Floyd-Hoare Logic

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Axiomatic Semantics

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1. Language for making assertions about programs

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2. Rules for establishing, i.e. proving the assertions

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Some typical languages of assertions:

- First-order logic
- Other logics (e.g., temporal logic)

TODAY'S PLAN

1. **Define** a small language
2. **Define** a logic for verifying assertions

IMP: An Imperative Language

syntax and operational
semantics

IMP Syntactic Entities

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- Int integer literals `n`

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- Aexp arithmetic expressions **`e`**
- Bexp boolean expressions **`b`**
- Comm commands **`c`**

Abstract Syntax: Arith Expressions (Aexp)

$\mathbf{e} ::=$	$\mathbf{n}$	for $\mathbf{n} \in \mathrm{Int}$
	$\mid \mathbf{x}$	for $\mathbf{x} \in \mathrm{Loc}$
	$\mid \mathbf{e}_1 + \mathbf{e}_2$	for $\mathbf{e}_1, \mathbf{e}_2 \in \mathrm{Aexp}$
	$\mid \mathbf{e}_1 - \mathbf{e}_2$	for $\mathbf{e}_1, \mathbf{e}_2 \in \mathrm{Aexp}$
	$\mid \mathbf{e}_1 * \mathbf{e}_2$	for $\mathbf{e}_1, \mathbf{e}_2 \in \mathrm{Aexp}$

Note:
- Variables are not declared
- All variables have integer type
- There are no side-effects

Abstract Syntax: Bool Expressions (Bexp)

$$
\begin{array}{lll}
\texttt{true} ::= & \texttt{true} & \\
 & \mid\ \texttt{false} & \\
 & \mid\ \texttt{e}_1 = \texttt{e}_2 & \text{for } \texttt{e}_1, \texttt{e}_2 \in \text{Aexp} \\
 & \mid\ \texttt{e}_1 < \texttt{e}_2 & \text{for } \texttt{e}_1, \texttt{e}_2 \in \text{Aexp} \\
 & \mid\ !\,\texttt{b} & \text{for } \texttt{b} \in \text{Bexp} \\
 & \mid\ \texttt{b}_1 \mid\mid \texttt{b}_2 & \text{for } \texttt{e}_1,\ \texttt{e}_2 \in \text{Bexp} \\
 & \mid\ \texttt{b}_1\ \&\ \texttt{b}_2 & \text{for } \texttt{e}_1,\ \texttt{e}_2 \in \text{Bexp}
\end{array}
$$

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`c ::= skip`	
`| x:= e`	for $\mathbf{x} \in$ L & $\mathbf{e} \in$ Aexp
`|` $c_1;c_2$	for $c_1, c_2 \in$ Comm
`| if b then` c_1 `else` c_2	for $\mathbf{b} \in$ Bexp & $c_1, c_2 \in$ Comm

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| $x := e$ for $x \in L$ & $e \in \text{Aexp}$

| $c_1 ; c_2$ for $c_1, c_2 \in \text{Comm}$

| **if** b **then** c_1 **else** c_2 for $b \in \text{Bexp}$ & $c_1, c_2 \in \text{Comm}$

| **while** b **do** c for $c \in \text{Comm}$ & $b \in \text{Bexp}$

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Note:

- Typing rules embedded in syntax definition
  - Other checks may not be context-free
  - need to be specified separately (e.g., variables are declared)

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| $\mathtt{if\ b\ then\ c_1\ else\ c_2}$	for $\mathtt{b} \in Bexp$ & $\mathtt{c_1}, \mathtt{c_2} \in Comm$
| $\mathtt{while\ b\ do\ c}$	for $\mathtt{c} \in Comm$ & $\mathtt{b} \in Bexp$

Note:

- Typing rules embedded in syntax definition
  - Other checks may not be context-free
  - need to be specified separately (e.g., variables are declared)
- Commands contain all the side-effects in the language

Semantics of IMP : States

- Meaning of IMP expressions depends on the values of variables
- A state σ is a function from Loc to Int
  - Value of variables at a given moment
  - Set of all states is Σ = Loc –> Int

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- Only if there is a unique derivation …

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Define $\sigma[\mathbf{x} := n]$ as:

