, $bd > 0$ and $Z^{in} \in \mathbb{R}^2$. Then, one can check that $Z(t)$ satisfy

$$E(t) = \sqrt{\frac{d}{b}}|Z_1|^2(t) + \sqrt{\frac{b}{d}}|Z_2|^2(t) + 2\frac{a}{\sqrt{db}}Z_1(t)Z_2(t) = E(0).$$

In particular, if $\sqrt{bd} > a$ then a trajectory in the phase space is an ellipse determined by the equation above. In the non-autonomous case, we cannot expect to have immediately a conserved quantity, however we have the following lemma which play a crucial role in our subsequent analysis.

Lemma 3.2.3. *Let $Z(t)$ be a solution to* (3.40) *with $\widehat{\Xi}^{in} = 0$. Define*

$$a(t) = \frac{1}{4}\frac{\partial_t p}{p}, \qquad b(t) = \frac{\sqrt{p}}{M}, \qquad d(t) = \frac{\sqrt{p}}{M} + \frac{2Mk^2}{p^{3/2}}. \tag{3.43}$$

and

$$E(t) = \left(\sqrt{\frac{d}{b}}|Z_1|^2\right)(t) + \left(\sqrt{\frac{b}{d}}|Z_2|^2\right)(t) + 2\left(\frac{a}{\sqrt{db}}\mathrm{Re}(Z_1\bar{Z}_2)\right)(t). \tag{3.44}$$

Then, there exists constants $c_1, C_1, c_2, C_2 > 0$ independent of k, η such that

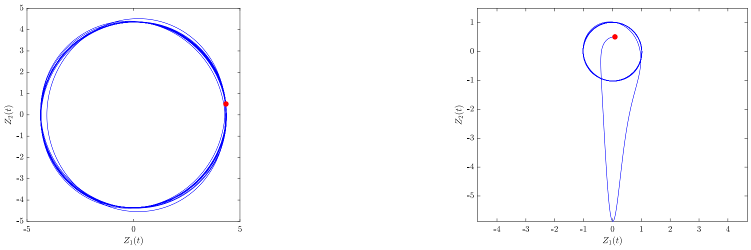
$$c_1E(0) \leq E(t) \leq C_1E(0), \tag{3.45}$$

and

$$c_2|Z^{in}| \leq \left|\Phi_L(t,0)Z^{in}\right| \leq C_2|Z^{in}|. \tag{3.46}$$

In addition, let $\mathrm{Re}(Z_1(t)) = r(t)\cos(\theta(t))$ *and* $\mathrm{Re}(Z_2(t)) = r(t)\sin(\theta(t))$ *(or the imaginary part), then we have*

$$\frac{\mathrm{d}}{\mathrm{d}t}\theta(t) = \frac{\sqrt{p}}{M} + \frac{2Mk^2}{p^{3/2}}\cos(\theta(t))^2 + \frac{1}{4}\frac{\partial_t p}{p}\sin(2\theta(t)). \tag{3.47}$$



(a) $M = 1$ (b) $M = 50$

Figure 3.2: Numerical simulations of the homogeneous problem associated to (3.40) for $k = 3, \eta = 21$ at different values of the Mach number. The red circle is the initial value, where we have considered $\widehat{R}^{in} = 20, \widehat{A}^{in} = 50$. In figure (a) the final time is 10, in figure (b) is 50.

This Lemma shows that the trajectories of the homogeneous problem associated to (3.40) are contained inside an annular region of the Z-plane and rotate with an angular velocity given by θ. In particular, since $d/b \to 1$ and $a/\sqrt{db} \to 0$ as $t \to \infty$, the limit cycle is a circle, see the Figure 3.2.By approximating (3.47) and retaining the leading order terms one may infer a dispersion relation like $M^{-1}\sqrt{k^2 + (\eta - kt)^2}$, which was also observed in [5, 92] and is the one suggested by the WKB approximation, see (3.38). However, dispersive properties require a more delicate analysis which we do not pursue in this thesis.

We now present the proof of the Lemma 3.2.3.

Proof. First of all, we define

$$\zeta = \sqrt{\frac{d}{b}}, \qquad \beta = \sqrt{bd}. \tag{3.48}$$

Then, with the notation introduced in (3.43), for $\widehat{\Xi}^{in} = 0$ the system (3.40) become

$$\frac{\mathrm{d}}{\mathrm{d}t} Z_1 = -aZ_1 - bZ_2, \tag{3.49}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} Z_2 = dZ_1 + aZ_2. \tag{3.50}$$

Notice that

$$1 \leq \zeta^2 = 1 + \frac{2M^2k^2}{p^2} \leq 1 + 2M^2, \tag{3.51}$$

hence, multiplying (3.49) by ζ and dividing (3.50) by ζ we obtain that

$$\zeta \frac{\mathrm{d}}{\mathrm{d}t} Z_1 = -a\zeta Z_1 - \beta Z_2, \tag{3.52}$$

$$\frac{1}{\zeta}\frac{\mathrm{d}}{\mathrm{d}t} Z_2 = \beta Z_1 + \frac{a}{\zeta} Z_2. \tag{3.53}$$

Now we multiply (3.52) by $\bar{Z}_1$, (3.53) by $\bar{Z}_2$ and we add the two equations to have that

$$\frac{\zeta}{2}\frac{\mathrm{d}}{\mathrm{d}t}|Z_1|^2 + \frac{1}{2\zeta}\frac{\mathrm{d}}{\mathrm{d}t}|Z_2|^2 = -a\left(\zeta|Z_1|^2 - \frac{1}{\zeta}|Z_2|^2\right). \tag{3.54}$$

Then, since the matrix L is trace-free, just observe the following

$$\frac{\mathrm{d}}{\mathrm{d}t}\operatorname{Re}(Z_1\bar{Z}_2)=\beta\big(\zeta|Z_1|^2-\frac{1}{\zeta}|Z_2|^2\big). \tag{3.55}$$

Plugging (3.55) into (3.54) we get that

$$\frac{\zeta}{2}\frac{\mathrm{d}}{\mathrm{d}t}|Z_1|^2+\frac{1}{2\zeta}\frac{\mathrm{d}}{\mathrm{d}t}|Z_2|^2+\frac{a}{\beta}\frac{\mathrm{d}}{\mathrm{d}t}\operatorname{Re}(Z_1\bar{Z}_2)=0.$$

Hence, in view of (3.48), we have that $E(t)$, see (3.44), satisfy

$$\frac{\mathrm{d}}{\mathrm{d}t}E=\frac{\mathrm{d}}{\mathrm{d}t}\bigg(\log(\zeta)\bigg)\zeta|Z_1|^2+\frac{\mathrm{d}}{\mathrm{d}t}\bigg(\log(\frac{1}{\zeta})\bigg)\frac{1}{\zeta}|Z_2|^2+2\frac{\mathrm{d}}{\mathrm{d}t}\bigg(\frac{a}{\beta}\bigg)\operatorname{Re}(Z_1\bar{Z}_2). \tag{3.56}$$

Then, since $|\partial_t p|\leq 2|k|\sqrt{p}$, observe that

$$\frac{|a|}{\beta}\leq\frac{1}{2}\frac{|k|}{\sqrt{p}}\left(\frac{p}{M^2}+\frac{2k^2}{p}\right)^{-\frac{1}{2}}\leq\frac{1}{2}\frac{|k|}{\sqrt{p}}\frac{\sqrt{p}}{\sqrt{2}|k|}=\frac{1}{2\sqrt{2}}. \tag{3.57}$$

Therefore, by calling $\widetilde{E}(t)=\zeta|Z_1|^2+\zeta^{-1}|Z_2|^2$, we obtain

$$-\frac{1}{2}\widetilde{E}(t)\leq 2(\frac{|a|}{\beta}\operatorname{Re}(Z_1\bar{Z}_2))(t)\leq\frac{1}{2}\widetilde{E}(t),$$

consequently it follows that

$$\frac{1}{2}\widetilde{E}(t)\leq E(t)\leq\frac{3}{2}\widetilde{E}(t). \tag{3.58}$$

By combining (3.56) with (3.58), we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E\leq&\frac{3}{2}\left(\left|\frac{\mathrm{d}}{\mathrm{d}t}\log(\zeta)\right|+\left|\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{a}{\beta}\right)\right|\right)\widetilde{E},\\ \leq&\frac{9}{4}\left(\left|\frac{\mathrm{d}}{\mathrm{d}t}\log(\zeta)\right|+\left|\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{a}{\beta}\right)\right|\right)E.\end{aligned} \tag{3.59}$$

Analogously, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}E\geq-\frac{1}{4}\left(\left|\frac{\mathrm{d}}{\mathrm{d}t}\log(\zeta)\right|+\left|\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{a}{\beta}\right)\right|\right)E. \tag{3.60}$$

In order to apply the Grönwall's Lemma, it remains to provide a uniform bound for the integral in time of the terms in bracket of (3.60). For the first one, observe that since $\partial_t\zeta^2=-4M^2k^2(\partial_t p)p^{-3}$ changes sign only in $t=\eta/k$ one has

$$\begin{aligned}\int_0^\infty\left|\frac{\mathrm{d}}{\mathrm{d}t}\log(\zeta)\right|d\tau=&\frac{1}{2}\int_0^\infty\left|\frac{\mathrm{d}}{\mathrm{d}t}\log(\zeta^2)\right|d\tau\\ =&\frac{1}{2}\log\left(\frac{\zeta^2(\eta/k)}{\zeta^2(0)}\right)+\frac{1}{2}\log\left(\frac{\zeta^2(\eta/k)}{\zeta^2(+\infty)}\right)\\ \leq&\ \log(1+2M^2),\end{aligned} \tag{3.61}$$

where we have also used (3.51). For the one involving a/β, since the bound (3.57) is uniform with respect to M, we simply observe that being a/β a bounded rational function may change sign only a finite number of times n_0, so that

$$\int_0^{+\infty} \left| \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{a}{\beta}\right)\right| d\tau \leq \frac{n_0}{\sqrt{2}}. \tag{3.62}$$

Therefore, by combining (3.59), (3.60) with (3.61) and (3.62), applying Grönwall's Lemma we infer

$$c_1 E(0) \leq E(t) \leq C_1 E(0), \tag{3.63}$$

whence proving (3.45). In addition, in view of (3.58), from (3.63) we get

$$\widetilde{c}_2 \widetilde{E}(0) \leq \widetilde{E}(t) \leq \widetilde{C}_2 \widetilde{E}(0). \tag{3.64}$$

Then, thanks to (3.51) we know that

$$(1+2M^2)^{-1}\widetilde{E}(t) \leq |\Phi_L(t,0)Z^{in}|^2 = |Z|^2(t) \leq (1+2M^2)\widetilde{E}(t), \tag{3.65}$$

and combining the inequalities above with (3.64) we prove (3.46).

To prove (3.47) observe that the coefficients of the system (3.40) are all real valued. Therefore, being the system linear, the real and imaginary part decouples. Then (3.47) directly follows by the fact that $r^2\dot{\theta} = x\dot{y} - \dot{x}y$. □

Remark 3.2.4. From the proof of Lemma 3.2.3, in view of the bounds (3.61) and (3.65), we also observe that the constants appearing in (3.46) satisfy $c_2, C_2 \approx \langle M\rangle^\beta$ for some $\beta > 1$ explicitly computable.

Upper and lower bounds

We can now present a more precise statement of the Theorem 3.0.1 in the inviscid case by considering separately the upper and lower bounds. Regarding the upper bounds we have the following.

Theorem 3.2.5. *Let ρ^{in}, $\omega^{in} \in H^1_x H^2_y$ and $\alpha^{in} \in H_x^{-\frac{1}{2}} L^2_y$ be the initial data of* (3.3)-(3.5)*. Then the following inequality hold*

$$\|(Q[\boldsymbol{v}] - Q[\boldsymbol{v}]_0)(t)\|_{L^2} + \frac{1}{M}\|(\rho-\rho_0)(t)\|_{L^2} \lesssim \langle t\rangle^{\frac{1}{2}} \left(\left\|\frac{\rho^{in}}{M}\right\|_{L^2} + \|\alpha^{in}\|_{H^{-1}} + \|\rho^{in}+\omega^{in}\|_{H^1}\right). \tag{3.66}$$

For the solenoidal component of the velocity we have

$$\begin{aligned}\|(P[\boldsymbol{v}]^x - P[\boldsymbol{v}]^x_0)(t)\|_{L^2} \lesssim{}& \frac{M}{\langle t\rangle^{\frac{1}{2}}}\left(\left\|\frac{\rho^{in}}{M}\right\|_{H_x^{-\frac{1}{2}}L^2_y} + \|\alpha^{in}\|_{H_x^{-\frac{1}{2}}H_y^{-1}} + \|\rho^{in}+\omega^{in}\|_{H_x^{-\frac{1}{2}}H_y^{\frac{1}{2}}}\right)\\ &+ \frac{1}{\langle t\rangle}\|\rho^{in}+\omega^{in}\|_{H_x^{-1}H^1_y},\end{aligned} \tag{3.67}$$

$$\|P[\boldsymbol{v}]^y\|_{L^2} \lesssim \frac{M}{\langle t\rangle^{\frac{3}{2}}}\bigg(\left\|\frac{\rho^{in}}{M}\right\|_{H_x^{-\frac{1}{2}}H_y^1} + \left\|\alpha^{in}\right\|_{H_x^{-\frac{1}{2}}L_y^2} + \left\|\rho^{in}+\omega^{in}\right\|_{H_x^{-\frac{1}{2}}H_y^{\frac{3}{2}}}\bigg) + \frac{1}{\langle t\rangle^2}\left\|\rho^{in}+\omega^{in}\right\|_{H_x^{-1}H_y^2}. \tag{3.68}$$

In view of the analysis in the frequency space that can be given through the Lemma 3.2.3, it is also possible to give an estimate on general Sobolev norms. Consequently, we could choose a suitable Sobolev space where also the acoustic part decays, which in particular implies a mixing phenomenon.

Corollary 3.2.6. *Let $s \geq 1/2$, $\rho^{in}, \omega^{in} \in H^{s-\frac{1}{2}}$ and $\alpha^{in} \in H^{s-\frac{3}{2}}$ be the initial data of* (3.3)-(3.5). *Then*

$$\|(Q[\boldsymbol{v}]-Q[\boldsymbol{v}]_0)(t)\|_{H^{-s}} + \frac{1}{M}\|(\rho-\rho_0)(t)\|_{H^{-s}} \leq \frac{1}{\langle t\rangle^{s-\frac{1}{2}}}C\big(\rho^{in},\alpha^{in},\omega^{in}\big),$$

where the constant depends upon Sobolev norms of the initial data.

We will not prove the corollary above since its proof can be directly deduced by the proof of Theorem 3.2.5, which we present in detail.

We now turn our attention to the lower bound (3.12), where we do not need to have regularity of the initial data as in the previous case. To state the results it is convenient to introduce

$$\Gamma(t, Z^{in}, \Xi^{in}) = \widehat{Z}^{in} + \int_0^t \Phi_L(0,s)F(s)\widehat{\Xi}^{in}ds. \tag{3.69}$$

We then have the following.

Theorem 3.2.7. *Let ρ^{in}, $\omega^{in} \in L_x^2 H_y^{-\frac{1}{2}}$ and $\alpha^{in} \in H_x^{-\frac{3}{2}}H_y^{-2}$. Then the solution of* (3.3)-(3.5) *with initial data $\rho^{in}, \alpha^{in}, \omega^{in}$ satisfy*

$$\|(Q[\boldsymbol{v}]-Q[\boldsymbol{v}]_0)(t)\|_{L^2} + \frac{1}{M}\|(\rho-\rho_0)(t)\|_{L^2} \gtrsim \langle t\rangle^{\frac{1}{2}}\left\|\Gamma(t,Z^{in},\Xi^{in})\right\|_{L_x^2H_y^{-1/2}},$$

where Z^{in} is defined as in (3.39) *and $\Xi^{in} = \rho^{in}+\omega^{in}$.*

Clearly, looking at (3.69), if $\Xi^{in} = 0$, namely $\rho^{in} = -\omega^{in}$, we immediately have a growth in time for non trivial initial conditions. When $\Xi^{in} \neq 0$, it may happen that the right-hand side of the inequality above become zero for some t. For this reason, in the following proposition we show that the set of initial data for which the right-hand side of the bound in Theorem 3.2.7 vanishes at some time has empty interior in any Sobolev space in which the initial data are taken.

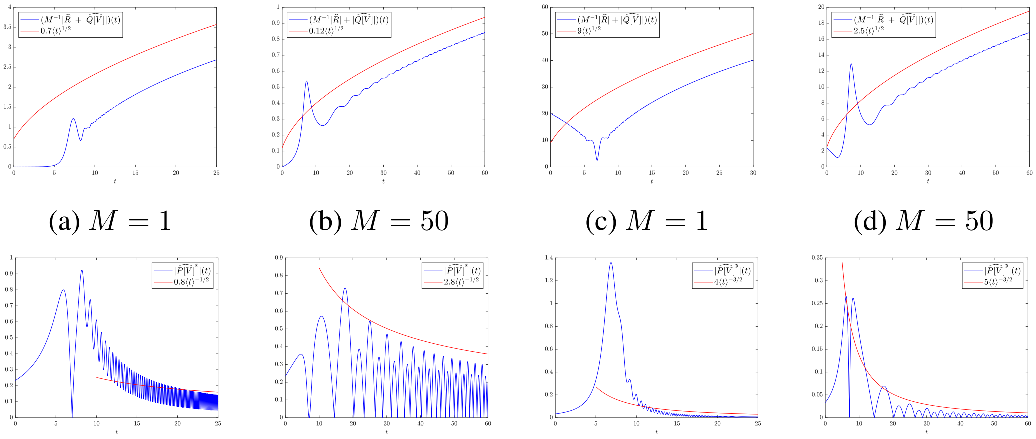
Proposition 3.2.8. *Let $s_1 \in \mathbb{R}$ and $s_2 \geq -1/2$. Given $\rho^{in}, \omega^{in} \in H_x^{s_1}H_y^{s_2}$ and $\alpha^{in} \in H_x^{s_1-\frac{3}{2}}H_y^{s_2-\frac{3}{2}}$, let $\Gamma(t, Z^{in}, \Xi^{in})$ be defined as in* (3.69)*, where Z^{in} is defined as in* (3.39) *and $\Xi^{in} = \rho^{in}+\omega^{in}$.*

Then, for any $\epsilon > 0$ sufficiently small, there exists $\rho_\epsilon^{in}, \alpha_\epsilon^{in}, \omega_\epsilon^{in}$ such that

$$\left\|\rho^{in}-\rho_\epsilon^{in}\right\|_{H_x^{s_1}H_y^{s_2}} + \left\|\omega^{in}-\omega_\epsilon^{in}\right\|_{H_x^{s_1}H_y^{s_2}} + \left\|\alpha^{in}-\alpha_\epsilon^{in}\right\|_{H_x^{s_1-\frac{3}{2}}H_y^{s_2-\frac{3}{2}}} \leq 2\epsilon, \tag{3.70}$$

and, by defining $Z_\epsilon^{in}, \Xi_\epsilon^{in}$ accordingly, the following inequality holds

$$\inf_{t\geq 0}\left\|\Gamma(t, Z_\epsilon^{in}, \Xi_\epsilon^{in})\right\|_{L_x^2H_y^{-1/2}} \geq \frac{\epsilon}{2}. \tag{3.71}$$



(a) $M = 1$ (b) $M = 50$ (c) $M = 1$ (d) $M = 50$

(e) $M = 1$ (f) $M = 50$ (g) $M = 1$ (h) $M = 50$

Figure 3.3: Numerical simulations of the system (3.3)-(3.5) at fixed frequencies $k = 3, \eta = 21$ at different values of the Mach number. The red lines are the asymptotic behaviours. In figures (a), (b) and (e)-(h) the initial values are $\widehat{R}^{in} = \widehat{A}^{in} = 0$ and $\widehat{\Xi}^{in} = 5$. In figures (c)-(d) we consider $\widehat{R}^{in} = 20, \widehat{A}^{in} = 50$ and $\widehat{\Xi}^{in} = 5$. In the figures at top we show the growth of the compressible part of the fluid. At the bottom we have the inviscid damping for the solenoidal component of the velocity field. When the initial density and divergence are not zero, namely in (c)-(d), we see the transient decay, mentioned in Remark 3.2.1, up to times close to $t = \eta/k = 7$.

Remark 3.2.9. In the proof of Proposition 3.2.8, given at the end of this section, we construct the perturbation $(\rho^{in}_\epsilon, \alpha^{in}_\epsilon, \omega^{in}_\epsilon)$ at any fixed frequency k, η, satisfying a non-degeneracy condition related to (3.71).

In the following, we prove Theorem 3.2.5 and Theorem 3.2.7, where we recall we are always assuming $\rho^{in}_0 = \alpha^{in}_0 = \omega^{in}_0 = 0$.

Proof of Theorem 3.2.5. We first prove the bounds for the solenoidal component of the velocity field, namely (3.67) and (3.68). By (3.33), we have

$$|\widehat{\Omega}|(t, k, \eta) \leq |\widehat{R}|(t, k, \eta) + |\widehat{\Xi}^{in}|(k, \eta).$$

Then, thanks to the Biot-Savart law, we prove (3.67) as follows,

$$\begin{aligned}
\|P[\boldsymbol{v}]^x\|^2_{L^2} &= \left\|\partial_y \Delta^{-1}\omega\right\|^2_{L^2} \\
&= \sum_k \int \frac{(\eta - kt)^2}{p^2}|\widehat{\Omega}|^2 d\eta \\
&\lesssim \sum_k \int M^2 \frac{(\eta - kt)^2}{p^{3/2}}\left|\frac{\widehat{R}}{Mp^{1/4}}\right|^2 + \frac{(\eta - kt)^2}{p^2}|\widehat{\Xi}^{in}|^2 d\eta \\
&\lesssim \sum_k \int \frac{M^2}{\sqrt{p}}(|Z^{in}|^2 + |\widehat{\Xi}^{in}|^2) + \frac{1}{p}|\widehat{\Xi}^{in}|^2 d\eta.
\end{aligned}$$

Now since $\langle \eta/k - t\rangle \langle \eta/k\rangle \gtrsim \langle t\rangle$ observe that

$$\frac{1}{\sqrt{p}} = \frac{1}{|k| \langle \eta/k - t\rangle} \frac{\langle \eta/k\rangle}{\langle \eta/k\rangle} \lesssim \frac{1}{\langle t\rangle} \frac{\langle \eta\rangle}{\langle k\rangle}.$$

Hence, recalling the definition of Z^{in}, see (3.42), we infer

$$\begin{aligned}\|P[\boldsymbol{v}]^x\|_{L^2}^2 \lesssim & \frac{M^2}{\langle t\rangle}\left(\left\|\frac{\rho^{in}}{M}\right\|_{H_x^{-\frac{1}{2}} L_y^2}^2 + \left\|\alpha^{in}\right\|_{H_x^{-\frac{1}{2}} H_y^{-1}}^2 + \left\|\rho^{in} + \omega^{in}\right\|_{H_x^{-\frac{1}{2}} H_y^{\frac{1}{2}}}^2 \right) \\ & + \frac{1}{\langle t\rangle^2} \left\|\rho^{in} + \omega^{in}\right\|_{H_x^{-1} H_y^1}^2,\end{aligned}$$

Similarly for $P[\boldsymbol{v}]^y$, we prove (3.68) as follows

$$\begin{aligned}\|P[\boldsymbol{v}]^y\|_{L^2}^2 = & \left\|\partial_x \Delta^{-1} \omega\right\|_{L^2}^2 \\ \lesssim & \sum_k \int M^2 \frac{k^2}{p^{3/2}} \left|\frac{\widehat{R}}{M p^{1/4}}\right|^2 + \frac{k^2}{p^2} |\widehat{\Xi}^{in}|^2 d\eta \\ \lesssim & \frac{M^2}{\langle t\rangle^3}\left(\left\|\frac{\rho^{in}}{M}\right\|_{H_x^{-\frac{1}{2}} H_y^1}^2 + \left\|\alpha^{in}\right\|_{H_x^{-\frac{1}{2}} L_y^2}^2 + \left\|\rho^{in} + \omega^{in}\right\|_{H_x^{-\frac{1}{2}} H_y^{\frac{3}{2}}}^2 \right) \\ & + \frac{1}{\langle t\rangle^4} \left\|\rho^{in} + \omega^{in}\right\|_{H_x^{-1} H_y^2}^2.\end{aligned}$$

To prove (3.66) first of all observe that, thanks to Lemma 3.2.3, we have

$$\begin{aligned}\int_0^\infty |\Phi_L(t,s) F(s)| ds \lesssim & \int_0^\infty |F(s)| ds \\ \lesssim & \frac{1}{|k|^{\frac{3}{2}}} \int_0^\infty \frac{ds}{(1 + (\eta/k - s)^2)^{\frac{7}{4}}} \lesssim \frac{1}{|k|^{\frac{3}{2}}}.\end{aligned} \tag{3.72}$$

Hence, by recalling the definition of Z, see (3.75), combining Lemma 3.2.3 with (3.72) we infer

$$|\widehat{Z}(t,k,\eta)| \lesssim |Z^{in}(k,\eta)| + |\widehat{\Xi}^{in}(k,\eta)| \quad \text{for any } t \geq 0. \tag{3.73}$$

Then, by the Helmholtz decomposition we have

$$\begin{aligned}\|Q[\boldsymbol{v}]\|_{L^2}^2 + \frac{1}{M^2}\|\rho\|_{L^2}^2 = & \left\|\partial_x \Delta^{-1} \alpha\right\|_{L^2}^2 + \left\|\partial_y \Delta^{-1} \alpha\right\|_{L^2}^2 + \frac{1}{M^2}\|\rho\|_{L^2}^2 \\ = & \sum_k \int \frac{|\widehat{\alpha}|^2(t,k,\eta)}{k^2 + \eta^2} + \frac{1}{M^2} |\widehat{\rho}|^2(t,k,\eta) d\eta \\ = & \sum_k \int \frac{|\widehat{A}|^2}{p}(t,k,\eta) + \frac{1}{M^2} |\widehat{R}|^2(t,k,\eta) d\eta.\end{aligned}$$

Therefore, by (3.73) and the fact that $p \leq \langle t\rangle^2 \langle k,\eta\rangle^2$, we get

$$\begin{aligned}
\|Q[\boldsymbol{v}]\|_{L^2}^2+\frac{1}{M^2}\|\rho\|_{L^2}^2=&\sum_k\int\sqrt{p}\bigg(\bigg|\frac{\widehat{A}}{p^{3/4}}\bigg|^2+\bigg|\frac{\widehat{R}}{Mp^{1/4}}\bigg|^2\bigg)d\eta \\
=&\sum_k\int\sqrt{p}|Z|^2d\eta \\
\lesssim&\langle t\rangle\left(\left\|Z^{in}\right\|_{H^1}^2+\left\|\rho^{in}+\omega^{in}\right\|_{H^1}^2\right) \\
=&\langle t\rangle\left(\left\|\frac{\rho^{in}}{M}\right\|_{L^2}^2+\left\|\alpha^{in}\right\|_{H^{-1}}^2+\left\|\rho^{in}+\omega^{in}\right\|_{H^1}^2\right),
\end{aligned}\tag{3.74}$$

hence concluding the proof ot Theorem 3.2.5. □

We now prove Theorem 3.2.7.

Proof of Theorem 3.2.7. Recall that the solution of (3.40) is given by the Duhamel's formula as

$$Z(t)=\Phi_L(t,0)\bigg(Z^{in}+\int_0^t\Phi_L(0,s)F(s)\widehat{\Xi}^{in}ds\bigg)=\Phi_L(t,0)\Gamma(t,Z^{in},\Xi^{in}),\tag{3.75}$$

where we have also used the definition of Γ given in (3.69). By Lemma 3.2.3 we have

$$|Z(t)|\geq c|\Gamma(t,Z^{in},\Xi^{in})|,$$

hence, combining the bound above with the identity in (3.74) we get

$$\begin{aligned}
\|Q[\boldsymbol{v}]\|_{L^2}^2+\frac{1}{M^2}\|\rho\|_{L^2}^2\gtrsim&\sum_k\int\sqrt{p}|\Gamma(t,Z^{in},\Xi^{in})|^2d\eta \\
\gtrsim&\sum_k\int\langle\eta-kt\rangle|\Gamma(t,Z^{in},\Xi^{in})|^2d\eta \\
\gtrsim&\langle t\rangle\sum_k\int\frac{1}{\langle\eta\rangle}|\Gamma(t,Z^{in},\Xi^{in})|^2d\eta,
\end{aligned}$$

where in the last two lines we have used that $\sqrt{p}\geq\langle\eta-kt\rangle$ and $\langle\eta-kt\rangle\langle\eta\rangle\gtrsim\langle kt\rangle$. Therefore we have

$$\|Q[\boldsymbol{v}])\|_{L^2}^2+\frac{1}{M^2}\|\rho\|_{L^2}^2\gtrsim\langle t\rangle\left\|\Gamma(t,Z^{in},\Xi^{in})\right\|_{L^2_xH^{-1/2}_y}^2$$

and the proof is over. □

Finally, we present the proof of the Proposition 3.2.8.

Proof of Proposition 3.2.8. With a slight abuse of notations, from the definition (3.69) we have

$$\widehat{\Gamma}(t,k,\eta)=\widehat{Z}^{in}(k,\eta)+\int_0^t\Phi_L(0,s)F(s,k,\eta)\widehat{\Xi}^{in}(k,\eta)ds.\tag{3.76}$$

Now, let us fix the frequencies k, η. In this way, $t \mapsto \widehat{\Gamma}(t)$ is a regular curve in $\mathbb{C}^2$. We now want to construct a perturbation of the initial data. If $\widehat{\Xi}^{in}(k,\eta) = 0$ there is nothing to prove. So we assume that $\widehat{\Xi}^{in}(k,\eta) \neq 0$.

First of all, by a computation similar to (3.72), we know that $\lim_{t\to\infty} \Gamma(t,k,\eta) = \Gamma^\infty(k,\eta)$. Let us first consider the case $\Gamma^\infty \neq 0$.

We claim that in this case $\Gamma(t,k,\eta)$ vanishes at most in a finite number of times t_i for $i = 0, \dots, n$.

