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Price of connectivity of graph parameters

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Title: Price of connectivity of graph parameters

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Abstract: A vertex cover of a given graph is a vertex set including at least one endpoint from every edge. A vertex cover number τ is the size of a minimum vertex cover. If the vertices from a vertex cover are required to induce a connected subgraph, the resulting set is called a connected vertex cover. The corresponding parameter τ_c is called a connected vertex cover number. The decision versions of both problems are NP-complete.

To better understand a relation between these two vertex cover numbers, Cardinal and Levy define the price of connectivity as a ratio between τ_c and τ. It is not surprising that determining whether the price of connectivity of a given graph is at most t is NP-hard. The notion of price of connectivity can be extended for more graph properties, such as for the dominating set. The price of connectivity has already been investigated in several papers, with some focusing on critical graphs whose price of connectivity is strictly greater than the price of connectivity of every induced subgraph.

This thesis provides an overview of the current state of research into the price of connectivity. Moreover, we focus on the structural properties of critical graphs for the price of connectivity for vertex cover and discuss a possible characterization of graphs in which the price of connectivity for a dominating set is at most $\frac{7}{3}$ for every induced subgraph.

Keywords: Graph theory, Price of connectivity, Connected vertex cover, Connected dominating set

Contents

1 Introduction **2**
1.1 Contents of this thesis . . . 3
1.2 Notation . . . 3
1.3 Vertex cover . . . 4
1.4 Connected vertex cover . . . 5
1.5 Price of connectivity . . . 7
1.6 Complexity . . . 7

2 Polynomial time algorithms using price of connectivity **9**

3 Price of connectivity for vertex cover **13**
3.1 PoC-perfect graphs for vertex cover . . . 13
3.2 PoC-near-perfect graphs for vertex cover . . . 14
3.3 PoC critical graphs for vertex cover . . . 15

4 Price of connectivity for dominating set **24**
4.1 Dominating set and independence . . . 24
4.2 Connected dominating set . . . 25
4.3 Price of connectivity . . . 27
4.4 Conclusions . . . 36

Bibliography **37**

1. Introduction

We say that a set of vertices is a *vertex cover* of a graph if it contains at least one endpoint from each edge. The size of the smallest vertex cover is called the vertex cover number. Karp in 1972 placed the decision version of vertex cover among first 21 NP-complete problems. Many variants of vertex cover have been studied since in literature; one of those is *connected vertex cover*. A connected vertex cover of a graph is a vertex cover that induces a single connected subgraph. The connected vertex cover number is the size of a minimum connected vertex cover. This problem aside its theoretical importance has many real-world applications for example in design of wireless networks.

Let us consider the following model: the nodes in the network are connected via transmission links. The signal decreses with transmission distance. To counteract this we can equip nodes with relay stations amplifying the signal. We want to place relay stations in such a way that every transmission link is adjacent to a relay station and any pair of relay stations has to be connected by a path containing only other relay stations. Our budget is limited so we wish to minimize the number of relay stations used. The solution of this problem is exactly the minimum connected vertex cover.

The mininum connected vertex cover can be also used to solve top right access point minimum length corridor problem (TRA-MLC for short.)[1]. On the input is a rectangle F partioned into rectangular polygons $P_1, \dots, P_k$. A set S of line segments lying along the boundary of F and along boundaries of polygons is called a *corridor*. The *length of the corridor S* is a sum of lengths of all line segments included in S. Our goal is to find a corridor with a minimum length such that line segments form a tree and contain at least one point from F and each P_i. A variant of this problem, when S has to include top-right corner of the boundary of F, is called *top right access point minimum length corridor problem*. TRA-MLC has applications in laying electrical wiring or optic-fiber cables in floor plans.

We can imagine that F represents a floor plan and P_i are individual rooms. We have to lay optic-fiber cables in order to grant internet access to every room. Such cables are expensive, so we would like the length of the used cable to be as small as possible.

A natural question to investigate is the relation between vertex cover number and connected vertex cover number. For that purpose Cardinal and Levy define the *price of connectivity* to be a ratio between the connected vertex cover number and the vertex cover number. Deciding for a given graph whether price of connectivity is at least r is NP-hard. The price of connectivity can be generalized to any graph problem with a meaningful connected variant. Authors of [2] demonstrate how to use known price of connectivity of various problems to design polynomial time algorihms.

Camby and Schaudt in [3] coin the term *PoC-Near-perfect graphs*. These are graphs in which price of connectivity for vertex cover is bounded by a fixed number t for any induced subgraph. They propose a characterization of such graphs for $t \leq \frac{3}{2}$ in the terms of forbidden induced subgraphs. Also they define critical graphs, which are graphs with price of connectivity stricly greater than their every induced subgraph. The same researchers in [4] in a similar fashion study the price of connectivity of dominating set.

The results from [5] imply that there are no PoC-near-perfect graphs for the dominating set with price of connectivity strictly between 2 and $\frac{7}{3}$. Motivated by these results one can ask if there are no PoC-near-perfect graphs with price of connectivity in the interval $(3-\frac{2}{k-1}, 3-\frac{2}{k})$. In this work, we show that is not true by constructioning critical graphs with the price of connectivity equal to $3-\frac{3}{k}$.

1.1 Contents of this thesis

In the rest if this chapter we will provide basic definitions, examine properties of vertex cover and formally define the price of connectivity. In the Chapter 2 we will show applications of price of connectivity in polynomial algorithms for minimum connected vertex cover and minimum connected feedback vertex set from [2]. In the Chapter 3 we will closely examine price of connectivity of vertex cover. To be more specific we will discuss results by Camby and Schaudt. The second half of the chapter focuses on deriving new structural properties of critical graphs. In the Chapter 4 we would like to refresh basic properties of dominating sets and present results on the price connectivity of dominating set. Finally we will present construction of critical graphs with price of connectivity equal to $3-\frac{3}{k}$.

1.2 Notation

We consider only finite undirected graphs without multiple edges and loops. Let $G = (V, E)$ be a graph, with V being set of vertices and E set of edges. Let $n = |V|$. The *neighborhood* of a vertex u is defined as $N(u) := \{v \in V | uv \in E\}$. The *closed neighborhood* of u is $N[u] := N(u) \cup \{u\}$. A vertex u is a *cut vertex* if its removal increases the number of connected components. Similarly, an edge uv is a *bridge* if its deletion increases the number of connected components. For a set $S \subseteq V$ we write $G[S]$ to denote the subgraph of G induced by S, i.e. a graph with the vertex set S, where vertices are adjacent if and only if they are adjacent in G. A vertex subset S is *independent* if every pair of vertices from S is non-adjacent. The independence number $\alpha(G)$ is the size of a maximum independent set.

We use the following standard notation to denote some special graphs:

- P_k is a path on k vertices.
- C_k is a cycle on k vertices.
- $K_{m,n}$ is a complete bipartite graph with parts of size m and n.
- K_m is a clique on m vertices.
- $H+G$ is the union of two disjoint graphs H and G. In particular sG denotes the union of s disjoint copies of G.

In this work we will focus mainly on H-free graphs.

Definition 1 (H-free graph)**.** *Let G and H be two graphs. We say that G is H-free if it does not contain graph H as its induced subgraph.*

Moreover, if G neither contains H_1 nor H_2 as induced subgraphs we say that G is (H_1, H_2)-free.

Let us show a few examples. First, let us prove that graph G is P_3-free if and only if every connected component of G is a clique. If G does not contain P_3 it is clear that G is a union of cliques. For the converse suppose that there is a component C that is not a clique. Pick two non-adjacent vertices from C. The shortest path between these two vertices contains at least two edges and thus P_3.

Next, we claim that complete bipartite graphs can be characterized as set of all $(K_3, \overline{P_3})$-free graphs. From the previous proof follows that complement of P_3-free graph is a graph, in which vertices can be partitioned into independent sets such that edges are between any pair of vertices from different independent set. To ensure that the number of these independent sets is two, it is enough to forbid K_3 as an induced subgraph.

Finally the class of P_4-free graphs are cographs. The proof of this is more involved and we refer to [6].

1.3 Vertex cover

Definition 2 (Vertex cover)**.** *A* vertex cover *of a graph G is a set of vertices including at least one endpoint of every edge.*

The vertex cover of the smallest size is called *minimum*. The vertex cover number $\tau(G)$ is the size of a minimum vertex cover. Vertex cover is *minimal* if its every proper subset is not vertex cover. A minimum vertex cover is always minimal, but not every minimal vertex cover is minimum. For an illustration see Example 1.3.

Vertex cover is closely related to independent set.

Observation 1. *A vertex set S is a vertex cover if and only if $V(G) \setminus S$ is an independent set.*

Proof. If $V(G) \setminus S$ is independent, then there are no edges with both endpoints in $G \setminus S$ and thus each edge is incident to a vertex from S. Conversely, suppose that S is a vertex cover. By definition all edges from $E(G)$ have endpoints in S so $V(G) \setminus S$ is independent.

□

Observation 2. *A vertex cover S is minimal if and only if $V(G) \setminus S$ is a maximal independent set.*

Proof. By the previous observation S is a vertex cover if and only $V(G) \setminus S$ is independent. Suppose that S is a minimal vertex cover and $V(G) \setminus S$ is not a maximal independent set. Then there is a vertex v that $(V(G) \setminus S) \cup \{v\}$ is also independent. However, $S \setminus \{v\}$ is vertex cover; contradiction with minimality of S. On the other hand assume that $G \setminus S$ is maximal independent set and that G

is not minimal vertex cover. Since S is not a minimal vertex cover, we can find a vertex v s.t. $S \setminus v$ is vertex cover of G. The set $(V(G) \setminus S) \cup \{v\}$ is independent because v does not have a neighbor in $V(G) \setminus S$. That contradicts the maximality of $V(G) \setminus S$.

□

In other words we have just shown that finding the minimum vertex cover is equivalent to finding maximum independent set. As an immediate corollary, we see that the number of vertices of G is equal to $\tau(G) + \alpha(G)$.

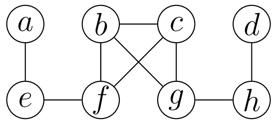


Figure 1.1

Example. *Let G be a graph from Figure 1.3.*

1. *Set $\{e, b, c, h\}$ is a minimum vertex cover of G: $\tau(G) = 4$.*
2. *Set $\{a, f, g, d\}$ is a maximum indepedent set of G: $\alpha(G) = 4$.*
3. *Set $\{a, f, c, g, d\}$ is a minimal vertex cover of G.*

The size of minimum vertex cover can be bounded from below by the size of maximum matching. It is not difficult to see that the size of the minimum vertex cover is at least the maximum number of edges in a matching. Each vertex in the minimum vertex cover is incident with at most one edge from a matching. For bipartite graphs these two numbers are equal by the König's theorem. Recall that maximum matching can be found in polynomial time in any graph G by Edmonds's blossom algorithm [7]. This famous result implies that in bipartite graphs finding a minimum vertex cover is polynomially solvable.

1.4 Connected vertex cover

Now we formally define the notion of connected vertex cover which is central term in our work.

Definition 3 (Connected vertex cover)**.** *A* connected vertex cover *is a vertex cover that induces a connected subgraph of G. We denote by $\tau_c(G)$ size of a minimum connected vertex cover.*

The size of minimum connected vertex cover is at least equal to the size of minimum vertex cover. We can imagine that it is possible to find a minimum connected vertex cover by only extending minimum vertex covers; however that is not the case. See Figure 1.2 depicting a graph in which the minimum connected vertex cover does not include any minimum vertex cover.