$\sigma[\mathbf{x} := n](\mathbf{x}) = n$

$\sigma[\mathbf{x} := n](\mathbf{y}) = \sigma(\mathbf{y})$

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$$\frac{\langle \texttt{e}, \sigma \rangle \Downarrow n}{\langle \texttt{x := e}, \sigma \rangle \Downarrow \sigma[\mathbf{x} := n]}$$

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- Other logics (e.g., temporal logic)

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History : Program Verification

- Turing 1949: Checking a large routine
- Floyd 1967: Assigning meaning to programs
- Hoare 1971: An "axiomatic basis for computer programming"

- Program Verifiers (70's – 80's)
- PREfix: Symbolic Execution for bug-hunting (WinXP)
- Software Validation tools

Foundation for Software Verification

- Deductive Verifiers: ESCJava, Spec#, Verifast, Y0, ...
- Model Checkers: SLAM, BLAST,...
- Test Generators: DART, CUTE, EXE,...

Hoare Triples

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- Partial correctness assertion: $\{A\}$ **c** $\{B\}$

  If A holds in state σ *and exists* σ' *s.t.* $\langle$**c**$, \sigma \rangle \Downarrow \sigma'$

  then B holds in σ'

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- [A] is called precondition, [B] is called postcondition

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  then there exists σ' s.t. $<\mathtt{c}, \sigma> \Downarrow \sigma'$ *and B holds in σ'*

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- Total correctness assertion: [A] `c` [B]

  If A holds in state σ

  then there exists σ' s.t. $<$`c`$, \sigma > \Downarrow \sigma'$ *and B holds in σ'*

- [A] is called precondition, [B] is called postcondition

- Example: { y=x } `z := x; z := z+1` { y < z }

The Assertion Language

- Arith Exprs + First-order Predicate logic

$$
\begin{aligned}
A ::= \;& \text{true} \mid \text{false} \\
&\mid e_1 = e_2 \mid e_1 \leq e_2 \\
&\mid \neg A \mid A_1 \mathbin{\&\&} A_2 \mid A_1 \mathbin{||} A_2 \mid A_1 \Rightarrow A_2 \\
&\mid \backslash\text{exists } x.A \mid \backslash\text{forall } x.A
\end{aligned}
$$

- IMP boolean expressions are assertions

Semantics of Assertions

- Judgment $\sigma \models A$ means assertion holds in given state

$\sigma \models \text{true}$	always
$\sigma \models e_1 = e_2$	iff $\langle e_1, \sigma\rangle \Downarrow n_1$, $\langle e_2,\sigma\rangle \Downarrow n_2$ and $n_1 = n_2$
$\sigma \models e_1 <= e_2$	iff $\langle e_1, \sigma\rangle \Downarrow n_1$, $\langle e_2,\sigma\rangle \Downarrow n_2$ and $n_1 <= n_2$
$\sigma \models A_1$ && A_2	iff $\sigma \models A_1$ and $\sigma \models A_2$
$\sigma \models A_1 \mid\mid A_2$	iff $\sigma \models A_1$ or $\sigma \models A_2$
$\sigma \models A_1 => A_2$	iff $\sigma \models A_1$ implies $\sigma \models A_2$
$\sigma \models$ \exists x.A	iff for *some* n in Z. $\sigma[x := n] \models A$
$\sigma \models$ \forall x. A	iff for *all* n in Z. $\sigma[x := n] \models A$

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Formal definition of partial correctness assertion:

$\models \{ A \}\ c\ \{ B \}$

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forall σ in Σ. $\sigma \models A$

implies [forall σ' in Σ. $\langle c, \sigma \rangle \Downarrow \sigma'$ implies $\sigma' \models B$]

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Derivation Rules for Hoare Triples

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- Write $\vdash \{A\}$ `c` $\{B\}$ when we can derive the triple using derivation rules

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$$\frac{A' \Rightarrow A \qquad \vdash \{A\}\ c\ \{B\} \qquad B \Rightarrow B'}{\vdash \{A'\}\ c\ \{B'\}}$$

Deriv. Rules for Hoare Logic |− {A} c {B}

Rules for each language construct

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$$\frac{}{\vdash \{A\}\ \texttt{skip}\ \{A\}}$$

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And the rule of consequence…