Indeed, since $|\Gamma^\infty| > 0$ and the integral in (3.76) is converging, see (3.72), there is a $T(\Gamma^\infty, k, \eta) > 0$ such that

$$|\Gamma(t,k,\eta)| > \frac{1}{2}|\Gamma^\infty(k,\eta)| \quad \text{for } t \geq T(\Gamma^\infty, k, \eta). \tag{3.77}$$

Hence, we know that Γ may vanish only for $t \in [0, T(\Gamma^\infty, k, \eta)]$. Then, by (3.76) and the bound (3.46) in Lemma 3.2.3, we have

$$\begin{aligned}|\partial_t \Gamma(t,k,\eta)| =&|\Phi_L(0,t)F(t,k,\eta)\widehat{\Xi}^{in}(k,\eta)| \geq C|F(t,k,\eta)\widehat{\Xi}^{in}(k,\eta)| \\ \geq& C(T,k,\eta)|\widehat{\Xi}^{in}(k,\eta)|,\end{aligned}$$

where, from the definition of F given in (3.41), we define

$$C(T,k,\eta) = C \min_{t\in[0,T]} |F(t,k,\eta)| = 2C\left(|k|^{\frac{3}{2}} \max_{t\in[0,T]} \langle \eta/k - t\rangle^{\frac{7}{2}}\right)^{-1} > 0,$$

and the last inequality follows since (k,η) are fixed, $|k| \geq 1$ and $T < +\infty$. Consequently, since in the compact set $[0, T(\Gamma^\infty)]$ there are no points t^* such that $\Gamma(t^*) = \partial_t\Gamma(t^*) = 0$, exploiting also the continuity of $\partial_t\Gamma$, we have that Γ vanishes at most in a finite number of times in the interval $[0, T(\Gamma^\infty)]$.
Now we can construct the perturbation of the initial data. Consider the set $\{\partial_t\Gamma(t_i,k,\eta)\}$ for $i = 0, \dots, n$. In view of the continuity of $\partial_t\Gamma$, there is an $\epsilon < \min\{|\Gamma^\infty|/2, 1\}$ and at least one unit vector $\nu_\epsilon(k,\eta) = (\nu_\epsilon^1(k,\eta), \nu_\epsilon^2(k,\eta))$ which is not parallel to any $\partial_t\Gamma(t_i,k,\eta)$ such that

$$|\Gamma(t,k,\eta) + \epsilon e^{-(k^2+\eta^2)}\nu_\epsilon(k,\eta)| > \frac{1}{2}\epsilon e^{-(k^2+\eta^2)}. \tag{3.78}$$

By choosing

$$\begin{aligned}\widehat{\alpha}^{in}_\epsilon(k,\eta) &= \widehat{\alpha}^{in}(k,\eta) + \epsilon(k^2+\eta^2)^{\frac{3}{4}} e^{-(k^2+\eta^2)}\nu_\epsilon^2(k,\eta), \\ \widehat{\rho}^{in}_\epsilon(k,\eta) &= \widehat{\rho}^{in}(k,\eta) + \epsilon M(k^2+\eta^2)^{\frac{1}{4}} e^{-(k^2+\eta^2)}\nu_\epsilon^1(k,\eta), \\ \widehat{\omega}^{in}_\epsilon(k,\eta) &= \widehat{\omega}^{in}(k,\eta) - \epsilon M(k^2+\eta^2)^{\frac{1}{4}} e^{-(k^2+\eta^2)}\nu_\epsilon^1(k,\eta),\end{aligned}$$

we clearly satisfy (3.70). In addition, we have

$$\widehat{Z}^{in}_\epsilon(k,\eta) = \widehat{Z}^{in}(k,\eta) + \epsilon e^{-(k^2+\eta^2)}\nu_\epsilon(k,\eta), \qquad \widehat{\Xi}^{in}_\epsilon(k,\eta) = \widehat{\Xi}^{in}(k,\eta).$$

Consequently

$$\begin{aligned}\widehat{\Gamma}^\epsilon(t,k,\eta) &= \widehat{Z}^{in}_\epsilon(k,\eta) + \int_0^t \Phi_L(0,s)F(s,k,\eta)\widehat{\Xi}^{in}_\epsilon(k,\eta)ds \\ &= \Gamma(t,k,\eta) + \epsilon e^{-(k^2+\eta^2)}\nu_\epsilon(k,\eta).\end{aligned}$$

By combining (3.77) with (3.78) we get that

$$|\Gamma^{\epsilon}(t,k,\eta)| \geq \frac{1}{2}\min\left(|\Gamma^{\infty}(k,\eta)|, \epsilon e^{-(k^2+\eta^2)}\right) \qquad \text{for any } t>0. \tag{3.79}$$

Let us now turn to the case $\Gamma^{\infty}(k,\eta)=0$. First we choose

$$\alpha_1^{in} = \alpha^{in} + \epsilon(k^2+\eta^2)^{\frac{3}{4}} e^{-(k^2+\eta^2)},$$

so that for the corresponding Γ^1 we get $|\Gamma^{1,\infty}(k,\eta)| = \epsilon e^{-(k^2+\eta^2)}$. At this point, we can repeat the previous argument.

Resuming, by using Plancherel's Theorem, from the bound (3.79) we obtain (3.71), hence the proof is over. $\square$

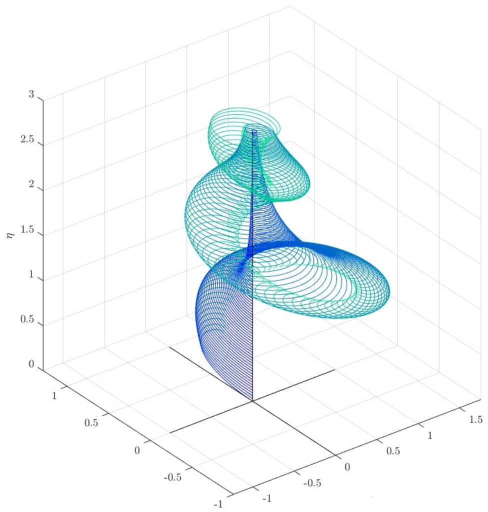


Figure 3.4: In the picture we show $\Gamma(t,1,\eta)$ for $t\in[0,10]$, $Z^{in}=0$ and $\widehat{\Xi}^{in}(1,\eta)=1$ for $\eta\in[0,3]$.

3.3 *The viscous case*

In this section, we study the system (3.3)-(3.5) in presence of viscosity, namely we assume that $\nu>0$ and $\lambda\geq 0$. As done in the previous section, we will prove the results only in the case $\rho_0^{in}=\alpha_0^{in}=\omega_0^{in}=0$, since the dynamics of the zero mode decouples with respect to fluctuations around it, see Section 3.1.

To remove the transport terms, we again consider the change of coordinates (3.16) and we use the notation introduced in (3.17)-(3.20). Defining $\mu=\nu+\lambda$, by taking the Fourier transform of the system (3.3)-(3.5) in the moving frame, we have

$$\partial_t \widehat{R} = -\widehat{A}, \tag{3.80}$$

$$\partial_t \widehat{A} = \frac{\partial_t p}{p}\widehat{A} - \mu p \widehat{A} + \frac{1}{M^2} p\widehat{R} - \frac{2k^2}{p}\widehat{\Omega}, \tag{3.81}$$

$$\partial_t \widehat{\Omega} = \widehat{A} - \nu p \widehat{\Omega}. \tag{3.82}$$

In the inviscid case, the conservation of $\Xi = R + \Omega$ was crucial in order to have a closed system in terms of R, A, which allow us to deal with a 2×2 non-autonomous system of ODE's in the Fourier space. Also in the viscous case it turns out that it is convenient to replace Ω with another auxiliary variable, however, the conservation of Ξ no longer hold since

$$\partial_t \widehat{\Xi} = -\nu p \widehat{\Xi} + \nu p \widehat{R}. \tag{3.83}$$

In addition, the last term in the right-hand side of (3.83) may be a problem to perform energy estimates. Indeed, it is not possible to directly control $\nu p \widehat{R}$ in a straightforward energy estimate, since we do not have a similar dissipative term for R. This problem relies essentially on the fact that we cannot hope to have any smoothing effect for the density. To overcome this difficulty, we observe that

$$\begin{aligned}
\partial_t(\widehat{\Xi} - \nu M^2 \widehat{A}) = & -\nu p(\widehat{\Xi} - \nu M^2 \widehat{A}) + \nu(\mu - \nu) M^2 p \widehat{A} \\
& -\nu M^2 \frac{\partial_t p}{p} \widehat{A} + 2\nu M^2 \frac{k^2}{p} \left(\widehat{\Xi} - \nu M^2 \widehat{A} \right) \\
& -2\nu M^2 \frac{k^2}{p} \widehat{R} + 2\nu^2 M^4 \frac{k^2}{p} \widehat{A},
\end{aligned} \tag{3.84}$$

where we have also used $\Omega = \Xi - R$. Although the equation (3.84) looks more complicated with respect to (3.83), it has a better structure to make use of $\Xi - \nu M^2 A$ as an auxiliary variable in an energy estimate. This because the first term in the right-hand side of (3.84) give us dissipation, the second term scales as the available dissipation for A (notice that if $\lambda = 0$ this term does not appear) and the remaining ones are lower order.

Remark 3.3.1. One of the main difficulties to obtain an enhanced dissipation estimate is the absence of a diffusive term in the continuity equation, because otherwise it would have been sufficient to combine the equation (3.83) with an adaptation of the energy estimates given in the inviscid case. Instead, we need to take advantage of the coupling between the density and the divergence to gain a dissipative term for the density. A similar strategy, inspired by the classical paper of Matsumura and Nishida [126], has been already exploited by Guo and Wang [89] to prove decay time rates for the compressible Navier-Stokes equations with smooth and small initial data.

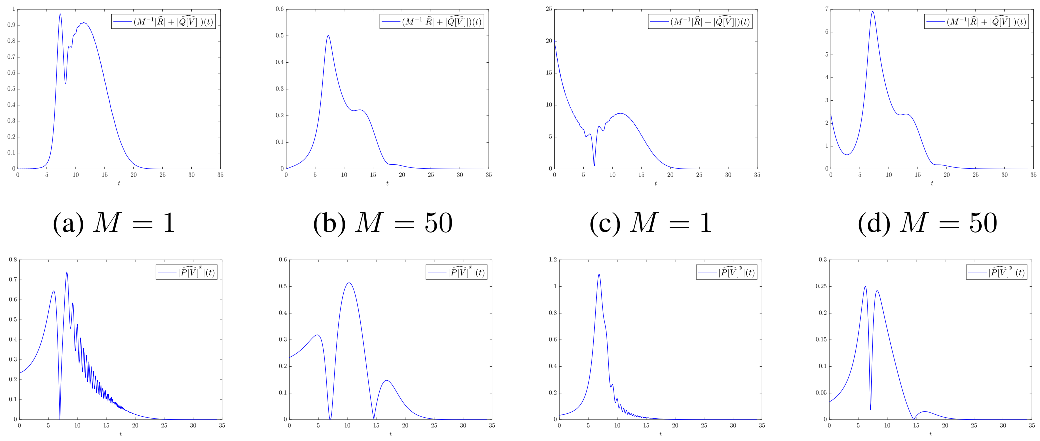
This section is organized as follows. In Subsection 3.3.1 we prove Theorem 3.0.1 whereas in Subsection 3.3.2 we prove Theorem 3.0.7.

Throughout all this section we make use of the following notation

$$C_{in,s} = \frac{1}{M} \left\| \rho^{in} \right\|_{H^{s+1}} + \left\| \alpha^{in} \right\|_{H^s} + \left\| \Xi^{in} - \nu M^2 \alpha^{in} \right\|_{H^s} \tag{3.85}$$

3.3.1 Combining the dissipation enhancement with the inviscid mechanism

In this subsection, we prove Theorem 3.0.1, which combines the dissipation enhancement generated by the presence of the background shear flow with the inviscid dynamics. More precisely, our goal is to obtain estimates such that in the limit $\nu \to 0$ we



(a) $M = 1$ (b) $M = 50$ (c) $M = 1$ (d) $M = 50$

(e) $M = 1$ (f) $M = 50$ (g) $M = 1$ (h) $M = 50$

Figure 3.5: Numerical simulations obtained from the system (3.80)-(3.82) at fixed frequencies $k = 3, \eta = 21$ at different values of the Mach number. The viscosities are $\nu = 0.0002$ and $\lambda = 0.001$. The initial conditions are the same as in the inviscid case shown in Figures 3.3a-3.3h. The inviscid dynamics starts being damped on a time-scale $O(\nu^{-1/3}) \approx 17$ (the standard diffusive one is $\nu^{-1} = 5000$).

recover the dynamics of the inviscid case, compare for instance Figure 3.5 with Figure 3.3. This implies, for example, that we need to weight the density and the divergence as done in the inviscid case, see (3.39), since for the weighted quantities we are able to infer an energy estimate. However, as previously discussed, when viscosity is present there is the loss of a conservation law, meaning that we will need to consider a 3×3 system in the Fourier space.

The key estimate which allow us to prove Theorem 3.0.1 is given in the following proposition.

Proposition 3.3.2. *Let $s \geq 0$, $\mu \leq 1/2$, $M > 0$ be such that $M \leq \min\{\mu^{-\frac{1}{2}}, \nu^{-\frac{1}{3}}\}$. If $\rho^{in} \in H^{s+1}(\mathbb{T} \times \mathbb{R})$ and $\alpha^{in}, \omega^{in} \in H^s(\mathbb{T} \times \mathbb{R})$ then*

$$\begin{aligned} \frac{1}{M} \left\| (p^{-\frac{1}{4}} \widehat{R})(t) \right\|_{H^s} + \left\| (p^{-\frac{3}{4}} \widehat{A})(t) \right\|_{H^s} \\ + \left\| (p^{-\frac{3}{4}} (\widehat{\Xi} - \nu M^2 \widehat{A}))(t) \right\|_{H^s} \lesssim e^{-\frac{1}{32}\nu^{\frac{1}{3}} t} C_{in,s}, \end{aligned} \tag{3.86}$$

where $C_{in,s}$ is defined in (3.85).

In accordance with the inviscid case, one should not expect any weight on the auxiliary variable $\widehat{\Xi} - \nu M^2 \widehat{A}$. However, the weight $p^{-\frac{3}{4}}$ is introduced for technical reasons, since it helps to control the second term in the right-hand side of (3.84), which is not present if $\lambda \neq 0$. We discuss more about this point in Remark 3.3.7.

To recover a bound on the vorticity one could exploit the fact that $\Omega = (\Xi - \nu M^2 A) + \nu M^2 A - R$ and use the previous proposition to infer estimates on $P[\boldsymbol{v}]$. This procedure, since A and $\Xi - \nu M^2 A$ have slower decay rates with respect to R (formally, think of p^{-1} as t^{-2}), would lead to worst decay rates with respect to the one given in Theorem 3.0.1 for the solenoidal component of the velocity. In particular, one cannot recover the estimates in the nonviscous and incompressible case by performing the formal limits $\nu \to 0$ and $M \to 0$ respectively.

Instead, by solving the equation for Ξ, see (3.83), via Duhamel's formula, from Proposition 3.3.2 we infer the following.

Corollary 3.3.3. *Let $s \geq 0$, $\mu \leq 1/2$, $M > 0$ be such that $M \leq \min\{\mu^{-\frac{1}{2}}, \nu^{-\frac{1}{3}}\}$. If $\rho^{in} \in H^{s+\frac{7}{2}}(\mathbb{T} \times \mathbb{R})$ and $\alpha^{in}, \omega^{in} \in H^{s+\frac{5}{2}}(\mathbb{T} \times \mathbb{R})$ then*

$$\begin{aligned} \|\Omega(t)\|_{H^s} \lesssim{}& M\langle t\rangle^{\frac{1}{2}} e^{-\frac{1}{32}\nu^{\frac{1}{3}}t} C_{in,s+\frac{1}{2}} + M \langle t\rangle^{\frac{1}{2}} e^{-\frac{1}{64}\nu^{\frac{1}{3}}t} C_{in,s+\frac{5}{2}} \\ &+ e^{-\frac{1}{12}\nu^{\frac{1}{3}}t} \left\|\omega^{in} + \rho^{in}\right\|_{H^s} \end{aligned} \tag{3.87}$$

Remark 3.3.4. Observe that in Proposition 3.3.2 and in Corollary 3.3.3 we are losing derivatives. The loss in (3.86) comes from the technical obstruction that forces us to introduce the weight $p^{-\frac{3}{4}}$ for the variable $\Xi - \nu M^2 A$. For $\lambda = 0$ one does not have this loss of derivatives, see Remark 3.3.7. Instead, the loss of derivatives in (3.87) seems to be necessary in view of the last term in the right-hand side of (3.83), where we can control time-growth by paying regularity.

In the following, appealing to Proposition 3.3.2 and Corollary 3.3.3 we first prove Theorem 3.0.1, while the proofs of the proposition and the corollary are postponed to the end of this subsection.

Proof of Theorem 3.0.1. We start with the proof of (3.9). From the Helmholtz decomposition (3.2) we have

$$\begin{aligned} \|Q[\boldsymbol{v}](t)\|_{L^2}^2 + \frac{1}{M^2}\|\rho(t)\|_{L^2}^2 &= \left\|(-\Delta)^{-\frac{1}{2}}\alpha(t)\right\|_{L^2}^2 + \frac{1}{M^2}\|\rho(t)\|_{L^2}^2 \\ &= \left\|(-\Delta_L)^{-\frac{1}{2}}A(t)\right\|_{L^2}^2 + \frac{1}{M^2}\|R(t)\|_{L^2}^2, \end{aligned}$$

where in the last line we have done the change of variables $X = x - yt,\ Y = y$. By the Plancherel's Theorem and the fact that $p \leq \langle t\rangle^2 \langle k,\eta\rangle^2$, we get

$$\begin{aligned} \|Q[\boldsymbol{v}](t)\|_{L^2}^2 + \frac{1}{M^2}\|\rho(t)\|_{L^2}^2 &= \left\|p^{\frac{1}{2}}(t)(p^{-\frac{3}{4}}\widehat{A})(t)\right\|_{L^2}^2 + \frac{1}{M^2}\left\|p^{\frac{1}{2}}(t)(p^{-\frac{1}{2}}\widehat{R})(t)\right\|_{L^2}^2 \\ &\lesssim \langle t\rangle \left(\left\|(p^{-\frac{3}{4}}\widehat{A})(t)\right\|_{H^1}^2 + \frac{1}{M^2}\left\|(p^{-\frac{1}{2}}\widehat{R})(t)\right\|_{H^1}^2\right) \\ &\lesssim \langle t\rangle e^{-\frac{1}{16}\nu^{\frac{1}{3}}t}(C_{in,1})^2, \end{aligned}$$

where in the last line we have used (3.86), hence proving (3.9).

We now turn our attention to the solenoidal component of the velocity in order to prove (3.10) and (3.11). By using again the Helmholtz decomposition, we have

$$\begin{aligned} \|P[\boldsymbol{v}]^x(t)\|_{L^2} &= \left\|\partial_y\Delta^{-1}\omega(t)\right\|_{L^2} = \left\|(\partial_Y - t\partial_X)(\Delta_L^{-1}\Omega)(t)\right\|_{L^2} \\ &\leq \left\|((-\Delta_L)^{-\frac{1}{2}}\Omega)(t)\right\|_{L^2}. \end{aligned}$$

Therefore, since $p^{\frac{1}{2}}\langle kt\rangle \geq \langle \eta - kt\rangle \langle kt\rangle \gtrsim \langle \eta\rangle$, we get

$$\|P[\boldsymbol{v}]^x(t)\|_{L^2} \lesssim \frac{1}{\langle t\rangle}\|\Omega(t)\|_{H^1}$$

and combining the previous bound with (3.87) we prove (3.10). The bound (3.11) follows analogously. □

In order to prove Proposition 3.3.2, we have to define a weighted energy functional to control. From the bounds on this energy functional the proof of Proposition 3.3.2 readily follows.

The weighted energy functional

We need to introduce the following Fourier multiplier, already used in [15, 16, 19, 166],

$$\begin{aligned}\partial_t m(t,k,\eta) =& \frac{2\nu^{\frac{1}{3}}}{\nu^{\frac{2}{3}}\left(\frac{\eta}{k}-t\right)^2+1} m(t,k,\eta),\\ m(0,k,\eta) =& \exp(2\arctan(\nu^{\frac{1}{3}}\frac{\eta}{k}))\end{aligned}$$

which is explicitly given by

$$m(t,k,\eta) = \exp(2\arctan(\nu^{\frac{1}{3}}(t-\frac{\eta}{k}))). \tag{3.88}$$

Clearly m and m^{-1} are bounded Fourier multipliers, therefore they generates an equivalent norm to the standard L^2.

The multiplier m is introduced since it enjoys the following crucial property

$$\nu p(t,k,\eta) + \frac{\partial_t m}{m}(t,k,\eta) \geq \nu^{\frac{1}{3}} \qquad \text{for any } t\geq 0,\ k\in\mathbb{Z}\setminus\{0\},\ \eta\in\mathbb{R}, \tag{3.89}$$

which compensates the slow down of the enhanced dissipation mechanism close to the critical times $t=\eta/k$. Indeed, $p\approx k^2$ for $t\approx \eta/k$. We then consider the system given by $(R, A, \Xi-\nu M^2 A)$, namely the equations (3.80), (3.81) when replacing $\widehat{\Omega}$ with $(\widehat{\Xi}-\nu M^2\widehat{A})+\nu M^2\widehat{A}-\widehat{R}$, and (3.84). Clearly, for the system under consideration the dynamics decouples in k,η, therefore we can perform estimates at each fixed frequency. Let $s\geq 0$, we define the following weighted variables

$$\begin{aligned}&Z_1(t) = \frac{1}{M}\langle k,\eta\rangle^s(m^{-1}p^{-\frac{1}{4}}\widehat{R})(t), \quad Z_2(t) = \langle k,\eta\rangle^s(m^{-1}p^{-\frac{3}{4}}\widehat{A})(t),\\ &Z_3(t) = \langle k,\eta\rangle^s(m^{-1}p^{-\frac{3}{4}}(\widehat{\Xi}-\nu M^2\widehat{A}))(t).\end{aligned} \tag{3.90}$$

Besides the multiplier $\langle k,\eta\rangle^s m^{-1}$, we remark that Z_1, Z_2 are the quantities also used in the non viscous case in order to symmetrize the system, see Section 3.2. Instead, Z_3, as explained, is introduced as an auxiliary variable to close the energy estimate.

Then, let $0 < \gamma = \gamma(M,\nu) \le 1/4$ be a parameter to be chosen later and consider the following energy functional

$$E(t) = \frac{1}{2}\Bigg(\left(1 + M^2 \frac{(\partial_t p)^2}{p^3}\right) |Z_1|^2(t) + |Z_2|^2(t) + |Z_3|^2(t) \tag{3.91}$$

$$+ (\frac{M}{2}\frac{\partial_t p}{p^{\frac{3}{2}}} \mathrm{Re}(\bar{Z}_1 Z_2))(t) - (2\gamma p^{-\frac{1}{2}} \mathrm{Re}(\bar{Z}_1 Z_2))(t) \Bigg). \tag{3.92}$$

Since $|\partial_t p| < p$, it is immediate to check that the previous energy functional is coercive, namely

$$E(t) \ge \frac{1}{4}\left((1 + M^2 \frac{(\partial_t p)^2}{p^3})|Z_1|^2 + |Z_2|^2 + 2|Z_3|^2 \right)(t) \tag{3.93}$$

$$E(t) \le \left((1 + M^2 \frac{(\partial_t p)^2}{p^3})|Z_1|^2 + |Z_2|^2 + |Z_3|^2 \right)(t). \tag{3.94}$$

Since m is a bounded Fourier multiplier, we also have

$$\sum_{k\neq 0} \int E(t) d\eta \approx \frac{1}{M^2} \left\| p^{-\frac{1}{4}} \widehat{R}(t) \right\|_{H^s}^2 + \left\| p^{-\frac{3}{4}} \widehat{A}(t) \right\|_{H^s}^2 + \left\| p^{-\frac{3}{4}} (\widehat{\Xi} - \nu M^2 \widehat{A})(t) \right\|_{H^s}^2 .$$

The latter equivalence tells us that a suitable estimate on $E(t)$ will imply the bound (3.86) in Proposition 3.3.2. In particular, we aim at proving the following

Lemma 3.3.5. *Under the assumptions of Proposition 3.3.2, let $E(t)$ be defined as in* (3.91) *and $\gamma = \dfrac{\nu^{\frac{1}{3}} M}{4}$, then*

$$\sum_k \int E(t) d\eta \lesssim e^{-\frac{\nu^{\frac{1}{3}}}{16} t} (C_{in,s})^2, \tag{3.95}$$

where $C_{in,s}$ is defined in (3.85).

Thanks to the previous Lemma, we conclude the proof of Proposition 3.3.2 as follows

$$\frac{1}{M^2} \left\| p^{-\frac{1}{4}} \widehat{R}(t) \right\|_{H^s}^2 + \left\| p^{-\frac{3}{4}} \widehat{A}(t) \right\|_{H^s}^2 + \left\| p^{-\frac{3}{4}} (\widehat{\Xi} - \nu M^2 \widehat{A})(t) \right\|_{H^s}^2$$
$$\lesssim \sum_k \int E(t) d\eta \lesssim e^{-\frac{\nu^{\frac{1}{3}}}{16} t} (C_{in,s})^2.$$

We now have to prove Lemma 3.3.5.

Proof of Lemma 3.3.5. We are going to prove the bound (3.95) via a Grönwall's inequality. Therefore, we have to first compute the time derivative of $E(t)$.

First of all, observe that

$$\partial_t Z_1 = -\frac{\partial_t m}{m} Z_1 - \frac{1}{4}\frac{\partial_t p}{p} Z_1 - \frac{1}{M} p^{\frac{1}{2}} Z_2, \tag{3.96}$$

$$\begin{aligned}\partial_t Z_2 = &-\left(\frac{\partial_t m}{m} Z_2 + \mu p\right) Z_2 + \frac{1}{4}\frac{\partial_t p}{p} Z_2 + \left(\frac{1}{M} p^{\frac{1}{2}} + 2M\frac{k^2}{p^{\frac{3}{2}}}\right) Z_1 \\ &- 2\frac{k^2}{p} Z_3 - 2\nu M^2 \frac{k^2}{p} Z_2,\end{aligned} \tag{3.97}$$

$$\begin{aligned}\partial_t Z_3 = &-\left(\frac{\partial_t m}{m} + \nu p\right) Z_3 - \frac{3}{4}\frac{\partial_t p}{p} Z_3 + \nu(\mu-\nu) M^2 p Z_2 - \nu M^2 \frac{\partial_t p}{p} Z_2 \\ &- 2\nu M^3 \frac{k^2}{p^{\frac{3}{2}}} Z_1 + 2\nu M^2 \frac{k^2}{p} Z_3 + 2\nu^2 M^4 \frac{k^2}{p} Z_2.\end{aligned} \tag{3.98}$$

From (3.96) we directly compute that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|Z_1|^2 = -\frac{\partial_t m}{m}|Z_1|^2 - \frac{1}{4}\frac{\partial_t p}{p}|Z_1|^2 - \frac{1}{M} p^{\frac{1}{2}} \mathrm{Re}(\bar{Z}_1 Z_2), \tag{3.99}$$

and

$$\begin{aligned}\frac{M^2}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left|\frac{\partial_t p}{p^{\frac{3}{2}}} Z_1\right|^2 = &M^2\left(\frac{2k^2\partial_t p}{p^3} - \frac{3}{2}\frac{(\partial_t p)^3}{p^4}\right)|Z_1|^2 - M^2\frac{\partial_t m}{m}\frac{(\partial_t p)^2}{p^3}|Z_1|^2 \\ &- \frac{M^2}{4}\frac{(\partial_t p)^3}{p^4}|Z_1|^2 - M\frac{(\partial_t p)^2}{p^{\frac{5}{2}}}\mathrm{Re}(\bar{Z}_1 Z_2).\end{aligned} \tag{3.100}$$

By (3.97) we get

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|Z_2|^2 = &-\left(\frac{\partial_t m}{m} + \mu p\right)|Z_2|^2 + \frac{1}{4}\frac{\partial_t p}{p}|Z_2|^2 + \frac{1}{M} p^{\frac{1}{2}} \mathrm{Re}(Z_1 \bar{Z}_2) \\ &+ 2M\frac{k^2}{p^{\frac{3}{2}}}\mathrm{Re}(Z_1\bar{Z}_2) - 2\frac{k^2}{p}\mathrm{Re}(Z_3\bar{Z}_2) - 2\nu M^2\frac{k^2}{p}|Z_2|^2.\end{aligned} \tag{3.101}$$

From (3.98) we have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|Z_3|^2 = &-\left(\frac{\partial_t m}{m} + \nu p\right)|Z_3|^2 - \frac{3}{4}\frac{\partial_t p}{p}|Z_3|^2 \\ &+ \nu(\mu-\nu)M^2 p \mathrm{Re}(Z_2\bar{Z}_3) - \nu M^2 \frac{\partial_t p}{p}\mathrm{Re}(Z_2\bar{Z}_3) \\ &- 2\nu M^3\frac{k^2}{p^{\frac{3}{2}}}\mathrm{Re}(Z_1\bar{Z}_3) + 2\nu M^2\frac{k^2}{p}|Z_3|^2 + 2\nu^2 M^4\frac{k^2}{p}\mathrm{Re}(Z_2\bar{Z}_3).\end{aligned} \tag{3.102}$$

Now we compute the time derivative of the mixed terms appearing in (3.92). The first term in (3.92) is introduced to cancel the terms with $\frac{1}{4}\partial_t p/p$ coming from the summation of (3.99) with (3.101), indeed observe that

$$
\begin{aligned}
\frac{M}{4}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{\partial_t p}{p^{\frac{3}{2}}}\mathrm{Re}(\bar{Z}_1 Z_2)\Big) =&\frac{M}{4}\left(\frac{2k^2}{p^{\frac{3}{2}}}-\frac{3}{2}\frac{(\partial_t p)^2}{p^{\frac{5}{2}}}\right)\mathrm{Re}(\bar{Z}_1 Z_2)-\frac{M}{2}\frac{\partial_t m}{m}\frac{\partial_t p}{p^{3/2}}\mathrm{Re}(\bar{Z}_1 Z_2) \qquad (3.103)\\
&-\frac{1}{4}\frac{\partial_t p}{p}(|Z_2|^2-|Z_1|^2)-\mu\frac{M}{4}\frac{\partial_t p}{p^{\frac{1}{2}}}\mathrm{Re}(\bar{Z}_1 Z_2)\\
&+M^2\frac{k^2\partial_t p}{2p^3}|Z_1|^2-M\frac{k^2\partial_t p}{2p^{\frac{5}{2}}}\mathrm{Re}(\bar{Z}_1 Z_3)\\
&-\nu M^3\frac{k^2\partial_t p}{2p^{\frac{5}{2}}}\mathrm{Re}(\bar{Z}_1 Z_2)
\end{aligned}
$$

The second term in (3.92) give us a dissipative term for Z_1 as follows

$$
\begin{aligned}
-\gamma\frac{\mathrm{d}}{\mathrm{d}t}\left(p^{-\frac{1}{2}}\mathrm{Re}(\bar{Z}_1 Z_2)\right) =&\frac{\gamma}{2}\frac{\partial_t p}{p^{\frac{3}{2}}}\mathrm{Re}(\bar{Z}_1 Z_2)+2\gamma\frac{\partial_t m}{m}p^{-\frac{1}{2}}\mathrm{Re}(\bar{Z}_1 Z_2) \qquad (3.104)\\
&+\frac{\gamma}{M}|Z_2|^2-\frac{\gamma}{M}\left(1+2M^2\frac{k^2}{p^2}\right)|Z_1|^2\\
&+\gamma\mu p^{\frac{1}{2}}\mathrm{Re}(\bar{Z}_1 Z_2)\\
&+2\gamma\frac{k^2}{p^{\frac{3}{2}}}\mathrm{Re}(\bar{Z}_1 Z_3)+2\gamma\nu M^2\frac{k^2}{p^{\frac{3}{2}}}\mathrm{Re}(\bar{Z}_1 Z_2).
\end{aligned}
$$

Hence, by rearranging the terms appearing in (3.99), (3.100), (3.101), (3.102), (3.103) and (3.104) we have the following identity

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}E(t) =&-\left(\frac{\partial_t m}{m}+\mu p\right)|Z_2|^2-\left(\frac{\partial_t m}{m}+\frac{\gamma}{M}\big(1+2M^2\frac{k^2}{p^2}\big)\right)|Z_1|^2 \qquad (3.105)\\
&-\left(\frac{\partial_t m}{m}+\nu p\right)|Z_3|^2+\sum_{i=1}^{5}\mathcal{D}_i+\sum_{i=1}^{6}\mathcal{I}_i,
\end{aligned}
$$

where we define the *dissipative error terms* as

$$
\begin{aligned}
\mathcal{D}_1 =&\frac{\gamma}{M}|Z_2|^2, \quad \mathcal{D}_2=\gamma\mu p^{\frac{1}{2}}\mathrm{Re}(\bar{Z}_1 Z_2), \quad \mathcal{D}_3=\nu(\mu-\nu)M^2 p\mathrm{Re}(\bar{Z}_2 Z_3) \qquad (3.106)\\
\mathcal{D}_4 =&-\nu M^2\frac{\partial_t p}{p}\mathrm{Re}(\bar{Z}_2 Z_3), \quad \mathcal{D}_5=-\frac{\mu M}{4}\frac{\partial_t p}{p^{\frac{1}{2}}}\mathrm{Re}(\bar{Z}_1 Z_2), \qquad (3.107)
\end{aligned}
$$

which we need to control with the negative terms appearing in (3.105). Instead, the *integrable error terms* are given by

$$\mathcal{I}_1 = M^2 \left(\frac{5k^2 \partial_t p}{2p^3} - \frac{7}{4} \frac{(\partial_t p)^3}{p^4} \right) |Z_1|^2, \tag{3.108}$$

$$\mathcal{I}_2 = M \left(\frac{1}{2}(\frac{\gamma}{M} - \frac{\partial_t m}{m}) \frac{\partial_t p}{p^{\frac{3}{2}}} + (\frac{5}{2} + 2\gamma\nu M) \frac{k^2}{p^{\frac{3}{2}}} - \frac{11}{8} \frac{(\partial_t p)^2}{p^{\frac{5}{2}}} \right) \mathrm{Re}(\bar{Z}_1 Z_2)$$
$$- \nu \frac{k^2 M^3 \partial_t p}{2p^{\frac{5}{2}}} \mathrm{Re}(\bar{Z}_1 Z_2) + 2\gamma \frac{\partial_t m}{m} p^{-\frac{1}{2}} \mathrm{Re}(\bar{Z}_1 Z_2),$$

$$\mathcal{I}_3 = -\frac{3}{4} \frac{\partial_t p}{p} |Z_3|^2 + 2\nu M^2 \frac{k^2}{p} |Z_3|^2, \tag{3.109}$$

$$\mathcal{I}_4 = M \left(2(\frac{\gamma}{M} - \nu M^2) \frac{k^2}{p^{\frac{3}{2}}} - \frac{k^2 \partial_t p}{2p^{\frac{5}{2}}} \right) \mathrm{Re}(\bar{Z}_1 Z_3),$$

$$\mathcal{I}_5 = \left(-2\frac{k^2}{p} + 2\nu^2 M^4 \frac{k^2}{p} \right) \mathrm{Re}(\bar{Z}_2 Z_3),$$

$$\mathcal{I}_6 = -M^2 \frac{\partial_t m}{m} \frac{(\partial_t p)^2}{p^3} |Z_1|^2 - 2\nu \frac{k^2 M^2}{p} |Z_2|^2, \tag{3.110}$$

which all involve Fourier multipliers integrable in time, as we explain below.