Observation 3. *A graph G on n vertices has a connected vertex cover of size k if and only if G contains the star $K_{1,n-k}$ as a contraction.*

Proof. Let S be a connected vertex cover of size k. Contracting every edge in S transforms G into $K_{1,n-k}$. If G contains $K_{1,n-k}$ as a contraction, then G can be partitioned into induced connected subgraphs $A, B_1, B_2, \ldots, B_{n-k}$ in such a way that for each B_i there is at least one edge incident to a vertex from A and no edges between vertices from different B_i. Move each vertex from B_i adjacent to vertex from A to A until all subgraphs B_i contain only one vertex. Such modified set A is connected and vertices from A covers all edges from G. Notice that $|V(A)| \leq k$. So A is now connected vertex cover of G with size at most k.

□

By using this observation we can reduce the size of input graph while looking for a minimum vertex cover. Edge uv can be contracted whenever both its endpoints belong to a connected vertex cover. This idea was used in the algorithm designed in [8].

Decision versions of both problems are defined as follows:

Vertex Cover (VC)
Input: A graph $G = (V, E)$ and a positive integer k
Question: Does G has a vertex cover S, such that, $|S| \leq k$?

Connected Vertex Cover (CVC)
Input: A graph $G = (V, E)$ and a positive integer k
Question: Does G has a connected vertex cover S, such that, $|S| \leq k$?

Finding a minimum vertex cover is NP-complete problem in planar graphs. On the other hand it becomes polynomially solvable for special classes of graphs: bipartite graphs (as we mentioned earlier), chordal graphs [9], series-parallel graphs [10].

Connected vertex cover has been in 1977 introduced and closely investigated by Garey and Johnson [11], where they established its NP-completeness for planar graphs of maximal degree at most 4. Fernau and Manlove [12] strengthen this result for planar bipartite graphs with degree at most 4. NP-completeness of a connected vertex cover was proved for 2-connected planar graphs with maximal degree 4 by Priyadarsini and Hemalatha [13]. Same result was showed by Watanabe et al. [14] even for 3-connected graphs. Minimum connected vertex cover is polynomially solvable for chordal graphs [15], graphs with maximal degree at most three and trees [16].

Figure 1.2: A graph in which the minimum connected vertex cover contains no minimum vertex cover. Minimum connected vertex cover is depicted by black vertices on the right. Black and white vertices on the left-hand picture represent two minimum vertex covers.

In the brief overview from above we can notice that for bipartite graphs complexities of these two problems differ. Interesting question to study is to find for which graph classes complexities of minimum vertex cover and connected minimum vertex cover coincide.

Now we turn our attention to the graphs with one forbidden subgraph: H-free graphs. Munaro [17] prove that CVC is NP-complete in H-free graphs if H contains induced cycle or claw ($K_{1,3}$). On the other hand VC becomes tractable if H contains a claw [18][19]. Even among H-free graphs we can found class for which complexities of both problems are not the same.

It remains to examine graphs without induced linear forest e.g. P_r-free graphs. Recently Johnson et al. [8] create polynomial algorithm for CVC in $(sP_1 + P_5)$-free graphs and Grzesik et al. [20] prove polynomiality of VC in $(sP_1 + P_6)$-free graphs. Complexity of CVC for $r \geq 6$ and of VC $r \geq 7$ is still unknown.

1.5 Price of connectivity

Different complexity results for these two problems rise the question how graph invariants τ_c and τ are related. To investigate that we define concept of the price of connectivity.

Definition 4 (Price of connectivity)**.** *The* price of connectivity *for a class of graphs $\mathcal{G}$ is defined as the maximal ratio $\frac{\tau_c(G)}{\tau(G)}$ over all graphs $G \in \mathcal{G}$.*

The notion of the price of connectivity has been introduced by Cardinal and Levy [21] as defined above. Also they proved that PoC is upper-bounded by $\frac{2}{1+\epsilon}$ in graphs with average degree ϵn. The price of connectivity can be generalized for any graph property for which its connected variant makes sense.

Definition 5 (Price of connectivity for property π)**.** *For a class of graphs $\mathcal{G}$ the* price of connectivity for property π *is defined as* $\max_{G\in\mathcal{G}}(\frac{\pi(G)}{\pi_c(G)})$*, where $\pi(G)$ denotes the size of a minimum set satisfying property π and similarly $\pi_c(G)$ is the size of a minimum connected set with property π.*

The price of connectivity was further studied for dominating set, feedback vertex set, face hitting set and odd cycle transversal set. Later we will thoroughly investigate the price of connectivity of vertex cover and dominating set.

1.6 Complexity

Deciding whether the price of connectivity for vertex cover is at most t for given graphs is NP-hard. However, theorem stated below implies that is not clear whether it lies in NP.

A complexity class Θ_2^p contains decision problems which are computable in polynomial time by a deterministic Turing machine which is allowed to use NP-oracle $O(\log(n))$ times, where n is the size of the input. Examples of Θ_2^p-complete problems are Odd-max-true-3SAT and Min-card-vertex-cover compare defined below. For more details about the class Θ_2^p we refer to [22].

Odd-max-true-3SAT
Input: 3-CNF formula F
Question: Is the maximum number of 1's in satisfying truth assignments for F odd?

Min-card-vertex cover compare
Input: Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$
Question: Is true that $\tau(G_1) < \tau(G_2)$?

It is easy to see that PoC problem belongs to Θ_2^p. We can use oracle to determine $\tau(G)$ and $\tau_c(G)$ by binary search.

Theorem 1 (Camby et al. [3])**.** *Given any connected graph G deciding whether $\frac{\tau(G)}{\tau_c(G)} \leq r$ is Θ_2^p-complete.*

Authors of [3] prove this result by reduction from Min-Card-Vertex-cover compare.

2. Polynomial time algorithms using price of connectivity

For some graph classes it has been shown that the numbers τ_c and τ differ on by an additive constant. Similar results exist for graph parameters different from τ. Chiarelli, Hartinger, Johnson, Milanič, Paulusma in [2] present a general technique that utilizes a constant difference between τ_c and τ. to design polynomial time algorithms.

Lemma 2 (Chiarelli et al. [2])**.** *Let π be a property of a set of vertices, which if it holds for a set S then it holds for every superset. Suppose that for every G from a class $\mathcal{G}$ all minimal vertex sets satisfying the property π can be enumerated in polynomial time. Also the size of a minimum connected set of vertices with the property π is at most $\pi(G) + c$ for a constant c. Then a minimum connected vertex set with the property π can be found in polynomial time for every $G \in \mathcal{G}$.*

The vertex cover and the dominating set are examples of graph properties closed under supersets on the other hand independent set is an example of a property which is not.

Proof. The algorithm works as follows. Enumerate all minimal vertex sets with property π. For each minimal set consider all possibilities of adding at most c vertices and check if the new set is connected. The smallest connected set of vertices found is returned. Notice that this algorithm is polynomial.

Suppose that a set S is a minimum connected set with property π. We need to show that the algorithm will return S or another minimum set with the same size. Pick from S a minimal set with property π and denote it by S'. The set S' will be one of the enumerated sets by definition. Due to the facts that $|S| \leq \pi(G) + c$ and $|S'| \geq \pi(G)$ we have $|S| - |S'| \leq c$. Hence, S will be found.

□

We continue with demonstration of making use of Lemma 2 to create polynomial algorithms for connected vertex cover and connected feedback vertex set following ideas from [2]. In this part we focus on a class of sP_2-free graphs, where sP_2 denotes a disjoint union of s paths on two vertices. (See Figure 2.1.)

Theorem 3 (Balas and Yu [23])**.** *For every constant $s \geq 1$, the number of maximal independent sets of an sP_2-free graph on n vertices is at most $n^{2s} + 1$.*

Theorem 4 (Tsukiyama et al. [24])**.** *For every constant $s \geq 1$, it is possible to enumerate all maximal independent sets of a graph G on n vertices and m edges with a delay of $O(nm)$. The delay is defined here as the maximal number of steps before the first and between any two consecutive outputs.*

Recall that vertices not included in a maximal independent set belong to a minimal vertex cover. Both theorems together imply, that it is possible to enumerate all minimal vertex covers in sP_2-free graphs in polynomial time. To fulfill all assumptions of Lemma 2 we need to show, that the numbers τ and τ_c differ only by an additive constant.

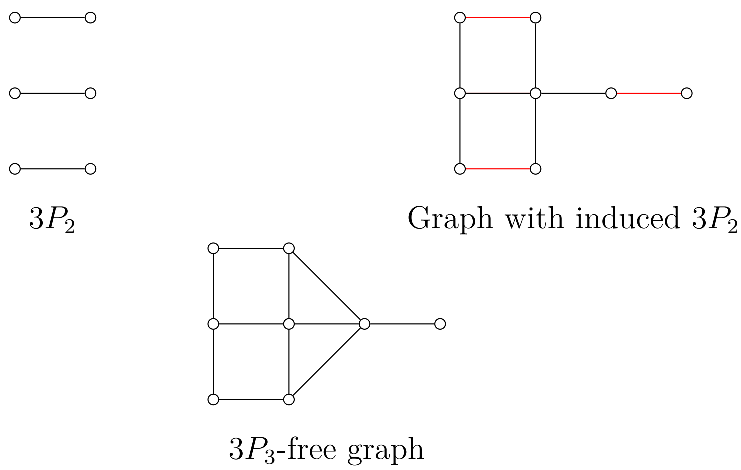


Figure 2.1: Examples illustrating $3P_2$-free graphs.

Theorem 5 (Hartinger et al. [25]). *Let $s \geq 1$ and let G be a connected sP_3-free graph. Then the size of a minimum connected vertex cover of G is at most $\tau + 4s^2 + 2s - 10$.*

Theorems 3, 4, 5 and Lemma 2 imply the following theorem.

Theorem 6 (Chiarelli et al. [2]). *For every constant $s \geq 1$ a minimum connected vertex cover can be found in polynomial time in sP_2-free graphs.*

In the same fashion as before we will use Lemma 2 to design an algorithm for the minimum connected feedback vertex set.

Definition 6 (Feedback vertex set). *A set of vertices S is called a* feedback vertex set *if the graph $G \setminus S$ does not contain any cycles.*

Theorem 7 (Chiarelli et al. [2]). *Minimum connected feedback vertex set is polynomial time solvable for a class of sP_2-free graphs.*

Theorem 8 (Belmonte et al. [26]). *Let $s \geq 1$ and let G be a connected P_3-free graph. Let f be the size of a minimum feedback vertex set of G. Then the size of a minimum connected feedback vertex set of G is at most $f + 12s^2 - 2s - 2$.*

Theorem 9 (Schwikowski and Speckenmeyer [27]). *It is possible to enumerate all minimal feedback vertex sets of a graph G on n vertices and m edges with a delay of $O(n^3 + n^2m)$. Delay is defined the same as in Theorem 4.*

Lemma 10 (Chiarelli et al. [2]). *For every $s \geq 1$ exists a constant c_s such that number of minimal feedback vertex sets in any sP_2-free graph on n vertices is $O(n^{c_s})$.*

Proof. Let S denote the minimal feedback vertex of a graph G. The complementary graph $G - S$ is a forest F_S. We define a *skeleton* F'_S of a forest F_S as the following subgraph. From the components of F_S isomorphic to P_2, we delete an arbitrary vertex; from the remaining components delete all leaves. We observe that the set of vertices $l(F'_S) := V(F_S) \setminus V(F'_S)$ is an independent set in G and each vertex has at most one neighbor in F'_S. (See Figure 2.2 with an example.) Let $J(F'_S)$ be a subset of vertices $V(G) \setminus V(F'_S)$ which have at most one neighbor in F'_S.