Free and Bound Variables

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Key idea in logic/PL: scoping & substitution

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Prove $\vdash$ {true} `x:=e` { x = e }

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$$\dfrac{\text{true} \Rightarrow e = e \qquad \dfrac{\mathbf{x} \text{ does not appear in } \mathbf{e}}{\vdash \{e = e\}\ \mathbf{x:=e}\ \{x = e\}}}{\vdash \{\text{true}\}\ \mathbf{x:=e}\ \{x = e\}}$$

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x<=0 => x<=6 |- {x<=6} **`W`** {x<=6 & x>5}

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Soundness of Axiomatic Semantics

Formal Statement of Soundness:

If $\vdash \{A\}\ c\ \{B\}$ then $\models \{A\}\ c\ \{B\}$

Equivalently

If $H :: \vdash \{A\}\ c\ \{B\}$ then

forall σ if $\sigma \models A$ and $D :: \langle c, \sigma \rangle \Downarrow \sigma'$ then $\sigma' \models B$

Proof:

Simultaneous induction on structure of D and H

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1. When to apply the rule of consequence ?
2. What invariant to use for while ?
3. How to prove implications (conseq. rule)?

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2. What invariant to use for while ?
3. How to prove implications (conseq. rule)?

Hint:

Algorithmic Verification

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2. What invariant to use for while ?
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Hint:

(3) involves ... SMT

(2) invariants are the hardest problem

(1) lets see how to deal with ...

Making Floyd-Hoare Algorithmic:

Predicate Transformers

Technique: Weakest Preconditions

Technique: Weakest Preconditions

After what preconditions does postcond. `x>0` hold?

Technique: Weakest Preconditions

$\vdash \{ y > 10 \}$ `x := y` $\{x > 0\}$

After what preconditions does postcond. $x>0$ hold?

Technique: Weakest Preconditions

$\vdash \{ y > 10 \}$ `x := y` $\{x > 0\}$

$\vdash \{ y > 100 \}$ `x := y` $\{x > 0\}$

After what preconditions does postcond. $x>0$ hold?

Technique: Weakest Preconditions

$\vdash$ { y >10 } `x := y` {x > 0}

$\vdash$ { y >100 } `x := y` {x > 0}

$\vdash$ { x=2 & y=5 } `x := y` {x > 0}

After what preconditions does postcond. x>0 hold?

Technique: Weakest Preconditions

$\vdash$ { y >10 } `x := y` {x > 0}

$\vdash$ { y >100 } `x := y` {x > 0}

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After what preconditions does postcond. x>0 hold?

WP(`c`,B): weakest predicate s.t. {WP(`c`,B)} `c` {B}

- For any A we have {A} `c` {B} iff A => WP(`c`, B)

Technique: Weakest Preconditions

|– { y >10 } `x := y` {x > 0}

|– { y >100 } `x := y` {x > 0}

|– { x=2 & y=5 } `x := y` {x > 0}

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- For any A we have {A} `c` {B} iff A => WP(`c`, B)

How to **verify** |– {A} `c` {B} ?

Technique: Weakest Preconditions

$\vdash$ { y >10 } `x := y` {x > 0}

$\vdash$ { y >100 } `x := y` {x > 0}

$\vdash$ { x=2 & y=5 } `x := y` {x > 0}

After what preconditions does postcond. x>0 hold?

WP(`c`,B): weakest predicate s.t. {WP(`c`,B)} `c` {B}

- For any A we have {A} `c` {B} iff A => WP(`c`, B)

How to **verify** $\vdash$ {A} `c` {B} ?

1. **Compute:** WP(`c`,B)
2. **Prove:** A => WP(`c`,B)

Weakest Preconditions

Weakest Preconditions

Define wp(c, B) using Hoare rules

Weakest Preconditions

Define wp(c, B) using Hoare rules

$$\frac{\vdash \{A\}\ \mathtt{c_1}\ \{B\} \qquad \vdash \{B\}\ \mathtt{c_2}}{\vdash \{A\}\ \mathtt{c_1;}\ \ \mathtt{c_2}\ \{C\}}$$

Weakest Preconditions

Define wp(c, B) using Hoare rules

wp(c_1;c_2, B)

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Define wp(c, B) using Hoare rules