Now we proceed by providing suitable bounds on the terms $\mathcal{D}_i$. To control $\mathcal{D}_1$, in view of the property (3.89), we need to choose γ such that $\gamma/M < \nu^{\frac{1}{3}}$. Therefore, we define

$$\gamma = \frac{M\nu^{\frac{1}{3}}}{4} \tag{3.111}$$

and notice that $\gamma \leq 1/4$ by assumptions on M. To control the remaining terms, we also need to exploit the hypothesis

$$\mu M^2 \leq 1. \tag{3.112}$$

Since $\gamma \leq 1/4$ and $M\mu \leq 1$, we bound $\mathcal{D}_2$ as

$$|\mathcal{D}_2| \leq \frac{\gamma}{2} \mu p |Z_2|^2 + \frac{\gamma}{2M}(M\mu)|Z_1|^2 \leq \frac{\mu}{8} p |Z_2|^2 + \frac{\gamma}{2M}|Z_1|^2. \tag{3.113}$$

To control $\mathcal{D}_3$, in view of the restriction $\mu M^2 \leq 1$, we have

$$|\mathcal{D}_3| \leq \frac{\nu}{2} p |Z_2|^2 + \frac{\nu}{2} p |Z_3|^2. \tag{3.114}$$

Since $|\partial_t p| \leq 2|k| p^{\frac{1}{2}}$, the bounds on $\mathcal{D}_4, \mathcal{D}_5$ are given by

$$|\mathcal{D}_4| \leq \frac{\nu}{4}|Z_3|^2 + \nu M^4 \frac{(\partial_t p)^2}{p^2} |Z_2|^2 \leq \frac{\nu}{4}|Z_3|^2 + 4M^2 \frac{k^2}{p} |Z_2|^2, \tag{3.115}$$

$$|\mathcal{D}_5| \leq \frac{\mu}{16} p |Z_2|^2 + \frac{\mu M^2 k^2}{p} |Z_1|^2 \tag{3.116}$$

Notice that the last terms in the right-hand side of the last two inequalities need not to be absorbed with the negative terms in (3.105), being $k^2 p^{-1}$ integrable in time.

We now turn our attention to provide bounds for the terms $\mathcal{I}_i$, where we can exploit integrability in time of the Fourier multipliers. More precisely, we have two main contributions, one given by $\partial_t m/m$, which is clearly integrable in time and can also be absorbed with the negative terms appearing in (3.105). The second one is the multiplier k^2p^{-1}, appearing for example in $\mathcal{I}_3, \mathcal{I}_5$, whose integral in time is uniformly bounded with respect to k, η, namely

$$\int_0^t \frac{k^2}{p(\tau)}d\tau = \int_0^t \frac{d\tau}{(\frac{\eta}{k}-\tau)^2+1} = \left(\arctan(\frac{\eta}{k}-t) - \arctan(\frac{\eta}{k})\right).$$

In addition, for the term $\mathcal{I}_3$, since $\partial_t p > 0$ for $t > \eta/k$, we have

$$|\mathcal{I}_3| \leq -\frac{3}{4}\frac{\partial_t p}{p}\chi_{t<\frac{\eta}{k}}|Z_3|^2 + \frac{2k^2M^2}{p}|Z_3|^2, \tag{3.117}$$

therefore we can also integrate in time the first term in the right-hand side of the last inequality. However, this will be the source of a loss of regularity as it will be clear later on.

Then, since $|\partial_t p| \leq 2|k|p^{\frac{1}{2}}$ and recalling (3.112), we roughly estimate the remaining terms as follows

$$\begin{aligned}
|\mathcal{I}_1| &\leq C_1M^2\frac{k^2}{p}|Z_1|^2, &(3.118)\\
|\mathcal{I}_2| &\leq \left(C_2M\frac{k^2}{p} + \frac{1}{2}\frac{\partial_t m}{m}\right)(|Z_1|^2+|Z_2|^2),\\
|\mathcal{I}_4| &\leq C_4M\frac{k^2}{p}(|Z_1|^2+|Z_3|^2),\\
|\mathcal{I}_5| &\leq 2\frac{k^2}{p}(|Z_2|^2+|Z_3|^2),\\
\mathcal{I}_6 &\leq 0, &(3.119)
\end{aligned}$$

where we perform the last trivial bound since we cannot gain much from $\mathcal{I}_6$.

Therefore, thanks to the choice of γ in (3.111), the properties (3.89) and (3.93), by combining (3.113), (3.114), (3.115), (3.118)-(3.119) with (3.105) we infer

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}E(t) \leq &-\frac{\nu^{\frac{1}{3}}}{16}\left((1+4M^2\frac{k^2}{p^2})|Z_1|^2+|Z_2|^2+|Z_3|^2\right)\\
&+4\left(C_M\frac{k^2}{p}+\frac{1}{2}\frac{\partial_t m}{m}\right)E(t) - \frac{3}{4}\frac{\partial_t p}{p}\chi_{t<\frac{\eta}{k}}|Z_3|^2,
\end{aligned} \tag{3.120}$$

where C_M is explicitly computable from the previous bounds. Then, since

$$M^2\frac{(\partial_t p)^2}{p^3} \leq 4M^2\frac{k^2}{p^2}, \tag{3.121}$$

and from (3.94) we know that $|Z_3|^2 \leq 2E(t)$, by (3.120) we get

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) \leq -\frac{\nu^{\frac{1}{3}}}{16}E(t) + \left(-\frac{3}{2}\frac{\partial_t p}{p}\chi_{t<\frac{\eta}{k}} + 4C_M\frac{k^2}{p} + 2\frac{\partial_t m}{m}\right)E(t),$$

hence, applying Grönwall's Lemma we have

$$E(t) \leq \widetilde{C}_M e^{-\frac{\nu^{\frac{1}{3}}}{16}t} \frac{p(0)^{\frac{3}{2}}}{p(t)^{\frac{3}{2}}\chi_{t<\frac{\eta}{k}} + p(\eta/k)^{\frac{3}{2}}\chi_{t>\frac{\eta}{k}}} E(0) \leq \widetilde{C}_M e^{-\frac{\nu^{\frac{1}{3}}}{16}t} \langle k,\eta\rangle^3 E(0), \tag{3.122}$$

where $\widetilde{C}_M = \exp(5\pi C_M)$. Clearly the term $\langle k,\eta\rangle^3$ is the one which cause the loss of regularity, coming from the bound (3.117) as we have stressed previously.

To conclude the proof of Lemma 3.3.5, by summing in k and integrating in η in (3.122), thanks to (3.94) and (3.90) we have

$$\sum_{k\neq 0}\int E(t)d\eta \lesssim e^{-\frac{\nu^{\frac{1}{3}}}{16}t}\left(\frac{1}{M^2}\left\|R^{in}\right\|^2_{H^{s+1}} + \left\|A^{in}\right\|^2_{H^s} + \left\|\Xi^{in} - \nu M^2 A^{in}\right\|^2_{H^s}\right),$$

therefore, in view of (3.85), the proof of Lemma 3.3.5 is over. □

Remark 3.3.6. Combining the choice of γ with the restrictions (3.112) we immediately recover the hypothesis on the Mach number made in Theorem 3.0.1, namely $M \leq \min\{\mu^{-\frac{1}{2}}, \nu^{-\frac{1}{3}}\}$. However, by choosing $\gamma = \delta M\nu^{\frac{1}{3}}/4$ for $0 < \delta \leq 1$, it would be sufficient that $M \leq \min\{\mu^{-\frac{1}{2}}, \delta^{-1}\nu^{-\frac{1}{3}}\}$, while in the exponential bound (3.95) a factor δ will appear, namely we have $e^{-\delta\frac{\nu^{\frac{1}{3}}}{16}t}$. Therefore, we could slightly improve the range of available Mach numbers by deteriorating the decay rates.

In addition, from (3.122) we also see that the constants hidden when using the symbol $\lesssim$ grows exponentially fast with respect to M, which is clearly irrelevant for $M \approx 1$ but deteriorates extremely the bounds for larger values of M. It should be possible to improve this dependency up to constants $O(\langle M\rangle^{\beta})$ for some $\beta > 1$ by considering exactly the energy functional used in the inviscid case, see Lemma 3.2.3 and Remark 3.2.4, plus the terms due to the viscosity.

Remark 3.3.7 (Regularity in absence of bulk viscosity)**.** When $\lambda = 0$, namely $\mu = \nu$, it is sufficient to consider the auxiliary variable

$$\widetilde{Z}_3 = \langle k,\eta\rangle^s m^{-1}(\widehat{\Xi} - \nu M^2\widehat{A}),$$

which satisfy the following equation

$$\begin{aligned}\partial_t\widetilde{Z}_3 = &-\left(\frac{\partial_t m}{m} + \nu p\right)\widetilde{Z}_3 - \nu M^2\frac{\partial_t p}{p^{\frac{1}{4}}}Z_2\\ &- 2\nu M^3\frac{k^2}{p^{\frac{3}{4}}}Z_1 + 2\nu M^2\frac{k^2}{p}\widetilde{Z}_3 + 2\nu^2M^4\frac{k^2}{p^{\frac{1}{4}}}Z_2,\end{aligned}$$

where Z_1 and Z_2 are defined in (3.90). We can then proceed as in the proof of Lemma 3.3.5, clearly by defining the new error terms accordingly. For example, the most dangerous one can be controlled as follows

$$\begin{aligned}\nu M^2\frac{|\partial_t p|}{p^{\frac{1}{4}}}\mathrm{Re}(\bar{Z}_2\widetilde{Z}_3) \leq& 4\nu M^4\frac{|\partial_t p|}{p^{\frac{5}{4}}}|Z_2|^2 + \frac{\nu}{16}p|\widetilde{Z}_3|^2\\ \leq& 4\nu M^4\frac{|k|}{p^{\frac{3}{4}}}|Z_2|^2 + \frac{\nu}{16}p|\widetilde{Z}_3|^2,\end{aligned}$$

since $|k|p^{-\frac{3}{4}}$ is integrable in time. However, by using $\widetilde{Z}_3$ we will not have the error term containing the multiplier $-\frac{3}{4}\partial_t p/p$, see (3.109), meaning that in the bound analogous to (3.122) there is not $\langle k,\eta\rangle$, hence in Lemma 3.3.5 we do not lose derivatives (consequently also in Proposition 3.3.2)

We now turn our attention to the proof of Corollary 3.3.3.

Proof of Corollary 3.3.3. In order to prove (3.87), let $\mathcal{L}_\nu(t,k,\eta)=\nu\int_0^t p(\tau,k,\eta)d\tau$. As observed in (1.14), we have that

$$\left\|e^{\mathcal{L}_\nu(t)}f\right\|_{H^s}\leq e^{-\frac{1}{12}\nu t^3}\left\|f\right\|_{H^s}\leq e^{-\frac{1}{12}\nu^{\frac{1}{3}}t}\left\|f\right\|_{H^s}, \tag{3.123}$$

Therefore, solving (3.83) via Duhamel's formula we have

$$\widehat{\Xi}(t)=e^{\mathcal{L}_\nu(t)}\Xi^{in}+\nu\int_0^t e^{\mathcal{L}_\nu(t-\tau)}p(\tau)\widehat{R}(\tau)d\tau.$$

Appealing to (3.123), we get

$$\left\|\widehat{\Xi}(t)\right\|_{H^s}\leq e^{-\frac{1}{12}\nu^{\frac{1}{3}}t}\left\|\widehat{\Xi}^{in}\right\|_{H^s}+\nu\int_0^t e^{-\frac{1}{12}\nu^{\frac{1}{3}}(t-\tau)}\left\|p(\tau)\widehat{R}(\tau)\right\|_{H^s}d\tau. \tag{3.124}$$

To bound the integrand of the last equation, we exploit the bound obtained on $M^{-1}p^{-1/4}\widehat{R}$, see (3.86). In particular, by using that $p\leq\langle t\rangle^2\langle k,\eta\rangle^2$, we have

$$\begin{aligned}\left\|p(\tau)\widehat{R}(\tau)\right\|_{H^s}=&M\left\|p^{\frac{5}{4}}(\tau)(M^{-1}p^{-\frac{1}{4}}\widehat{R})(\tau)\right\|_{H^s}\\ \lesssim&M\langle\tau\rangle^{\frac{5}{2}}\left\|(M^{-1}p^{-\frac{1}{4}}\widehat{R})(\tau)\right\|_{H^{s+\frac{5}{2}}}\\ \lesssim&M\langle\tau\rangle^{\frac{5}{2}}e^{-\frac{1}{32}\nu^{\frac{1}{3}}\tau}C_{in,s+\frac{5}{2}}\end{aligned}$$

where we recall the definition of $C_{in,s}$ given in (3.85). Consequently we get

$$\begin{aligned}\nu\int_0^t e^{-\frac{1}{12}\nu^{\frac{1}{3}}(t-\tau)}\left\|p(\tau)\widehat{R}(\tau)\right\|_{H^s}d\tau\lesssim&\ MC_{in,s+\frac{5}{2}}\nu\int_0^t e^{-\frac{1}{12}\nu^{\frac{1}{3}}(t-\tau)}\langle\tau\rangle^{\frac{5}{2}}e^{-\frac{1}{32}\nu^{\frac{1}{3}}\tau}d\tau\\ \lesssim MC_{in,s+\frac{5}{2}}\nu\langle t\rangle^{\frac{1}{2}}\int_0^t e^{-\frac{1}{12}\nu^{\frac{1}{3}}(t-\tau)}\nu^{-\frac{2}{3}}(\nu^{\frac{1}{3}}\langle\tau\rangle)^2e^{-\frac{1}{32}\nu^{\frac{1}{3}}\tau}d\tau&\\ \lesssim MC_{in,s+\frac{5}{2}}\nu^{\frac{1}{3}}\langle t\rangle^{\frac{1}{2}}\int_0^t e^{-\frac{1}{12}\nu^{\frac{1}{3}}(t-\tau)}e^{-\frac{1}{64}\nu^{\frac{1}{3}}\tau}d\tau&\\ \lesssim MC_{in,s+\frac{5}{2}}\nu^{\frac{1}{3}}\langle t\rangle^{\frac{1}{2}}e^{-\frac{1}{64}\nu^{\frac{1}{3}}t}\int_0^t e^{-(\frac{1}{12}-\frac{1}{64})\nu^{\frac{1}{3}}(t-\tau)}d\tau&\\ \lesssim MC_{in,s+\frac{5}{2}}\langle t\rangle^{\frac{1}{2}}e^{-\frac{1}{64}\nu^{\frac{1}{3}}t}.&\end{aligned}$$

Combining the previous estimate with (3.124) we obtain

$$\left\|\Xi(t)\right\|_{H^s}\lesssim e^{-\frac{1}{12}\nu^{\frac{1}{3}}t}\left\|\Xi^{in}\right\|_{H^s}+M\langle t\rangle^{\frac{1}{2}}e^{-\frac{1}{64}\nu^{\frac{1}{3}}t}C_{in,s+\frac{5}{2}} \tag{3.125}$$

Finally, we directly recover the bound on Ω as follows

$$\begin{aligned}\|\Omega(t)\|_{H^s} &\leq \|\Xi(t)\|_{H^s} + \|R(t)\|_{H^s}\\ &= \|\Xi(t)\|_{H^s} + M\left\|p^{\frac{1}{4}}(M^{-1}p^{-\frac{1}{4}}R)(t)\right\|_{H^s}\\ &\lesssim \|\Xi(t)\|_{H^s} + M\langle t\rangle^{\frac{1}{2}}\left\|M^{-1}p^{-\frac{1}{4}}R(t)\right\|_{H^{s+\frac{1}{2}}}.\end{aligned}$$

The proof of the Corollary 3.3.3 then follows by combining the last bound with (3.86) and (3.125). □

3.3.2 Dissipation enhancement without loss of derivates

The purpose of this subsection is to prove Theorem 3.0.7, where again we assume that $\rho_{in,0} = \alpha_{in,0} = \omega_{in,0} = 0$. In the following, we first present a toy model to introduce the key Fourier multiplier which is crucial to avoid the loss of derivative encountered in Proposition 3.3.2, see also Remark 3.3.4. Then, in analogy with the previous subsection, we present a weighted estimate that follows by the control of a suitable energy functional. Theorem 3.0.7 is a consequence of the weighted estimate.

The key Fourier multiplier

In the inviscid case and in the previous subsection it was crucial to properly symmetrize the system by weighting the density and the divergence with some negative powers of the Laplacian, see (3.39) and (3.90) and recall that p is the symbol associated to $-\Delta_L$. This was essential to balance the growth given by the term $\partial_t p/p$ present in the equation for the divergence, see also Remark 3.2.1. On the one hand, using negative powers of the Laplacian is important to obtain sharp decay rates and it seems to be necessary in the inviscid case. On the other hand, forces us to weight the auxiliary variable $\Xi - \nu M^2 A$ with the multiplier $p^{-3/4}$, which then cause the loss of derivatives, see Remark 3.3.4.

However, in the viscous case it is possible to balance the growth given by $\partial_t p/p$ by using the dissipation. To explain how, we consider a toy model introduced by Bedrossian, Germain and Masmoudi in [15], where they study nonlinear asymptotic stability properties of the Couette flow in the 3D incompressible Navier-Stokes case. In particular, consider the following

$$\partial_t f = \frac{\partial_t p}{p} f - \nu p f, \tag{3.126}$$

which is clearly a relevant toy model also in our case, since the first two terms in the right-hand side of (3.81) have exactly this structure. One may explicitly solve the previous equation since $\partial_t(p^{-1}f) = -\nu p(p^{-1}f)$, however, as said, we want to avoid the use of negative powers of the Laplacian. First of all, for $t \leq \eta/k$ we know that $\partial_t p = -2k(\eta - kt) \leq 0$, hence we may ignore this term in an energy estimate. Instead, for $t \geq \eta/k$ we have $\partial_t p \geq 0$ leading to a growth on f that it is not balanced by the dissipative term, indeed near the critical times $t = \eta/k$ one has $\nu p \approx \nu k^2$. More precisely, we cannot hope to have a uniform estimate like $\partial_t p/p \lesssim \nu p$ for $t \in$

$[\eta/k, \eta/k + C_\nu]$ for some C_ν. If we are sufficiently far away from the critical times dissipation overcomes the growth, namely for any $\beta > 0$ one has

$$\nu p(t,k,\eta) \geq \beta^2 \nu^{\frac{1}{3}}, \qquad \text{if } |t - \frac{\eta}{k}| \geq \beta \nu^{-\frac{1}{3}}, \tag{3.127}$$

$$\frac{\partial_t p}{p}(t,k,\eta) \leq \frac{2}{\sqrt{1 + (\frac{\eta}{k} - t)^2}} \leq 2\beta^{-1}\nu^{\frac{1}{3}} \qquad \text{if } |t - \frac{\eta}{k}| \geq \beta \nu^{-\frac{1}{3}}, \tag{3.128}$$

so that for $\beta > 2$ we see that $\partial_t p/p \leq \nu p/4$ if $|t - \eta/k| \geq \beta \nu^{-\frac{1}{3}}$.

In order to control the growth near the critical times, for a fixed $\beta > 2$ to be specified later we introduce the following Fourier multiplier

$$\begin{aligned} (\partial_t w)(t,k,\eta) &= \begin{cases} 0 & \text{if } t \notin [\frac{\eta}{k}, \frac{\eta}{k} + \beta\nu^{-\frac{1}{3}}] \\ \left(\dfrac{\partial_t p}{p} w\right)(t,k,\eta) & \text{if } t \in [\frac{\eta}{k}, \frac{\eta}{k} + \beta\nu^{-\frac{1}{3}}] \end{cases} \\ w(0,k,\eta) &= 1, \end{aligned} \tag{3.129}$$

which is explicitly given by

$$w(t,k,\eta) = \begin{cases} 1 & \text{if } \eta k \geq 0 \text{ and } 0 \leq t \leq \frac{\eta}{k}, \\ 1 & \text{if } \eta k < 0,\ |\frac{\eta}{k}| \geq \beta\nu^{-\frac{1}{3}} \text{ and } t \geq 0, \\ \dfrac{p(t,k,\eta)}{k^2} & \text{if } \frac{\eta}{k} \leq t \leq \frac{\eta}{k} + \beta\nu^{-\frac{1}{3}}, \\ 1 + \beta^2 \nu^{-\frac{2}{3}} & \text{in all the other cases.} \end{cases} \tag{3.130}$$

This multiplier appeared in [15]. In order to understand why the multiplier defined above will be useful, let us state some properties of it.

Lemma 3.3.8. *Let w and m be the Fourier multipliers defined in* (3.130) *and* (3.88) *respectively. Then, for any $t \geq 0$, $\eta \in \mathbb{R}$ and $k \in \mathbb{Z} \setminus \{0\}$ the following inequalities holds:*

$$1 \leq w(t,k,\eta) \leq \beta^2 \nu^{-\frac{2}{3}}, \tag{3.131}$$

$$(wp^{-1})(t,k,\eta) \leq \frac{1}{k^2}. \tag{3.132}$$

In addition, for any $\max\{2(\beta(\beta^2-1))^{-1}, 4\beta^{-1}\} < \delta_\beta \leq 1$ *one has*

$$\left(\delta_\beta(\frac{\partial_t m}{m} + \nu p) + \frac{\partial_t w}{w} - \frac{\partial_t p}{p}\right)(t,k,\eta) \geq \delta_\beta \nu^{\frac{1}{3}}, \tag{3.133}$$

$$\left(\delta_\beta(\frac{\partial_t m}{m} + \nu^{\frac{1}{3}}) + \frac{\partial_t w}{w} - \frac{\partial_t p}{p}\right)(t,k,\eta) \geq \frac{\delta_\beta}{2} \nu^{\frac{1}{3}}, \tag{3.134}$$

Observe that the bound (3.131) is exactly the maximal growth expected by solving explicitly (3.126). Indeed, solving (3.126) one has

$$(p^{-1}f)(t,k,\eta) = \frac{1}{\langle k,\eta\rangle^2} e^{-\nu \int_0^t p(\tau,k,\eta)d\tau} f^{in}(k,\eta),$$

so that, since $p \leq \langle t \rangle^2 \langle k, \eta \rangle^2$, by using (3.123) we get

$$\|f\|_{H^s} = \left\|p(p^{-1}f)\right\|_{H^s} \leq \langle t \rangle^2 e^{-\frac{1}{12}\nu^{\frac{1}{3}}t} \left\|f^{in}\right\|_{H^s} \lesssim \nu^{-\frac{2}{3}} e^{-\frac{1}{24}\nu^{\frac{1}{3}}t} \left\|f^{in}\right\|_{H^s}.$$

Whereas if we multiply (3.126) by $m^{-2}w^{-2}f$ we obtain

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|w^{-1}m^{-1}f|^2 =& -\left(\frac{\partial_t m}{m} + \frac{\partial_t w}{w} + \nu p - \frac{\partial_t p}{p}\right)|w^{-1}m^{-1}f|^2 \\ \leq& -\nu^{\frac{1}{3}}|w^{-1}m^{-1}f|^2,\end{aligned}$$

where in the last line we have used (3.133). Hence, one has $\|w^{-1}m^{-1}f\|_{H^s} \leq e^{-\nu^{\frac{1}{3}}t}\|f^{in}\|_{H^s}$. Then, since $m \approx 1$, see (3.88), from (3.131) we infer

$$\|f\|_{H^s} = \left\|wm(w^{-1}m^{-1}f)\right\|_{H^s} \lesssim \nu^{-\frac{2}{3}} \left\|w^{-1}m^{-1}f\right\|_{H^s} \lesssim \nu^{-\frac{2}{3}} e^{-\nu^{\frac{1}{3}}t} \left\|f^{in}\right\|_{H^s}.$$

Remark 3.3.9. As shown in the computations above, we see that by using the weight p^{-1} or $(mw)^{-1}$ we obtain the same asymptotic behaviour. The advantage of the weight w with respect to p it is clearly the uniform bound (3.131), so that we do not have to pay regularity to translate the estimates from weighted to unweighted quantities. Notice also that, in view of (3.134), to control the growth given by $\partial_t p/p$ it is enough to have a dissipative term with constant coefficients. This property will be crucial for the density, where we can hope to recover a similar dissipative term by exploiting its coupling with the divergence, see for example (3.105).

Let us prove Lemma 3.3.8

Proof. The proof of (3.131) and (3.132) readily follows by the definition of w given in (3.130).

To obtain (3.133) and (3.134), when $\partial_t p \leq 0$, namely $0 \leq t \leq \eta/k$, in account of the property (3.89) there is nothing to prove. When $t \in [\eta/k, \eta/k + \beta\nu^{-\frac{1}{3}}]$ we make use of the definition of $\partial_t w/w$, see (3.129), and the property (3.89). In all the other cases we have $|t - \eta/k| \geq \beta\nu^{-\frac{1}{3}}$. Therefore, appealing to (3.127) and (3.128), we infer

$$\begin{aligned}\delta_\beta \nu p - \frac{\partial_t p}{p} &\geq \nu^{\frac{1}{3}}(\delta_\beta \beta^2 - 2\beta^{-1}) \geq \delta_\beta \nu^{\frac{1}{3}}, \\ \delta_\beta \nu^{\frac{1}{3}} - \frac{\partial_t p}{p} &\geq \nu^{\frac{1}{3}}(\delta_\beta - 2\beta^{-1}) \geq \frac{\delta_\beta}{2}\nu^{\frac{1}{3}},\end{aligned}$$

where we have also used that $\beta > 2$ and $\max\{2(\beta(\beta^2-1))^{-1}, 4\beta^{-1}\} < \delta_\beta \leq 1$, hence the proof is over. $\square$

We are now ready to introduce the weighted energy functional.

The weighted estimate and the proof of Theorem 3.0.7

Having defined the weight w in (3.130), in the following proposition we present the weighted estimate which allow us to prove Theorem 3.0.7.

Proposition 3.3.10. *Let $s \geq 0$, $\mu \leq 1/2$, $M > 0$ be such that $M \leq \min\{\mu^{-\frac{1}{2}}, \nu^{-\frac{1}{3}}\}$. If $\rho^{in} \in H^{s+1}(\mathbb{T}\times\mathbb{R})$ and $\alpha^{in}, \omega^{in} \in H^s(\mathbb{T}\times\mathbb{R})$ then*

$$\begin{aligned}\frac{1}{M}\left\|(w^{-\frac{3}{4}}p^{\frac{1}{2}}\widehat{R})(t)\right\|_{H^s} &+ \left\|(w^{-\frac{3}{4}}\widehat{A})(t)\right\|_{H^s} + \left\|(w^{-\frac{3}{4}}(\widehat{\Xi}-\nu M^2\widehat{A}))(t)\right\|_{H^s} \\ &\lesssim e^{-\frac{1}{64}\nu^{\frac{1}{3}}t}(\|\nabla\rho^{in}\|_{H^s} + \|\alpha^{in}\|_{H^s} + \|\rho^{in}+\omega^{in}-\nu M^2\alpha^{in}\|_{H^s}).\end{aligned} \tag{3.135}$$

Notice that with respect to Proposition (3.3.2) we have replaced the weight $p^{-\frac{3}{4}}$ with $w^{-\frac{3}{4}}$ for A and $\Xi - \nu M^2 A$. Instead, for the density, in view of (3.132) the same asymptotic behaviour is expected since $w^{-\frac{3}{4}}p^{\frac{1}{2}} \gtrsim p^{-\frac{1}{4}}$.

Appealing to Proposition 3.3.10, we first prove Theorem 3.0.7. Then, we present the proof of Proposition (3.3.10).

Proof of Theorem 3.0.7. To prove (3.13), by the change of coordinates $X = x - yt,\ Y = y$ we have

$$\begin{aligned}\|\alpha(t)\|_{L^2} + \frac{1}{M}\|\nabla\rho(t)\|_{L^2} + \|\omega(t)\|_{L^2} &= \|A(t)\|_{L^2} + \frac{1}{M}\|(\nabla_L R)(t)\|_{L^2} + \|\Omega(t)\|_{L^2} \\ &= \left\|\widehat{A}(t)\right\|_{L^2} + \frac{1}{M}\left\|(p^{\frac{1}{2}}\widehat{R})(t)\right\|_{L^2} + \left\|\widehat{\Omega}(t)\right\|_{L^2}.\end{aligned}$$

Then, observe that

$$|\widehat{\Omega}| \leq |\widehat{\Xi}-\nu M^2\widehat{A}| + |\widehat{R}| + \nu M^2|\widehat{A}| \leq |\widehat{\Xi}-\nu M^2\widehat{A}| + M\frac{1}{M}p^{\frac{1}{2}}|\widehat{R}| + \nu M^2|\widehat{A}|. \tag{3.136}$$

Hence we get

$$\begin{aligned}\|\alpha(t)\|_{L^2} + &\frac{1}{M}\|\nabla\rho(t)\|_{L^2} + \|\omega(t)\|_{L^2} \\ &\lesssim \left\|\widehat{A}(t)\right\|_{L^2} + \frac{1}{M}\left\|(p^{\frac{1}{2}}\widehat{R})\right\|_{L^2} + \left\|(\widehat{\Xi}-\nu M^2\widehat{A})(t)\right\|_{L^2} \\ &\lesssim \nu^{-\frac{1}{2}}\Bigg(\left\|(w^{-\frac{3}{4}}\widehat{A})(t)\right\|_{L^2} + \frac{1}{M}\left\|(w^{-\frac{3}{4}}p^{\frac{1}{2}}\widehat{R})(t)\right\|_{L^2} \\ &\qquad + \left\|(w^{-\frac{3}{4}}(\widehat{\Xi}-\nu M^2\widehat{A}))(t)\right\|_{L^2}\Bigg),\end{aligned}$$

where in the last line we have used (3.131), namely $w^{\frac{3}{4}} \lesssim \nu^{-\frac{1}{2}}$. Therefore, (3.13) follows by combining the bound above with (3.136) and (3.135).