Claim 11. *The set $l(F'_S)$ is a maximal independent set of $G[J(F'_S)]$.*

To prove the claim suppose that in $J(F'_S) \setminus l(F'_S)$ there exists a vertex v non-adjacent to vertices in $l(F'_S)$. Then $G[F_S \cup \{v\}]$ is a forest; thus, $S \setminus \{v\}$ is also a feedback vertex set. That contradicts the minimality of S. In other words, the set $J(F'_S)$ consists only of $l(F'_S)$ and vertices from S. The claim is proved.

To find all minimal feedback vertex sets in a given graph G, we consider every skeleton and check every maximal independent set on the rest of the vertices incident to at most one vertex of the skeleton. Next we test whether the vertices neither included in the skeleton nor the maximal independent set are minimal feedback vertex set of G. The number of all maximal independent sets is upper-bounded by the expression $n^{2s}+1$ according to Theorem 3. To complete the proof it suffices to show that the number of all skeletons is polynomial in n, because that the number of minimal feedback vertex sets is also polynomial in n.

Claim 12. $|V(F'_S)| \leq 3s^2 - 5s + 2$.

One way how to estimate the number of vertices in a forest F'_S is to decompose each component C into paths going from its leaves to a fixed vertex. These paths contain all vertices in C, eventhough some vertices can be counted multiple times. Roughly speaking, using the assumption that G is sP_2-free, we show that number of leaves is at most $s-1$ and the length of the longest path in C is at most $3s-2$. The latter is true for any induced path of G, as path on $3s-1$ vertices contains s disjoint copies of P_2. Let us define and estimate the number of leaves. Set A of vertices of F'_S contains all isolated vertices of F'_S, from each component isomorphic to P_2 one arbitrary vertex and lastly from the rest of the components all vertices of degree 1. Each vertex from A has at least one neighbor in $l(F'_S)$ by definition of F'_S, for each vertex we pick one such neighbor. These pairs induced disjoint set of P_2. Number of these pairs is at most $s-1$. Now consider paths from vertices in A to a fixed vertex in the same component. This concludes the proof of the claim.

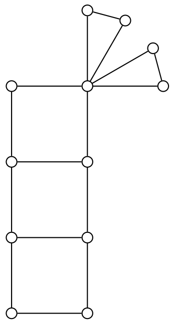
The number of all possible choices of a skeleton is $O(n^{3s^2})$ and the number of all choices for a maximal independent set is $O(n^{2s})$. Therefore, there is a constant c_s depending on s such that the number of all minimal feedback vertex sets in G is $O(n^{c_s})$.

□

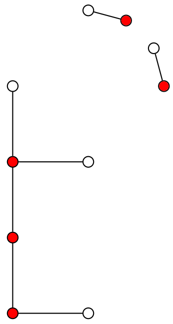
Proof of Theorem 7. The previous lemma and Theorem 9 imply that the minimum feedback vertex set can be found in polynomial time by enumerating all minimal feedback vertex sets in sP_2-free graphs. The property of being a feedback vertex set holds for supersets. By Theorem 8 the difference between both parameters is constant. All assumptions of Lemma 2 are satisfied, so there is an algorithm that finds a minimum connected feedback vertex in polynomial time in sP_2-free graphs. □

Using Lemma 2 directly fails if either the graphs have exponentially many minimal sets, or the difference between a minimal and a connected minimal set is no longer constant.

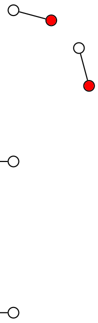
It is easy to construct connected $2P_3$-free graph with exponentially many maximal independent sets. We will now show this construction. Let number of



$5P_2$-free graph G.



Set F_S and set F'_S in red.



Set A depicted in red.

Figure 2.2: Illustration to the proof of Lemma 10.

vertices be $n = 3l+1$. Divide $3l$ vertices into l triples. The vertices in every triple induce a triangle. The remaining vertex v is set to be adjacent to every other vertex. From each triple exactly one vertex can be added to the independent set, so the number of all maximal independent sets is $3^{\frac{n-1}{3}}$. Our graph is $2P_3$-free because every induced P_3 needs to go through the central vertex v. Even though structural Theorems 5 and 8 still hold for sP_3-free graphs, we cannot assume that the number of minimal feedback vertex sets and maximal independent sets is polynomial in a number of vertices of G.

Despite this fact there are still polynomial results for these problems using various techniques. In the case of minimum size feedback vertex set Dabrowski et al. [28] constructed polynomial algorithm for (sP_1+P_3)-free and P_4-free graphs using structural properties of these classes. For vertex cover, the authors of [20] use potential maximal cliques.

3. Price of connectivity for vertex cover

Theorem 13. *For every graph G with at least one edge it holds that* $1 \leq \frac{\tau_c(G)}{\tau(G)} < 2$.

Proof. Let C be any vertex cover of a connected graph G and c the number of components of $G[C]$. Then we can add at most $c-1$ vertices to C to construct a connected vertex cover of G. Thus $\tau_c(G) \leq 2\tau(G) - 1$. The size of a minimum connected vertex cover is at least the size of a minimum vertex cover. For the ratio $\frac{\tau_c(G)}{\tau(G)}$ to be defined G must contain at least one edge, otherwise both numbers τ and τ_c are equal to zero. $\square$

Example of graphs for which $\tau_c(G) = 2\tau(G) - 1$ holds are odd length paths and cycles with an even number of vertices. For this class of graphs we can say, that the upper bound from the previous theorem is asymptotically sharp because:

$$\lim_{k \to \infty} \frac{\tau_c(G)}{\tau(G)} = \lim_{k \to \infty} \frac{2k-1}{k} = 2.$$

Earlier we discussed that deciding whether the price of connectivity for a graph G is bounded by a number $t \in (1, 2)$ is Θ_2^p-complete. One can ask how to weaken this problem as little as possible to make it polynomially solvable. Let us focus on classes of graphs for which the number $\frac{\tau_c(G)}{\tau(G)}$ is bounded by a fixed number t for every induced subgraph of G. Classes which are closed under induced subgraphs are called hereditary.

Camby et al. in [3] characterize hereditary graph classes with PoC upper-bounded by 1, $\frac{4}{3}$ and $\frac{3}{2}$. All these classes were determined by a finite list of forbidden induced subgraphs. Such characterization gives us a polynomial time recognition algorithm. For a given graph G and every graph H_i from the list of forbidden subraphs it is enough to check each vertex subset of G with the size $|V(H_i)|$ if it induces graph H_i. The length of the list is finite and independent from the size of G and the number of all checked subsets is polynomial in the number of vertices in G.

The following two sections 3.1 and 3.2 are devoted to the characterization of hereditary classes given in [3].

3.1 PoC-perfect graphs for vertex cover

Definition 7. *A graph G is called* PoC-perfect *if every subgraph H of G satisfies $\tau_c(H) = \tau(H)$.*

Class of PoC-perfect graphs is described in this theorem.

Theorem 14 (Camby et al. [3])**.** *The following assertions are equivalent for every graph G :*

- *For every induced subgraph H of G it holds that $\tau_c(H) = \tau(H)$.*

- *G is (P_5, C_5, C_4)-free.*
- *G is chordal and P_5-free.*

It is easy to observe that $\tau(C_4) = \tau(P_5) = 2$ and $\tau_c(C_4) = \tau_c(P_5) = 3$. The price of connectivity of C_5 is $\frac{4}{3}$. Since these graphs are not PoC-perfect then they cannot be induced subgraphs of PoC-perfect graphs.

3.2 PoC-near-perfect graphs for vertex cover

Definition 8. *A graph G is* PoC-near-perfect *with a threshold $t \in (1, 2)$ if every subgraph H of G satisfies the inequality $\frac{\tau_c(H)}{\tau(H)} \leq t$.*

Theorem 15 ([3])**.** *The following assertions are equivalent for every graph G :*

- *For every induced subgraph H of G it holds that $\tau_c(H) \leq \frac{4}{3}\tau(H)$.*
- *G is (P_5, C_4)-free.*

Theorem 16 ([3])**.** *The following assertions are equivalent for every graph G:*

- *For every induced subgraph H of G it holds that $\tau_c(H) \leq \frac{3}{2}\tau(H)$.*
- *G is $(P_7, C_6, \Delta_1, \Delta_2)$-free.*

The graph Δ_1 is constructed from two disjoint copies of C_4 by identifying two vertices each from a different copy of C_4. Δ_2 is obtained from Δ_1 by deleting an edge incident to a vertex with degree four. For illustration see the Figure 3.1. We will present two lemmata which are used in proofs of Theorem 15 and Theorem 16. These structural properties may be useful while trying to derive similar results for different values of the parameter t.

Lemma 17 (Camby et al. [3])**.** *Let G be a connected graph and let C be a vertex cover of G. Suppose that $(\mathcal{A}, \mathcal{B})$ is a bipartition of the connected components of C with $\mathcal{A}, \mathcal{B} \neq \emptyset$, then there exist two components A and B each from different parts such that the number of vertices on the shortest path from A to B is 3.*

Figure 3.1: Graphs Δ_1 and Δ_2 from Theorem 16.

Proof. Let us pick two components $A \in \mathcal{A}$ and $B \in \mathcal{B}$ with the minimum distance. Such pair exists because C has a finite number of connected components. Path $x_1, \dots, x_k$ is the shortest path between A and B. Vertex $x_1 \in A$ and $x_k \in B$. On the contrary we suppose that $k > 3$. None of the vertices $x_2, \dots x_{k-1}$ belong to C. Otherwise, B is not the nearest component to A. Edge x_2, x_3 is not covered, but that contradicts that C is a vertex cover.

□

Let C be a vertex cover of a connected graph G and S be a connected component of C. Then $P_C(S)$ denotes a set of vertices $v \in V(G)$ such that $N(v) \cap C \subseteq V(S)$. In other words, $P_C(S)$ contains S and vertices whose entire neighborhood is in S. In particular $P_C(S)$ induces connected subgraph in G.

Lemma 18 (Camby et al. [3])**.** *Let $S_1, S_2, \dots, S_k$ be connected components of a vertex cover C. There exists at least one $P_C(S_i)$ which is not a cutset of G, i.e. $G[V(G) \setminus P_C(S_i)]$ is connected.*

Proof. Sets $P_C(S_i)$ are disjoint by definition. We define a new graph H with vertices corresponding to the sets $P_C(S_i)$. Two vertices $P_C(S_i)$ and $P_C(S_j)$ are adjacent in H if neighborhoods of $P_C(S_i)$ and $P_C(S_j)$ share at least one vertex. Formally:

$$V(H) = \{P_C(S_i) : i \in \{1, \dots, k\}\}$$

and

$$E(H) = \{P_C(S_i), P_C(S_j) | P_C(S_i) \cap P_C(S_j) \neq \emptyset\}.$$

Since G is connected and C is a vertex cover, graph H is connected. Every connected graph contains at least one vertex that is not a cut vertex. Thus, in G there exists a set $P_C(S_i)$ that is not a cutset.

□

3.3 PoC critical graphs for vertex cover

Definition 9. *A graph G is* critical *if every induced subgraph H of G has strictly smaller price of connectivity than G.*

Critical graphs are precisely those which can be listed in forbidden subgraphs characterization. Note that all previous examples of forbidden subgraphs are also critical. In this section we will investigate structure of critical graphs in more detail.

Many proofs from this section will use the following inequality to derive a contradiction.

Lemma 19. *Let $\frac{a_1}{b_1}, \dots, \frac{a_i}{b_i}, \dots \frac{a_n}{b_n}$ be real fractions with positive denominator. Denote the largest and the smallest fraction M and m respectively. Then*

$$m \leq \frac{\sum_{i=1}^{n} a_i}{\sum_{i=1}^{n} b_i} \leq M$$

.