$\mathrm{wp}(\mathtt{c_1;c_2}, B)$

$= \mathrm{wp}(\mathtt{c_1}, \mathrm{wp}(\mathtt{c_2}, B))$

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$$\frac{}{\vdash \{[e/x]A\}\ \mathtt{x:=e}\ \{A\}}$$

Weakest Preconditions

Define wp(c, B) using Hoare rules

$$\mathrm{wp}(\mathtt{c_1;c_2}, B) = \mathrm{wp}(\mathtt{c_1}, \mathrm{wp}(\mathtt{c_2}, B))$$

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Weakest Preconditions

Define wp(c, B) using Hoare rules

$\text{wp}(\mathbf{c}_1 ; \mathbf{c}_2, B)$

$= \text{wp}(\mathbf{c}_1, \text{wp}(\mathbf{c}_2, B))$

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$\text{wp}(\mathbf{x := e}\ , B)$

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Weakest Preconditions

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$\mathrm{wp}(\mathtt{x:=e}\ , B)$

$= [e/x]B$

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Weakest Preconditions

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$= [e/x]B$

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$\mathrm{wp}(\mathbf{if\ e\ then\ c_1\ else\ c_2}, B)$

Weakest Preconditions

Define wp(c, B) using Hoare rules

$wp(\mathbf{c_1;c_2}, B)$

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$$\frac{\vdash \{A\}\ \mathbf{c_1}\ \{B\} \quad \vdash \{B\}\ \mathbf{c_2}}{\vdash \{A\}\ \mathbf{c_1;\ c_2}\ \{C\}}$$

$wp(\mathbf{x:=e}\ , B)$

$= [e/x]B$

$$\frac{}{\vdash \{[e/x]A\}\ \mathbf{x:=e}\ \{A\}}$$

$wp(\mathbf{if\ e\ then\ c_1\ else\ c_2}, B)$

$= e \Rightarrow wp(\mathbf{c_1}, B)\ \&\&\ !e \Rightarrow wp(\mathbf{c_2}, B)$

$$\frac{\vdash \{A\&b\}\ \mathbf{c_1}\ \{B\} \quad \vdash \{A\ \&\ !b\}\ \mathbf{c_2}\ \{B\}}{\vdash \{A\}\ \mathbf{if\ b\ then\ c_1\ else\ c_2}\ \{B\}}$$

Weakest Preconditions for Loops

Weakest Preconditions for Loops

Start from the equivalence

`while b do c` =

  `if b then (c; while b do c) else skip`

Let W = wp(**`while b do c`**, B)

It must be that: W = [b => wp(**`c`**, W) & !b=>B]

But this is a recursive equation! How to compute?!

- We'll return to finding loop WPs later …

Technique: Strongest Postconditions

Technique: Strongest Postconditions

What postcond. is guaranteed after prec. y>100 ?

Technique: Strongest Postconditions

|- { y > 100 } `x := y` {x > 10}

What postcond. is guaranteed after prec. y>100 ?

Technique: Strongest Postconditions

$\vdash \{ y > 100 \}$ `x := y` $\{x > 10\}$

$\vdash \{ y > 100 \}$ `x := y` $\{x > 20\}$

What postcond. is guaranteed after prec. $y>100$?

Technique: Strongest Postconditions

$\vdash \{\, y > 100 \,\}$ `x := y` $\{x > 10\}$

$\vdash \{\, y > 100 \,\}$ `x := y` $\{x > 20\}$

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What postcond. is guaranteed after prec. $y>100$?

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|- { y > 100 } `x := y` {x > 100}

What postcond. is guaranteed after prec. y>100 ?

SP(`c`,A): **strongest predicate** s.t. {A} `c` {SP(`c`,A)}

- For any B we have {A} `c` {B} iff SP(`c`,A) => B

Technique: Strongest Postconditions

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$\vdash \{ y > 100 \}$ `x := y` $\{x > 100\}$

What postcond. is guaranteed after prec. $y>100$?

SP(`c`,A): **strongest predicate** s.t. {A} `c` {SP(`c`,A)}

- For any B we have {A} `c` {B} iff SP(`c`,A) => B

How to verify {A} `c` {B} ?