The inequality (3.14) instead is obtained as follows. By the Helmholtz decomposition and the change of variable (3.16), we first observe that

$$\begin{aligned}\|\boldsymbol{v}(t)\|_{L^2} + \frac{1}{M}\|\rho(t)\|_{L^2} &\leq \left\|(\nabla_L\Delta_L^{-1}A)(t)\right\|_{L^2} + \left\|(\nabla_L^{\perp}\Delta_L^{-1}\Omega)(t)\right\|_{L^2} + \frac{1}{M}\|R(t)\|_{L^2} \\ &\leq \left\|(p^{-\frac{1}{2}}\widehat{A})(t)\right\|_{L^2} + \left\|(p^{-\frac{1}{2}}\widehat{\Omega})(t)\right\|_{L^2} + \frac{1}{M}\left\|(p^{-\frac{1}{2}}p^{\frac{1}{2}}\widehat{R})(t)\right\|_{L^2}.\end{aligned}$$

Then, by using (3.131)-(3.132) we have

$$\begin{aligned}\|\boldsymbol{v}(t)\|_{L^2}+\frac{1}{M}\|\rho(t)\|_{L^2}\leq&\left\|(w^{\frac14}w^{\frac12}p^{-\frac12}(w^{-\frac34}\widehat{A}))(t)\right\|_{L^2}+\left\|(w^{\frac14}w^{\frac12}p^{-\frac12}(w^{-\frac34}\widehat{\Omega}))(t)\right\|_{L^2}\\&+\frac{1}{M}\left\|(w^{\frac14}w^{\frac12}p^{-\frac12}(w^{-\frac34}p^{\frac12}\widehat{R}))(t)\right\|_{L^2}\\\lesssim&\nu^{-\frac16}\Bigg(\left\|(w^{-\frac34}\widehat{A})(t)\right\|_{L^2}+\left\|(w^{-\frac34}\widehat{\Omega})(t)\right\|_{L^2}\\&+\left\|(w^{-\frac34}p^{\frac12}\widehat{R})(t)\right\|_{L^2}\Bigg),\end{aligned}$$

whence concluding the proof by combining the bound above with (3.136) and (3.135). □

It thus remain to prove Proposition (3.3.10). We do not present the proof in detail since it will be similar to the one of Proposition (3.3.2).

Proof of Proposition (3.3.10). Consider the weight defined in (3.88), we introduce the following weighted variables

$$\begin{aligned}&Z_1^w(t)=\frac{1}{M}\langle k,\eta\rangle^s(m^{-1}w^{-\frac34}p^{\frac12}\widehat{R})(t),\quad Z_2^w(t)=\langle k,\eta\rangle^s(m^{-1}w^{-\frac34}\widehat{A})(t),\\&Z_3^w(t)=\langle k,\eta\rangle^s(m^{-1}w^{-\frac34}(\Xi-\nu M^2A))(t).\end{aligned}\tag{3.137}$$

Notice that with respect to (3.90), for Z_2^w and Z_3^w we have replaced $p^{-\frac34}$ with $w^{-\frac34}$. Then, in account of (3.80), (3.81) and (3.84), we observe that

$$\partial_t Z_1^w=-\left(\frac{\partial_t m}{m}+\frac34(\frac{\partial_t w}{w}-\frac{\partial_t p}{p})\right)Z_1^w-\frac14\frac{\partial_t p}{p}Z_1^w-\frac{1}{M}p^{\frac12}Z_2^w,\tag{3.138}$$

$$\begin{aligned}\partial_t Z_2^w=&-\left(\frac{\partial_t m}{m}+\mu p+\frac34(\frac{\partial_t w}{w}-\frac{\partial_t p}{p})\right)Z_2^w+\frac14\frac{\partial_t p}{p}Z_2^w\\&+\left(\frac{1}{M}p^{\frac12}+\frac{2Mk^2}{p^{\frac32}}\right)Z_1^w-2\frac{k^2}{p}Z_3^w-2\nu M^2\frac{k^2}{p}Z_2^w,\end{aligned}\tag{3.139}$$

$$\begin{aligned}\partial_t Z_3^w=&-\left(\frac{\partial_t m}{m}+\nu p\right)Z_3^w-\frac34\frac{\partial_t w}{w}Z_3^w+\nu(\mu-\nu)M^2pZ_2^w\\&-\nu M^2\frac{\partial_t p}{p}Z_2^w-2\nu M^3\frac{k^2}{p^{\frac32}}Z_1^w+2\nu M^2\frac{k^2}{p}Z_3^w+2\nu^2M^4\frac{k^2}{p}Z_2^w.\end{aligned}\tag{3.140}$$

Besides the first term on the left-hand side, the equations (3.138)-(3.139) and (3.96)-(3.97) have the same structure. The only difference between the equation (3.140) and (3.98) is that in (3.140) we have $-\frac34\frac{\partial_t w}{w}$ whereas in (3.98) there is $-\frac34\frac{\partial_t p}{p}$. Hence, we define the energy functional as done in (3.91)-(3.92), namely

$$\begin{aligned}E^w(t)=\frac12\Bigg(&\left(1+M^2\frac{(\partial_t p)^2}{p^3}\right)|Z_1^w|^2(t)+|Z_2^w|^2(t)+|Z_3^w|^2(t)\\&+(\frac{M}{2}\frac{\partial_t p}{p^{\frac32}}\mathrm{Re}(\bar{Z}_1^wZ_2^w))(t)-\frac{M\nu^{\frac13}}{2}(p^{-\frac12}\mathrm{Re}(\bar{Z}_1^wZ_2^w))(t)\Bigg),\end{aligned}$$

which is clearly coercive and satisfy the same bounds given in (3.93)-(3.94). By analogous computations done to obtain (3.105), we have that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E^w(t) =& -\left(\frac{\partial_t m}{m}+\mu p+\frac{3}{4}(\frac{\partial_t w}{w}-\frac{\partial_t p}{p})\right)|Z_2^w|^2\\&-\left(\frac{\partial_t m}{m}+\frac{\nu^{\frac13}}{4}(1+2M^2\frac{k^2}{p^2})+\frac{3}{4}(\frac{\partial_t w}{w}-\frac{\partial_t p}{p})\right)|w_1|^2\\&-\left(\frac{\partial_t m}{m}+\nu p\right)|Z_3|^2+\sum_{i=1}^{5}\mathcal{D}_i^w+\sum_{i=1}^{6}\mathcal{I}_i^w,\end{aligned}$$

where $\mathcal{D}_i^w$, for $i=1,\dots,5$, are defined as in (3.106)-(3.107) by replacing Z_j with Z_j^w for $j=1,2,3$. Analogously, $\mathcal{I}_i^w$, for $i=1,\dots,6$ and $i\neq 3$, are defined as in (3.108)-(3.110). Instead, the term $\mathcal{I}_3^w$ is given by

$$\mathcal{I}_3^w=-\frac{3}{4}\frac{\partial_t w}{w}|Z_3^w|^2+2\nu M^2\frac{k^2}{p}|Z_3^w|^2. \tag{3.141}$$

In particular, with respect to (3.109), there is the great advantage that $\partial_t w/w\geq 0$, meaning that we can bound $\mathcal{I}_3^w$ just with the last term in the right-hand side of (3.141). Therefore, thanks to (3.141), by making the same estimates given in (3.113)-(3.116) and (3.118)-(3.119) we infer

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E^w(t) \leq& -\left(\frac{1}{16}(\frac{\partial_t m}{m}+\mu p)+\frac{3}{4}(\frac{\partial_t w}{w}-\frac{\partial_t p}{p})\right)|Z_2^w|^2\\&-\left(\frac{1}{16}\left(\frac{\partial_t m}{m}+\nu^{\frac13}\right)+\frac{3}{4}(\frac{\partial_t w}{w}-\frac{\partial_t p}{p})\right)|Z_1^w|^2\\&-\frac{\nu^{\frac13}}{8}\frac{M^2(\partial_t p)^2}{p^3}|Z_1^w|^2 \qquad (3.142)\\&-\frac{1}{16}\left(\frac{\partial_t m}{m}+\nu p\right)|Z_3^w|^2+4\left(C_M\frac{k^2}{p}+\frac{1}{2}\frac{\partial_t m}{m}\right)E^w(t), \qquad (3.143)\end{aligned}$$

where to obtain (3.142) we have used (3.121), whereas to get the last term in (3.143) we have used the coercivity properties of the functional, see (3.93)-(3.94). We now have to exploit the properties (3.133)-(3.134). In particular, in our case we have $\delta_\beta=1/12$, hence choosing $\beta>4$ in the definition of w, see (3.130), we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E^w(t) \leq& -\frac{1}{16}\nu^{\frac13}|Z_2^w|^2-\frac{1}{32}\nu^{\frac13}|Z_1^w|^2-\frac{\nu^{\frac13}}{8}\frac{M^2(\partial_t p)^2}{p^3}|Z_1^w|^2\\&-\frac{1}{16}\nu^{\frac13}|Z_3^w|^2+4\left(C_M\frac{k^2}{p}+\frac{1}{2}\frac{\partial_t m}{m}\right)E^w(t).\\ \leq& -\frac{1}{32}\nu^{\frac13}E^w(t)+4\left(C_M\frac{k^2}{p}+\frac{1}{2}\frac{\partial_t m}{m}\right)E^w(t),\end{aligned}$$

where in the last line we have used (3.94). Therefore, by applying Grönwall's Lemma we obtain

$$E^w(t)\lesssim e^{-\frac{1}{32}\nu^{\frac13}t}E^w(0). \tag{3.144}$$

Then, in view of the definition (3.137) and the coercivity of the functional, we also know that

$$\begin{aligned}\sum_k \int E^w(t) d\eta \approx & \frac{1}{M} \left\| (w^{-\frac{3}{4}} p^{\frac{1}{2}} \widehat{R})(t) \right\|_{H^s}^2 + \left\| (w^{-\frac{3}{4}} \widehat{A})(t) \right\|_{H^s}^2 \\ & + \left\| (w^{-\frac{3}{4}} (\widehat{\Xi} - \nu M^2 \widehat{A}))(t) \right\|_{H^s}^2 ,\end{aligned}$$

hence, thanks to (3.144), the proof is over. □

CHAPTER 4

Linear inviscid damping for shear flows near Couette in the 2D exponentially stratified regime

This chapter, based on a joint work with R. Bianchini and M. Coti Zelati [22], is concerned with the study of the stability of a particular class of stratified shear flows (1.5). The study of the effects of density stratification in fluid flows it is of great interest in oceanography and meteorology [62, 159], where it is usual to consider a non-homogeneous incompressible fluid under the action of gravity, also called a buoyancy driven fluid. In particular, we consider the following equations

$$\begin{aligned} &\partial_t\widetilde{\rho} + \boldsymbol{u}\cdot\nabla\widetilde{\rho} = 0, \quad \text{in } \mathbb{T}\times\mathbb{R},\ t\geq 0,\\ &\widetilde{\rho}(\partial_t\boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}) + \nabla\widetilde{p} = -\widetilde{\rho}\,\boldsymbol{g},\\ &\operatorname{div}(\boldsymbol{u}) = 0,\end{aligned} \tag{4.1}$$

where $\boldsymbol{g} = (0, \mathfrak{g})$ is (minus) gravity. In a heterogeneous fluid, the presence of an external force may strongly influence the dynamics, for instance gravity will send the heavier fluid to the bottom. In this case, the density at the equilibrium has to satisfy $\bar{\rho}'(y) \leq 0$, which is also called a *stably stratified regime*.

The linearized system for perturbations around the equilibrium (1.5), with pressure $\widetilde{p}(t, x, y) = \bar{p}(y) + p(t, x, y)$, is given by

$$\begin{aligned} &\partial_t\rho + U\partial_x\rho + v^y\bar{\rho}' = 0, \quad \text{in } \mathbb{T}\times\mathbb{R},\ t\geq 0,\\ &\bar{\rho}(\partial_t\boldsymbol{v} + U\partial_x\boldsymbol{v}) + \bar{\rho}\begin{pmatrix} U'v^y \\ 0\end{pmatrix} + \nabla p = -\rho\,\boldsymbol{g},\\ &\operatorname{div}(\boldsymbol{v}) = 0,\end{aligned} \tag{4.2}$$

where we have assumed the equilibrium pressure to be $\bar{p}'(y) = -\mathfrak{g}\bar{\rho}(y)$. It is convenient to rewrite the system above in vorticity formulation, namely

$$(\partial_t + U\partial_x)\rho + \bar{\rho}'\partial_x\psi = 0, \quad \text{in } \mathbb{T}\times\mathbb{R},\ t\geq 0, \tag{4.3a}$$

$$\bar{\rho}(\partial_t + U\partial_x)\omega + \bar{\rho}'(\partial_t + U\partial_x)\partial_y\psi - (\bar{\rho}U')'\partial_x\psi = -\mathfrak{g}\partial_x\rho, \tag{4.3b}$$

$$\boldsymbol{v} = \nabla^\perp\psi, \quad \Delta\psi = \omega,$$

where we recall $\nabla^\perp = (-\partial_y, \partial_x)$ and $\omega = \nabla^\perp \cdot v$.

As mentioned in Chapter 1, a first stability criterion for stratified shear flows is due to Miles [128] and Howard [97], that we are going to discuss in detail below.

The Miles-Howard criterion

In analogy to what was done to prove the Rayleigh's inflection point Theorem, see Chapter 1, we proceed via a normal-mode analysis. Consider perturbations of the form

$$\rho(t,x,y) = r(y)e^{\lambda t + ikx} \qquad \psi(t,x,y) = \varphi(y)e^{\lambda t + ikx},$$

so that from (4.3) we have

$$(\lambda + ikU)r + ik\bar{\rho}'\varphi = 0, \tag{4.4}$$

$$\bar{\rho}(\lambda + ikU)(-k^2\varphi + \varphi'') + \bar{\rho}'(\lambda + ikU)\varphi' - ik(\bar{\rho}U')'\varphi = -ik\mathfrak{g}r. \tag{4.5}$$

Defining $\gamma = \gamma(y) = \lambda + ikU(y)$ and assuming $\mathrm{Re}(\lambda) > 0$, by (4.4) we know that $r = -\gamma^{-1}ik\bar{\rho}'\varphi$. Hence, rearranging (4.5) we get the *Taylor-Goldstein equation*

$$(\bar{\rho}\varphi')' - \left(\bar{\rho}k^2 + \frac{ik}{\gamma}(\bar{\rho}U')' - \frac{k^2}{\gamma^2}\mathfrak{g}\bar{\rho}'\right)\varphi = 0, \tag{4.6}$$

which was derived independently by G. I. Taylor [145] and Goldstein [83] in 1931 [1]. Observe that (4.6) is the Rayleigh equation (1.8) when $\bar{\rho} = 1$. One may directly infer an integral identity from (4.6), however the resulting equation does not lead to obvious conclusions. Instead, as done in [97], we introduce the new variable $\phi = \gamma^{-1/2}\varphi$ which satisfies

$$(\bar{\rho}\gamma\phi')' - k^2\bar{\rho}\gamma\phi + \left(\frac{1}{4}\bar{\rho}(U')^2 + \mathfrak{g}\bar{\rho}'\right)\frac{k^2}{\gamma}\phi - \frac{ik}{2}(\bar{\rho}U')'\phi = 0.$$

Multiplying the equation above by the complex conjugate of ϕ, integrating in y and retaining only the real part we get

$$\mathrm{Re}(\lambda)\int_{-\infty}^{+\infty} \bar{\rho}(|\phi'|^2 + k^2|\phi|^2) + \left(\mathrm{Ri} - \frac{1}{4}\right)\frac{k^2}{|\gamma|^2}|\phi|^2 dy = 0, \tag{4.7}$$

where the *Richardson number* is defined as

$$\mathrm{Ri}(y) = \mathfrak{g}\left(\frac{-\bar{\rho}'}{\bar{\rho}(U')^2}\right)(y). \tag{4.8}$$

The identity (4.7) cannot hold if $\mathrm{Ri} \geq 1/4$, meaning that $\mathrm{Re}(\lambda) > 0$ is not possible under this assumption. This proves the spectral stability of stratified shear flows for which $\mathrm{Ri} \geq 1/4$, namely the famous *Miles-Howard criterion*.

[1] G. I. Taylor writes in the introduction of [145] the following: "The chief part of the work described in this paper was done in 1914 and formed part of the essay for which the Adams Prize was awarded in 1915. During the war years it was laid aside, and since then I have delayed publication, hoping to be able to undertake experiments designed to verify, or otherwise, the results. Lately, however, Mr. Goldstein has told me that he is engaged on similar problems and he has encouraged me to publish the work without waiting for experimental results".

Remark 4.0.1. The factor $1/4$ comes from the specific choice of the power of γ when defining the new variable $\phi = \gamma^{-1/2}\varphi$. The exponent $-1/2$ can be deduced by the following argument. Define $\phi_\alpha = \gamma^{-\alpha}\varphi$ for some $\alpha \geq 0$. This new variable satisfies

$$
\begin{aligned}
&(\bar\rho\gamma\phi_\alpha')' - k^2\bar\rho\gamma\phi_\alpha + \big(\alpha(1-\alpha)\bar\rho(U')^2 + \mathfrak{g}\bar\rho'\big)\,\frac{k^2}{\gamma}\phi_\alpha - ik(1-\alpha)(\bar\rho U')'\phi_\alpha \\
&+ ik(2\alpha-1)\bar\rho U'\phi_\alpha' = 0.
\end{aligned}
$$

If we then multiply the identity above by the complex conjugate of ϕ_α and integrate in y, we see that the real part of the last term does not have a definite sign. However, this term clearly disappears for $\alpha = 1/2$.

Remark 4.0.2. The Miles-Howard criterion is not sharp, in the sense that the linear stability is possible even when $\mathrm{Ri} < 1/4$. This fact was essentially observed by Case in 1960 [35] and Hartman in 1975 [91] and rigorously proved for the Couette flow by Yang and Lin in 2017 [157].

We have already observed that the Taylor-Goldstein equation (4.6) reduce to the Rayleigh's one (1.8) when $\bar\rho' = 0$. Then, one may think that for a homogeneous shear flow in a heterogeneous fluid, under the action of gravity, the Rayleigh's inflection point Theorem is still valid. This would imply an immediate stability criterion when $\mathrm{Ri} = 0$. Unfortunately, this conclusion is false. Indeed, to derive (4.6) we have first solved (4.4). When $\bar\rho' = 0$, (4.4) is given by $(\lambda + ikU)r = 0$, meaning that either $\lambda + ikU = 0$, hence one cannot divide by $\lambda + ikU$ to get the inflection point theorem, or $r = 0$, so that one is left with a homogeneous fluid. However, when $\bar\rho' = 0$ by (4.3a) one has that the density is transported along the shear flow, leading to an explicit source term in the equation for the vorticity (4.3b). In the Couette case, linear stability can be proved also for $\mathrm{Ri} = 0$ [157] and in the Boussinesq approximation the linear problem can be explicitly solved [54, 166].

Statement of the results

We are going to consider an *exponential stratification*, where the stable background density is

$$\bar\rho(y) = e^{-\beta y}, \qquad \text{for some } \beta > 0.$$

This choice is particularly convenient from the mathematical point of view. Indeed, we can divide (4.3a) and (4.3b) by $\bar\rho'$ and exploit the fact that $\bar\rho'/\bar\rho = -\beta$ to simplify the equations. More precisely, defining the scaled density

$$q = \frac{\rho}{\beta\bar\rho}$$

we have

$$(\partial_t + U(y)\partial_x)q = \partial_x\psi, \quad \text{in } \mathbb{T}\times\mathbb{R},\ t \geq 0, \tag{4.9a}$$

$$(\partial_t + U(y)\partial_x)(\omega - \beta\partial_y\psi) - (U''(y) - \beta U'(y))\partial_x\psi = -R\partial_x q, \tag{4.9b}$$

$$\boldsymbol{v} = \nabla^\perp\psi, \quad \Delta\psi = \omega, \tag{4.9c}$$

where we have defined

$$R = \beta\mathfrak{g}.$$

The latter parameter is called *Brunt-Väisälä frequency* and it satisfies $R = (U')^2\mathrm{Ri}$, where Ri is defined in (4.8). In particular, in the Couette case R is equal to the Richardson number.

Remark 4.0.3. As we explain in Subection 4.0.1, it is useful to consider $R > 0$ and $\beta \geq 0$ as two *independent parameters*, since the linearization around a shear flow, with a linear stratification and in the Boussinesq approximation can be recovered when setting $\beta = 0$ in the system (4.9).

Notice that the x-averages are conserved along the dynamics. More precisely, recall the notation $f_0(y) = (2\pi)^{-1}\int_{\mathbb{T}} f(x,y)dx$. By the incompressibility condition we know $v_0^y = 0$. Hence, from (4.2) we deduce that the x-averages of the density and the velocity are conserved quantities for the dynamics.

The main result of this chapter is a quantification of linear inviscid damping (Section 1.2) for shear flows which are close to Couette.

Theorem 4.0.4. *Let $R > 1/4$ and $\beta \geq 0$ be arbitrarily fixed. Let $\omega^{in} \in H^{\frac{3}{2}}(\mathbb{T}\times\mathbb{R})$ and $q^{in} \in H^2(\mathbb{T}\times\mathbb{R})$ be the initial data of (4.9). There exists a small constant $\varepsilon_0 = \varepsilon_0(\beta, R) \in (0,1)$ with the following property. If $\varepsilon \in (0, \varepsilon_0]$ and*

$$\|U' - 1\|_{H^6} + \|U''\|_{H^5} \leq \varepsilon, \tag{4.10}$$

then for every $t \geq 0$ we have

$$\left\|q(t) - q_0^{in}\right\|_{L^2} + \left\|\boldsymbol{v}^x(t) - \boldsymbol{v}_0^{in,x}\right\|_{L^2} \lesssim \frac{1}{\langle t\rangle^{\frac{1}{2}-\delta_\varepsilon}}\left(\left\|\omega^{in}\right\|_{H^{\frac{1}{2}}} + \left\|q^{in}\right\|_{H^{\frac{3}{2}}}\right), \tag{4.11}$$

$$\|\boldsymbol{v}^y(t)\|_{L^2} \lesssim \frac{1}{\langle t\rangle^{\frac{3}{2}-\delta_\varepsilon}}\left(\left\|\omega^{in}\right\|_{H^{\frac{3}{2}}} + \left\|q^{in}\right\|_{H^2}\right), \tag{4.12}$$

where $\delta_\varepsilon = \sqrt{\varepsilon}/2$.

Assumption (4.10) is a quantification of how close the background shear is to the linear profile given by the Couette flow $U(y) = y$, which corresponds to $\varepsilon = 0$. Theorem 4.0.4 is a consequence of more general stability estimates in H^s, for any $s \geq 0$ (see Theorems 4.2.1 and 4.3.1 below). It is worth mentioning here that the constants hidden in the symbol $\lesssim$ in (4.11)-(4.12) blow up exponentially fast as $R \to 1/4$. Moreover, the choice of δ_ε is not dictated by any scaling reasons: it can be actually taken proportional to ε itself, at the cost of a few more technicalities (see Remark 4.3.9).

We have discussed in detail the inviscid damping mechanism in Section 1.2. Of interest to the case treated here, we mention that the mixing phenomenon behind inviscid damping has been observed also in stratified shear flows [33].

However, in the density-dependent case, vorticity mixing is not the only relevant effect. This is clarified in the special case of the Couette flow $U(y) = y$, where a proof which is *point-wise* in frequency space can be obtained, and gives the following instability result for the vorticity.

Corollary 4.0.5. *Let $U(y) = y$ in* (4.9)*, and assume that $R > 1/4$ and $\beta \geq 0$. If $\omega^{in} \in L^2$ and $q^{in} \in H^{\frac{3}{2}}$, then*

$$\begin{aligned}\left\|\omega(t)-\omega_0^{in}\right\|_{L^2}+\left\|\nabla q(t)-(\nabla q^{in})_0\right\|_{L^2} &\gtrsim \langle t\rangle^{\frac{1}{2}}\Big(\left\|(-\Delta)^{-\frac{1}{4}}(\omega^{in}-\omega_0^{in})\right\|_{L^2_x H_y^{-\frac{1}{2}}}\\ &\qquad+\left\|q^{in}-q_0^{in}\right\|_{L^2}\Big),\\ \left\|\omega(t)-\omega_0^{in}\right\|_{L^2}+\left\|\nabla q(t)-(\nabla q^{in})_0\right\|_{L^2} &\lesssim \langle t\rangle^{\frac{1}{2}}\Big(\left\|\omega^{in}-\omega_0^{in}\right\|_{L^2}\\ &\qquad+\left\|(\nabla q^{in})-(\nabla q^{in})_0\right\|_{H^{\frac{1}{2}}}\Big),\end{aligned} \tag{4.13}$$

for every $t \geq 0$.

Corollary 4.0.5 highlights the presence of a linear Lyapunov-type instability in L^2. We have already observed a similar instability in the compressible case, see Chapter 3, and it was also recently proved in the Boussinesq approximation [166] (just for the vorticity). We point out that this instability result is the reason why the inviscid damping rates in Theorem 4.0.4 for the velocity field are slower than the ones of the constant density case, see Section 1.2. However, notice that for the scaled density q we have better bounds with respect to the one that we can get if q is simply advected by the Couette flow. This is evident by the decay in L^2, but in the corollary above we also see that the gradient can grow as $t^{\frac{1}{2}}$, whereas for a transported quantity the gradient can grow as t. Hence, on the one hand we have an instability mechanism for the vorticity, on the other hand we have a stabilizing mechanism for the scaled density.

Comparison with previous works and proof strategy

The case of the Couette flow $U(y) = y$ has been considered by Yang and Lin [157] for an arbitrary $R \geq 0$. In the case $R > 1/4$, an improvement of the statement of [157] is recovered by Theorem 4.0.4 by simply setting $\varepsilon = 0$. In particular no (exponential) weights are needed in the norms. However, it was shown in [157] that for $R \leq 1/4$ the decay rates undergo (at least) a logarithmic correction. The methods of proof in our work are completely different from those employed in [157]. There, the analysis is based on the use of hypergeometric functions (an approach that dates back to Hartman [91]), which were needed to solve the system (4.9) in frequency space. In fact, by performing the change of variables to follow the background shear, it is possible to rewrite (4.9) as a second order linear ODE in terms of ψ that can be solved with the use of hypergeometric functions.

We instead follow the approach introduced in the inviscid compressible case, see Chapter 3. Namely, we consider (4.9) as a 2×2 non-autonomous dynamical system at each fixed frequency. Then, by exploiting a proper symmetrization, we present a proof of Theorem 4.0.4 in the Couette case (i.e. for $\varepsilon = 0$), which will be propaedeutic to the study of shears close to Couette.

Finally, our method of proof in the case of shears close to Couette, relies on the use of an energy functional endowed with certain suitable weights, a method already used by Zillinger (see [163] and references therein) in the constant density case for shears close to Couette. Notice that the analysis of this model requires a delicate tracking of the constants depending on β, R, which will be involved in the definition of one of the weights of our energy functional. In fact, this is the idea which allows us to handle the general case of exponentially stratified fluids near Couette. We would also like to point out that the shears-near-Couette case could be seen as a step forward aiming at investigating *nonlinear* stability of the Couette flow for stratified fluids, as a good understanding of the dynamics of the background shear flow was a key point for the nonlinear stability in the constant density case, see [17].

4.0.1 Stratified flows in the Boussinesq approximation

In this subsection we discuss how the system (4.9) is related to the linearization around a stably stratified equilibrium in the *Boussinesq approximation* [28, 85, 139, 161], which is a common approximation for buoyancy driven fluids. In particular, the equations (4.1) are considered too difficult to deal with for practical purposes. The Boussinesq approximation is based on the observation that for some physical non-homogeneous fluids the variations of the density profile are negligible compared to its (constant) average. Hence, "*density is assumed constant except when it directly causes buoyant forces*"-[85]. Namely, the continuity equation does not change while the momentum equation become

$$\bar{\rho}_c(\partial_t \boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}) + \nabla\widetilde{p} = -\widetilde{\rho}\boldsymbol{g},$$

where $\bar{\rho}_c$ is the constant density. For a physical derivation of this approximation we refer to the classical papers [85, 139] and the more recent review [161].

Then, under the Boussinesq approximation, the linearized system near a shear flow $(U(y), 0)$ with a stratified density profile $\bar{\rho} = \bar{\rho}(y)$, in vorticity formulation, is given by

$$\begin{aligned}
&(\partial_t + U(y)\partial_x)\rho = -\bar{\rho}'\partial_x\psi,\\
&(\partial_t + U(y)\partial_x)\omega - U''(y)\partial_x\psi = -\mathfrak{g}\frac{\partial_x\rho}{\bar{\rho}_c},\\
&\boldsymbol{v} = \nabla^\perp\psi, \qquad \Delta\psi = \omega,
\end{aligned}$$

which has a simpler form with respect to (4.3). For simplicity of notation we will now assume that $\bar{\rho}_c = 1$. Observe that having an exponentially stratified density does not immediately simplify the equations. Hence, a further simplification is to assume to have a linear density profile, namely

$$\bar{\rho} = 1 - \gamma y, \quad \text{for some } 0 < \gamma \ll 1, \tag{4.14}$$

as done for example in [91, 71, 157]. Defining the scaled density

$$q = \frac{\rho}{\gamma},$$

we end up with

$$\begin{aligned}
&(\partial_t + U(y)\partial_x)q = \partial_x\psi,\\
&(\partial_t + U(y)\partial_x)\omega - U''(y)\partial_x\psi = -R\partial_x q,\\
&\boldsymbol{v} = \nabla^\perp\psi, \qquad \Delta\psi = \omega,
\end{aligned} \tag{4.15}$$

where

$$R = \gamma \mathfrak{g}$$

in this case. The system above, with a completely analogous derivation, is the one considered by Yang and Lin in [157]. Notice that (4.15) is exactly (4.9) with $\beta = 0$, so that Theorem 4.0.4 gives asymptotic stability for system (4.15) in the regime $R > 1/4$.

Remark 4.0.6 (About the choice of a linear density profile)**.** To derive the system (4.15) one has to assume that background density profile is given by (4.14). Locally we know that $\bar{\rho} \approx e^{-\gamma y}$. However, since we are considering a domain which has an infinite extent in y, the linear density profile has to be considered as a first approximation towards the understanding of more physically relevant cases.

Outline of the chapter

The next Section 4.1 is dedicated to the set-up of the functional-analytic tools needed to study our problem.

In Section 4.2, relying on the point-wise in frequency approach adopted in Chapter 3, we prove optimal inviscid damping decay rates of the Couette flow $U(y) = y$. Next, we are able to extend the result (up to a small loss in terms of the decay rates) to shears of general profiles, which are close to Couette in a suitable sense. The latter result is obtained by means of a different method, which is based on the use of a properly weighted energy functional, and is presented in Section 4.3.

