Sol.

WTS: $e^{-\frac{1}{2}\left(\left(\sum_{i=1}^{n} e(x_i - \theta)^2\right) + \tau(\theta - \theta_0)^2\right)} \propto e^{-\frac{1}{2}(\tau + ne)\left(\theta - \frac{1}{\tau + ne}\left(\tau\theta_0 + e\sum_{i=1}^{n} x_i\right)\right)^2}$

$\Rightarrow -\frac{1}{2}\left(\left(\sum_{i=1}^{n} e(x_i - \theta)^2\right) + \tau(\theta - \theta_0)^2\right) = -\frac{1}{2}\left(\left(\sum_{i=1}^{n} x_i^2 e - 2x_i \theta e + \theta^2 e\right) + \theta^2 \tau - 2\theta\theta_0\tau + \theta_0^2 \tau\right)$

$= -\frac{1}{2}\left(e\sum_{i=1}^{n} x_i^2 - 2\theta e\sum_{i=1}^{n} x_i + n\theta^2 e + \theta^2 \tau - 2\theta\theta_0\tau + \theta_0^2 \tau\right)$

$= -\frac{1}{2}\left(n\theta^2 e + \theta^2 \tau - 2\theta e\sum_{i=1}^{n} x_i - 2\theta\theta_0\tau + e\sum_{i=1}^{n} x_i^2 + \theta_0^2 \tau\right)$

$= -\frac{1}{2}\left(\theta^2(ne + \tau) - 2\theta\left(e\sum_{i=1}^{n} x_i - \theta_0\tau\right) + e\sum_{i=1}^{n} x_i^2 + \theta_0^2 \tau\right)$

$= -\frac{1}{2}\left(a\theta^2 - 2b\theta + c\right),$ $\qquad$ $a = (ne + \tau), \quad b = \left(e\sum_{i=1}^{n} x_i - \theta_0\tau\right), \quad c = e\sum_{i=1}^{n} x_i^2 + \theta_0^2 \tau$

$= -\frac{1}{2}\left(a\left(\theta + \frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4a}\right)\right)$ $\qquad$ $\frac{b}{2a} = \frac{2b}{2a} = -\frac{e\sum_{i=1}^{n} x_i - \theta_0\tau}{ne + \tau}, \quad -\frac{b^2}{4a} = -\frac{\left(e\sum_{i=1}^{n} x_i - \theta_0\tau\right)^2}{ne + \tau}$

$= -\frac{1}{2}\left((ne + \tau)\left(\theta - \frac{e\sum_{i=1}^{n} x_i - \theta_0\tau}{ne + \tau}\right)^2 + e\sum_{i=1}^{n} x_i^2 + \theta_0^2 \tau - \frac{\left(e\sum_{i=1}^{n} x_i - \theta_0\tau\right)^2}{ne + \tau}\right)$

$\Rightarrow e^{-\frac{1}{2}\left((ne + \tau)\left(\theta - \frac{e\sum_{i=1}^{n} x_i - \theta_0\tau}{ne + \tau}\right)^2 + e\sum_{i=1}^{n} x_i^2 + \theta_0^2 \tau - \frac{\left(e\sum_{i=1}^{n} x_i - \theta_0\tau\right)^2}{ne + \tau}\right)} = e^{-\frac{1}{2}(ne + \tau)\left(\theta - \frac{e\sum_{i=1}^{n} x_i - \theta_0\tau}{ne + \tau}\right)^2} \cdot e^{-\frac{1}{2}e\sum_{i=1}^{n} x_i^2} \cdot e^{\frac{1}{2}\frac{\left(e\sum_{i=1}^{n} x_i - \theta_0\tau\right)^2}{ne + \tau}}$

$\propto e^{-\frac{1}{2}(ne + \tau)\left(\theta - \frac{e\sum_{i=1}^{n} x_i - \theta_0\tau}{ne + \tau}\right)^2}$ $\qquad\qquad$ $\square$