Technique: Strongest Postconditions

$\vdash \{ y > 100 \}$ `x := y` $\{x > 10\}$

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What postcond. is guaranteed after prec. $y>100$?

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- For any B we have {A} `c` {B} iff SP(`c`,A) => B

How to verify {A} `c` {B} ?

1. **Compute:** SP(`c`,A)

Technique: Strongest Postconditions

$\vdash$ { y > 100 } `x := y` {x > 10}

$\vdash$ { y > 100 } `x := y` {x > 20}

$\vdash$ { y > 100 } `x := y` {x > 100}

What postcond. is guaranteed after prec. y>100 ?

SP(`c`,A): **strongest predicate** s.t. {A} `c` {SP(`c`,A)}

- For any B we have {A} `c` {B} iff SP(`c`,A) => B

How to verify {A} `c` {B} ?

1. **Compute:** SP(`c`,A)
2. **Prove:** SP(`c`,A) => B

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Define sp(c, B) following Hoare rules

$sp(c_1; c_2, A) =$

$sp(c_2, sp(c_1, A))$

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Define sp(c, B) following Hoare rules

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$sp(\mathbf{c_1;c_2}, A) =$

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$sp(\mathbf{x:=e}\ , A) =$

Strongest Postconditions

Define sp(c, B) following Hoare rules

sp(c_1;c_2, A) =

sp(c_2, sp(c_1,A))

sp(x:=e , A) =

\exists x_0. [x_0/x]A && x=[x_0/x]e

Strongest Postconditions

Define sp(c, B) following Hoare rules

sp(`c`$_1$`;c`$_2$, A) =

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sp(**x:=e** , A) =

\exists x_0. [x_0/x]A && x=[x_0/x]e

sp(**if e then** $\mathbf{c_1}$ **else** $\mathbf{c_2}$, A) =

Strongest Postconditions

Define sp(c, B) following Hoare rules

sp(`c1;c2`, A) =

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sp(`x:=e` , A) =

\exists x_0. $[x_0/x]A$ && $x=[x_0/x]e$

$$\frac{}{\vdash \{[e/x]A\}\ \texttt{x:=e}\ \{A\}}$$

sp(`if e then c1 else c2`, A) =

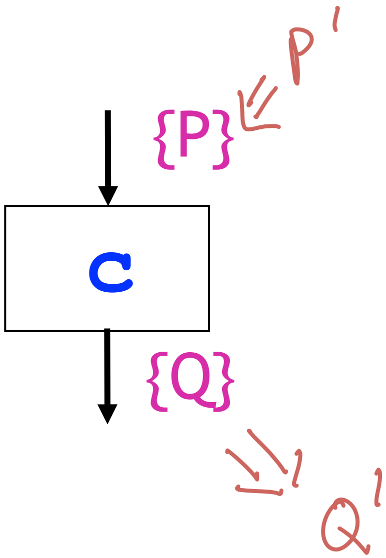
sp(`c1`, A & e) || sp(`c2`, A & !e)

$$\frac{\vdash \{A\&b\}\ c_1\ \{B\} \quad \vdash \{A\ \&\ !b\}\ c_2\ \{B\}}{\vdash \{A\}\ \texttt{if b then c}_1\ \texttt{else c}_2\ \{B\}}$$