4.1 Preliminaries

In this section, we first introduce the change of variables which follow the background shear flow. Next, since the new spatial coordinates as well as partial derivatives depend on derivatives of the shear flow, we need to define the action of differential operators in the moving coordinates. Related results are provided in Proposition 4.1.1-4.1.2 at the end of this section.

4.1.1 Change of coordinates and decoupling

It is best to further re-write the system (4.9) by defining the auxiliary variable

$$\theta = \omega - \beta \partial_y \psi = (I - \beta \partial_y \Delta^{-1})\omega. \tag{4.16}$$

Since the operator $I - \beta \partial_y \Delta^{-1}$ is well-defined and invertible for any $\beta \geq 0$, we can write (4.9) in terms of θ and q alone as

$$\begin{aligned} (\partial_t + U(y)\partial_x)\theta &= -R\partial_x q + (U''(y) - \beta U'(y))\partial_x \Delta^{-1} \left(I - \beta \partial_y \Delta^{-1}\right)^{-1} \theta, \\ (\partial_t + U(y)\partial_x) q &= \partial_x \Delta^{-1} \left(I - \beta \partial_y \Delta^{-1}\right)^{-1} \theta, \end{aligned} \tag{4.17}$$

Due to the transport nature of (4.17), the convenient set of coordinates is that of a moving frame that follows the background shear. Define

$$X = x - U(y)t, \qquad Y = U(y)$$

and the corresponding new unknowns

$$\begin{aligned}&\Theta(t,X,Y)=\theta(t,x,y), \quad Q(t,X,Y)=q(t,x,y),\\&\Omega(t,X,Y)=\omega(t,x,y).\end{aligned} \tag{4.18}$$

The differential operators will change accordingly, in analogy with the cases already studied in the incompressible literature [163, 162, 49, 104]. In particular, by defining

$$g(Y):=U'(U^{-1}(Y)), \qquad b(Y):=U''(U^{-1}(Y)), \tag{4.19}$$

we obtain the rules

$$\partial_t\to\partial_t-U(y)\partial_X, \qquad \partial_x\to\partial_X, \qquad \partial_y\to g(Y)(\partial_Y-t\partial_X),$$

and

$$\Delta\to\Delta_t:=\partial_{XX}+g^2(Y)(\partial_Y-t\partial_X)^2+b(Y)(\partial_Y-t\partial_X). \tag{4.20}$$

As we shall see, if g, b are small in a way made more precise later (see Proposition 4.1.1), the operator Δ_t is invertible. Similarly (see Proposition 4.1.2), the operator

$$B_t=\left(I-\beta g(Y)(\partial_Y-t\partial_X)\Delta_t^{-1}\right)^{-1}. \tag{4.21}$$

is well-defined for any β. Moreover, it follows from (4.16) and (4.18) that

$$\Theta:=B_t^{-1}\Omega.$$

In the moving frame, the equations (4.17), written in terms of Θ and Q read

$$\begin{aligned}&\partial_t\Theta=-R\partial_XQ+\left(b(Y)-\beta g(Y)\right)\partial_X\Delta_t^{-1}B_t\Theta,\\&\partial_tQ=\partial_X\Delta_t^{-1}B_t\Theta.\end{aligned} \tag{4.22}$$

It is apparent that the above system decouples only in the X-frequency for a general shear flow, while in the Couette case it decouples also in the Y-frequency (as in Section 1.2 and Chapter 3). We then adopt the the following convention. Given a function $f=f(X,Y)$, we can write

$$f(X,Y)=\sum_{k\in\mathbb{Z}}f_k(Y)\mathrm{e}^{ikX}, \qquad f_k(Y)=\frac{1}{2\pi}\int_{\mathbb{T}}f(X,Y)\mathrm{e}^{-ikX}\mathrm{d}X, \tag{4.23}$$

where throughout the paper $k\in\mathbb{Z}$ will denote the X-Fourier variable. Applying this reasoning to Θ and Q, we can interpret (4.22) as a coupled system of infinitely many equations that read

$$\begin{aligned}&\partial_t\Theta_k=-ikRQ_k+ik\left(b(Y)-\beta g(Y)\right)\Delta_t^{-1}B_t\Theta_k,\\&\partial_tQ_k=ik\Delta_t^{-1}B_t\Theta_k.\end{aligned} \tag{4.24}$$

In the above equations, we identified Δ_t^{-1} and B_t with their X-Fourier localizations. It is clear that for $k=0$ there is nothing to prove, and therefore we will always assume $k\neq 0$.

4.1.2 Definitions of the operators

The Fourier transform in Y of a function $f = f(X,Y)$ is defined by

$$\widehat{f}(X,\eta) = \int_{\mathbb{R}} f(X,Y)\mathrm{e}^{-i\eta Y}\mathrm{d}Y, \qquad \eta \in \mathbb{R}. \tag{4.25}$$

For any $k \in \mathbb{Z}$ and and $\eta \in \mathbb{R}$, we recall that $\langle \eta \rangle = \sqrt{1+\eta^2}$ and $\langle k,\eta \rangle = \sqrt{1+k^2+\eta^2}$. We use the following notation for the scalar product in the Sobolev space $H^s(\mathbb{T}\times\mathbb{R})$ for any $s \in \mathbb{R}$

$$\langle f,h\rangle_s = \langle \widehat{f},\widehat{h}\rangle_s = \sum_{k\in\mathbb{Z}}\int_{\mathbb{R}} \langle k,\eta\rangle^{2s}\widehat{f}_k(\eta)\overline{\widehat{h}_k(\eta)}\mathrm{d}\eta.$$

From now on, we denote the associated norm as

$$\|f\|_s^2 = \|\widehat{f}\|_s^2 = \sum_{k\in\mathbb{Z}}\int_{\mathbb{R}} \langle k,\eta\rangle^{2s}|\widehat{f}_k(\eta)|^2\mathrm{d}\eta.$$

By the above notation, we mean that we make tacit use of Plancherel's theorem. When $s=0$, the above reduce to the standard L^2 scalar product and norm, and we will use the explicit notation $\|\cdot\|_{L^2}$ for the norm. An important role will be played by the operator

$$\Delta_L := \partial_{XX} + (\partial_Y - t\partial_X)^2,$$

which was used thoroughly in Chapter 3 and it is exactly what one obtains from Δ_t in (4.20) when setting $g \equiv 1$ and $b \equiv 0$. We recall that the Fourier symbol of $-\Delta_L$ is denoted by

$$p(t,k,\eta) = k^2 + (\eta - kt)^2, \tag{4.26}$$

and the symbol associated to $-2\partial_X(\partial_Y - t\partial_X)$ is

$$\partial_t p(t,k,\eta) = -2k(\eta - kt).$$

To see Δ_t as a perturbation of Δ_L, it is convenient to write

$$\Delta_t = [I - T_\varepsilon]\,\Delta_L, \tag{4.27}$$

with

$$T_\varepsilon = \left[(g^2(Y)-1)(\partial_Y - t\partial_X)^2 + b(Y)(\partial_Y - t\partial_X)\right](-\Delta_L^{-1}). \tag{4.28}$$

In particular,

$$T_\varepsilon = T_\varepsilon^g + T_\varepsilon^b,$$

where

$$T_\varepsilon^g = (g^2(Y)-1)(\partial_Y - t\partial_X)^2(-\Delta_L^{-1}), \qquad T_\varepsilon^b = b(Y)(\partial_Y - t\partial_X)(-\Delta_L^{-1}). \tag{4.29}$$

The following proposition holds true.

Proposition 4.1.1. *Let $s \geq 0$ be arbitrarily fixed. There exists a constant $c_s \geq 1$ with the following property. Assume that*

$$\|g-1\|_{s+1} + \|b\|_{s+1} \leq \varepsilon,$$

for a positive $\varepsilon < 1/c_s$. Then Δ_t is invertible on H^s and

$$\Delta_t^{-1} = \Delta_L^{-1} T_L, \tag{4.30}$$

where

$$T_L = \sum_{n=0}^{\infty} T_\varepsilon^n = I + T_\varepsilon T_L. \tag{4.31}$$

The convergence above is in the H^s-operator norm, and

$$\|T_L\|_{H^s \to H^s} \leq \frac{1}{1 - c_s \varepsilon}. \tag{4.32}$$

Proof. We begin to prove the convergence of the Neumann series (4.31) by obtaining a suitable bound on

$$\|T_\varepsilon f\|_s \leq \|T_\varepsilon^g f\|_s + \|T_\varepsilon^b f\|_s,$$

for any $f : \mathbb{R} \to \mathbb{C}$ in H^s. To estimate the first part, notice that the Fourier symbol of the operator $(\partial_Y - t\partial_X)^2(-\Delta_L^{-1})$ is uniformly bounded above by 1. Therefore, by Young's convolution inequality we have

$$\begin{aligned} \|T_\varepsilon^g f\|_s &\leq \left\| |\widehat{g^2-1}| * |\widehat{f}| \right\|_s \leq \left\| \langle \cdot \rangle^s \widehat{g^2-1} \right\|_{L^1} \|\langle \cdot \rangle^s f\|_{L^2} \\ &\lesssim \|g-1\|_{s+1} \|f\|_s, \end{aligned}$$

where the last inequality follows from

$$\begin{aligned} \left\| \langle \cdot \rangle^s \widehat{g^2-1} \right\|_{L^1} &= \int_{\mathbb{R}} \langle \eta \rangle^s \left| \widehat{g^2-1} \right| \mathrm{d}\eta = \int_{\mathbb{R}} \frac{1}{\langle \eta \rangle} \langle \eta \rangle^{s+1} \left| \widehat{g^2-1} \right| \mathrm{d}\eta \\ &\lesssim \left\| g^2 - 1 \right\|_{s+1} \lesssim \|g-1\|_{s+1}. \end{aligned}$$

In the same way,

$$\|T_\varepsilon^b f\|_s \lesssim \|b\|_{s+1} \|f\|_s.$$

Hence, there exists a constant c_s such that

$$\|T_\varepsilon f\|_s \leq c_s \left(\|g-1\|_{s+1} + \|b\|_{s+1} \right) \|f\|_s \leq c_s \varepsilon \|f\|_s. \tag{4.33}$$

If ε is chosen such that $c_s \varepsilon < 1$, the Neumann series (4.31) converges. In particular,

$$T_L = [I - T_\varepsilon]^{-1},$$

so that from (4.27) we also obtain the relation (4.30), and the proof is over. $\square$

We also need to describe the action of B_t on H^s. Its definition relies on the operator

$$B_L := (I - \beta(\partial_Y - t\partial_X)\Delta_L^{-1})^{-1}, \tag{4.34}$$

which is precisely B_t in the Couette case where $g \equiv 1$ and $b \equiv 0$ (compare with (4.21)). Adopting the same notation as for the expansion of Δ_t^{-1} in (4.30)-(4.31), we prove in the following proposition that we can write

$$B_t = T_B B_L,$$

where T_B is a proper Neumann series (see Proposition 4.1.2). Notice that the operator B_L is a Fourier multiplier, and

$$B_L^{-1}(t,k,\eta) = 1 + \frac{i\beta(\eta - kt)}{k^2 + (\eta - kt)^2}, \tag{4.35}$$

so everything is well-defined. It is now an easy task to show that, for any fixed frequency, the following bound holds true

$$\frac{1}{\sqrt{1+\beta^2}} \leq |B_L(t;k,\eta)| \leq 1. \tag{4.36}$$

For further reference, we also need to introduce

$$B_\varepsilon := \beta B_L \left((g(Y)-1)(\partial_Y - t\partial_X)\Delta_L^{-1} + g(Y)(\partial_Y - t\partial_X)\Delta_L^{-1}T_\varepsilon T_L\right). \tag{4.37}$$

The action of the above operators on H^s is detailed in the proposition below.

Proposition 4.1.2. *Let $s \geq 0$ and $\beta \geq 0$ be arbitrarily fixed. There exists $0 < \varepsilon_0 < 1$ with the following property. Assume that*

$$\|g-1\|_{s+1} + \|b\|_{s+1} \leq \varepsilon, \tag{4.38}$$

for a positive $\varepsilon \in (0, \varepsilon_0]$. Then we have

$$\|B_\varepsilon\|_{H^s \to H^s} \lesssim \varepsilon. \tag{4.39}$$

The operator B_t on H^s is well-defined and explicitly given by

$$B_t = (I - B_\varepsilon)^{-1} B_L = T_B B_L, \tag{4.40}$$

where

$$T_B = \sum_{n=0}^{\infty} B_\varepsilon^n = I + B_\varepsilon T_B. \tag{4.41}$$

Moreover,

$$\|T_B\|_{H^s \to H^s} \leq 2. \tag{4.42}$$

Proof of Proposition 4.1.2. The expansion of B_t in (4.21) in terms of the Neumann series (4.40) follows from the definition of Δ_t^{-1} in (4.30)-(4.31) and the identity

$$
\begin{aligned}
B_t &= (I-\beta g(\partial_Y - t\partial_X)\Delta_t^{-1})^{-1}\\
&= (I-\beta g(\partial_Y - t\partial_X)(\Delta_L^{-1}+\Delta_L^{-1}T_\varepsilon T_L))^{-1}\\
&= (I-\beta(\partial_Y - t\partial_X)\Delta_L^{-1} - \beta(g-1)(\partial_Y - t\partial_X)\Delta_L^{-1} - \beta g(\partial_Y - t\partial_X)\Delta_L^{-1}T_\varepsilon T_L)^{-1}\\
&= (B_L^{-1} - B_L^{-1}B_\varepsilon)^{-1} = (1-B_\varepsilon)^{-1}B_L = T_B B_L,
\end{aligned}
$$

where B_L, B_ε are in (4.34) and (4.37). The above expansion is rigorous provided that $I - B_\varepsilon$ is invertible, which is what we are going to prove next. We first need to rearrange the expression of B_ε in (4.37) as follows

$$
\begin{aligned}
B_\varepsilon =& B_L(\beta(g-1)(\partial_Y - t\partial_X)\Delta_L^{-1} + \beta(g-1)(\partial_Y - t\partial_X)\Delta_L^{-1}T_\varepsilon T_L\\
&+ \beta(\partial_Y - t\partial_X)\Delta_L^{-1}T_\varepsilon T_L).
\end{aligned}
$$

We use the upper bound on the multiplier B_L in (4.36) and the fact that $|\eta - kt|p^{-1} \leq p^{-1/2} \leq 1$ to obtain

$$
\begin{aligned}
\|B_\varepsilon f\|_s &\leq \beta \left\| |\widehat{g-1}| * |\eta - kt|p^{-1}|\widehat{f}| \right\|_s + \beta \left\| |\widehat{g-1}| * |\eta - kt|p^{-1}|\widehat{T_\varepsilon T_L f}| \right\|_s\\
&\quad + \beta \left\| |\eta - kt|p^{-1}|\widehat{T_\varepsilon T_L f} \right\|_s\\
&\leq \beta \left\| |\widehat{g-1}| * |\widehat{f}| \right\|_s + \beta \left\| |\widehat{g-1}| * |\widehat{T_\varepsilon T_L f}| \right\|_s + \beta \|T_\varepsilon T_L f\|_s\\
&\lesssim \beta \left\| \langle \cdot \rangle^s \widehat{g-1} \right\|_{L^1} \|f\|_s + \left\| \langle \cdot \rangle^s \widehat{g-1} \right\|_{L^1} \|T_\varepsilon T_L f\|_s + \beta \|T_\varepsilon T_L f\|_s\\
&\lesssim \beta \|g-1\|_{s+1} \|f\|_s + \beta(1 + \|g-1\|_{s+1}) \|T_\varepsilon T_L f\|_s,
\end{aligned}
$$

where we applied the Young's convolution inequality exactly as in the proof of Proposition 4.1.1. It remains to deal with the last term above. Thus we first appeal to (4.33) and after we employ (4.32), so that

$$
\begin{aligned}
\|T_\varepsilon T_L f\|_s &\leq c_s(\|g-1\|_{s+1} + \|b\|_{s+1}) \|T_L f\|_s\\
&\leq \frac{c_s}{1-c_s\varepsilon_0}(\|g-1\|_{s+1} + \|b\|_{s+1}) \|f\|_s,
\end{aligned}
$$

which yields (4.39) thanks to the smallness assumptions (4.38), where $\varepsilon_0 \geq \varepsilon$ satisfies

$$
\beta\varepsilon_0\left(1 + \frac{4c_s}{1-c_s\varepsilon_0}\right) < 1.
$$

Hence, T_B in (4.41) is well-defined and $I - B_\varepsilon$ is invertible. The bound (4.42) directly follows from its definition (4.41) and the bound on B_ε. The proof is over.

□

Notation and conventions

We highlight the following conventions, which will be used throughout the chapter.

- We occasionally identify operators with their symbols in order to avoid additional notations (this is for instance the case of the operator Δ_t^{-1}, as pointed out in (4.24) and the lines below).
- We drop the subscript k related to the X-Fourier-localization of the variables defined in (4.23) and we also identify functions with their Y-Fourier transform in (4.25) when there is no confusion. The adoption of this convection is usually pointed out at the beginning of the proofs.
- We use the notation $f_1 \lesssim f_2$ if there exists a constant $C = C(R, \beta, s)$ such that $f_1 \leq C f_2$, where C is independent of k, ε and β. We denote $f_1 \approx f_2$ if $f_1 \lesssim f_2$ and $f_2 \lesssim f_1$.

4.2 The Couette flow

We begin our analysis with the Couette flow, i.e. $U(y) = y$. In this case, in (4.19) we have $g \equiv 1$ and $b \equiv 0$, so that B_t in (4.21) coincides with B_L in (4.34). The system (4.24) takes the simpler form

$$\begin{aligned} \partial_t \Theta_k &= -ikRQ_k - ik\beta\Delta_L^{-1} B_L \Theta_k, \\ \partial_t Q_k &= ik\Delta_L^{-1}\Theta_k + ik\Delta_L^{-1}(B_L - 1)\Theta_k, \end{aligned} \tag{4.43}$$

where

$$\Theta_k = B_L^{-1}\Omega_k.$$

We remark again that B_L is just a Fourier multiplier, see (4.35), and hence commutes with Δ_L^{-1}. The lower and upper bounds on B_L were already provided in (4.36).

To investigate the system (4.43) we adopt the point of view introduced in Chapter 3. In particular, we see system (4.43) as a non-autonomous dynamical system for each fixed frequency k, η. Although the proof carried out in the next Section 4.3 for more general shears applies in this case as well, we prefer to argue point-wise in both k and η since we can deduce more properties in the simple case of the Couette flow. Our result reads as follows.

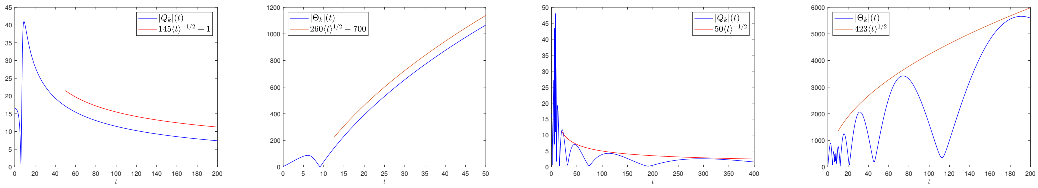
Theorem 4.2.1. *Let $R > 1/4$, $\beta \geq 0$ and $k \neq 0$. For any $t \geq 0$, the solution to* (4.43) *satisfies the uniform bounds*

$$|p^{-\frac{1}{4}}\widehat{\Theta}_k(t)|^2 + |p^{\frac{1}{4}}\widehat{Q}_k(t)|^2 \approx |(k^2+\eta^2)^{-\frac{1}{4}}\widehat{\Theta}_k^{in}|^2 + |(k^2+\eta^2)^{\frac{1}{4}}\widehat{Q}_k^{in}|^2, \tag{4.44}$$

point-wise in $\eta \in \mathbb{R}$. In particular, thanks to (4.36)*,*

$$|p^{-\frac{1}{4}}\widehat{\Omega}_k(t)|^2 + |p^{\frac{1}{4}}\widehat{Q}_k(t)|^2 \approx |(k^2+\eta^2)^{-\frac{1}{4}}\widehat{\Omega}_k^{in}|^2 + |(k^2+\eta^2)^{\frac{1}{4}}\widehat{Q}_k^{in}|^2, \tag{4.45}$$

point-wise in $\eta \in \mathbb{R}$.



(a) $R = 1/4 + 0.1$ (b) $R = 1/4 + 0.1$ (c) $R = 10$ (d) $R = 10$

Figure 4.1: Numerical simulations of the system 4.43 at fixed frequencies $k = 3, \eta = 21$. The value of β is given by $R/\mathfrak{g}$, hence we have $\beta = 0.0357$ in (a)-(b) and $\beta = 1.0204$ in (c)-(d). The initial conditions are $\Theta_k^{in} = 2 + 4i$, $Q_k^{in} = 4 + 16i$. The red lines are the expected asymptotic behaviours.

The pre-factors $p^{\pm\frac{1}{4}}$ in (4.45) appear in view of a natural symmetrization of system (4.43) that we carry out in the sequel (compare also with Chapter 3). Now we show that the proof of the main Theorem 4.0.4 in the Couette flow case (namely, $\varepsilon = 0$) follows from (4.45).

Proof of Theorem 4.0.4 when $U(y) = y$. Recall that from (1.9b) the velocity field (in the moving frame) is given by

$$\widehat{\boldsymbol{V}}_k = (\widehat{V}_k^x, \widehat{V}_k^y) = (-(\partial_Y - t\partial_X)\Delta_L^{-1}\widehat{\Omega}_k, \partial_X \Delta_L^{-1}\widehat{\Omega}_k).$$

Thus, using that $|\eta - kt|^2 p^{-1} \leq 1$, from (4.45) we deduce

$$\begin{aligned}
\|V_k^x(t)\|_{L^2}^2 &= \int_{\mathbb{R}} \frac{|\eta - kt|^2}{p^{\frac{3}{2}} \langle k, \eta\rangle} |p^{-\frac{1}{4}}\widehat{\Omega}_k(t)|^2 \langle k, \eta\rangle \mathrm{d}\eta \\
&\leq \int_{\mathbb{R}} \frac{1}{p^{\frac{1}{2}} \langle k, \eta\rangle} |p^{-\frac{1}{4}}\widehat{\Omega}_k(t)|^2 \langle k, \eta\rangle \mathrm{d}\eta \\
&\lesssim \frac{1}{\langle kt\rangle} \left[\|\Omega_k^{in}\|_{L^2}^2 + \|Q_k^{in}\|_1^2\right].
\end{aligned}$$

Similarly,

$$\|V_k^y(t)\|_{L^2}^2 \lesssim \frac{1}{\langle kt\rangle^3} \left[\|\Omega_k^{in}\|_{\frac{1}{4}}^2 + \|Q_k^{in}\|_{\frac{5}{4}}^2\right],$$

and

$$\|Q_k(t)\|_{L^2}^2 \lesssim \frac{1}{\langle kt\rangle} \left[\|\Omega_k^{in}\|_{L^2}^2 + \|Q_k^{in}\|_1^2\right].$$

Thus, Theorem 4.0.4 is proven.

□

We can now prove the Corollary 4.0.5

Proof of Corollary 4.0.5. By the change of coordinates $X = x - yt,\ Y = y$ we have

$$\|\omega\|_{L^2}^2 + \|\nabla q\|_{L^2}^2 = \|\Omega\|_{L^2}^2 + \left\|(-\Delta_L)^{\frac{1}{2}} Q\right\|_{L^2}^2.$$

From Plancherel's Theorem (recalling that we are assuming $k \neq 0$) we deduce

$$\begin{aligned}\|\omega\|_{L^2}^2 + \|\nabla q\|_{L^2}^2 &= \sum_{k\neq 0} \int |\widehat{\Omega}_k|^2 + p|\widehat{Q}_k|^2 d\eta \\ &= \sum_{k\neq 0} \int p^{\frac{1}{2}} \left(|p^{-\frac{1}{4}}\widehat{\Omega}_k|^2 + |p^{\frac{1}{4}}\widehat{Q}_k|^2\right) d\eta.\end{aligned}$$

Since $p^{\frac{1}{2}} \geq \langle \eta - kt \rangle \gtrsim \langle t \rangle \langle \eta \rangle^{-1}$, in account of the lower bound (4.45) we have

$$\begin{aligned}\|\omega\|_{L^2}^2 + \|\nabla q\|_{L^2}^2 &\gtrsim \langle t \rangle \left(\left\|(-\Delta)^{-\frac{1}{4}} \Omega^{in}\right\|_{L_x^2 H_y^{-\frac{1}{2}}}^2 + \left\|Q^{in}\right\|_{L^2}^2 \right) \\ &= \langle t \rangle \left(\left\|(-\Delta)^{-\frac{1}{4}} \omega^{in}\right\|_{L_x^2 H_y^{-\frac{1}{2}}}^2 + \left\|q^{in}\right\|_{L^2}^2 \right),\end{aligned}$$

and this proves (4.13). The upper bound follows by using that $p^{\frac{1}{2}} \lesssim \langle t \rangle \langle k, \eta \rangle$, hence the proof of Corollary 4.0.5 is over. □

It thus remain to prove Theorem 4.2.1.

Proof of Theorem 4.2.1. Let $k \neq 0$ and $\eta \in \mathbb{R}$ be fixed. Define the auxiliary variables

$$Z_1(t) := (p^{-\frac{1}{4}} \widehat{\Theta})_k(t, \eta), \qquad Z_2(t) := i\sqrt{R}(p^{\frac{1}{4}} \widehat{Q})_k(t, \eta). \tag{4.46}$$

Taking into account (4.40), we then find

$$\partial_t \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} = \begin{pmatrix} -\dfrac{1}{4}\dfrac{\partial_t p}{p} & -k\sqrt{R}p^{-\frac{1}{2}} \\ k\sqrt{R}p^{-\frac{1}{2}} & \dfrac{1}{4}\dfrac{\partial_t p}{p} \end{pmatrix} \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} + \begin{pmatrix} \beta\dfrac{ik}{p}B_L & 0 \\ k\sqrt{R}p^{-\frac{1}{2}}(B_L - 1) & 0 \end{pmatrix} \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}.$$

The proof of Theorem 4.2.1 is reduced in proving a bound for $|Z_1|^2 + |Z_2|^2$. Hence, we define the following energy functional

$$E(t) = \frac{1}{2}\left[|Z_1(t)|^2 + |Z_2(t)|^2 + \frac{1}{2k\sqrt{R}}\mathrm{Re}\left(\frac{\partial_t p}{p^{\frac{1}{2}}} Z_1 \overline{Z_2}\right)(t)\right]. \tag{4.47}$$

Now using that $|\partial_t p| p^{-\frac{1}{2}} \leq 2|k|$, we deduce

$$\frac{1}{2|k|\sqrt{R}}\left|\frac{\partial_t p}{p^{\frac{1}{2}}} Z_1 \overline{Z_2}\right| \leq \frac{1}{\sqrt{R}}|Z_1||Z_2| \leq \frac{1}{2\sqrt{R}}\left(|Z_1|^2 + |Z_2|^2\right).$$

As a consequence, the functional E is coercive whenever $R > 1/4$, namely

$$\frac{1}{2}\left(1 - \frac{1}{2\sqrt{R}}\right)\left[|Z_1|^2 + |Z_2|^2\right] \leq E \leq \frac{1}{2}\left(1 + \frac{1}{2\sqrt{R}}\right)\left[|Z_1|^2 + |Z_2|^2\right]. \tag{4.48}$$

Hence, the proof of Theorem 4.2.1 can be obtained by estimating $E(t)$. We now have to compute the time-derivative of E. First notice that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|Z_1|^2 = -\frac{1}{4}\frac{\partial_t p}{p}|Z_1|^2 - \beta\frac{k}{p}\mathrm{Im}(B_L)|Z_1|^2 - \frac{k\sqrt{R}}{p^{\frac{1}{2}}}\mathrm{Re}(Z_1\overline{Z_2}).$$

Similarly,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|Z_2|^2 = \frac{1}{4}\frac{\partial_t p}{p}|Z_2|^2 + \frac{k\sqrt{R}}{p^{\frac{1}{2}}}\mathrm{Re}(Z_1\overline{Z_2}) + \frac{k\sqrt{R}}{p^{\frac{1}{2}}}\mathrm{Re}((B_L-1)Z_1\overline{Z_2}).$$

Regarding the last term, we find that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Re}\left(\frac{\partial_t p}{p^{\frac{1}{2}}}Z_1\overline{Z_2}\right) &= \left(\partial_t\left(\frac{\partial_t p}{p^{\frac{1}{2}}}\right)\right)\mathrm{Re}\left(Z_1\overline{Z_2}\right) + \beta k\frac{\partial_t p}{p^{\frac{3}{2}}}\mathrm{Re}\left(iB_L Z_1\overline{Z_2}\right)\\ &\quad - k\sqrt{R}\,\frac{\partial_t p}{p}\left[|Z_2|^2 - |Z_1|^2\right] + k\sqrt{R}\,\frac{\partial_t p}{p}\mathrm{Re}(B_L-1)|Z_1|^2.\end{aligned}$$

Therefore, the energy function E satisfies the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}E = \sum_{i=1}^{5}\mathcal{I}_i.$$

where the error terms are defined as

$$\begin{aligned}\mathcal{I}_1 &= \frac{1}{4k\sqrt{R}}\left(\partial_t\left(\frac{\partial_t p}{p^{\frac{1}{2}}}\right)\right)\mathrm{Re}\left(Z_1\overline{Z_2}\right)\\ \mathcal{I}_2 &= \frac{k\sqrt{R}}{p^{\frac{1}{2}}}\mathrm{Re}((B_L-1)Z_1\overline{Z_2}),\\ \mathcal{I}_3 &= \frac{1}{4}\frac{\partial_t p}{p}\mathrm{Re}(B_L-1)|Z_1|^2,\\ \mathcal{I}_4 &= -\beta\frac{k}{p}\mathrm{Im}(B_L)|Z_1|^2,\\ \mathcal{I}_5 &= \frac{\beta}{4\sqrt{R}}\frac{\partial_t p}{p^{\frac{3}{2}}}\mathrm{Re}\left(iB_L Z_1\overline{Z_2}\right),\end{aligned}$$

We now proceed to show that each $\mathcal{I}_i$ has good time-integrability, and therefore we can close a Grönwall estimate which allow us to prove Theorem 4.2.1.