Proof. We observe that the following iquality holds:

$$a_1 + a_2 + \cdots + a_n = b_1 q_1 + b_2 q_2 + \cdots + b_n q_n,$$

where q_i is equal to $\frac{a_i}{b_i}$. The expression on the right is clearly at most $(b_1 + \cdots + b_n)M$ and at least $(b_1 + \cdots + b_n)m$. To finish the proof it is enough to divide both sides of the inequality by the sum $b_1 + \cdots + b_n$.

□

Lemma 20 (Camby et al. [3]). *Let G be a critical graph. An edge with both endpoints in the minimum size vertex cover cannot be a bridge in G.*

Proof. Suppose for the sake of contradiction that G has a minimum vertex cover with vertices x, y, such that edge xy is a bridge. This means that graph $G \setminus (xy)$ has two connected components C_1 and C_2. C_1 is adjacent only to x and C_2 is adjacent only to y. Consider two induced subgraphs H_1 and H_2 defined like this. The subgraph H_i contains C_i and vertices x and y. The minimum vertex cover of G must have at least $\tau(H_1) + \tau(H_2)$ vertices, because both vertices x and y are in the minimum vertex cover of G and components C_1 and C_2 are disjoint. On the other hand, union of connected vertex covers of H_1 and H_2 is also a connected vertex cover of G because vertices x, y are in the union and edge xy is a bridge. So number $\tau_c(G)$ is at most $\tau_c(H_1) + \tau_c(H_1)$. These inequalities together with Lemma 19 yields:

$$\frac{\tau(G)}{\tau_c(G)} \leq \frac{\tau(H_1) + \tau(H_2)}{\tau_c(H_1) + \tau_c(H_2)} \leq \max\left\{\frac{\tau(H_1)}{\tau_c(H_1)}, \frac{\tau(H_2)}{\tau_c(H_2)}\right\}$$

That contradicts the assumption that G is critical.

□

The technique demonstrated above allows us to derive further structural results about critical graphs.

Lemma 21. *Let G be a critical graph, let a vertex x be an arbitrary vertex from a minimum size vertex cover. Then $N(x)$ is an independent set.*

Proof of Lemma 21. Let G be a critical graph and S a minimum vertex cover of G and vertex $x \in S$. For the sake of contradiction suppose there are two adjacent vertices u and v in G such that $u, v \in N(x)$. We will distinguish between these three cases.

1. The vertex x is not a cut vertex.

   Let H be the graph obtained by deleting of x. Clearly $\tau(G)$ is at most $\tau(H) + 1$ because minimum vertex cover of H together with vertex x is a vertex cover of G. The opposite inequality is also true. Notice that $S \cap V(H)$ is a minimum vertex cover of H. A minimum connected vertex cover of H together with vertex x is also a connected vertex cover of G. That is true because any vertex cover of H must include at least one vertex from the

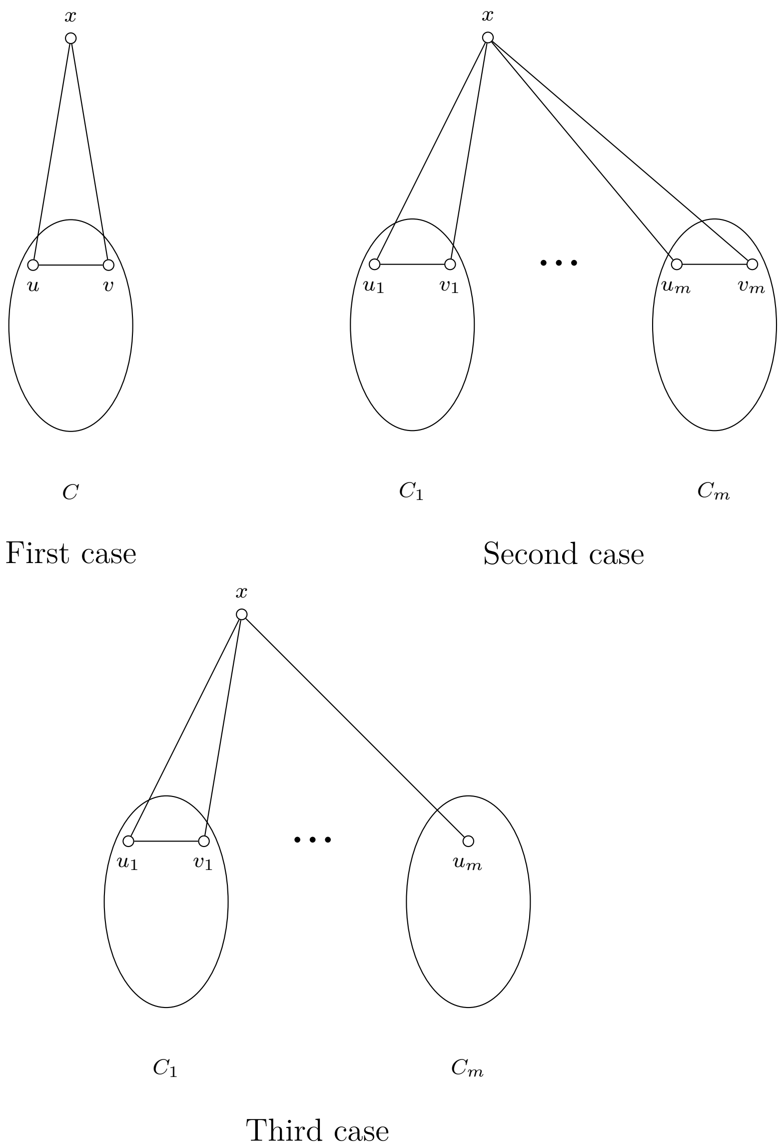


Figure 3.2: Illustration of the cases from the proof of Lemma 21.

edge uv and both vertices are neighbors of x. Thus, $\tau_c(G)$ can be bounded from above by this expression:

$$\tau_c(G) \leq \tau_c(H) + 1.$$

Using these two inequalities and inequality from Lemma 19 gives us:

$$\frac{\tau_c(G)}{\tau(G)} \leq \frac{\tau_c(H) + 1}{\tau(H) + 1} \leq \max\left\{\frac{\tau_c(H)}{\tau(H)}, 1\right\}.$$

For every graph G the ratio between τ and τ_c is at least one. That means that the induced subgraph H of G satisfies $\frac{\tau_c(G)}{\tau(G)} \leq \frac{\tau_c(H)}{\tau(H)}$. However, that contradicts the criticality of G.

2. The vertex x is a cut vertex and each component in the graph $G \setminus \{x\}$ has an edge in $N(x)$.

   Denote these components $C_1, C_2, \ldots, C_m$.

   Claim 22. $\tau(G) = \sum_{i=1}^{m} \tau(C_i) + 1$

   The inequality $\tau(G) \leq \sum_{i=1}^{m} \tau(C_i) + 1$ is easy to see as the union of minimum vertex covers of C_i with vertex x is a vertex cover of the entire graph.

   Observe that $S \cap V(C_i)$ is also a minimum vertex cover of C_i. This implies the desired inequality.

$$\tau(G) \geq \sum_{i=1}^{m} \tau(C_i) + 1.$$

   The claim is proved.

   We need to find an upper bound for a size of a minimum connected vertex cover of G. The union of minimum connected vertex covers of C_i and vertex x make a connected vertex cover of G. The reason is that each vertex cover of C_i contains at least one neighbor of x due to edges in $V(C_i) \cap N(x)$. This yields the following upper bound:

$$\tau_c(G) \leq \sum_{i=1}^{m} \tau_c(C_i) + 1.$$

   Let us combine all estimates together and then use Lemma 19:

$$\frac{\tau_c(G)}{\tau(G)} \leq \frac{\tau_c(C_1)+, \ldots, +\tau_c(C_m) + 1}{\tau(C_1)+, \ldots, +\tau(C_m) + 1} \leq \max\left\{\frac{\tau_c(C_i)}{\tau(C_i)}, 1\right\}.$$

   Again we have a contradiction with the assumption that G is critical.

3. The vertex x is a cut vertex and at least one component in $G \setminus \{x\}$ does not have an edge in $N(x)$.

   Additionally, we can assume that in G there is one component C_1 with an edge uv in $N(x)$. The proof is complete otherwise. We denote such components $C_1, \ldots, C_m$. Consider all these induced subgraphs of G: H_1, C_1, ..., C_m. Subgraph H_1 includes all components without an edge in $N(u)$

together with vertices x and u. Without loss of generality we can assume that x is in some minimum vertex cover of H_1. If it is not that the case that we know that in the minimum vertex cover is v. We can switch v with vertex x. Again this is a vertex cover of H_1 because v is a leaf in H_1. That implies that union of minimum vertex covers of H_1 and C_i is a vertex cover of G. Conversely the size of minimum vertex cover of G is at least $\tau(H_1) + \tau(C_1), + \ldots, \tau(C_m)$. $D \cap C_i$ is vertex cover of C_i and x is already counted in $\tau(H_1)$. Observe that vertex x must be in any connected vertex cover of H_1 because an edge xv needs to be covered and x is a cut vertex in H_1. Also leaf v is not in any minimum connected vertex cover of H_1. That means that in the sum of τ_c of our subgraphs vertices x and v are counted no more than once. Using this and inequality from Lemma 19 we can estimate the ratio between $\tau(G)$ and $\tau_c(G)$ in the following way:

$$\frac{\tau_c(G)}{\tau(G)} \leq \frac{\tau_c(H_1) + \tau_c(C_1)+, \ldots, \tau(C_m)}{\tau(H_1) + \tau(C_1), + \ldots, \tau(C_m)} \leq \max\left\{\frac{\tau_c(H_1)}{\tau(H_1)}, \frac{\tau_c(C_i)}{\tau(C_i)}\right\}.$$

This inequality implies that for one of the induced subgraphs H_1 or C_i their price of connectivity is at least equal to $\frac{\tau_c(G)}{\tau(G)}$, but that contradicts assumption about criticality of G.

□

Lemma 23. *Let G be a critical graph with $\frac{\tau_c(G)}{\tau(G)} \geq \frac{3}{2}$ and a minimum vertex cover contains two vertices x, y such that $G \setminus \{x, y\}$ is connected. Then $N(x) \cup N(y)$ is an independent set. In particular $xy \notin E$.*

Proof. Suppose that there is an edge between two vertices $u \in N(x)$ and $v \in N(y)$. Denote by G' graph obtained by deletion of vertices x and y from G. Observe that $\tau(G) \geq \tau(G') + 2$. Any vertex cover of G without vertices x and y is a vertex cover of G' and vertices x and y belong to some minimum vertex cover of G. From any connected vertex cover of G' it is possible to construct a connected vertex cover of G by adding vertices x, y, u, v. At least one of the vertices u and v is already in the connected minimum vertex cover so we can bound connected dominating number: $\tau_c(G) \leq \tau_c(G') + 3$.

$$\frac{\tau_c(G)}{\tau(G)} \leq \frac{\tau_c(G') + 3}{\tau(G') + 2} \leq \max\left\{\frac{\tau_c(G')}{\tau(G')}, \frac{3}{2}\right\}$$

□

If we know, that vertices x and y are adjacent, then condition for the ratio $\frac{\tau_c(G)}{\tau(G)}$ can be omitted.

Lemma 24. *Let G be critical and minimum vertex cover contains two vertices x, y such that $G \setminus \{x, y\}$ is connected. Then $N(x) \cup N(y)$ is an independent set.*

Proof. Proof is nearly the same as for previous Lemma 23. The only difference is, that to transform a connected vertex cover of G' into a connected cover of the whole graph, it is enough to add only vertices x, y and one vertex from the pair u, v. Since uv is an edge, at least one of the vertices u and v must be in any vertex cover of G'.

$$\frac{\tau_c(G)}{\tau(G)} \leq \frac{\tau_c(G')+2}{\tau(G')+2} \leq \max\left\{\frac{\tau_c(G')}{\tau(G')}, 1\right\}.$$

□

Now we turn our attention to the case, where graph $G \setminus \{x, y\}$ has more than one component.