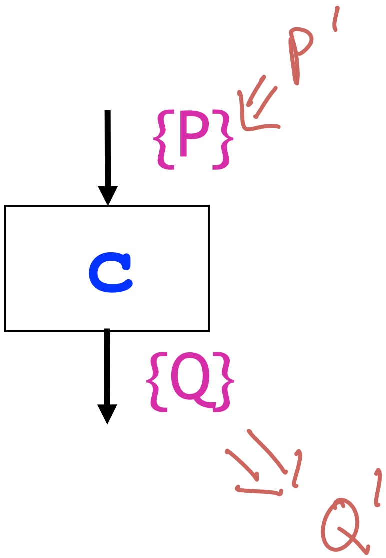
Axiomatic Semantics on Flow Graphs

Floyd’s Original Formulation

Axiomatic Semantics over Flow Graphs

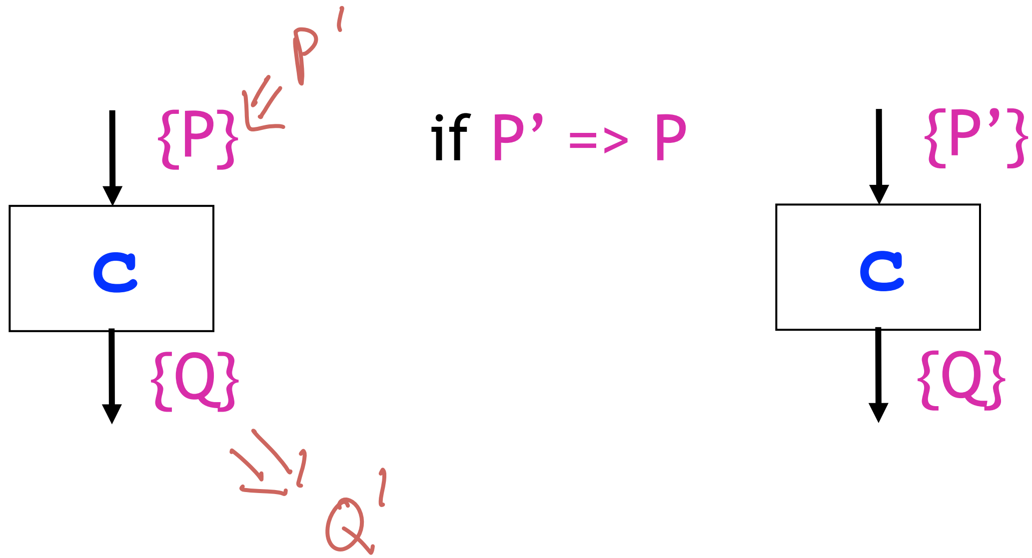


Axiomatic Semantics over Flow Graphs

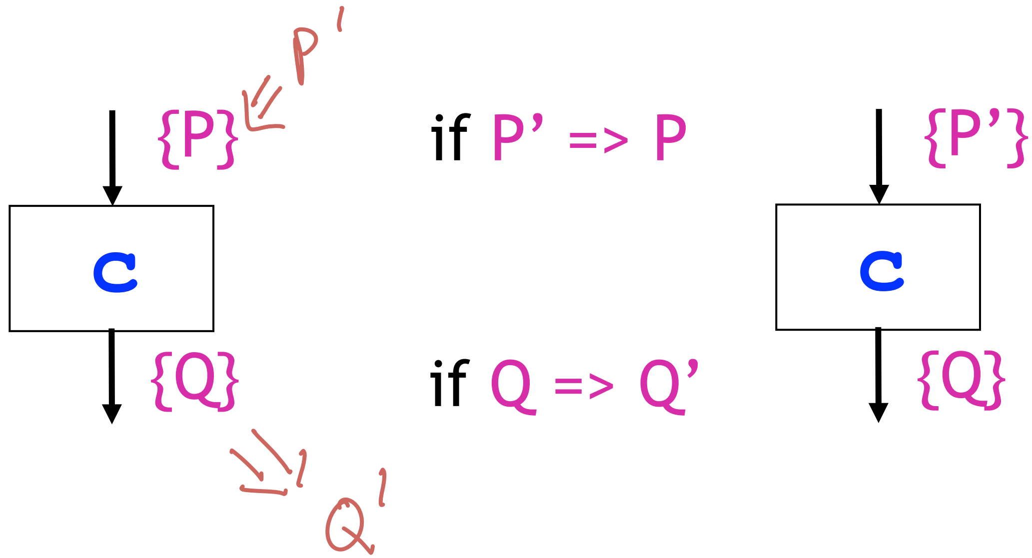


if P’ => P

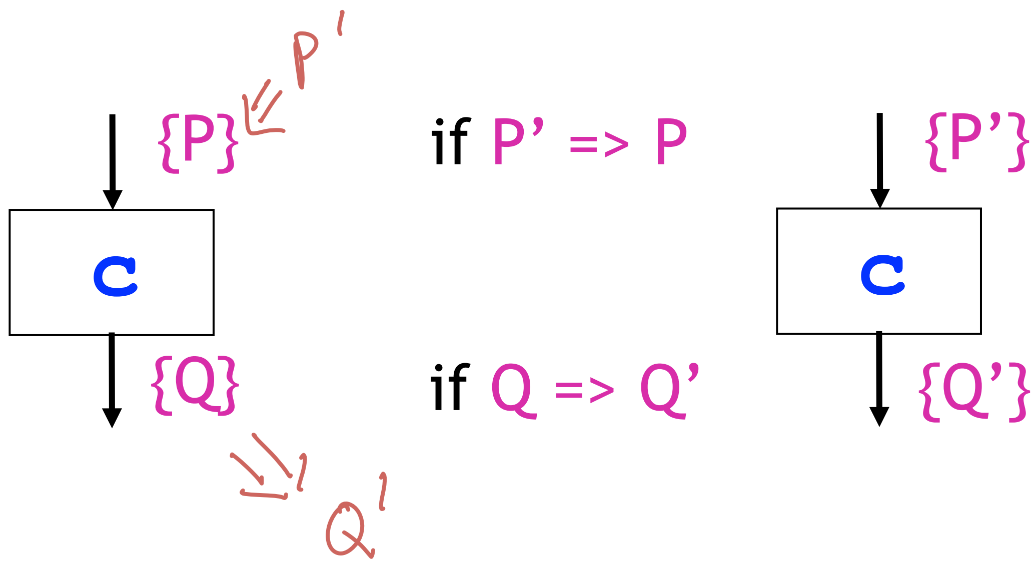
Axiomatic Semantics over Flow Graphs