To control $\mathcal{I}_1$, notice that

$$\partial_t\left(\frac{\partial_t p}{p^{\frac{1}{2}}}\right) = \frac{2k^2}{p^{\frac{1}{2}}} - \frac{1}{2}\frac{(\partial_t p)^2}{p^{\frac{3}{2}}} = \frac{2k^4}{p^{\frac{3}{2}}} = \frac{2|k|}{(1+|t-\frac{\eta}{k}|^2)^{\frac{3}{2}}}. \tag{4.49}$$

Therefore

$$|\mathcal{I}_1| \leq \frac{1}{4\sqrt{R}}\frac{1}{(1+|t-\frac{\eta}{k}|^2)^{\frac{3}{2}}}(|Z_1|^2+|Z_2|^2). \tag{4.50}$$

Then, observe that the multiplier B_L is contained in all the remaining terms. We can compute it explicitly from (4.35), namely, we know that $B_L^{-1} = 1 + i\beta(\eta - kt)p^{-1} = (p + i\beta(\eta - kt))p^{-1}$, hence

$$B_L = \frac{p^2}{p^2 + \beta^2(\eta - kt)^2} - i\beta p \frac{\eta - kt}{p^2 + \beta^2(\eta - kt)^2}.$$

In particular we have

$$\begin{aligned}
|B_L| &\leq 1 + \beta, \\
|\mathrm{Im}(B_L)| &\leq \frac{\beta}{p^{\frac{1}{2}}}, \\
|\mathrm{Re}(B_L - 1)| &= \beta^2 \frac{(\eta - kt)^2}{p^2 + \beta^2(\eta - kt)^2} \leq \frac{\beta^2}{p}, \\
|B_L - 1| &\leq |\mathrm{Re}(B_L - 1)| + |\mathrm{Im}(B_L)| \leq \frac{\beta + \beta^2}{p^{\frac{1}{2}}}.
\end{aligned} \tag{4.51}$$

By using the bounds above and the fact that $|\partial_t p| \leq 2|k|p^{\frac{1}{2}}$ we infer

$$\begin{aligned}
|\mathcal{I}_2| + |\mathcal{I}_5| &\lesssim \frac{|k|}{p}(|Z_1|^2 + |Z_2|^2), \\
|\mathcal{I}_3| + |\mathcal{I}_4| &\lesssim \frac{|k|}{p^{\frac{3}{2}}}|Z_1|^2.
\end{aligned}$$

Combining the bounds above with (4.50), we deduce that

$$|\sum_{i=1}^{5} \mathcal{I}_i| \lesssim \frac{1}{1 + (t - \frac{\eta}{k})^2} = \frac{k^2}{p},$$

where we have roughly bounded all the terms with $k^2 p^{-1}$. Consequently, by the coercivity properties of E, see (4.48), we get

$$-\frac{1}{2\sqrt{R} - 1}\frac{|k|^2}{p}E \lesssim \frac{\mathrm{d}}{\mathrm{d}t}E \lesssim \frac{1}{2\sqrt{R} - 1}\frac{|k|^2}{p}E. \tag{4.52}$$

Now, since $\int_0^\infty |k|^2 p^{-1} dt \leq \pi$, we can apply the Grönwall lemma to (4.52) and obtain

$$E(t) \approx E^{in}.$$

This translates immediately into (4.44) thanks to (4.48), thereby concluding the proof of Theorem 4.2.1. □

4.3 Shears close to Couette

This section deals with the full system (4.24). Using the expressions of Δ_t^{-1} and B_t in (4.30)-(4.31) and (4.40)-(4.41), the system reads

$$\partial_t \Theta_k = -ikRQ_k + ik\left(b(Y) - \beta g(Y)\right)\Delta_t^{-1} B_t \Theta_k, \tag{4.53}$$

$$\begin{aligned}
\partial_t Q_k = {} & ik\Delta_L^{-1}\Theta_k + ik\Delta_L^{-1}(B_L - 1)\Theta_k + ik\Delta_L^{-1} T_\varepsilon T_L B_L \Theta_k \\
& + ik\Delta_t^{-1} B_\varepsilon B_t \Theta_k.
\end{aligned} \tag{4.54}$$

We have expanded the right-hand side above to highlight the similarities with the Couette flow (4.43). In the first equation (4.53), the second term in the right-hand side is treated as an error term, just like in the previous section. In the second equation (4.54), we have expanded the operator Δ_t^{-1} and B_t in order to extract the Couette-like structure, while all the other terms are treated as errors. We have the following for shear flows near Couette.

Theorem 4.3.1. *Let $R > 1/4$, $\beta \geq 0$ and $s \geq 0$ be fixed. There exist $\varepsilon_0 \in (0,1)$ with the following property. If $\varepsilon \in (0, \varepsilon_0]$ and*

$$\|g-1\|_{s+5} + \|b\|_{s+4} \leq \varepsilon,$$

then for every $k \neq 0$ the solution to (4.53)-(4.54) *satisfies the uniform H^s-bound*

$$\left\|p^{-\frac{1}{4}}\Theta_k(t)\right\|_s^2 + \left\|p^{\frac{1}{4}}Q_k(t)\right\|_s^2 \lesssim \langle t\rangle^{\delta}\left(\left\|\Theta_k^{in}\right\|_s^2 + \left\|Q_k^{in}\right\|_{s+1}^2\right), \qquad \forall t \geq 0,$$

where $\delta = \sqrt{\varepsilon}$ and p is given by (4.26).

The bound on the vorticity follows directly.

Corollary 4.3.2. *Under the same assumptions of Theorem 4.3.1, for $k \neq 0$ the following inequality holds true*

$$\left\|p^{-\frac{1}{4}}\Omega_k(t)\right\|_s^2 + \left\|p^{\frac{1}{4}}Q_k(t)\right\|_s^2 \lesssim \langle t\rangle^{\delta}\left(\left\|\Omega_k^{in}\right\|_s^2 + \left\|Q_k^{in}\right\|_{s+1}^2\right), \qquad \forall t \geq 0.$$

Appealing to Theorem 4.3.1 and Corollary 4.3.2, which will be proved throughout the chapter, we first prove Theorem 4.0.4.

Proof of Theorem 4.0.4. In order to prove (4.11), first observe that since

$$p^{-\frac{1}{4}} \lesssim \langle kt\rangle^{-\frac{1}{2}} \langle k, \eta\rangle^{\frac{1}{2}}$$

we get

$$\|q_k(t)\|_{L^2} = \|Q_k(t)\|_{L^2} = \left\|p^{-\frac{1}{4}}p^{\frac{1}{4}}Q_k(t)\right\|_{L^2} \lesssim \frac{1}{\langle kt\rangle^{\frac{1}{2}}}\left\|p^{\frac{1}{4}}Q_k(t)\right\|_{\frac{1}{2}}$$

Hence, appealing to Corollary 4.3.2 and summing in k, we prove the bound for q in (4.11). For the velocity field, we know that its components in the coordinates driven by the flow read

$$V^x = -g(Y)(\partial_Y - t\partial_X)\Delta_t^{-1}\Omega, \quad V^y = \partial_X\Delta_t^{-1}\Omega.$$

We begin with the bound on V^x. First rewrite V^x as

$$\begin{aligned} V^x &= -(\partial_Y - t\partial_X)\Delta_t^{-1}\Omega - (g(Y)-1)(\partial_Y - t\partial_X)\Delta_t^{-1}\Omega \\ &:= V^{x,1} + V^{x,\varepsilon} \end{aligned}$$

Since $\Delta_t^{-1} = \Delta_L^{-1} T_L$ as in (4.30), we bound $V^{x,1}$ as follows

$$\begin{aligned}\left\|V_k^{x,1}\right\|_{L^2}^2 = &\int \frac{(\eta - kt)^2}{p^2} |\widehat{T_L \Omega_k}|^2 d\eta \le \int \frac{1}{p} |\widehat{T_L \Omega_k}|^2 d\eta \\ \lesssim &\frac{1}{\langle kt \rangle} \left\| p^{-\frac{1}{4}} (T_L \Omega_k)(t) \right\|_{\frac{1}{2}}^2 ,\end{aligned} \tag{4.55}$$

where in the last inequality we have used again $p^{-\frac{1}{2}} \lesssim \langle kt \rangle^{-1} \langle k, \eta \rangle$. Then, thanks to (4.31) we know that $T_L = I + T_\varepsilon T_L$, where T_ε is defined in (4.28). Therefore

$$\left\| p^{-\frac{1}{4}} T_L \Omega_k(t) \right\|_{\frac{1}{2}} \le \left\| p^{-\frac{1}{4}} \Omega_k(t) \right\|_{\frac{1}{2}} + \left\| p^{-\frac{1}{4}} T_\varepsilon T_L \Omega_k(t) \right\|_{\frac{1}{2}} \tag{4.56}$$

Now we would like to absorb the last term in the equation above in the left-hand side, similarly to what was done in the proof of Proposition 4.1.1. However, the multiplier $p^{-\frac{1}{4}}$ does not commute with T_ε. Hence, we have to use the inequality

$$p^{-\frac{1}{4}}(t, k, \eta) \lesssim \langle \eta - \xi \rangle^{\frac{1}{2}} p^{-\frac{1}{4}}(t, k, \xi),$$

see also Lemma 4.3.6, to exchange frequencies in the last term of the right-hand side of (4.56). In particular, we deduce

$$\begin{aligned}\left\| p^{-\frac{1}{4}} T_\varepsilon T_L \Omega_k \right\|_{\frac{1}{2}} \le &\left\| p^{-\frac{1}{4}} \left(|\widehat{g^2 - 1}| * |\widehat{T_L \Omega_k}| \right) \right\|_{\frac{1}{2}} + \left\| p^{-\frac{1}{4}} \left(|\widehat{b}| * |\widehat{T_L \Omega_k}| \right) \right\|_{\frac{1}{2}} \\ \lesssim &\left\| \langle \cdot \rangle |\widehat{g^2 - 1}| * \langle \cdot \rangle^{\frac{1}{2}} p^{-\frac{1}{4}} |\widehat{T_L \Omega_k}| \right\|_{L^2} + \left\| \langle \cdot \rangle |\widehat{b}| * \langle \cdot \rangle^{\frac{1}{2}} p^{-\frac{1}{4}} |\widehat{T_L \Omega_k}| \right\|_{L^2} .\end{aligned}$$

Thanks to Young's convolution inequality, by combining the previous bound with (4.56) we infer

$$\left\| p^{-\frac{1}{4}} T_L \Omega_k \right\|_{\frac{1}{2}} \lesssim \left\| p^{-\frac{1}{4}} \Omega_k \right\|_{\frac{1}{2}} + \varepsilon \left\| p^{-\frac{1}{4}} T_L \Omega_k \right\|_{\frac{1}{2}} ,$$

namely

$$\left\| p^{-\frac{1}{4}} T_L \Omega_k \right\|_{\frac{1}{2}} \lesssim \left\| p^{-\frac{1}{4}} \Omega_k \right\|_{\frac{1}{2}} . \tag{4.57}$$

Since

$$\left\| V_k^{x,\varepsilon} \right\|_{L^2} \le \|g - 1\|_{L^\infty} \left\| V_k^{x,1} \right\|_{L^2} \lesssim \varepsilon \left\| V_k^{x,1} \right\|_{L^2} ,$$

by combining (4.55) with (4.57) and Corollary 4.3.2, we get

$$\|V_k^x\|_{L^2} \lesssim \frac{1}{\langle kt \rangle^{\frac{1}{2}}} \left\| p^{-\frac{1}{4}} \Omega_k \right\|_{\frac{1}{2}} \lesssim \frac{1}{\langle kt \rangle^{\frac{1}{2} - \frac{\delta}{2}}} \left(\left\| \Omega_k^{in} \right\|_{\frac{1}{2}} + \left\| Q_k^{in} \right\|_{\frac{3}{2}} \right).$$

Applying the same reasoning for V_k^y, we get

$$\|V_k^y(t)\|_{L^2} \lesssim \frac{1}{\langle kt \rangle^{\frac{3}{2} - \frac{\delta}{2}}} \left(\|\Omega_k^{in}\|_{\frac{3}{2}} + \|Q_k^{in}\|_2 \right).$$

Hence, by summing up in k and defining $\delta_\varepsilon = \delta/2 = \sqrt{\varepsilon}/2$, the proof of Theorem 4.0.4 is concluded. $\square$

4.3.1 The weighted energy functional

Since (4.53)-(4.54) decouples in the X-Fourier variable, we fix a nonzero integer k and define the auxiliary variables

$$Z_1 := m^{-1}p^{-\frac{1}{4}}\Theta_k, \qquad Z_2 := m^{-1}p^{\frac{1}{4}}i\sqrt{R}Q_k, \tag{4.58}$$

with $m = m(t,k,\eta)$ a positive weight, that will be specified later, such that $\partial_t m > 0$. The choice made in (4.58), up to the weight m is exactly the one of the Couette case, see (4.46). In fact, one immediately sees that Z_1, Z_2 satisfy

$$\partial_t \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{4}\frac{\partial_t p}{p} & -k\sqrt{R}p^{-\frac{1}{2}} \\ k\sqrt{R}p^{-\frac{1}{2}} & \frac{1}{4}\frac{\partial_t p}{p} \end{pmatrix} \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} - \frac{\partial_t m}{m}\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} + \mathcal{R}(t)\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix},$$

where $\mathcal{R}(t)$ will be treated as a perturbation being indeed a matrix of remainders. The error terms which are hidden (up to now) in $\mathcal{R}(t)$ need to be controlled by means of the artificial dissipation introduced by the weight m.

Given a real number $s \geq 0$, we define the energy functional as

$$E_s(t) = \frac{1}{2}\left[\|Z_1(t)\|_s^2 + \|Z_2(t)\|_s^2 + \frac{1}{2k\sqrt{R}}\mathrm{Re}\left\langle \frac{\partial_t p}{p^{\frac{1}{2}}}Z_1(t), Z_2(t)\right\rangle_s\right],$$

which is exactly the one used for the Couette case, see (4.47), when integrated in the variable η.

Now using that $|\partial_t p|p^{-\frac{1}{2}} \leq 2|k|$, we deduce that

$$\frac{1}{2|k|\sqrt{R}}\left|\langle \frac{\partial_t p}{p^{\frac{1}{2}}}Z_1, Z_2\rangle_s\right| \leq \frac{1}{\sqrt{R}}\|Z_1\|_s\|Z_2\|_s \leq \frac{1}{2\sqrt{R}}\left(\|Z_1\|_s^2 + \|Z_2\|_s^2\right).$$

As a consequence, the functional is coercive whenever $R > 1/4$, namely

$$\frac{1}{2}\left(1 - \frac{1}{2\sqrt{R}}\right)\left[\|Z_1\|_s^2 + \|Z_2\|_s^2\right] \leq E_s \leq \frac{1}{2}\left(1 + \frac{1}{2\sqrt{R}}\right)\left[\|Z_1\|_s^2 + \|Z_2\|_s^2\right].$$

The rest of the paper aims at proving that $t \to E_s(t)$ is non-increasing.

Lemma 4.3.3. *The functional $E_s(t)$ satisfies*

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}E_s + \left(1 - \frac{1}{2\sqrt{R}}\right)\left[\left\|\sqrt{\frac{\partial_t m}{m}}Z_1\right\|_s^2 + \left\|\sqrt{\frac{\partial_t m}{m}}Z_2\right\|_s^2\right] \\ &\leq \frac{1}{4|k|\sqrt{R}}\left|\langle\left(\partial_t\left(\frac{\partial_t p}{p^{\frac{1}{2}}}\right)\right)Z_1, Z_2\rangle_s\right| + \sum_{i=1}^{8}\mathcal{R}_i, \end{aligned} \tag{4.59}$$

where the error terms $\mathcal{R}_i$ are

$$
\begin{aligned}
\mathcal{R}_1 &= |k| \left| \langle Z_1, m^{-1} p^{-\frac{1}{4}} \left((b-\beta g) \Delta_L^{-1} T_L T_B B_L \Theta \right) \rangle_s \right|, \\
\mathcal{R}_2 &= |k|\sqrt{R} \left| \langle p^{-\frac{1}{2}} (B_L - 1) Z_1, Z_2 \rangle_s \right|, \\
\mathcal{R}_3 &= |k|\sqrt{R} \left| \langle m^{-1} p^{-\frac{3}{4}} T_\varepsilon T_L B_L \Theta, Z_2 \rangle_s \right|, \\
\mathcal{R}_4 &= |k|\sqrt{R} \left| \langle m^{-1} p^{-\frac{3}{4}} T_L B_\varepsilon T_B B_L \Theta, Z_2 \rangle_s \right|, \\
\mathcal{R}_5 &= \frac{1}{4\sqrt{R}} \left| \langle (\partial_t p) p^{-\frac{3}{4}} m^{-1} \left((b-\beta g) \Delta_L^{-1} T_L T_B B_L \Theta \right), Z_2 \rangle_s \right|, \\
\mathcal{R}_6 &= \frac{1}{4} \left| \langle Z_1, \frac{\partial_t p}{p} (B_L - 1) Z_1 \rangle_s \right|, \\
\mathcal{R}_7 &= \frac{1}{4} \left| \langle \frac{\partial_t p}{p} Z_1, m^{-1} p^{-\frac{1}{4}} T_\varepsilon T_L B_L \Theta \rangle_s \right|, \\
\mathcal{R}_8 &= \frac{1}{4} \left| \langle \frac{\partial_t p}{p} Z_1, m^{-1} p^{-\frac{1}{4}} T_L B_\varepsilon T_B B_L \Theta \rangle_s \right|.
\end{aligned}
$$

Proof. Taking the time-derivative of the functional, from Z_1 we have

$$
\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|Z_1\|_s^2 = &- \left\| \sqrt{\frac{\partial_t m}{m}} Z_1 \right\|_s^2 - \frac{1}{4} \langle Z_1, \frac{\partial_t p}{p} Z_1 \rangle - k\sqrt{R}\,\mathrm{Re}\langle Z_1, p^{-\frac{1}{2}} Z_2 \rangle_s \\
&+ \mathrm{Re}\langle Z_1, ikm^{-1} p^{-\frac{1}{4}} \left((b-\beta g) \Delta_L^{-1} T_L T_B B_L \Theta \right) \rangle_s.
\end{aligned} \tag{4.60}
$$

About Z_2,

$$
\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|Z_2\|_s^2 = &- \left\| \sqrt{\frac{\partial_t m}{m}} Z_2 \right\|_s^2 + \frac{1}{4} \langle Z_2, \frac{\partial_t p}{p} Z_2 \rangle_s + k\sqrt{R}\,\mathrm{Re}\langle p^{-\frac{1}{2}} Z_1, Z_2 \rangle_s \\
&+ k\sqrt{R}\,\mathrm{Re}\langle p^{-\frac{1}{2}} (B_L - 1) Z_1, Z_2 \rangle_s \\
&+ k\sqrt{R}\,\mathrm{Re}\langle m^{-1} p^{-\frac{3}{4}} T_\varepsilon T_L B_L \Theta, Z_2 \rangle_s \\
&+ k\sqrt{R}\,\mathrm{Re}\langle m^{-1} p^{-\frac{3}{4}} T_L B_\varepsilon T_B B_L \Theta, Z_2 \rangle_s.
\end{aligned} \tag{4.61}
$$

For the mixed term we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Re}\langle\frac{\partial_t p}{p^{\frac{1}{2}}}Z_1, Z_2\rangle_s =&\mathrm{Re}\langle\left(\partial_t\left(\frac{\partial_t p}{p^{\frac{1}{2}}}\right)\right)Z_1, Z_2\rangle_s \qquad (4.62)\\ &-2\langle\frac{\partial_t p}{p^{\frac{1}{2}}}\frac{\partial_t m}{m}Z_1, Z_2\rangle_s - k\sqrt{R}\,\langle Z_2, \frac{\partial_t p}{p}Z_2\rangle_s\\ &+\mathrm{Re}\langle ik(\partial_t p)p^{-\frac{3}{4}}m^{-1}\left((b-\beta g)\Delta_L^{-1}T_L B_t\Theta\right), Z_2\rangle_s\\ &+k\sqrt{R}\,\langle Z_1, \frac{\partial_t p}{p}Z_1\rangle_s \qquad (4.63)\\ &+k\sqrt{R}\,\mathrm{Re}\langle Z_1, \frac{\partial_t p}{p}(B_L-1)Z_1\rangle_s\\ &+k\sqrt{R}\,\mathrm{Re}\langle\frac{\partial_t p}{p}Z_1, m^{-1}p^{-\frac{1}{4}}T_\varepsilon T_L B_L\Theta\rangle_s\\ &+k\sqrt{R}\,\mathrm{Re}\langle\frac{\partial_t p}{p}Z_1, m^{-1}p^{-\frac{1}{4}}T_L B_\varepsilon T_B B_L\Theta\rangle_s.\end{aligned}$$

Now, notice that the sum of the last term on the right-hand side of (4.60) and the last term on the right-hand side of (4.61) is zero. Next, the last term of (4.62) multiplied by $1/(4k\sqrt{R})$ and the second term in the right-hand side of (4.61) balance each other, while the sum of (4.63) multiplied by $1/(4k\sqrt{R})$ and the second term in the right-hand side of (4.60) vanishes. Thus, we end up with

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E_s(t) + \left\|\sqrt{\frac{\partial_t m}{m}}Z_1\right\|_s^2 + \left\|\sqrt{\frac{\partial_t m}{m}}Z_2\right\|_s^2 \\ \leq \frac{1}{4|k|\sqrt{R}}\left|\langle\left(\partial_t\left(\frac{\partial_t p}{p^{\frac{1}{2}}}\right)\right)Z_1, Z_2\rangle_s\right| \\ + \frac{1}{2|k|\sqrt{R}}\left|\langle\frac{\partial_t p}{p^{\frac{1}{2}}}\frac{\partial_t m}{m}Z_1, Z_2\rangle_s\right| + \sum_{i=1}^{8}\mathcal{R}_i.\end{aligned}$$

This way, using that $|(\partial_t p)p^{-\frac{1}{2}}| \leq 2|k|$, we get

$$\frac{1}{2|k|\sqrt{R}}\left|\langle\frac{\partial_t p}{p^{\frac{1}{2}}}\frac{\partial_t m}{m}Z_1, Z_2\rangle_s\right| \leq \frac{1}{2\sqrt{R}}\left(\left\|\sqrt{\frac{\partial_t m}{m}}Z_1\right\|_s^2 + \left\|\sqrt{\frac{\partial_t m}{m}}Z_2\right\|_s^2\right),$$

and the proof is concluded. □

4.3.2 Choice of weights and their properties

The crucial observation for a proper choice of the weights is that we need an additional (uniformly bounded) weight to control the error terms generated by the various $\mathcal{R}_i$'s. More precisely, as it will be clarified later on, the time-derivative of this new weight has to be comparable with $|\partial_t p|/p$. This implies that the decay rates proven in Theorem 4.0.4 have a small δ-correction compared to those in Couette. For these reasons, the weight m in the definition of Z_1 and Z_2 in (4.58) now takes the form

$$m = m_1 w^{\delta},$$

where we recall that $\delta = \sqrt{\epsilon}$. The weight w encodes the decay correction, so that

$$\frac{\partial_t w}{w} = \frac{1}{4}\frac{|\partial_t p|}{p}, \qquad w|_{t=0} = 1. \tag{4.64}$$

The choice of the weight m_1,

$$\frac{\partial_t m_1}{m_1} = C_\beta \frac{|k|^2}{p}, \qquad m_1|_{t=0} = 1, \tag{4.65}$$

where the constant C_β is given by

$$C_\beta = 256\sqrt{R}\left(\frac{2\sqrt{R}}{2\sqrt{R}-1}\right)(1+\beta^2), \tag{4.66}$$

plays a role in controlling the right-hand side of (4.59). It is worth pointing out that C_β blows up as $R \to 1/4$. Notice that

$$\frac{\partial_t m}{m} = \delta \frac{\partial_t w}{w} + \frac{\partial_t m_1}{m_1}. \tag{4.67}$$

Explicitly, for w we have

$$w(t,k,\eta) = \begin{cases} \left(\dfrac{k^2+\eta^2}{p(t,k,\eta)}\right)^{\frac{1}{4}}, & t < \dfrac{\eta}{k}, \\ \left(\dfrac{(k^2+\eta^2)p(t,k,\eta)}{k^4}\right)^{\frac{1}{4}}, & t \geq \dfrac{\eta}{k}, \end{cases} \tag{4.68}$$

while

$$m_1(t,k,\eta) = \exp\left[C_\beta\left(\arctan\left(t - \frac{\eta}{k}\right) - \arctan\left(\frac{\eta}{k}\right)\right)\right]. \tag{4.69}$$

In particular, the weight m_1 is uniformly bounded.

Remark 4.3.4. We have already used similar weights in Chapter 3. In addition, the weight m_1 is standard in the case of incompressible Euler/Navier-Stokes with constant density [20, 163, 162], while the weight w^4 has beed used in [117].

The goal of the remaining part of this section is to prove the following proposition.

Proposition 4.3.5. *Let $R > 1/4$, $\beta \geq 0$ and $s \geq 0$ be fixed. There exist $\varepsilon_0 \in (0,1)$ with the following property. If $\varepsilon \in (0, \varepsilon_0]$ and*

$$\|g-1\|_{s+5} + \|b\|_{s+4} \leq \varepsilon,$$

then

$$\frac{\mathrm{d}}{\mathrm{d}t}E_s + \frac{1}{4}\left(1 - \frac{1}{2\sqrt{R}}\right)\left[\left\|\sqrt{\frac{\partial_t m}{m}}Z_1\right\|_s^2 + \left\|\sqrt{\frac{\partial_t m}{m}}Z_2\right\|_s^2\right] \leq 0,$$

for every $t \geq 0$.

Theorem 4.3.1 follows from the above proposition. We prove this claim at the end of this chapter since we need some technical tool introduced in the sequel. The proof of this proposition is a consequence of the properties of the chosen weights (see Lemmas and 4.3.6 and 4.3.8 below), and the estimates on the various error terms, postponed in the next Section 4.3.3. We begin by computing how much it costs to exchange weights in the various convolutions appearing in the error terms.

Lemma 4.3.6. *Let $k \neq 0$ and $t \geq 0$ be fixed. For any $\eta, \xi \in \mathbb{R}$ we have*

$$p^{-1}(t,k,\eta) \lesssim \langle \eta - \xi \rangle^2 p^{-1}(t,k,\xi), \tag{4.70}$$

$$\frac{|\partial_t p|}{p}(t,k,\eta) \lesssim \langle \eta - \xi \rangle^2 \frac{|\partial_t p|}{p}(t,k,\eta) + |k| \langle \eta - \xi \rangle^3 p^{-1}(t,k,\eta), \tag{4.71}$$

$$m^{-1}(t,k,\eta) \lesssim \langle \eta - \xi \rangle^\delta m^{-1}(t,k,\xi). \tag{4.72}$$

Proof. In the course of the proof, we omit the dependency of all the quantities on k and t. Starting with (4.70), the inequality to prove is equivalent to

$$\langle \frac{\xi}{k} - t \rangle^2 \lesssim \langle \frac{\eta - \xi}{k} \rangle^2 \langle \frac{\eta}{k} - t \rangle^2.$$

Choosing $a = \dfrac{\xi}{k} - t$ and $b = \dfrac{\eta}{k} - t$, this follows from the general inequality

$$\langle a \rangle \lesssim \langle a - b \rangle \langle b \rangle.$$

Turning to (4.71), we write

$$\begin{aligned}
\frac{|\partial_t p|}{p}(\eta) &= \frac{2|\frac{\eta}{k} - t|}{\langle \frac{\eta}{k} - t \rangle^2} \leq \frac{2|\frac{\xi}{k} - t| + 2|\frac{\xi}{k} - \frac{\eta}{k}|}{\langle \frac{\eta}{k} - t \rangle^2} \\
&\leq \left[2|\frac{\xi}{k} - t| + 2|\frac{\xi}{k} - \frac{\eta}{k}| \right] \langle \frac{\eta}{k} - \frac{\xi}{k} \rangle^2 \frac{1}{\langle \frac{\xi}{k} - t \rangle^2} \\
&\leq \left[2|\frac{\xi}{k} - t| + 2|\frac{\xi}{k} - \frac{\eta}{k}| \right] \langle \eta - \xi \rangle^2 k^2 p^{-1}(\xi) \\
&\lesssim \frac{|\partial_t p|}{p}(\xi) \langle \eta - \xi \rangle^2 + |k| \langle \eta - \xi \rangle^3 p^{-1}(\xi).
\end{aligned}$$

We now prove (4.72). Recalling that

$$m^{-1} = (w^\delta m_1)^{-1},$$

where w, m_1 are given by (4.68), (4.69). Since m_1 is uniformly bounded, then

$$(m_1)^{-1}(\eta) \lesssim (m_1)^{-1}(\xi).$$

Thus we deal with $w^{-\delta}$. We know from (4.70) that

$$p^{-1}(\xi) \lesssim \langle \eta - \xi \rangle^2 p^{-1}(\eta).$$

Therefore

$$p(\eta) \lesssim \langle \eta - \xi \rangle^2 p(\xi).$$

Notice that

$$\frac{p(\eta)}{k^2+\eta^2} \lesssim \langle \eta - \xi \rangle^4 \frac{p(\xi)}{k^2+\xi^2}.$$

Recalling the expression of w in (4.68) for $t < \dfrac{\eta}{k}$, it follows that

$$w^{-\delta}(\eta) \le \langle \eta - \xi \rangle^{\delta} w^{-\delta}(\xi).$$

The complementing case $t \ge \dfrac{\eta}{k}$ is analogous, and the proof is over. $\square$

Remark 4.3.7. l It is worth pointing out that the constant value which is implicitly involved in the right-hand side of (4.72) (through the notation " $\lesssim$ ") is e^{2C_β}, up to additional constants (independent of ε, β). This exponential term e^{2C_β} comes directly from the explicit expression of the weight m_1 in (4.69).

We also need to compute how much it costs to commute $T_L, T_\varepsilon, T_B, B_\varepsilon$ with the weights.