Lemma 25. *Let G be critical with a minimum vertex cover S. Suppose that S contains a pair of adjacent vertices x, y. Then set of vertices $N(x) \cup N(y) \setminus \{x, y\}$ is an independent set.*

To be able to prove this lemma we need to find out more about the structure of G. More precisely we want to reduce this problem to the case, where $G \setminus x, y$ has only one component adjacent to both x and y.

Lemma 26. *Suppose that graph G is critical. Let S be a minimum vertex cover of G including vertices x and y. Assume that there is at least one component C in $G \setminus \{x, y\}$ such that a set of vertices $(N(x) \cup N(y)) \cap V(C)$ is independent. Then there is at most one component D with edges among vertices from $(N(x) \cup N(y)) \cap V(D)$, moreover vertices from $(N(x) \cup N(y)) \cap V(D)$ induce a complete bipartite subgraph in G.*

Proof. First let us show that $G \setminus \{x, y\}$ contains at most one component D adjacent to both vertices x and y with edges in $G[N(x) \cup N(y)]$. Suppose for the sake of contradiction that there are two such components, we will denote them D_1 and D_2. Pick one edge u_1v_1 with both endpoints in $(N(x) \cup N(y)) \cap V(D_1)$ analogously pick an edge u_2, v_2 in $(N(x) \cup N(y)) \cap V(D_2)$. Vertices u_1, u_2 are from $N(x)$ and v_1, v_2 belong to $N(y)$ (see Figure 3.3). A subgraph H is defined as:

$$H := G[V(G) \setminus V(D_1) \setminus V(D_2) \cup \{u_1, v_2\}]$$

. It consists of vertices u_1, u_2 and all remaining vertices outside the components D_i. Consider induced subgraphs H, D_1 and D_2.

Claim 27. $\tau(G) = \tau(H) + \tau(D_1) + \tau(D_2)$

We may choose minimum vertex covers of these subgraphs that are disjoint, moreover x and y are in a minimum vertex cover of H. Only problematic vertices are u_1 and v_2. Vertices u_1 and v_2 are leaves in H, so they can be replaced by vertices x and y. It is easy to see that the union of vertex covers of H, D_1 and D_2 is a vertex cover of G. Thus,

$$\tau(G) \leq \tau(H) + \tau(D_1) + \tau(D_2).$$

The opposite inequality follows from the observation that sets $S \cap V(D_1)$, $S \cap V(D_2)$ and $X \cap (H \setminus \{u_1, v_2\})$ are vertex covers of D_1, D_2 and H respectively.

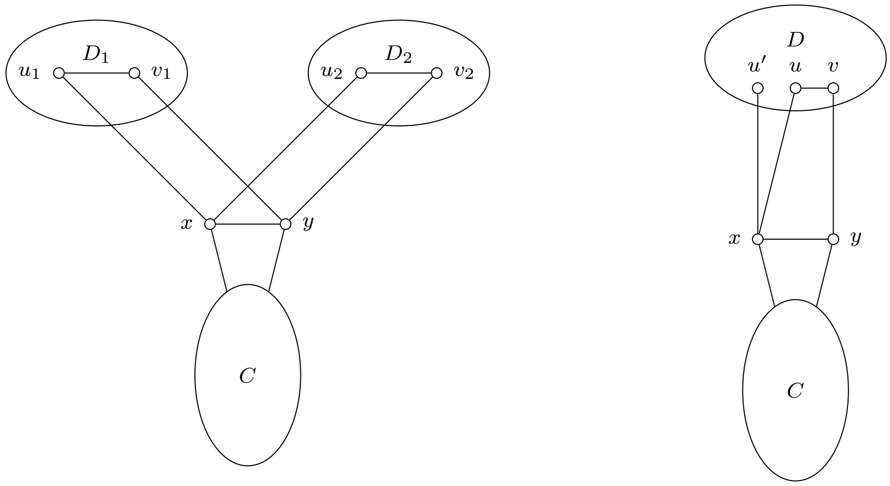


Figure 3.3: Graphs from proof of Lemma 26

Claim 28. $\tau_c(G) \leq \tau_c(H) + \tau_c(D_1) + \tau_c(D_2)$

The union of connected vertex covers of H, D_1 and D_2 gives a connected vertex cover of G. Vertices x and y are in every connected vertex cover of H due to vertices u_1, v_2. Also any vertex cover of D_i must include a vertex from $N(x) \cup N(y)$ because of edges $u_i v_i$. So the inequality from the claim holds.

Finally, we can use these estimates and the inequality from Lemma 19 to bound the price of connectivity of G:

$$\frac{\tau_c(G)}{\tau(G)} \leq \frac{\tau_c(H) + \tau_c(D_1) + \tau_c(D_2)}{\tau(H) + \tau(D_1) + \tau(D_2)} \leq \max\left\{\frac{\tau_c(H)}{\tau(H)}, \frac{\tau_c(D_i)}{\tau(D_i)}\right\}.$$

At least one of the induced subgraphs H, D_1 and D_2 has price of connectivity which is equal or greater than price of connectivity of G which contradicts the criticality of G.

It remains to prove, that vertices in $(N(x) \cup N(y)) \cap V(D)$ induce a complete bipartite graph. From Lemma 21 we know that vertices from one neighborhood cannot be adjacent to each other; thus, the graph has two parts: $N(x)$ and $N(y)$. Let us assume that there are there three vertices $u, u' \in N(x)$ and vertex $v \in N(y)$ s.t. $u'v \notin E$ and $vu \in E$ (see Figure 3.3). Induced subgraphs H includes vertices v, u' and every vertex from $V(G \setminus \{x, y\}) \setminus V(D)$. Similarly as before we consider induced subgraph H and D.

The rest of the proof follows the same strategy as before.

□

Lemma 29. *Let G be a critical graph with minimum vertex cover S. Pick any two vertices x and y from S. If $G \setminus \{x, y\}$ contains one component D adjacent to x and y with an edge on $N(x) \cup N(y)$, then there are no other components adjacent to both x and y.*

Proof. Suppose that there is a component C adjacent to x and y. In D there exist an edge u, v, where $u \in N(x)$ and $v \in N(y)$. Without loss of generality we

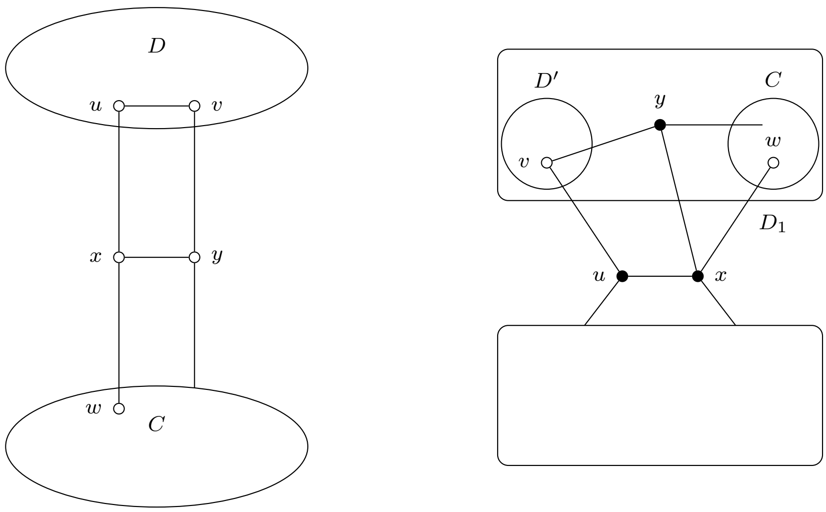


Figure 3.4: Illustration of situation in proof of Lemma 29.

can assume that vertex u is in S. Consider graph $G \setminus \{u, x\}$. Vertices x and u have neighbors connected with an edge namely v and y. Vertex x has a neighbor w in the component C non-adjacent to u. (See Figure 3.4.)

Let us take two induced subgraphs H and D_1 defined as follows. Subgraph D_1 contains entire component C, vertex y, vertex v and connected component of $D \setminus u$ containing v. Especially vertices x and u are not included in D_1. It is possible that $D \setminus u$ is no longer connected. Subgraph H is defined as: $H := G[V(G) \setminus V(D_1) \cup \{u, w\}]$. In particular H contains x, y, u, w.

Claim 30. *The union of connected vertex covers of H and D_1 is a connected vertex cover of G.*

By choice of H vertices u and x are both in every minimum connected vertex cover of H. Vertex y is a cut vertex in D_1, it links components C and part of component D included in D_1. Thus, union of minimum connected vertex covers of H and D is connected vertex cover of G and the claim is proved. See figure 3.4

To finish the proof we need to find an lower bound of $\tau(G)$. Notice that $S \cap D_1$ is vertex cover of D_1 and $S \cap H$ is vertex cover H. Sets $S \cap D_1$ and $S \cap H$ are disjoint. Then this inequality follows: $\tau(G) \geq \tau(H) + \tau(D_1)$.

Using these facts and Lemma 19 it is possible to estimate the price of connectivity of G as:

$$\frac{\tau_c(G)}{\tau(G)} \leq \frac{\tau_c(H) + \tau_c(D}{\tau(H) + \tau(D)} \leq \max\left\{\frac{\tau_c(H)}{\tau(H)}, \frac{\tau_c(D)}{\tau(D)}\right\}.$$

The last inequality yields contradiction with criticality of G.

□

Proof of Lemma 25. If the graph $G \setminus \{x, y\}$ is connected then we are done with the proof according to Lemma 24. By Lemma 20, Lemma 26 and Lemma 29 the graph G has the following structure. In $G \setminus \{x, y\}$ there is exactly one component

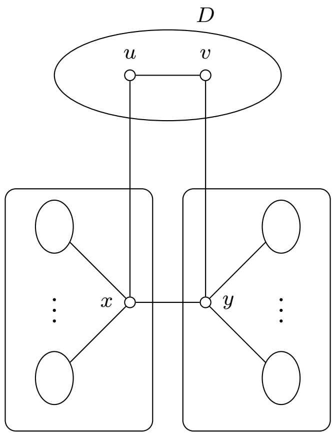


Figure 3.5: Illustration of situation in proof of Lemma 25.

D with an edge uv, where $u \in N(x)$ and $v \in N(y)$ and all other components of $G \setminus \{x, y\}$ are adjacent either to x or to y. (See Figure 3.5.) In the same manner as before we focus on three induced subgraphs H_1, H_2 and D such that the vertices x, y are in the union of connected vertex covers. Subgraph H_1 contains components adjacent to x, vertices x and u similarly, in H_2 there are vertices y, v and components seeing only y. Notice that sets $S \cap V(H_1)$, $S \cap V(H_2)$ and $S \cap V(D)$ are vertex covers of H_1, H_2, D, respectively. Thus, the vertex cover number of G is at least $\tau(H_1) + \tau(H_2) + \tau(D)$.