Lemma 4.3.8. *Let $R > 1/4$, $\beta \ge 0$ and $s \ge 0$ be fixed. There exist $\varepsilon_0 \in (0,1)$ with the following property. If $\varepsilon \in (0, \varepsilon_0]$ and*

$$\|g-1\|_{s+5} + \|b\|_{s+4} \le \varepsilon, \tag{4.73}$$

then, for every smooth function f and every $t \ge 0$, the following estimates hold true

$$\left\| \sqrt{\frac{\partial_t m_1}{m_1}} p^{-\frac14} m^{-1} T_\varepsilon f \right\|_s \lesssim \varepsilon \left\| \sqrt{\frac{\partial_t m_1}{m_1}} p^{-\frac14} m^{-1} f \right\|_s, \tag{4.74}$$

$$\left\| \sqrt{\frac{\partial_t m_1}{m_1}} p^{-\frac14} m^{-1} B_\varepsilon f \right\|_s \lesssim \varepsilon \left\| \sqrt{\frac{\partial_t m_1}{m_1}} p^{-\frac14} m^{-1} f \right\|_s, \tag{4.75}$$

and

$$\left\| \sqrt{\frac{\partial_t w}{w}} p^{-\frac14} m^{-1} T_\varepsilon f \right\|_s \lesssim \varepsilon \left(\left\| \sqrt{\frac{\partial_t w}{w}} p^{-\frac14} m^{-1} f \right\|_s + \left\| \sqrt{\frac{\partial_t m_1}{m_1}} p^{-\frac14} m^{-1} f \right\|_s \right) \tag{4.76}$$

$$\left\| \sqrt{\frac{\partial_t w}{w}} p^{-\frac14} m^{-1} B_\varepsilon f \right\|_s \lesssim \varepsilon \left(\left\| \sqrt{\frac{\partial_t w}{w}} p^{-\frac14} m^{-1} f \right\|_s + \left\| \sqrt{\frac{\partial_t m_1}{m_1}} p^{-\frac14} m^{-1} f \right\|_s \right) \tag{4.77}$$

In addition, the following inequalities hold

$$\left\| \sqrt{\frac{\partial_t m_1}{m_1}} p^{-\frac{1}{4}} m^{-1} T_L f \right\|_s \leq 2 \left\| \sqrt{\frac{\partial_t m_1}{m_1}} p^{-\frac{1}{4}} m^{-1} f \right\|_s , \tag{4.78}$$

$$\left\| \sqrt{\frac{\partial_t m_1}{m_1}} p^{-\frac{1}{4}} m^{-1} T_B f \right\|_s \leq 2 \left\| \sqrt{\frac{\partial_t m_1}{m_1}} p^{-\frac{1}{4}} m^{-1} f \right\|_s , \tag{4.79}$$

and

$$\left\| \sqrt{\frac{\partial_t w}{w}} p^{-\frac{1}{4}} m^{-1} T_L f \right\|_s \leq 2 \left(\left\| \sqrt{\frac{\partial_t w}{w}} p^{-\frac{1}{4}} m^{-1} f \right\|_s + \left\| \sqrt{\frac{\partial_t m_1}{m_1}} p^{-\frac{1}{4}} m^{-1} f \right\|_s \right) \tag{4.80}$$

$$\left\| \sqrt{\frac{\partial_t w}{w}} p^{-\frac{1}{4}} m^{-1} T_B f \right\|_s \leq 2 \left(\left\| \sqrt{\frac{\partial_t w}{w}} p^{-\frac{1}{4}} m^{-1} f \right\|_s + \left\| \sqrt{\frac{\partial_t m_1}{m_1}} p^{-\frac{1}{4}} m^{-1} f \right\|_s \right) \tag{4.81}$$

Proof. We start with T_L, T_ε. In order to prove (4.74), we first use the operators T_ε^g and T_ε^b defined in (4.29) and the identity

$$T_\varepsilon = T_\varepsilon^g + T_\varepsilon^b$$

to obtain the bound

$$\begin{aligned} \left\| \sqrt{\frac{\partial_t m_1}{m_1}} p^{-\frac{1}{4}} m^{-1} T_\varepsilon f \right\|_s \leq & \left\| \sqrt{\frac{\partial_t m_1}{m_1}} p^{-\frac{1}{4}} m^{-1} T_\varepsilon^g f \right\|_s \\ & + \left\| \sqrt{\frac{\partial_t m_1}{m_1}} p^{-\frac{1}{4}} m^{-1} T_\varepsilon^b f \right\|_s . \end{aligned} \tag{4.82}$$

Now, by Lemma 4.3.6 we find that

$$\left(\sqrt{\frac{\partial_t m_1}{m_1}} p^{-\frac{1}{4}} m^{-1} \right)(\eta) \lesssim \langle \xi - \eta \rangle^{\frac{3}{2}+\delta} \left(\sqrt{\frac{\partial_t m_1}{m_1}} p^{-\frac{1}{4}} m^{-1} \right)(\xi), \tag{4.83}$$

where we recall that the hidden constant in the right-hand side of the previous bound (4.83) is proportional to e^{2C_β}, see Remark 4.3.7. Therefore,

$$\begin{aligned} \left\| \sqrt{\frac{\partial_t m_1}{m_1}} p^{-\frac{1}{4}} m^{-1} T_\varepsilon^g f \right\|_s \leq & \left\| \langle \cdot \rangle^s \sqrt{\frac{\partial_t m_1}{m_1}} p^{-\frac{1}{4}} m^{-1} (|\widehat{g^2 - 1}| * |\widehat{f}|) \right\|_{L^2} \\ \lesssim & \left\| \langle \cdot \rangle^{s+\frac{3}{2}+\delta} |\widehat{g^2 - 1}| * \langle \cdot \rangle^s \sqrt{\frac{\partial_t m_1}{m_1}} p^{-\frac{1}{4}} m^{-1} |\widehat{f}| \right\|_{L^2} \\ \lesssim & \, \|g^2 - 1\|_{s+\frac{5}{2}+\delta} \left\| \sqrt{\frac{\partial_t m_1}{m_1}} p^{-\frac{1}{4}} m^{-1} f \right\|_s . \end{aligned}$$

In a similar fashion,

$$\left\| \sqrt{\frac{\partial_t m_1}{m_1}} p^{-\frac{1}{4}} m^{-1} T_\varepsilon^b f \right\|_s \lesssim \|b\|_{s+\frac{5}{2}+\delta} \left\| \sqrt{\frac{\partial_t m_1}{m_1}} p^{-\frac{1}{4}} m^{-1} f \right\|_s .$$

Plugging the above two bounds in (4.82) and using (4.73) we obtain (4.74). Looking at (4.76), we have again that

$$\left\|\sqrt{\frac{\partial_t w}{w}}p^{-\frac{1}{4}}m^{-1}T_\varepsilon f\right\|_s \leq \left\|\sqrt{\frac{\partial_t w}{w}}p^{-\frac{1}{4}}m^{-1}T_\varepsilon^g f\right\|_s + \left\|\sqrt{\frac{\partial_t w}{w}}p^{-\frac{1}{4}}m^{-1}T_\varepsilon^b f\right\|_s.$$

As above, Lemma 4.3.6 now gives us

$$\begin{aligned}\left(\sqrt{\frac{\partial_t w}{w}}p^{-\frac{1}{4}}m^{-1}\right)(\eta) \lesssim& \langle \eta-\xi\rangle^{\frac{3}{2}+\delta}\left(\sqrt{\frac{\partial_t w}{w}}p^{-\frac{1}{4}}m^{-1}\right)(\xi)\\ &+\langle \eta-\xi\rangle^{2+\delta}\left(\sqrt{\frac{\partial_t m_1}{m_1}}p^{-\frac{1}{4}}m^{-1}\right)(\xi).\end{aligned}$$

Therefore,

$$\begin{aligned}\left\|\sqrt{\frac{\partial_t w}{w}}p^{-\frac{1}{4}}m^{-1}T_\varepsilon^g f\right\|_s &\leq \left\|\langle\cdot\rangle^s\sqrt{\frac{\partial_t w}{w}}p^{-\frac{1}{4}}m^{-1}(|\widehat{g^2-1}|*|\widehat{f}|)\right\|_{L^2}\\ &\lesssim \left\|\langle\cdot\rangle^{s+\frac{3}{2}+\delta}|\widehat{g^2-1}|*\langle\cdot\rangle^s\sqrt{\frac{\partial_t w}{w}}p^{-\frac{1}{4}}m^{-1}|\widehat{f}|\right\|_{L^2}\\ &\quad+\left\|\langle\cdot\rangle^{s+2+\delta}|\widehat{g^2-1}|*\langle\cdot\rangle^s\sqrt{\frac{\partial_t m_1}{m_1}}p^{-\frac{1}{4}}m^{-1}|\widehat{f}|\right\|_{L^2}\\ &\lesssim \|g^2-1\|_{s+3+\delta}\left\|\sqrt{\frac{\partial_t w}{w}}p^{-\frac{1}{4}}m^{-1}f\right\|_s\\ &\quad+\|g^2-1\|_{s+3+\delta}\left\|\sqrt{\frac{\partial_t m_1}{m_1}}p^{-\frac{1}{4}}m^{-1}f\right\|_s.\end{aligned}$$

The part with T_ε^b is similar, so that

$$\left\|\sqrt{\frac{\partial_t w}{w}}p^{-\frac{1}{4}}m^{-1}T_\varepsilon^b f\right\|_s \lesssim \|b\|_{s+3+\delta}\left(\left\|\sqrt{\frac{\partial_t w}{w}}p^{-\frac{1}{4}}m^{-1}f\right\|_s+\left\|\sqrt{\frac{\partial_t m_1}{m_1}}p^{-\frac{1}{4}}m^{-1}f\right\|_s\right).$$

By combining the previous two bounds we prove (4.76). About (4.78), since $T_L = I + T_\varepsilon T_L$, see (4.31), by using (4.74) we get

$$\left\|\sqrt{\frac{\partial_t m_1}{m_1}}p^{-\frac{1}{4}}m^{-1}T_L f\right\|_s \leq \left\|\sqrt{\frac{\partial_t m_1}{m_1}}p^{-\frac{1}{4}}m^{-1}f\right\|_s + C\varepsilon_0\left\|\sqrt{\frac{\partial_t m_1}{m_1}}p^{-\frac{1}{4}}m^{-1}T_L f\right\|_s,$$

where C is properly chosen so that the constant value which is hidden in the right-hand side of (4.74) is compensated. Therefore, for ε_0 small enough we prove (4.78).

To deal with (4.80), we combine (4.31) with (4.76) to get

$$\left\|\sqrt{\frac{\partial_t w}{w}}p^{-\frac{1}{4}}m^{-1}T_L f\right\|_s \leq \left\|\sqrt{\frac{\partial_t w}{w}}p^{-\frac{1}{4}}m^{-1}f\right\|_s + C\varepsilon_0\left\|\sqrt{\frac{\partial_t m_1}{m_1}}p^{-\frac{1}{4}}m^{-1}T_L f\right\|_s + C\varepsilon_0\left\|\sqrt{\frac{\partial_t w}{w}}p^{-\frac{1}{4}}m^{-1}T_L f\right\|_s .$$

For ε_0 small, we can absorb the last term using the left-hand side. Applying (4.78) we end up with (4.80).

Now we deal with B_ε and T_B, starting with (4.75). From the definition of B_ε in (4.37), since $|B_L| \leq 1$ as in (4.36), we get

$$\begin{aligned}\left\|\sqrt{\frac{\partial_t m_1}{m_1}}p^{-\frac{1}{4}}m^{-1}B_\varepsilon f\right\|_s &\lesssim \left\|\sqrt{\frac{\partial_t m_1}{m_1}}p^{-\frac{1}{4}}m^{-1}\left(\widehat{g-1} * \frac{\partial_t p}{2k}p^{-1}\widehat{f}\right)\right\|_s \\ &+ \left\|\sqrt{\frac{\partial_t m_1}{m_1}}p^{-\frac{1}{4}}m^{-1}\frac{\partial_t p}{2k}p^{-1}\widehat{T_\varepsilon T_L f}\right\|_s \\ &+ \left\|\sqrt{\frac{\partial_t m_1}{m_1}}p^{-\frac{1}{4}}m^{-1}\left(\widehat{g-1} * \frac{\partial_t p}{2k}p^{-1}\widehat{T_\varepsilon T_L f}\right)\right\|_s .\end{aligned} \tag{4.84}$$

Considering the term in the right-hand side of (4.84), by using (4.83) we get

$$\begin{aligned}&\left\|\sqrt{\frac{\partial_t m_1}{m_1}}p^{-\frac{1}{4}}m^{-1}\left(\widehat{g-1} * \frac{\partial_t p}{2k}p^{-1}\widehat{f}\right)\right\|_s \\ &\qquad\lesssim \left\|\langle\cdot\rangle^{s+\frac{3}{2}+\delta}|\widehat{g-1}| * \langle\cdot\rangle\sqrt{\frac{\partial_t m_1}{m_1}}p^{-\frac{1}{4}}m^{-1}\widehat{f}\right\|_{L^2} \\ &\qquad\lesssim \|g-1\|_{s+\frac{5}{2}+\delta}\left\|\sqrt{\frac{\partial_t m_1}{m_1}}p^{-\frac{1}{4}}m^{-1}f\right\|_s .\end{aligned} \tag{4.85}$$

Thanks to (4.74) and (4.78) we obtain

$$\left\|\sqrt{\frac{\partial_t m_1}{m_1}}p^{-\frac{1}{4}}m^{-1}\frac{\partial_t p}{2k}p^{-1}\widehat{T_\varepsilon T_L f}\right\|_s \lesssim \varepsilon\left\|\sqrt{\frac{\partial_t m_1}{m_1}}p^{-\frac{1}{4}}m^{-1}f\right\|_s . \tag{4.86}$$

Proceeding as done to get (4.85), by using the previous bound we also have

$$\left\|\sqrt{\frac{\partial_t m_1}{m_1}}p^{-\frac{1}{4}}m^{-1}\left(\widehat{g-1} * \frac{\partial_t p}{2k}p^{-1}\widehat{T_\varepsilon T_L f}\right)\right\|_s \lesssim \varepsilon\left\|\sqrt{\frac{\partial_t m_1}{m_1}}p^{-\frac{1}{4}}m^{-1}f\right\|_s . \tag{4.87}$$

Therefore, combining (4.84) with (4.85), (4.86) and (4.87) we have that (4.75) holds true.

From (4.37) and the upper bound on B_L in (4.36), it is easy to check that (4.77) follows by arguing as already done to obtain (4.76). Turning to (4.79), by using (4.40) we have

$$\left\|\sqrt{\frac{\partial_t m_1}{m_1}}p^{-\frac{1}{4}}m^{-1}T_B f\right\|_s \leq \left\|\sqrt{\frac{\partial_t m_1}{m_1}}p^{-\frac{1}{4}}m^{-1}f\right\|_s + \left\|\sqrt{\frac{\partial_t m_1}{m_1}}p^{-\frac{1}{4}}m^{-1}B_\varepsilon T_B f\right\|_s ,$$

therefore, since $|B_L| \leq 1$, by using (4.75) we get

$$\left\| \sqrt{\frac{\partial_t m_1}{m_1}} p^{-\frac{1}{4}} m^{-1} T_B f \right\|_s \leq \left\| \sqrt{\frac{\partial_t m_1}{m_1}} p^{-\frac{1}{4}} m^{-1} f \right\|_s + C\varepsilon_0 \left\| \sqrt{\frac{\partial_t m_1}{m_1}} p^{-\frac{1}{4}} m^{-1} T_B f \right\|_s,$$

which proves (4.79) choosing ε_0 small enough. The proof of (4.81) is similar. □

4.3.3 Proof of Proposition 4.3.5

The starting point is the differential inequality (4.59). Exploiting the identity (4.49), we have that $\partial_t((\partial_t p)p^{-\frac{1}{2}}) \lesssim C_\beta^{-1}|k|\partial_t m_1/m_1$, hence we get

$$\frac{1}{4|k|\sqrt{R}} \left| \langle \left(\partial_t \left(\frac{\partial_t p}{p} \right) \right) Z_1, Z_2 \rangle_s \right| \leq \frac{1}{2C_\beta} \left(\left\| \sqrt{\frac{\partial_t m_1}{m_1}} Z_1 \right\|_s^2 + \left\| \sqrt{\frac{\partial_t m_1}{m_1}} Z_2 \right\|_s^2 \right),$$

where we recall C_β is given in (4.66).

It remains to deal with the $\mathcal{R}_i$'s. The idea is to absorb the estimates of these terms by using the second (positive) term in the left-hand side of (4.59). These computations are developed below.

Estimate on $\mathcal{R}_1$. We want to show that

$$\begin{aligned} \mathcal{R}_1 &= |k| \left| \langle Z_1, m^{-1} p^{-\frac{1}{4}} \left((b - \beta g) \Delta_L^{-1} T_L T_B B_L \Theta \right) \rangle_s \right| \\ &\leq \frac{1}{32} \left(1 - \frac{1}{2\sqrt{R}} \right) \left\| \sqrt{\frac{\partial_t m_1}{m_1}} Z_1 \right\|_s^2. \end{aligned} \tag{4.88}$$

By using that $\partial_t m_1/m_1 = C_\beta |k|^2/p$ we have

$$\begin{aligned} \mathcal{R}_1 \leq& \langle |Z_1|, m^{-1} p^{-\frac{1}{4}} \left((|\widehat{b}| + \beta|\widehat{g-1}|) * |k| p^{-1} |T_L \widehat{T_B B_L} \Theta| \right) \rangle_s & (4.89) \\ &+ \beta \langle |Z_1|, m^{-1} p^{-\frac{1}{4}} |k| p^{-1} |T_L \widehat{T_B B_L} \Theta| \rangle_s & (4.90) \\ \leq& \frac{1}{C_\beta |k|} \langle |Z_1|, m^{-1} p^{-\frac{1}{4}} \left((|\widehat{b}| + \beta|\widehat{g-1}|) * \partial_t m_1 m_1^{-1} |T_L \widehat{T_B B_L} \Theta| \right) \rangle_s & (4.91) \\ &+ \frac{\beta}{C_\beta |k|} \langle |Z_1|, m^{-1} p^{-\frac{1}{4}} \partial_t m_1 m_1^{-1} |T_L \widehat{T_B B_L} \Theta| \rangle_s \\ :=& \mathcal{R}_1^1 + \mathcal{R}_1^2. \end{aligned}$$

Appealing to Lemma 4.3.6, since $\sqrt{\partial_t m_1/m_1} = \sqrt{C_\beta}|k|/\sqrt{p}$ and $\langle k, \eta \rangle^s \lesssim \langle \eta - \xi \rangle^s \langle k, \xi \rangle^s$, then

$$\left(\langle k, \eta \rangle^s m^{-1} p^{-\frac{1}{4}} \right)(\eta) \sqrt{\frac{\partial_t m_1}{m_1}}(\xi) \lesssim \langle \eta - \xi \rangle^{s+\frac{3}{2}+\delta} \left(\langle k, \xi \rangle^s m^{-1} p^{-\frac{1}{4}} \right)(\xi) \sqrt{\frac{\partial_t m_1}{m_1}}(\eta).$$

Plugging in (4.91) the previous inequality and using Cauchy-Schwarz,

$$\mathcal{R}_1^1 \lesssim \left\| \sqrt{\frac{\partial_t m_1}{m_1}} Z_1 \right\|_s \left\| \langle \cdot \rangle^{s+\frac{3}{2}+\delta} (|\widehat{b}| + \beta|\widehat{g-1}|) * \langle \cdot \rangle^s \sqrt{\frac{\partial_t m_1}{m_1}} m^{-1} p^{-\frac{1}{4}} |T_L \widehat{T_B B_L} \Theta| \right\|_{L^2}.$$

Applying Young's convolution inequality we have

$$\mathcal{R}_1^1 \lesssim \left(\|b\|_{s+\frac{5}{2}+\delta} + \|g-1\|_{s+\frac{5}{2}+\delta}\right) \left\|\sqrt{\frac{\partial_t m_1}{m_1}} Z_1\right\|_s \left\|\sqrt{\frac{\partial_t m_1}{m_1}} m^{-1} p^{-\frac{1}{4}} T_L T_B B_L \Theta\right\|_s .$$

Using (4.78) and (4.79), since $|B_L| \leq 1$, see (4.36), we bound the last term as follows

$$\left\|\sqrt{\frac{\partial_t m_1}{m_1}} m^{-1} p^{-\frac{1}{4}} T_L T_B B_L \Theta\right\|_s \leq 2 \left\|\sqrt{\frac{\partial_t m_1}{m_1}} m^{-1} p^{-\frac{1}{4}} T_B B_L \Theta\right\|_s \leq 4 \left\|\sqrt{\frac{\partial_t m_1}{m_1}} Z_1\right\|_s . \tag{4.92}$$

Thus we get

$$\mathcal{R}_1^1 \lesssim \varepsilon \left\|\sqrt{\frac{\partial_t m_1}{m_1}} Z_1\right\|_s^2 . \tag{4.93}$$

To bound $\mathcal{R}_1^2$, thanks to (4.92) we get

$$\mathcal{R}_1^2 \leq \frac{4\beta}{C_\beta} \left\|\sqrt{\frac{\partial_t m_1}{m_1}} Z_1\right\|_s^2 , \tag{4.94}$$

therefore, by the choice of C_β, see (4.66), combining (4.93) with (4.94) and choosing ε_0 small enough, the proof of (4.88) is over.

Estimate on $\mathcal{R}_2$. We prove that

$$\begin{aligned} \mathcal{R}_2 &= |k|\sqrt{R} \left|\langle p^{-\frac{1}{2}} (B_L - 1) Z_1, Z_2 \rangle_s\right| \\ &\leq \frac{1}{32}\left(1 - \frac{1}{2\sqrt{R}}\right) \left(\left\|\sqrt{\frac{\partial_t m_1}{m_1}} Z_1\right\|_s^2 + \left\|\sqrt{\frac{\partial_t m_1}{m_1}} Z_2\right\|_s^2 \right) . \end{aligned} \tag{4.95}$$

Recall the bound given in (4.51), namely

$$|B_L - 1| \leq \beta(1+\beta) p^{-\frac{1}{2}},$$

from the definition of m_1, see (4.65), we get

$$\mathcal{R}_2 \leq \frac{\sqrt{R}\beta(1+\beta)}{C_\beta} \left\|\sqrt{\frac{\partial_t m_1}{m_1}} Z_1\right\|_s \left\|\sqrt{\frac{\partial_t m_1}{m_1}} Z_2\right\|_s .$$

Hence (4.95) follows from the definition of C_β, see (4.66).

Estimate on $\mathcal{R}_3$. We aim at proving that

$$\begin{aligned} \mathcal{R}_3 =& |k|\sqrt{R} \left|\langle m^{-1} p^{-\frac{3}{4}} T_\varepsilon T_L B_L \Theta, Z_2 \rangle_s\right| \\ \leq& \frac{1}{32}\left(1 - \frac{1}{2\sqrt{R}}\right) \left(\left\|\sqrt{\frac{\partial_t m}{m}} Z_1\right\|_s^2 + \left\|\sqrt{\frac{\partial_t m}{m}} Z_2\right\|_s^2 \right) . \end{aligned} \tag{4.96}$$

First of all, notice that the following inequality holds true

$$
\begin{aligned}
|k|p^{-\frac{1}{2}}(t,k,\eta) &\lesssim (|\partial_t p|p^{-1})(t,k,\eta) + |k|^2 p^{-1}(t,k,\eta) \\
&\lesssim \frac{\partial_t w}{w}(t;k,\eta) + \frac{\partial_t m_1}{m_1}(t;k,\eta).
\end{aligned}
$$

Therefore, we get

$$
\begin{aligned}
\mathcal{R}_3 \lesssim &\left\| \sqrt{\frac{\partial_t w}{w}} m^{-1} p^{-\frac{1}{4}} T_\varepsilon T_L B_L \Theta \right\|_s \left\| \sqrt{\frac{\partial_t w}{w}} Z_2 \right\|_s \\
&+ \left\| \sqrt{\frac{\partial_t m_1}{m_1}} m^{-1} p^{-\frac{1}{4}} T_\varepsilon T_L B_L \Theta \right\|_s \left\| \sqrt{\frac{\partial_t m_1}{m_1}} Z_2 \right\|_s .
\end{aligned}
\tag{4.97}
$$

By applying (4.74), (4.76) followed by (4.78), (4.80), since $|B_L| \leq 1$ we infer

$$
\begin{aligned}
\mathcal{R}_3 \lesssim &\varepsilon \left(\left\| \sqrt{\frac{\partial_t w}{w}} Z_1 \right\|_s \left\| \sqrt{\frac{\partial_t w}{w}} Z_2 \right\|_s + \left\| \sqrt{\frac{\partial_t m_1}{m_1}} Z_1 \right\|_s \left\| \sqrt{\frac{\partial_t w}{w}} Z_2 \right\|_s \right) \\
&+ \varepsilon \left\| \sqrt{\frac{\partial_t m_1}{m_1}} Z_1 \right\|_s \left\| \sqrt{\frac{\partial_t m_1}{m_1}} Z_2 \right\|_s
\end{aligned}
$$

Finally, by the identity in (4.67) we have

$$
\mathcal{R}_3 \lesssim \sqrt{\varepsilon} \left(\left\| \sqrt{\frac{\partial_t m}{m}} Z_1 \right\|_s^2 + \left\| \sqrt{\frac{\partial_t m}{m}} Z_2 \right\|_s^2 \right), \tag{4.98}
$$

hence by choosing ε_0 sufficiently small we get (4.96).

Remark 4.3.9. Notice that in the previous bound (4.98), it would have been sufficient to choose $\delta = C_0(\beta, R, s)\varepsilon$ for a proper constant C_0 big enough.

Estimate on $\mathcal{R}_4$. The goal is again to show that

$$
\begin{aligned}
\mathcal{R}_4 &= |k|\sqrt{R} \left| \langle m^{-1} p^{-\frac{3}{4}} T_L B_\varepsilon T_B B_L \Theta, Z_2 \rangle_s \right| \\
&\leq \frac{1}{32}\left(1 - \frac{1}{2\sqrt{R}}\right) \left(\left\| \sqrt{\frac{\partial_t m}{m}} Z_1 \right\|_s^2 + \left\| \sqrt{\frac{\partial_t m}{m}} Z_1 \right\|_s^2 \right).
\end{aligned}
\tag{4.99}
$$

Applying the reasoning that for $\mathcal{R}_3$ gives (4.97), here we get

$$
\begin{aligned}
\mathcal{R}_4 \lesssim &\left\| \sqrt{\frac{\partial_t w}{w}} m^{-1} p^{-\frac{1}{4}} T_L B_\varepsilon T_L B_L \Theta \right\|_s \left\| \sqrt{\frac{\partial_t w}{w}} Z_2 \right\|_s \\
&+ \left\| \sqrt{\frac{\partial_t m_1}{m_1}} m^{-1} p^{-\frac{1}{4}} T_L B_\varepsilon T_L B_L \Theta \right\|_s \left\| \sqrt{\frac{\partial_t m_1}{m_1}} Z_2 \right\|_s .
\end{aligned}
$$

Therefore, by applying first (4.80), (4.75) and again (4.80), since $|B_L| \leq 1$ and (4.67) holds true, we get

$$\mathcal{R}_4 \lesssim \sqrt{\varepsilon} \left(\left\| \sqrt{\frac{\partial_t m}{m}} Z_1 \right\|_s^2 + \left\| \sqrt{\frac{\partial_t m}{m}} Z_2 \right\|_s^2 \right),$$

The proof of (4.99) is over.

Estimate on $\mathcal{R}_5$. We show that

$$\begin{aligned}\mathcal{R}_5 =& \frac{1}{4\sqrt{R}} \left| \langle (\partial_t p) p^{-\frac{3}{4}} m^{-1} \left((b - \beta g) \Delta_L^{-1} T_L T_B B_L \Theta \right), Z_2 \rangle_s \right| \\ \leq& \frac{1}{32} \left(1 - \frac{1}{2\sqrt{R}} \right) \left(\left\| \sqrt{\frac{\partial_t m_1}{m_1}} Z_1 \right\|_s^2 + \left\| \sqrt{\frac{\partial_t m_1}{m_1}} Z_1 \right\|_s^2 \right). \end{aligned} \tag{4.100}$$

Since $|\partial_t p| \leq 2|k|p^{\frac{1}{2}}$, by using (4.30) we get

$$\begin{aligned}\mathcal{R}_5 \leq& \frac{1}{2\sqrt{R}} \langle m^{-1} p^{-\frac{1}{4}} \left((|\widehat{b}| + \beta |\widehat{g-1}|) * |k| p^{-1} |T_L \widehat{T_B B_L} \Theta| \right), |Z_2| \rangle_s \\ &+ \frac{\beta}{2\sqrt{R}} \langle m^{-1} p^{-\frac{1}{4}} |k| p^{-1} |T_L \widehat{T_B B_L} \Theta|, |Z_2| \rangle_s. \end{aligned}$$

Up to the constant $2\sqrt{R}$, the previous bound has the structure of (4.89)-(4.90) with Z_1 replaced by Z_2. Thus we repeat the computations performed for $\mathcal{R}_1$ to obtain

$$\mathcal{R}_5 \leq \frac{1}{C_\beta} \frac{2\beta + C\varepsilon}{\sqrt{R}} \left\| \sqrt{\frac{\partial_t m_1}{m_1}} Z_1 \right\|_s \left\| \sqrt{\frac{\partial_t m_1}{m_1}} Z_2 \right\|_s,$$

which implies (4.100) thanks to the choice of C_β, see (4.66).

Estimate on $\mathcal{R}_6$. Since $|\partial_t p| \leq |k|p^{\frac{1}{2}}$, analogously to what was done for the term $\mathcal{R}_2$ we get

$$\mathcal{R}_6 = \frac{1}{4} \left| \langle Z_1, \frac{\partial_t p}{p} (B_L - 1) Z_1 \rangle_s \right| \leq \frac{1}{32} \left(1 - \frac{1}{2\sqrt{R}} \right) \left\| \sqrt{\frac{\partial_t m_1}{m_1}} Z_1 \right\|_s^2.$$

Estimate on $\mathcal{R}_7$. We want to prove that

$$\begin{aligned}\mathcal{R}_7 =& \frac{1}{4} \left| \langle \frac{\partial_t p}{p} Z_1, m^{-1} p^{-\frac{1}{4}} T_\varepsilon T_L B_L \Theta \rangle_s \right| \\ \leq& \frac{1}{32} \left(1 - \frac{1}{2\sqrt{R}} \right) \left\| \sqrt{\frac{\partial_t m_1}{m_1}} Z_1 \right\|_s^2. \end{aligned} \tag{4.101}$$

Recall the definition of $\partial_t w / w$, see (4.64) to have that

$$\mathcal{R}_7 \leq \left\| \sqrt{\frac{\partial_t w}{w}} m^{-1} p^{-\frac{1}{4}} T_\varepsilon T_L B_L \Theta \right\|_s \left\| \sqrt{\frac{\partial_t w}{w}} Z_1 \right\|_s.$$

Consequently, by using (4.74), (4.80) with the fact that $|B_L| \leq 1$, we get

$$\mathcal{R}_7 \lesssim \varepsilon \left(\left\| \sqrt{\frac{\partial_t w}{w}} Z_1 \right\|_s^2 + \left\| \sqrt{\frac{\partial_t m_1}{m_1}} Z_1 \right\|_s^2 \right) \lesssim \sqrt{\varepsilon} \left\| \sqrt{\frac{\partial_t m}{m}} Z_1 \right\|_s^2,$$

where in the last line we have used (4.67), hence (4.101) follows upon choosing ε_0 sufficiently small.

Estimate on $\mathcal{R}_8$. The last step is to prove that

$$\begin{aligned} \mathcal{R}_8 &= \frac{1}{4} \left| \langle \frac{\partial_t p}{p} Z_1, m^{-1} p^{-\frac{1}{4}} T_L B_\varepsilon T_B B_L \Theta \rangle_s \right| \\ &\leq \frac{1}{32} \left(1 - \frac{1}{2\sqrt{R}} \right) \left\| \sqrt{\frac{\partial_t m}{m}} Z_1 \right\|_s^2 . \end{aligned} \tag{4.102}$$

Using (4.80) and (4.75) we obtain that

$$\mathcal{R}_8 \lesssim \varepsilon \left(\left\| \sqrt{\frac{\partial_t w}{w}} m^{-1} p^{-\frac{1}{4}} T_B B_L \Theta \right\|_s + \left\| \sqrt{\frac{\partial_t m_1}{m_1}} m^{-1} p^{-\frac{1}{4}} T_B B_L \Theta \right\|_s \right) \left\| \sqrt{\frac{\partial_t w}{w}} Z_1 \right\|_s .$$

Then, thanks to (4.81), (4.79) and $|B_L| \leq 1$, we get we get

$$\mathcal{R}_8 \lesssim \varepsilon \left(\left\| \sqrt{\frac{\partial_t w}{w}} Z_1 \right\|_s^2 + \left\| \sqrt{\frac{\partial_t m_1}{m_1}} Z_1 \right\|_s^2 \right) \lesssim \sqrt{\varepsilon} \left\| \sqrt{\frac{\partial_t m}{m}} Z_1 \right\|_s^2,$$

where in the last line we have used (4.67). This proves (4.102).

Proof of Theorem 4.3.1 and Corollary 4.3.2

Having at hand Proposition (4.3.5), we can now prove Theorem 4.3.1 and Corollary 4.3.2.

Proof of Theorem 4.3.1. By means of Proposition 4.3.5, we know that

$$\|Z_1(t)\|_s^2 + \|Z_2(t)\|_s^2 \lesssim \left\|Z_1^{in}\right\|_s^2 + \left\|Z_2^{in}\right\|_s^2 .$$

Then, from the explicit definition of w, see (4.68), we deduce

$$w \lesssim \langle t \rangle^{\frac{1}{2}} \langle k, \eta \rangle .$$

Hence, recalling the definitions of Z_1, Z_2 given in (4.58) and since $m = w^{-\delta} m_1^{-1}$, being m_1 uniformly bounded, we have that

$$\begin{aligned} \left\| p^{-\frac{1}{4}} \Theta_k(t) \right\|_s^2 + \left\| p^{\frac{1}{4}} Q_k(t) \right\|_s^2 &= \left\| m_1 w^\delta Z_1(t) \right\|_s^2 + \left\| m_1 w^\delta Z_2(t) \right\|_s^2 \\ &\lesssim \langle t \rangle^\delta \left(\|Z_1(t)\|_{s+\delta}^2 + \|Z_2(t)\|_{s+\delta}^2 \right) \\ &\lesssim \langle t \rangle^\delta \left(\left\|Z_1^{in}\right\|_{s+\delta}^2 + \left\|Z_2^{in}\right\|_{s+\delta}^2 \right) \end{aligned}$$

where the constant hidden in the last inequalities depends on the uniform bound on m_1. Exploiting the definition of Z_1^{in}, Z_2^{in}, by the rough bound $\delta \leq \frac{1}{2}$ the proof of Theorem 4.3.1 is over. □

Proof of Corollary 4.3.2. It is enough to prove that

$$\left\| p^{-\frac{1}{4}}\Omega(t) \right\|_s \lesssim \left\| p^{-\frac{1}{4}}\Theta(t) \right\|_s, \tag{4.103}$$

$$\left\| \Theta^{in} \right\|_s \lesssim \left\| \Omega^{in} \right\|_s. \tag{4.104}$$

Since $\Omega = B_t\Theta = T_B B_L \Theta$, thanks to (4.41) and $|B_L| \leq 1$ we have

$$\left\| p^{-\frac{1}{4}}\Omega(t) \right\|_s \leq \left\| p^{-\frac{1}{4}}\Theta(t) \right\|_s + \left\| p^{-\frac{1}{4}} B_\varepsilon T_B B_L \Omega(t) \right\|_s. \tag{4.105}$$

To treat the second term, we argue as in (4.75) and (4.79) and commute $p^{-\frac{1}{4}}$ with $B_\varepsilon T_B$. We therefore obtain

$$\left\| p^{-\frac{1}{4}} B_\varepsilon T_B B_L \Omega(t) \right\|_s \lesssim \varepsilon \left\| p^{-\frac{1}{4}}\Omega(t) \right\|_s.$$

As ε_0 is small enough, we can absorb the last term in the left-hand side of (4.105), hence obtaining (4.103). The proof of (4.104) simply comes from the fact that $\Theta^{in} = (I - \beta\partial_y \Delta^{-1})\Omega^{in}$. $\square$

Bibliography

[1] G. ALBERTI, G. CRIPPA, AND A. L. MAZZUCATO, *Exponential self-similar mixing by incompressible flows*, J. Amer. Math. Soc., 32 (2019), pp. 445–490.

[2] P. ANTONELLI, M. DOLCE, AND P. MARCATI, *Linear stability analysis for 2d shear flows near Couette in the isentropic compressible euler equations*, arXiv preprint arXiv:2003.01694, (2020).

[3] V. I. ARNOLD, *Conditions for non-linear stability of stationary plane curvilinear flows of an ideal fluid*, in Vladimir I. Arnold-Collected Works, Springer, 1965, pp. 19–23.

[4] K. BAJER, A. P. BASSOM, AND A. D. GILBERT, *Accelerated diffusion in the centre of a vortex*, Journal of Fluid Mechanics, 437 (2001), pp. 395–411.

[5] N. A. BAKAS, *Mechanisms underlying transient growth of planar perturbations in unbounded compressible shear flow*, Journal of Fluid Mechanics, 639 (2009), pp. 479–507.

[6] C. BARDOS, Y. GUO, AND W. STRAUSS, *Stable and unstable ideal plane flows*, Chinese Annals of Mathematics, 23 (2002), pp. 149–164.

[7] A. P. BASSOM AND A. D. GILBERT, *The spiral wind-up of vorticity in an inviscid planar vortex*, J. Fluid Mech., 371 (1998), pp. 109–140.

[8] G. BATCHELOR, *An introduction to fluid dynamics*, Cambridge university press, 2000.

[9] M. BECK AND C. E. WAYNE, *Metastability and rapid convergence to quasi-stationary bar states for the two-dimensional Navier-Stokes equations*, Proc. Roy. Soc. Edinburgh Sect. A, 143 (2013), pp. 905–927.

[10] J. BEDROSSIAN AND M. COTI ZELATI, *Enhanced dissipation, hypoellipticity, and anomalous small noise inviscid limits in shear flows*, Arch. Ration. Mech. Anal., 224 (2017), pp. 1161–1204.

[11] J. BEDROSSIAN, M. COTI ZELATI, AND N. GLATT-HOLTZ, *Invariant Measures for Passive Scalars in the Small Noise Inviscid Limit*, Comm. Math. Phys., 348 (2016), pp. 101–127.

[12] J. Bedrossian, M. Coti Zelati, and V. Vicol, *Vortex axisymmetrization, inviscid damping, and vorticity depletion in the linearized 2d euler equations*, Annals of PDE, 5 (2019), p. 4.

[13] J. Bedrossian, P. Germain, and N. Masmoudi, *Dynamics near the subcritical transition of the 3D Couette flow I: Below threshold case*, arXiv preprint arXiv:1506.03720, (2015).

[14] ——, *Dynamics near the subcritical transition of the 3D Couette flow II: Above threshold case*, arXiv preprint arXiv:1506.03721, (2015).

[15] ——, *On the stability threshold for the 3D Couette flow in Sobolev regularity*, Ann. of Math. (2), 185 (2017), pp. 541–608.

[16] ——, *Stability of the Couette flow at high reynolds numbers in two dimensions and three dimensions*, Bulletin of the American Mathematical Society, 56 (2019), pp. 373–414.

[17] J. Bedrossian and N. Masmoudi, *Inviscid damping and the asymptotic stability of planar shear flows in the 2D Euler equations*, Publ. Math. Inst. Hautes Études Sci., 122 (2015), pp. 195–300.

[18] J. Bedrossian, N. Masmoudi, and C. Mouhot, *Landau damping: paraproducts and gevrey regularity*, Annals of PDE, 2 (2016), p. 4.

[19] J. Bedrossian, N. Masmoudi, and V. Vicol, *Enhanced dissipation and inviscid damping in the inviscid limit of the Navier–Stokes equations near the two dimensional Couette flow*, Archive for Rational Mechanics and Analysis, 219 (2016), pp. 1087–1159.

[20] J. Bedrossian, V. Vicol, and F. Wang, *The Sobolev stability threshold for 2D shear flows near Couette*, J. Nonlinear Sci., 28 (2018), pp. 2051–2075.

[21] A. J. Bernoff and J. F. Lingevitch, *Rapid relaxation of an axisymmetric vortex*, Physics of Fluids, 6 (1994), pp. 3717–3723.

[22] R. Bianchini, M. Coti Zelati, and M. Dolce, *Linear inviscid damping for shear flows near Couette in the 2d stably stratified regime*, arXiv preprint arXiv:2005.09058, (2020).

[23] W. Blumen, *Shear layer instability of an inviscid compressible fluid*, Journal of Fluid Mechanics, 40 (1970), pp. 769–781.

[24] W. Blumen, P. Drazin, and D. Billings, *Shear layer instability of an inviscid compressible fluid. part 2*, Journal of Fluid Mechanics, 71 (1975), pp. 305–316.

[25] G. Bodo, G. Chagelishvili, G. Murante, A. Tevzadze, P. Rossi, and A. Ferrari, *Spiral density wave generation by vortices in keplerian flows*, Astronomy & Astrophysics, 437 (2005), pp. 9–22.

[26] H. M. Boffin, *Spiral waves in accretion discs—theory*, in Astrotomography, Springer, 2001, pp. 69–87.

[27] F. Bouchet and H. Morita, *Large time behavior and asymptotic stability of the 2d euler and linearized euler equations*, Physica D: Nonlinear Phenomena, 239 (2010), pp. 948–966.

[28] J. Boussinesq, *Théorie analytique de la chaleur mise en harmonic avec la thermodynamique et avec la théorie mécanique de la lumière: Tome I-[II]...*, vol. 2, Gauthier-Villars, 1903.

[29] A. Bressan, *A lemma and a conjecture on the cost of rearrangements*, Rendiconti del Seminario Matematico della Universita di Padova, 110 (2003), pp. 97–102.

[30] E. Bruè and Q.-H. Nguyen, *Advection diffusion equations with sobolev velocity field*, arXiv preprint arXiv:2003.08198, (2020).

[31] G. Buresti, *A note on stokes' hypothesis*, Acta Mechanica, 226 (2015), pp. 3555–3559.

[32] E. Caglioti and C. Maffei, *Time asymptotics for solutions of vlasov–poisson equation in a circle*, Journal of statistical physics, 92 (1998), pp. 301–323.

[33] R. Camassa and C. Viotti, *Transient dynamics by continuous-spectrum perturbations in stratified shear flows*, Journal of Fluid Mechanics, 717 (2013).

[34] D. R. Carlson, S. E. Widnall, and M. F. Peeters, *A flow-visualization study of transition in plane poiseuille flow*, Journal of Fluid Mechanics, 121 (1982), pp. 487–505.

[35] K. Case, *Stability of an idealized atmosphere. i. discussion of results*, The Physics of Fluids, 3 (1960), pp. 149–154.

[36] ——, *Stability of inviscid plane Couette flow*, The Physics of Fluids, 3 (1960), pp. 143–148.

[37] G. Chagelishvili, A. Rogava, and I. Segal, *Hydrodynamic stability of compressible plane Couette flow*, Physical Review E, 50 (1994), p. R4283.

[38] G. Chagelishvili, A. Tevzadze, G. Bodo, and S. Moiseev, *Linear mechanism of wave emergence from vortices in smooth shear flows*, Physical review letters, 79 (1997), p. 3178.

[39] S. Chandrasekhar, *Hydrodynamic and hydromagnetic stability*, Oxford Clarendon Press and Courier Corporation, 1954, 2013.

[40] Q. Chen, T. Li, D. Wei, and Z. Zhang, *Transition threshold for the 2-d Couette flow in a finite channel*, Archive for rational mechanics and analysis, 238 (2020), pp. 125–183.

[41] ——, *Transition threshold for the 3d Couette flow in a finite channel*, arXiv preprint arXiv:2006.00721, (2020).

[42] Q. CHEN, D. WEI, AND Z. ZHANG, *Linear stability of pipe poiseuille flow at high reynolds number regime*, arXiv preprint arXiv:1910.14245, (2019).

[43] P. CONSTANTIN, A. KISELEV, L. RYZHIK, AND A. ZLATOS, *Diffusion and mixing in fluid flow*, Ann. of Math. (2), 168 (2008), pp. 643–674.

[44] M. COTI ZELATI, M. DELGADINO, AND T. ELGINDI, *On the Relation between Enhanced Dissipation Timescales and Mixing Rates*, Comm. Pure Appl. Math., 73 (2020), pp. 1205–1244.

[45] M. COTI ZELATI AND M. DOLCE, *Separation of time-scales in drift-diffusion equations on* $\mathbb{R}^2$, Journal de Mathématiques Pures et Appliquées, in press, (2020).

[46] M. COTI ZELATI AND T. D. DRIVAS, *A stochastic approach to enhanced diffusion*, arXiv preprint arXiv:1911.09995, (2019).

[47] M. COTI ZELATI, T. M. ELGINDI, AND K. WIDMAYER, *Enhanced dissipation in the navier–stokes equations near the poiseuille flow*, Communications in Mathematical Physics, (2020), pp. 1–24.

[48] ——, *Stationary structures near the Kolmogorov and Poiseuille flows in the 2d Euler equations*, arXiv preprint arXiv:2007.11547, (2020).

[49] M. COTI ZELATI AND C. ZILLINGER, *On degenerate circular and shear flows: the point vortex and power law circular flows*, Comm. Partial Differential Equations, 44 (2019), pp. 110–155.

[50] M. COUETTE, *Oscillations tournantes d'un solide de révolution en contact avec un fluide visqueux*, Compt. Rend. Acad. Sci. Paris, 105 (1887), pp. 1064–1067.

[51] G. CRIPPA, R. LUCÀ, AND C. SCHULZE, *Polynomial mixing under a certain stationary Euler flow*, Phys. D, 394 (2019), pp. 44–55.

[52] P. DANCKWERTS, *The definition and measurement of some characteristics of mixtures*, Applied Scientific Research, Section A, 3 (1952), pp. 279–296.

[53] W. DENG, *Resolvent estimates for a two-dimensional non-self-adjoint operator*, Commun. Pure Appl. Anal., 12 (2013), pp. 547–596.

[54] W. DENG, J. WU, AND P. ZHANG, *Stability of Couette flow for 2d boussinesq system with vertical dissipation*, arXiv preprint arXiv:2004.09292, (2020).

[55] Y. DENG AND N. MASMOUDI, *Long time instability of the Couette flow in low gevrey spaces*, arXiv preprint arXiv:1803.01246, (2018).

[56] Y. DENG AND C. ZILLINGER, *Echo chains as a linear mechanism: Norm inflation, modified exponents and asymptotics*, arXiv preprint arXiv:1910.12914, (2019).

[57] ——, *On the smallness condition in linear inviscid damping: Monotonicity and resonance chains*, arXiv preprint arXiv:1911.02066, (2019).

[58] L. DESVILLETTES AND C. VILLANI, *On the trend to global equilibrium in spatially inhomogeneous entropy-dissipating systems: the linear Fokker-Planck equation*, Comm. Pure Appl. Math., 54 (2001), pp. 1–42.

[59] J. DOLBEAULT, C. MOUHOT, AND C. SCHMEISER, *Hypocoercivity for linear kinetic equations conserving mass*, Trans. Amer. Math. Soc., 367 (2015), pp. 3807–3828.

[60] R. J. DONNELLY, *Taylor–Couette flow: the early days*, Phys. Today, 44 (1991), pp. 32–39.

[61] P. DRAZIN AND A. DAVEY, *Shear layer instability of an inviscid compressible fluid. part 3*, Journal of Fluid Mechanics, 82 (1977), pp. 255–260.

[62] P. DRAZIN AND L. HOWARD, *Hydrodynamic stability of parallel flow of inviscid fluid*, in Advances in applied mechanics, vol. 9, Elsevier, 1966, pp. 1–89.

[63] P. G. DRAZIN AND W. H. REID, *Hydrodynamic stability*, Cambridge university press, 2004.

[64] B. DUBRULLE AND S. NAZARENKO, *On scaling laws for the transition to turbulence in uniform-shear flows*, Euro. Phys. Lett., 27 (1994), p. 129.

[65] P. W. DUCK, G. ERLEBACHER, AND M. Y. HUSSAINI, *On the linear stability of compressible plane Couette flow*, Journal of Fluid Mechanics, 258 (1994), pp. 131–165.

[66] C. ECKART, *An analysis of the stirring and mixing processes*, in SCIENCE, vol. 108, AMER ASSOC ADVANCEMENT SCIENCE 1200 NEW YORK AVE, NW, WASHINGTON, DC 20005, 1948, pp. 597–598.

[67] C. ECKART, *Extension of howard's circle theorem to adiabatic jets*, The Physics of Fluids, 6 (1963), pp. 1042–1047.

[68] T. M. ELGINDI AND A. ZLATOŠ, *Universal mixers in all dimensions*, Advances in Mathematics, 356 (2019), p. 106807.

[69] L. EULER, *Principes généraux du mouvement des fluides*, Mémoires de l'Académie des Sciences de Berlin, (1757), pp. 274–315.

[70] B. FARRELL AND P. IOANNOU, *Transient and asymptotic growth of two-dimensional perturbations in viscous compressible shear flow*, Physics of Fluids, 12 (2000), pp. 3021–3028.

[71] B. F. FARRELL AND P. J. IOANNOU, *Transient development of perturbations in stratified shear flow*, Journal of the atmospheric sciences, 50 (1993), pp. 2201–2214.

[72] R. FJØRTOFT, *Application of integral theorems in deriving criteria of stability for laminar flows and for the baroclinic circular vortex*, Grøndahl & søns boktr., I kommisjon hos Cammermeyers boghandel, 1950.

[73] S. FRIEDLANDER, W. STRAUSS, AND M. VISHIK, *Nonlinear instability in an ideal fluid*, in Annales de l'Institut Henri Poincare (C) Non Linear Analysis, vol. 14, Elsevier, 1997, pp. 187–209.

[74] U. FRISCH, *Translation of Leonhard Euler's: General principles of the motion of fluids*, arXiv preprint arXiv:0802.2383, (2008).

[75] I. GALLAGHER, T. GALLAY, AND F. NIER, *Spectral asymptotics for large skew-symmetric perturbations of the harmonic oscillator*, Int. Math. Res. Not. IMRN, (2009), pp. 2147–2199.

[76] T. GALLAY, *Enhanced dissipation and axisymmetrization of two-dimensional viscous vortices*, Arch. Ration. Mech. Anal., 230 (2018), pp. 939–975.

[77] ——, *Estimations pseudo-spectrales et stabilité des tourbillons plans*, Séminaire BOURBAKI, (2019), p. 72e.

[78] ——, *Stability of vortices in ideal fluids: the legacy of kelvin and rayleigh*, arXiv preprint arXiv:1901.02815, (2019).

[79] M.-H. GIGA, Y. GIGA, AND J. SAAL, *Nonlinear partial differential equations*, vol. 79 of Progress in Nonlinear Differential Equations and their Applications, Birkhäuser Boston, Inc., Boston, MA, 2010.

[80] W. GLATZEL, *The linear stability of viscous compressible plane Couette flow*, Journal of Fluid Mechanics, 202 (1989), pp. 515–541.

[81] P. GOLDREICH AND D. LYNDEN-BELL, *I. Gravitational stability of uniformly rotating disks*, Monthly Notices of the Royal Astronomical Society, 130 (1965), pp. 97–124.

[82] ——, *II. Spiral arms as sheared gravitational instabilities*, Monthly Notices of the Royal Astronomical Society, 130 (1965), pp. 125–158.

[83] S. GOLDSTEIN, *On the stability of superposed streams of fluids of different densities*, Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character, 132 (1931), pp. 524–548.

[84] R. E. GRAVES AND B. M. ARGROW, *Bulk viscosity: past to present*, Journal of Thermophysics and Heat Transfer, 13 (1999), pp. 337–342.

[85] D. D. GRAY AND A. GIORGINI, *The validity of the boussinesq approximation for liquids and gases*, International Journal of Heat and Mass Transfer, 19 (1976), pp. 545–551.

[86] E. GRENIER, T. T. NGUYEN, AND I. RODNIANSKI, *Landau damping for analytic and gevrey data*, arXiv preprint arXiv:2004.05979, (2020).

[87] E. GRENIER, T. T. NGUYEN, F. ROUSSET, AND A. SOFFER, *Linear inviscid damping and enhanced viscous dissipation of shear flows by using the conjugate operator method*, J. Funct. Anal., 278 (2020), pp. 108339, 27.

[88] S. GROSSMANN, *The onset of shear flow turbulence*, Reviews of modern physics, 72 (2000), p. 603.

[89] Y. GUO AND Y. WANG, *Decay of dissipative equations and negative sobolev spaces*, Communications in Partial Differential Equations, 37 (2012), pp. 2165–2208.

[90] A. HANIFI, P. J. SCHMID, AND D. S. HENNINGSON, *Transient growth in compressible boundary layer flow*, Physics of Fluids, 8 (1996), pp. 826–837.

[91] R. HARTMAN, *Wave propagation in a stratified shear flow*, Journal of Fluid Mechanics, 71 (1975), pp. 89–104.

[92] J.-N. HAU, G. CHAGELISHVILI, G. KHUJADZE, M. OBERLACK, AND A. TEVZADZE, *A comparative numerical analysis of linear and nonlinear aerodynamic sound generation by vortex disturbances in homentropic constant shear flows*, Physics of Fluids, 27 (2015), p. 126101.

[93] F. HÉRAU, *Short and long time behavior of the Fokker-Planck equation in a confining potential and applications*, J. Funct. Anal., 244 (2007), pp. 95–118.

[94] F. HÉRAU AND F. NIER, *Isotropic hypoellipticity and trend to equilibrium for the Fokker-Planck equation with a high-degree potential*, Arch. Ration. Mech. Anal., 171 (2004), pp. 151–218.

[95] B. HOF, A. JUEL, AND T. MULLIN, *Scaling of the turbulence transition threshold in a pipe*, Physical review letters, 91 (2003), p. 244502.

[96] D. D. HOLM, J. E. MARSDEN, T. RATIU, AND A. WEINSTEIN, *Nonlinear stability of fluid and plasma equilibria*, Physics reports, 123 (1985), pp. 1–116.

[97] L. N. HOWARD, *Note on a paper of John W. Miles*, Journal of Fluid Mechanics, 10 (1961), pp. 509–512.

[98] S. HU AND X. ZHONG, *Linear stability of viscous supersonic plane Couette flow*, Physics of Fluids, 10 (1998), pp. 709–729.

[99] S. IBRAHIM, Y. MAEKAWA, AND N. MASMOUDI, *On Pseudospectral Bound for Non-selfadjoint Operators and Its Application to Stability of Kolmogorov Flows*, Ann. PDE, 5 (2019), p. Paper No. 14.

[100] A. D. IONESCU AND H. JIA, *Axi-symmetrization near point vortex solutions for the 2d euler equation*, arXiv preprint arXiv:1904.09170, (2019).

[101] ——, *Inviscid damping near the Couette flow in a channel*, Communications in Mathematical Physics, (2019), pp. 1–82.

[102] ——, *Nonlinear inviscid damping near monotonic shear flows*, arXiv preprint arXiv:2001.03087, (2020).

[103] G. IYER, A. KISELEV, AND X. XU, *Lower bounds on the mix norm of passive scalars advected by incompressible enstrophy-constrained flows*, Nonlinearity, 27 (2014), p. 973.

[104] H. JIA, *Linear inviscid damping in gevrey spaces*, Archive for Rational Mechanics and Analysis, 235 (2020), pp. 1327–1355.

[105] Y. KAGEI, *Asymptotic behavior of solutions of the compressible navier–stokes equation around the plane Couette flow*, Journal of Mathematical Fluid Mechanics, 13 (2011), pp. 1–31.

[106] ——, *Asymptotic behavior of solutions to the compressible navier–stokes equation around a parallel flow*, Archive for Rational Mechanics and Analysis, 205 (2012), pp. 585–650.

[107] L. KELVIN, *Stability of fluid motion: rectilinear motion of viscous fluid between two parallel plates*, Phil. Mag, 24 (1887), pp. 188–196.

[108] G. KREISS, A. LUNDBLADH, AND D. S. HENNINGSON, *Bounds for threshold amplitudes in subcritical shear flows*, Journal of Fluid Mechanics, 270 (1994), pp. 175–198.

[109] L. LANDAU AND E. LIFSHITZ, *Theoretical Physics, vol. 6, Fluid Mechanics*, Pergamon, London, 1987.

[110] M. LATINI AND A. BERNOFF, *Transient anomalous diffusion in Poiseuille flow*, Journal of Fluid Mechanics, 441 (2001), pp. 399–411.

[111] L. LEES AND C. C. LIN, *Investigation of the stability of the laminar boundary layer in a compressible fluid*, (1946).

[112] H. J. LEUTHEUSSER AND V. H. CHU, *Experiments on plane Couette flow*, Journal of the Hydraulics Division, 97 (1971), pp. 1269–1284.

[113] T. LI, D. WEI, AND Z. ZHANG, *Pseudospectral and spectral bounds for the Oseen vortices operator*, ArXiv e-prints, (2017).

[114] T. LI, D. WEI, AND Z. ZHANG, *Pseudospectral Bound and Transition Threshold for the 3D Kolmogorov Flow*, Comm. Pure Appl. Math., 73 (2020), pp. 465–557.

[115] Z. LIN, *Nonlinear instability of ideal plane flows*, International Mathematics Research Notices, 2004 (2004), pp. 2147–2178.

[116] Z. LIN AND C. ZENG, *Inviscid dynamical structures near Couette flow*, Archive for rational mechanics and analysis, 200 (2011), pp. 1075–1097.

[117] K. LISS, *On the sobolev stability threshold of 3d Couette flow in a uniform magnetic field*, Communications in Mathematical Physics, (2020), pp. 1–50.

[118] E. LUNASIN, Z. LIN, A. NOVIKOV, A. MAZZUCATO, AND C. R. DOERING, *Optimal mixing and optimal stirring for fixed energy, fixed power, or fixed palenstrophy flows*, J. Math. Phys., 53 (2012), pp. 115611, 15.

[119] T. LUNDGREN, *Strained spiral vortex model for turbulent fine structure*, The Physics of Fluids, 25 (1982), pp. 2193–2203.

[120] A. M. LYAPUNOV, *The general problem of the stability of motion*, International journal of control, 55 (1992), pp. 531–534.

[121] M. MALIK, J. DEY, AND M. ALAM, *Linear stability, transient energy growth, and the role of viscosity stratification in compressible plane Couette flow*, Physical Review E, 77 (2008), p. 036322.

[122] A. MALLOCK, *Iv. determination of the viscosity of water*, Proceedings of the Royal Society of London, 45 (1889), pp. 126–132.

[123] N. MASMOUDI AND W. ZHAO, *Stability threshold of the 2d Couette flow in sobolev spaces*, arXiv preprint arXiv:1908.11042, (2019).

[124] ——, *Enhanced dissipation for the 2d Couette flow in critical space*, Communications in Partial Differential Equations, (2020), pp. 1–20.

[125] ——, *Nonlinear inviscid damping for a class of monotone shear flows in finite channel*, arXiv preprint arXiv:2001.08564, (2020).

[126] A. MATSUMURA AND T. NISHIDA, *Initial boundary value problems for the equations of motion of compressible viscous and heat-conductive fluids*, Communications in Mathematical Physics, 89 (1983), pp. 445–464.

[127] C. J. MILES AND C. R. DOERING, *Diffusion-limited mixing by incompressible flows*, Nonlinearity, 31 (2018), pp. 2346–2350.

[128] J. W. MILES, *On the stability of heterogeneous shear flows*, Journal of Fluid Mechanics, 10 (1961), pp. 496–508.

[129] C. MOUHOT AND C. VILLANI, *On Landau damping*, Acta Math., 207 (2011), pp. 29–201.

[130] M. NISHIOKA, Y. ICHIKAWA, ET AL., *An experimental investigation of the stability of plane poiseuille flow*, Journal of Fluid Mechanics, 72 (1975), pp. 731–751.

[131] W. M. ORR, *The stability or instability of the steady motions of a perfect liquid and of a viscous liquid. part ii: A viscous liquid*, in Proceedings of the Royal Irish Academy. Section A: Mathematical and Physical Sciences, vol. 27, JSTOR, 1907, pp. 69–138.

[132] L. RAYLEIGH, *On the stability, or instability, of certain fluid motions*, Proc. London Math. Soc., 9 (1880), pp. 57–70.

[133] O. REYNOLDS, *Xxix. an experimental investigation of the circumstances which determine whether the motion of water shall be direct or sinuous, and of the law of resistance in parallel channels*, Philosophical Transactions of the Royal society of London, (1883), pp. 935–982.

[134] P. RHINES AND W. YOUNG, *How rapidly is a passive scalar mixed within closed streamlines?*, Journal of Fluid Mechanics, 133 (1983), pp. 133–145.

[135] V. A. ROMANOV, *Stability of plane-parallel Couette flow*, Functional analysis and its applications, 7 (1973), pp. 137–146.

[136] G. SAVONIJE, J. PAPALOIZOU, AND D. LIN, *On tidally induced shocks in accretion discs in close binary systems*, Monthly Notices of the Royal Astronomical Society, 268 (1994), pp. 13–28.

[137] P. J. SCHMID AND D. HENNINGSON, *Stability and transition in shear flows*, vol. 142 of Applied mathematical sciences, Springer, 2001.

[138] C. SEIS, *Maximal mixing by incompressible fluid flows*, Nonlinearity, 26 (2013), pp. 3279–3289.

[139] E. A. SPIEGEL AND G. VERONIS, *On the boussinesq approximation for a compressible fluid.*, The Astrophysical Journal, 131 (1960), p. 442.

[140] G. G. STOKES, *On the theories of the internal friction of fluids in motion, and of the equilibrium and motion of elastic solids*, Transactions of the Cambridge Philosophical Society, 8 (1880).

[141] M. SUBBIAH AND R. JAIN, *Stability of compressible shear flows*, Journal of mathematical analysis and applications, 151 (1990), pp. 34–41.

[142] S. P. SUTERA AND R. SKALAK, *The history of poiseuille's law*, Annual review of fluid mechanics, 25 (1993), pp. 1–20.

[143] J. L. SYNGE, *The stability of heterogeneous liquids*, Trans. R. Soc. Canada, 27 (1933), p. 1.

[144] G. I. TAYLOR, *Viii. stability of a viscous liquid contained between two rotating cylinders*, Philosophical Transactions of the Royal Society of London. Series A, Containing Papers of a Mathematical or Physical Character, 223 (1923), pp. 289–343.

[145] ——, *Effect of variation in density on the stability of superposed streams of fluid*, Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character, 132 (1931), pp. 499–523.

[146] J.-L. THIFFEAULT, *Using multiscale norms to quantify mixing and transport*, Nonlinearity, 25 (2012), p. R1.

[147] N. TILLMARK AND P. H. ALFREDSSON, *Experiments on transition in plane Couette flow*, Journal of Fluid Mechanics, 235 (1992), pp. 89–102.

[148] L. N. TREFETHEN, A. E. TREFETHEN, S. C. REDDY, AND T. A. DRISCOLL, *Hydrodynamic stability without eigenvalues*, Science, 261 (1993), pp. 578–584.

[149] C. VILLANI, *Hypocoercivity*, Mem. Amer. Math. Soc., 202 (2009), pp. iv+141.

[150] D. WEI, *Diffusion and mixing in fluid flow via the resolvent estimate*, arXiv e-prints, (2018).

[151] D. WEI AND Z. ZHANG, *Transition threshold for the 3D Couette flow in Sobolev space*, ArXiv e-prints, (2018).

[152] ——, *Enhanced dissipation for the Kolmogorov flow via the hypocoercivity method*, Sci. China Math., 62 (2019), pp. 1219–1232.

[153] D. WEI, Z. ZHANG, AND W. ZHAO, *Linear inviscid damping for a class of monotone shear flow in sobolev spaces*, Communications on Pure and Applied Mathematics, 71 (2018), pp. 617–687.

[154] ——, *Linear inviscid damping and vorticity depletion for shear flows*, Annals of PDE, 5 (2019), p. 3.

[155] ——, *Linear inviscid damping and enhanced dissipation for the kolmogorov flow*, Advances in Mathematics, 362 (2020), p. 106963.

[156] A. M. YAGLOM, *Hydrodynamic instability and transition to turbulence*, vol. 100, Springer Science & Business Media, 2012.

[157] J. YANG AND Z. LIN, *Linear inviscid damping for Couette flow in stratified fluid*, Journal of Mathematical Fluid Mechanics, 20 (2018), pp. 445–472.

[158] Y. YAO AND A. ZLATOS, *Mixing and un-mixing by incompressible flows*, J. Eur. Math. Soc. (JEMS), 19 (2017), pp. 1911–1948.

[159] C.-S. YIH, *Stratified flows*, Elsevier, 2012.

[160] M. C. ZELATI, *Stable mixing estimates in the infinite péclet number limit*, Journal of Functional Analysis, (2020), p. 108562.

[161] R. K. ZEYTOUNIAN, *Joseph boussinesq and his approximation: a contemporary view*, Comptes Rendus Mecanique, 331 (2003), pp. 575–586.

[162] C. ZILLINGER, *Linear inviscid damping for monotone shear flows in a finite periodic channel, boundary effects, blow-up and critical sobolev regularity*, Archive for Rational Mechanics and Analysis, 221 (2016), pp. 1449–1509.

[163] ——, *Linear inviscid damping for monotone shear flows*, Transactions of the American Mathematical Society, 369 (2017), pp. 8799–8855.

[164] ——, *Linear inviscid damping in sobolev and gevrey spaces*, arXiv preprint arXiv:1911.00880, (2019).

[165] ——, *On geometric and analytic mixing scales: comparability and convergence rates for transport problems*, Pure and Applied Analysis, 1 (2019), pp. 543–570.

[166] ——, *On enhanced dissipation for the boussinesq equations*, arXiv preprint arXiv:2004.08125, (2020).