The union of connected vertex covers of the three subgraph is a connected vertex cover of G. Vertex x is in every connected vertex cover of H_1, y is in every connected vertex cover of H_2 and at least one neighbor of x or y is in the connected vertex cover of C due to edge uv. This size of a minimum connected vertex cover of G is at most $\tau_c(H_1) + \tau_c(H_2) + \tau_c(D)$.

$$\frac{\tau_c(G)}{\tau(G)} \leq \frac{\tau_c(H_1) + \tau_c(H_2) + \tau_c(D)}{\tau(H_1) + \tau(H_2) + \tau(D)} \leq \max\left\{\frac{\tau_c(H_1)}{\tau(H_1)}, \frac{\tau_c(H_2)}{\tau(H_2)}, \frac{\tau_c(D)}{\tau(D)}\right\}.$$

Using our estimates and Lemma 19 we derived inequality implying that one of the induced subgraphs has PoC at least equal to $\frac{\tau_c(G)}{\tau(G)}$. However, that contradicts the assumption that G is critical. $\square$

4. Price of connectivity for dominating set

Definition 10. *A* dominating set *of a graph G is a subset of vertices D such that every vertex has a neighbor in D or is in D. The size of the smallest dominating set is denoted by $\gamma_c(G)$ and a dominating set of this size is called* minimum.

Dominating set (DS)
Input: A graph $G = (V, E)$ and a positive integer k
Question: Does G have a dominating set D, such that $|D| \leq k$?

The decision version of the dominating set problem for a given graph G is a well studied NP-problem. To get acquainted with the problem, we show its NP-completeness by a reduction from vertex cover.

Theorem 31. *The dominating set problem is NP-complete [29].*

Proof. We prove a slightly stronger statement: For every graph G there exists a graph G' such that every vertex cover of G of size k corresponds to a vertex cover in G' of size $k+1$, and that G' has a vertex cover of size at most c if and only if G' has a dominating set of size at most c.

We first handle isolated vertices of G, which always belong to a dominating set but never belong to a vertex cover. Starting with G, we construct a new graph G'' that contains edges and vertices from G plus two additional vertices x and y. Additionally, we add the edge xy and an edge xv for every vertex $v \in V(G)$.

Observe that G'' has a vertex cover of size at most $k+1$ iff G has a vertex cover of size at most k, and that in G'' there are no more isolated vertices.

We construct our final graph G' as follows: To start, the graph G' contains all vertices and edges from G''. For every edge uv from $E(G'')$, we add a new vertex w and edges uw, vw. We have to prove that G'' has a vertex cover of size at most k if and only if G' has a dominating set of size at most k. First, suppose that G'' has a vertex cover S of size k. The set S contains at least one endpoint from each edge, so the original vertices from G'' are dominated. The vertices representing edges of the original graph are adjacent to two vertices, one of which is in S. Thus, these vertices are also dominated.

Conversely, let D be a dominating set of G'. Notice that if D includes a vertex w added for an edge uv, we can replace it by u or v; we are allowed to do this as vertex w dominates only u, v and these three vertices induce a triangle. Since every additional vertex is dominated, such a modified dominating set D contains at least one endpoint from every edge. The set D' is vertex cover of size at most k of G''.

□

4.1 Dominating set and independence

The size of a minimum dominating set that is also independent is called the *independent domination number* and denoted by $i(G)$. The *upper domination* *number* $\Gamma(G)$ is the size of the largest minimal dominating set. From these definitions we can immediately see that $\gamma(G) \leq i(G) \leq \alpha(G)$.

Observation 4. *An independent set S is maximal independent if and only if it is independent and dominating.*

Proof. Suppose that S is a maximal independent set. For any vertex v from $V(G) \setminus S$ the set $S \cup \{v\}$ is no longer independent, which means that every vertex in $V(G) \setminus S$ is adjacent to at least one vertex from S. Hence, S is a dominating set.

On the contrary, let S be a dominating and independent set and suppose that it is not maximal. There is a vertex u from $V(G) \setminus S$ such that $S \cup \{u\}$ is independent. However, S is not dominating because u does not share any edge with vertex from S. This is a contradiction.

□

Berge in his book [30] observed that a vertex set is maximal independent if it is minimal dominating set.

Observation 5 ([30])**.** *Every maximal independent set in a graph G is a minimal dominating set of G.*

Proof. Let S be a maximal independent set in G. By Observation 4 S is a dominating set. For the sake of contradiction suppose that D is not a minimal dominating set. In S there is a vertex v such that the set $S \setminus \{v\}$ is dominating. If the set $S \setminus \{v\}$ dominates vertices from $V \setminus S \setminus \{v\}$, then at least one vertex from S must be adjacent to v and S is not independent, which leads to a contradiction.

□

A direct corollary of this observation is the following inequality which is a part of the so-called *domination chain*. For more information on the chain we refer to Chapter 3 of the book [31] written by Haynes, Hedetniemi and Slayter.

Corollary. *For any graph G*

$$\gamma(G) \leq i(G) \leq \alpha(G) \leq \Gamma(G).$$

The domination chain does not need to be strict; if there is a graph G for which every induced subgraph H satisfies $i(H) = \gamma(G)$, we call such graph is called *domination perfect*. The domination perfect graphs were defined by Sumner and Moore in 1979 [32] who were inspired by a notion of perfect graphs. The characterization of the class was open for some time. Finally, in 1995, Zverovich and Zverovich [33] found a characterization in terms of 17 forbidden induced subgraphs. Aside from the above, many other variants of domination are also investigated. The focus will be on one of them, namely connected domination.

4.2 Connected dominating set

Definition 11. *A* connected dominating set *of a graph G is a dominating set D such that $G[D]$ is connected. The size of the smallest dominating set is denoted* *by $\gamma_c(G)$ and a connected dominating set of the size $\tau_c(G)$ is called a minimum connected dominating set.*

A connected dominating set can exist only in connected graphs (a DS must include at least one vertex from each connected component). Sampathkumar and Walikar, who were among the first researchers investigating connected domination, observe in [34] the following property of the connected domination number:

Theorem 32 ([34])**.** *Let G be a graph and H its spanning subgraph. Then, $\gamma_c(G) \leq \gamma_c(H)$.*

Another interesting property of connected dominating sets regarding spanning trees is the following.

Theorem 33. *[35] Let G be a graph on $n \geq 3$ vertices. Denote the maximum number of leaves over all spanning trees of G by $\epsilon_T(G)$. Then, $\gamma_c(G) = (n - \epsilon_T(G)) \leq n - 2$.*

Proof. Let T be a spanning tree with $\epsilon_T(G)$ vertices. Since T without leaves is a connected dominating set of G the following inequality follows: $\gamma_c(G) \leq n - \epsilon_T(G)$.

Suppose that D is a connected dominating set of G. The set D is connected and thus it has a spanning tree T'. We can add all vertices from $V(G) \setminus D$ to T' in such a way that each remaining vertex has only one edge to T'. We created a new spanning tree T of the entire graph G. The number of added vertices is at most $\epsilon_T(G)$. The size of a minimum connected dominating set is at least $n - \epsilon_T(G)$. □

Camby and Schaudt discovered in [36] that any minimal connected dominating set in P_t-free graph is either isomorphic to P_{t-2}, or it does not contain P_{t-2} as an induced subgraph. This theorem was used in several papers to design polynomial algorithms in P_t-free graphs. Specifically, Johnson, Paesani and Paulusma [8] use it to construct an efficient algorithm for finding a minimum connected vertex cover in $(sP_1 + P_5)$-free graphs and Bonomo et. al. [37] apply it to create a polynomial algorithm determining whether a given P_7-free graph is three colorable. Moreover, in the same paper, Camby and Schaudt show that a connected dominating set with these properties can be found in polynomial time.

Theorem 34. *For all $t \geq 3$, any connected P_t-free graph has a connected dominating set whose induced subgraph is either P_{t-2}-free, or isomorphic to P_{t-2}.*

We will not show their proof in its entirety. Instead we will explain to a reader a proof of a weaker lemma which is used in the original proof by Camby and Schaudt. What is more, this lemma will come in handy later in proofs of Theorems 37 and 39 characterizing Near-PoC-perfect graphs for dominating set.

Lemma 35. *Let G be a connected graph that is (P_k, C_k)-free for some $k \geq 4$ and let X be a minimal connected dominating set of G. Then $G[X]$ is P_{k-2}-free.*

Proof. Let X be a minimum connected dominating set of G. On the contrary suppose that X contains an induce path $v_1, \ldots, v_{k-2}$. Look at the set $X \setminus \{v_1\}$.

Because X is a minimum dominating set, $X \setminus \{v_1\}$ is not dominating or not connected. In the first case in G there is a vertex v_1' adjacent only to one vertex from X:v_1.

In the second case vertices $v_2, \dots, v_{k-2}$ are all in one connected component. Vertex v_1 has a neighbor v_1' in a different connected component. In both cases v_1 has a neighbor that is not adjacent to any from the vertices $v_2, \dots v_{k_2}$. We can use the same reasoning for a vertex v_{k-2}. Vertices $v_1', v_1, \dots, v_{k-2}, v_{k-2}'$ induce path or cycle on k vertices which contradicts the fact that G is (P_k, C_k)-free.

□

4.3 Price of connectivity

We can define price of connectivity for dominating set in the same way as we did for vertex cover in Chapter 3.

Definition 12. *For a graph G, the price of connectivity (PoC) for dominating set is defined as the ratio $\frac{\gamma_c(G)}{\gamma(G)}$.*

Observation 6. *The price of connectivity for any graph lies in the interval* $[1, 3)$*.*

Proof. Let D be a dominating set of G such that $G[D]$ consists of c connected components. To make D connected it is enough to add at most $2c - 2$ vertices. The distance between vertices in D is at most 2, since D is dominating. Assume that minimum dominating set of D has size k. Using the previous observation:

$$\frac{\gamma_c(G)}{\gamma(G)} \leq \frac{k(2c-2)}{k} \leq \frac{c(2c-2)}{c} = \frac{3c-2}{c} < 3.$$

□

Paths and cycles are examples of graphs attaining these bounds. Analogously, we can define PoC-perfect and PoC-near-perfect graphs for the dominating set:

Definition 13. *Graph G is* PoC-perfect *if for every subgraph H of G it holds $\gamma_c(H) = \gamma(H)$.*

Definition 14. *Graph G is* PoC-near-perfect *with a threshold $t \in (1, 3)$ if for every subgraph H of G this inequality holds $\frac{\gamma_c(H)}{\gamma(H)} \leq t$.*

Zverovich in [38] provides characterization of PoC-perfect graphs.

Theorem 36 (Zverovich [38])**.** *The following assertions are equivalent:*

1. *For a given graph G and every connected subgraph H it holds: $\gamma_c(H) = \gamma(H)$.*

2. *Graph G is (P_5, C_5)-free.*

Camby and Schaudt continue this line of research and in [4] they give a forbidden subgraphs characterization of PoC-near-perfect graphs with $\gamma_c(H) \leq \gamma(H) + 1$:

Theorem 37 (Camby and Schaudt [4]). *For every connected subgraph H of a graph G the inequality $\gamma_c(H) \leq \gamma(H) + 1$ holds if and only if G is (P_6, C_6)-free.*

Proof. It is easy to see that $\gamma(C_6) = \gamma(P_6) = 2$ and $\gamma_c(C_6) = \gamma_c(P_6) = 4$. These two graphs violate the inequality $\gamma_c(H) \leq \gamma(H) + 1$ and therefore they cannot be induced subgraphs of G.

Assume that G is (P_6, C_6)-free. Consider any connected induced subgraph H of G. Observe that H is also (P_6, C_6)-free. To prove the sufficiency of our condition, we need to show that H satisfies $\gamma_c(H) \leq \gamma(H) + 1$. Let D be a minimum dominating set of H with connected components $D_1, \ldots, D_k$. The set D must have at least two connected components, otherwise $\gamma_c(H) = \gamma(H)$ and we are done. Consider the smallest set of vertices C such that $H[D \cup C]$ is connected. Denote X a minimum connected dominating set of $H[D \cup C]$. We know that $\gamma_c(H) \leq |X|$. Define the set of indices $I = \{i \subseteq \{1, \ldots, k\} : D_i \cap X = \emptyset\}$. For each $i \in I$ consider $x_i \in X$ such that x_i is adjacent to a vertex from D_i. Such vertex exists because X is dominating. The vertices x_i are all from C and they do not have to be distinct. Define the set $S \subseteq X$:

$$S = \bigcup_{i \notin I} D_i \cap X \cup \{x_j : j \in I\}.$$

The size of S is by definition at most

$$\sum_{i \notin I} |D_i \cap X| + |I|$$

and that is at most $|D|$. A minimum dominating set has to include at least one vertex from every component.

We distinguish between two cases according to connectivity of $H[S]$.

Suppose that $H[S]$ is connected. Then $H[D \cup \{x_j : j \in I\}]$ is connected and C contains only vertices x_j. Recall that C is a minimal set with this property. In this case $X = S$ and this yields:

$$\gamma_c(H) \leq |X| = |S| \leq \sum_{i \notin I} |D_i \cap X| + |I| \leq |D| = \gamma(H).$$

$H[S]$ is not connected. By Lemma 35 $H[X]$ is P_4-free. It is known that P_4-free graphs on at least two vertices are either disconnected or their complement is disconnected [6] Specifically, it means that the graph $\overline{H[S]}$ is connected. Since $H[X]$ is connected, the graph $\overline{H[X]}$ is disconnected and it contains the set S in one connected component and a nonempty set of vertices Y which are not adjacent to any vertex from S. What can we say about structure of graph $H[X]$? In X, each vertex in Y is adjacent to every vertex from S. Thus, we can pick any vertex from $y \in Y$ such that $H[D \cup \{x_j : j \in I\} \cup \{y\}]$ is connected. So $|C| = \{x_j : j \in I\} \cup \{y\}$ and $X = S \cup \{y\}$. This gives us:

$$\gamma_c(H) \leq |X| + 1 = |S| + 1 \leq \sum_{i \notin I} |D_i \cap X| + |I| + 1 \leq |D| + 1 = \gamma(H) + 1.$$

This concludes the proof.

∎

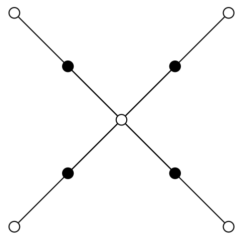


Figure 4.1: Graph F_k for $k = 4$ with minimum dominating set and minimum connected dominating set indicated by black vertices.

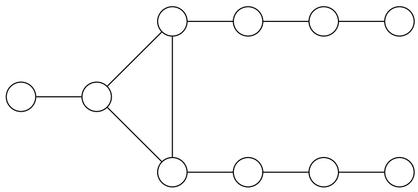


Figure 4.2: Graph F from Theorem 39.

To see that this bound is sharp, we consider the graphs F_k which are constructed from $K_{1,k}$ by subdividing every edge once. See Figure 4.1 for an illustration.

Camby and Schaudt [4] also attempt to characterize PoC-near-perfect graphs with a parameter $t = 2$, presenting the following theorem:

Theorem 38 (Camby and Schaudt [4])**.** *For every (P_8, C_8)-free graph G, it holds that $\gamma_c(G) \leq 2\gamma(G)$.*

The same authors in [4] and in [39] proposed a conjecture how to characterize all PoC-near-perfect graphs with parameter $t = 2$. Recently Bonamy et al. in [5] prove that their conjecture was correct:

Theorem 39 (Bonamy et al. [5])**.** *The following assertions are equivalent:*

1. *For a graph G and every induced subgraph H it holds: $\gamma_c(H) \leq 2\gamma(H)$.*
2. *The graph G is (C_9, P_9, F)-free.*

The remaining question is how to characterize PoC-near-perfect graphs for different values of parameter t. The construction of forbidden subgraphs from the previous theorems can be generalized in the following way:

Definition 15. *Class of graphs $\mathcal{T}_k$ consist of all graphs created as follows:*

- *Start with any tree on k vertices. Denote these nodes $v_1, v_2, \ldots, v_k$.*
- *Subdivide each edge twice.*
- *To each original leaf, we attach a new vertex (which becomes the new leaf).*
- *The vertices $N(v_i)$ may be connected by new edges if v_i is not adjacent to a leaf. If at any point $N(v_i)$ induces a connected graph, we add a new leaf to v_i and we do not add any more edges into $N(v_i)$.*

- *Finally, we may add a set M of edges among leaves if M is a (not necessarily maximal) matching.*

Observation 7. *For any graph G_k from $\mathcal{T}_k$, the domination number γ is equal to k and $\gamma_c(G_k) = 3k - 2$.*

Proof. Observe that vertices v_i dominate all vertices from $V(G_k)$, and so $\gamma \leq k$. Since the closed neighborhoods of individual vertices v_i are disjoint and any dominating set must contain at least one vertex from each $N[v_i]$, we have $\gamma \geq k$.

The size of a minimum connected dominating set is at most $3k-2$. Let us take a minimum dominating set that contains all vertices v_i. To make this dominating set connected, we need to add all vertices except leaves. Number of edges in the initial tree is $k - 1$, in the second step we added two vertices for each edge. The total sum is $k + (2k - 1) = 3k - 2$.

To finish the proof it remains to show that the size of a minimum connected dominating set is at least $3k - 2$. We consider a minimum connected dominating set D. If D contains all vertices v_i from the discussion above it follows that the size of D is at least $3k - 2$.

Assume that a vertex v_j is not in D. Moreover, we suppose that v_j is adjacent to a leaf l_1 and a vertex x_1. The situation is the following. The set D dominates l_1; thus, there is a leaf l_2 in D adjacent to l_1. From the construction of G_k we can see that all leaves are adjacent to the vertices v_i. In particular l_2 is adjacent to a vertex v_m. For an illustration see Figure 4.3. At least one vertex from a pair x_1, l_1 must be in D, otherwise v_j is not dominated by D. Vertices l_2, v_m and x_2 are in D. Since edges on leaves induce a matching we can modify D in such a way that it includes v_j and has at most the same size. Namely we D vertices v_j, x_1 and remove l_1.

Suppose that vertex v_j is not adjacent to any leaf. In that case $N(v_j)$ does not induce a connected subgraph. Vertex v_j is dominated by a vertex w. Denote by C_1 a connected component of $N(v_j)$. In the neighborhood of v_j there is another connected component C_2. Denote by w' a vertex from D that has a neighbor in C_2. Such vertex exists because vertices from C_2 needs to be dominated by D. Let w'' be a common neighbor of vertices v_j and w'. Since subgraph $H[D]$ is connected, it contains a path P from w to w'. The path P must contain an edge between two leaves l_1 and l_2. We remove l_1 and l_2 from D and add vertices v_j, w'' instead. Notice that such modified D is still a connected dominating set and the size is at most the same as before. For reference see Figure 4.3.

□

A direct corollary of this observation is that for any graph G_k from $\mathcal{T}_k$ the ratio $\frac{\gamma_c(G)}{\gamma(G)}$ is at most $3 - \frac{2}{k}$.

Conjecture 40 (Bonamy et al. [40])**.** *Let G be a graph. For each subgraph H of G it holds: $\frac{\gamma_c(H)}{\gamma(H)} \leq 3 - \frac{2}{k-1}$ if and only if G is $\mathcal{T}_k$-free.*

Let us look at graphs from $\mathcal{T}_k$ for various values of k.

- $k = 2$. The class $\mathcal{T}_2$ contains only graphs C_6 and P_6 which are forbidden subgraphs from Theorem 37.

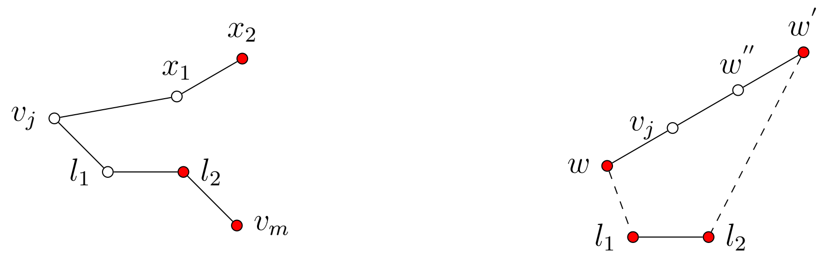


Figure 4.3: Illustration to the proof of Observation 7. Red vertices are in a connected dominating set D

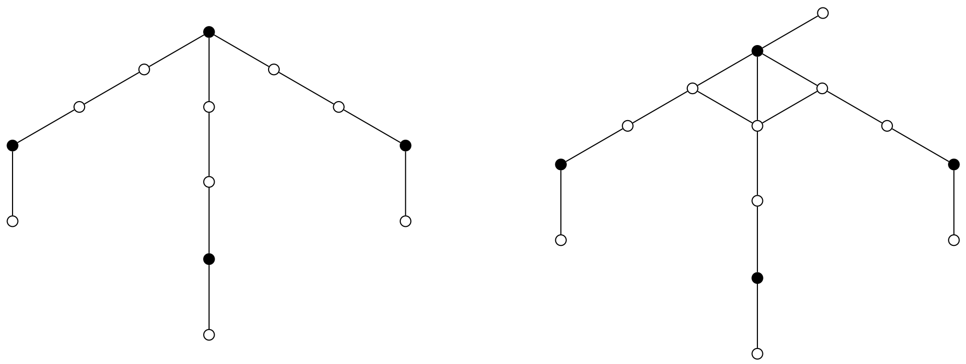


Figure 4.4: An example of graphs from $\mathcal{T}_4$ with vertices v_i indicated by black vertices.

- $k = 3$. The class $\mathcal{T}_3$ includes C_9, P_9, F. Their PoC is $\frac{7}{3}$ and they are forbidden subgraphs for graphs with PoC at most 2 from Theorem 38.
- $k = 4$. Some examples of graphs from $\mathcal{T}_4$ are P_{12}, C_{12} and graphs derived from a star on 4 vertices (see Figure 4.4).

Note that not all graphs from $\mathcal{T}_i$ are necessary critical. The construction from Definition 15 allows the existence of subgraphs with the PoC equal or greater than $3 - \frac{3}{k}$, namely an addition of edges to the neighborhood of v_i together with edges between leaves can create long induced paths. This fact does not contradict our conjecture, it only means that our characterization is not minimal.

Unfortunately we will show that Conjecture 40 is not true even for $k = 3$. To establish that we will show construction of critical graphs with PoC greater than $3 - \frac{3}{k-1}$ without forbidden subgraphs from $\mathcal{T}_i$.

Theorem 41. *Let G_d be a graph with $\gamma(G_d) = d$ constructed like this:*

- *Start with a star on k vertices with central vertex v.*
- *Add edges such that $N(v)$ induces $K_{\lceil\frac{d-1}{2}\rceil,\lfloor\frac{d-1}{2}\rfloor}$.*
- *To every vertex from the bipartite graph add a path on three vertices.*

Then for $d = 5$ and $d > 6$, the graph G_d is critical and we have $\frac{\gamma_c(G_d)}{\gamma(G_d)} = 3 - \frac{3}{d}$.

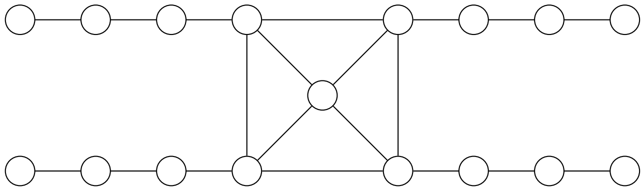


Figure 4.5: An example of a graph from Theorem 41 for $d = 5$.

Proof. The size of a minimum connected dominating set of graph G_d is $3d - 3$. It must include two vertices from each P_3 and the whole bipartite subgraph. The minimum dominating set contains one vertex from each P_3 and vertex v.

It remains to prove that G_d is a critical graph. In other words, every proper subgraph H must have a strictly smaller PoC than G_d. Due to relatively simple structure of G_d we analyze all possibilities of the subgraphs. For the rest of the proof we assume that the subgraph H is connected; the price of connectivity is defined only for connected graphs. Before diving deeper into the case analysis, we wish introduce some notation.

Vertices of graph G_d can be divided into the following sets:

- $P :=$ the set of vertices from P_3 added in the last step of the construction.
- $B_1, B_2 :=$ the sets of vertices from each part of the bipartite graph from the second step.
- $v :=$ the central vertex of G_k.

The number of induced paths with at least two vertices in $P \cap V(H)$ is denoted by l, the number of paths with one vertex in $P \cap V(H)$ is denoted by s.

We will distinguish between several cases according to the structure of the subgraph H.

Case 1. $H \cong P_9$.

The price of connectivity of P_9 is equal to $\frac{7}{3}$. To ensure criticality of G_d, we need to set the value d such that the PoC of P_9 is smaller than the PoC of G_d.

$$\frac{\gamma_c(G_d)}{\gamma(G_d)} > \frac{7}{3}.$$

This inequality implies that d must be greater than 4. We observe that P_9 is the longest possible induced path in G_d.

Case 2. $V(H) \cap P = \emptyset$.

In this case $\frac{\gamma_c(H)}{\gamma(H)} = 1$. Vertices from the bipartite subgraph are dominated by either the vertex v, or at most two adjacent vertices.

The set of vertices $H \cap P$ is nonempty. What does a minimum dominating set of H look like? In any dominating set, there must be at least one vertex from P for each long path and at least one vertex from B to dominate the short path. The minimum connected dominating set contains at most two vertices from P and one vertex from B for each path in $H \cap P$. We distinguish between the following cases according to the number of short paths and the presence of vertex v in H.

Case 3. $s = 0$ and $v \in H$. Every long path and the vertex v needs to be dominated, and so the size of a minimum dominating set of H is at least $l + 1$. Suppose that the vertices from a minimum CDS of H (which dominate the vertices on the long paths from $P \cap H$) include some from both parts B_1 and B_2. Thus $\gamma_c(H) \leq 3l$.

$$\frac{\gamma_c(H)}{\gamma(H)} \leq \frac{3l}{l+1} = 3 - \frac{3}{l+1}.$$

The last expression is increasing in l and $l \leq d + 1$, so it is at most $3 - \frac{3}{d}$. For smaller values of l the PoC of H is strictly smaller than $3 - \frac{3}{d}$. When l is equal to $d - 1$, at least one long path has one vertex not in H, otherwise $H \cong G$. Thus, we can improve the bound to:

$$\frac{3l-1}{l+1} \leq 3 - \frac{4}{d} < 3 - \frac{3}{d}.$$

In case that vertices from B adjacent to vertices from $P \cap H$ are from the same part B_i, the CDS contains one additional vertex that connects vertices from B_i. Thus, $\gamma_c(H) \leq 3l + 1$ and we have:

$$\frac{\gamma_c(H)}{\gamma(H)} \leq \frac{3l+1}{l+1} = 3 - \frac{2}{l+1}.$$

The number of paths with vertices from $V(H) \cap P$ is at most $\lceil \frac{d-1}{2} \rceil$, which we plug in:

$$3 - \frac{2}{l+1} \leq 3 - \frac{2}{\frac{d+2}{2}} = 3 - \frac{4}{d+2} < 3 - \frac{3}{d}.$$

The rightmost inequality is satisfied for $d > 6$. This bound is not sharp for odd d. The number of long paths is at most $\frac{d-1}{2}$. Using this bound yields the same result as before for $d > 3$.

Case 4. $s \geq 2$ and $v \in H$.

The size of the minimum dominating set is at least $l + s$. In the same manner as before, we distinguish several cases according to the size of a minimum connected dominating set.

- $\gamma_c(H) \leq 3l + s$. We compute:

$$\frac{\gamma_c(H)}{\gamma(H)} \leq \frac{3l+s}{l+s} = 3 - \frac{2s}{l+s} \leq 3 - \frac{4}{d-1} < 3 - \frac{3}{d}.$$

We have used that s is at least two.

- $\gamma_c(H) \leq 3l + s + 1$. In this situation all paths are adjacent to vertices from the same part of B:

$$\frac{\gamma_c(H)}{\gamma(H)} \leq \frac{3l+s+1}{l+s} = 3 - \frac{2s-1}{l+s} \leq 3 - \frac{3}{l+s}.$$

The number of paths is at most $\frac{d}{2}$.

$$3 - \frac{3}{l+s} \leq 3 - \frac{6}{d} < 3 - \frac{3}{d}.$$

Case 5. $s = 1$ and $v \in H$. We distinguish three subcases:

- All long paths on vertices $V(H) \cap P$ are adjacent to one part B_i.

  For a minimum connected dominating set, the inequality $\gamma_c(H) \leq 3l+2$ holds. By assuming $s = 1$, a minimum connected dominating set includes a vertex from B dominating the short path. Aside from this vertex, a minimum CDS also contains at most 3 vertices from each long path and at most one vertex from B. In particular, if one extra vertex from B is needed, then all paths on vertices $P \cap V(H)$ are adjacent to vertices either from B_1, or B_2. We calculate:

  $$\frac{\gamma_c(H)}{\gamma(H)} \leq \frac{3l+1}{l+1} = 3 - \frac{4}{d} \leq 3 - \frac{3}{d}.$$

  The last inequality holds for $d > 0$.

- $P \cap H$ induce two P_3 adjacent to different parts of B, or all paths $H[V(H) \cap P]$ are adjacent to B_i.

  This assumption implies that the size of minimum dominating set is at least $l + 2$. From our estimates it follows that

  $$\frac{\gamma_c(H)}{\gamma(H)} \leq \frac{3l+2}{l+2} = 3 - \frac{4}{d} < 3 - \frac{3}{d}.$$

- The paths of length 3 with vertices from $P \cap H$ are adjacent to one part of B. The size of a minimum connected dominating can be expressed by:

  $$\gamma_c(H) = 3l_3 + 2l_2 + 1,$$

  where l_3 is the number of P_3 and l_2 the number of P_2 in $H[V(H) \cap P]$. By our assumption, l_3 is at most $\lceil \frac{d-1}{2} \rceil \leq \frac{d}{2}$. Similarly $l_2 \leq \frac{d-1}{2}$.

  $$\gamma_c(H) = 3\frac{d}{2} + 2\frac{d-1}{2} - 1 = \frac{5}{2}d - 2$$

  .

  Vertex dominating short path cannot be counted twice. Let us bound price of connectivity of H

  $$\frac{\frac{5}{2}d - 2}{l_3 + l_2 + 1} \leq \frac{\frac{5}{2}d - 2}{d-1} = 2 - \frac{d}{2(d-1)}.$$

  The rightmost expression is strictly less than $3 - \frac{3}{d}$ for $d > 7$. For even d bound of $\gamma_c(H)$ can be improved because $l_3 \leq (d-1)/2$. Using this bound instead yields the same result for $d > 3$.

Case 6. $s = 0$ and $v \notin H$. We go through three subcases once more:

- The graph $H[V(H) \cap P]$ contains at least one P_3 and $\gamma_c(H) \leq 3l$. Due to the P_3 in $H[V(H) \cap P]$, the size of a dominating set is at least $l+1$.

$$\frac{\gamma_c(H)}{\gamma(H)} \leq \frac{3l}{l+1} = 3 - \frac{3}{l+1}$$

  For $l < d-1$, the rightmost expression is strictly less than price of connectivity of H. In the case when $l = d$ and at least one long path is shorter than 3, we are done by Case 3.

  Assume that each path from $H[V(H) \cap P]$ is P_3 and $l = d-1$. If this is the case then observe that size of minimum dominating set is equal to $l+2$.

$$\frac{\gamma_c(H)}{\gamma(H)} \leq \frac{3l}{l+2} = 3 - \frac{6}{d+1} < 3 - \frac{3}{d}$$

  The last inequality holds for all d.

- Paths from $H[V(H) \cap P]$ are adjacent to vertices from one part of B.

$$\frac{\gamma_c(H)}{\gamma(H)} \leq \frac{3l+1}{l+1} = 3 - \frac{2}{l+1} \leq 3 - \frac{2}{\frac{d+2}{2}} = 3 - \frac{4}{d+2} < 3 - \frac{3}{d}.$$

  Since all paths are adjacent to vertices from one part of B the number of long paths is at most $\frac{d}{2}$. The last inequality holds for $d > 6$. For odd d the number l is at most $\frac{d-1}{2}$. Using this bound instead we derive the same result for $d > 3$.

- All paths in $H[V(H) \cap P]$ have length 2.

$$\frac{\gamma_c(H)}{\gamma(H)} \leq \frac{2l+1}{l} = 2 - \frac{1}{l}$$

  Inequality $2 - \frac{1}{l} < 3 - \frac{3}{d}$ holds for $d > 4$.

Case 7. $s = 1$ and $v \notin H$.

Follows the same proof from Case 5.

Case 8. $s \geq 2$ and $v \notin H$.

This situation is identical to Case 4.

□

The price of connectivity of graphs from $\mathcal{T}_4$ is $\frac{5}{2}$. We proved that G_5 is critical; thus it cannot contain induced subgraphs with PoC greater than $\frac{12}{5}$. In particular it cannot contain graphs with PoC equal to $\frac{5}{2}$. The conjecture for $k = 4$ claims that PoC-near-perfect graphs for $t = \frac{7}{3}$ are exactly $\mathcal{T}_4$-free graphs. However, G_5 is a graph with price of connectivity greater than $\frac{7}{3}$. Hence, we found an counterexample.

In general we can derive the following corollary.

Corollary. *For $k > 5$ graphs G_{k+1} from Theorem 41 are $\mathcal{T}_k$-free.*

Proof. From Theorem 41 graphs G_{k+1} are critical with price of connectivity equal to $3 - \frac{3}{k+1}$. By Observation 7 the price of connectivity of graphs from $\mathcal{T}_k$ is equal to $3 - \frac{2}{k}$. The inequality

$$3 - \frac{3}{k+1} < 3 - \frac{2}{k}$$

holds for $k > 2$. Thus, graphs from $\mathcal{T}_k$ cannot be induced subgraphs of G_{k+1}. □

4.4 Conclusions

Chapter 3 discussed the characterization of PoC-near-perfect graphs for vertex cover for $t = 1, \frac{4}{3}, \frac{3}{2}$ as given by Camby and Schaudt. The same authors propose a possible characterization of the PoC-near-perfect graphs for $t \leq \frac{5}{3}$. One possible direction of future research may be try to prove their conjecture and find a list of forbidden subgraphs for different values of t. In the same chapter, we have discussed the structural properties of PoC-cricital graphs; namely we observe that vertices from any minimum vertex cover have an independent neighborhood. We derive a similar result for a pair of adjacent vertices from a minimum vertex cover in Lemma 25. The remaining question is whether the same property holds even for non-adjacent pair of vertices x, y.

In Chapter 4 we show critical graphs with the price of connectivity for a dominating set equal to $3 - \frac{3}{k}$. From the results listed in Chapter 4 about PoC-near-perfect graphs that in the interval $(1, 3)$, there exist rational values of the price of connectivity that cannot be attained by any PoC-near-perfect graph. Interesting research problem to examine further could be to specify for which values of the price of connectivity PoC-near-perfect graphs exist.

Notice that all PoC-near-perfect graphs do not contain long paths. Their study may be helpful in a better understanding to the class of P_r-graphs. Recall that one of the few general characterizations of P_r-graphs (Theorem 34) was established during an investigation of PoC-near-perfect graphs.

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