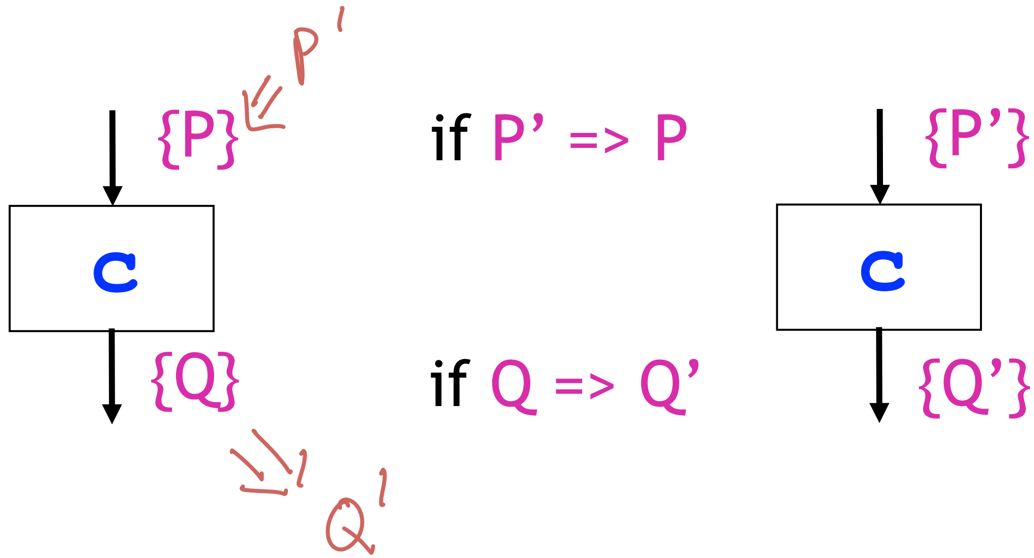
Axiomatic Semantics over Flow Graphs



Axiomatic Semantics over Flow Graphs



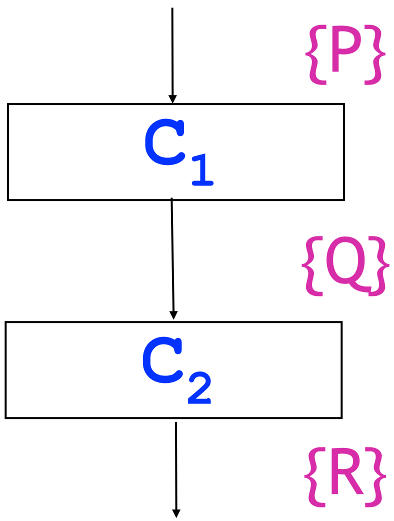
Axiomatic Semantics over Flow Graphs



Relaxing Specifications via Consequence

Will revisit later as **subtyping**

Sequential Composition



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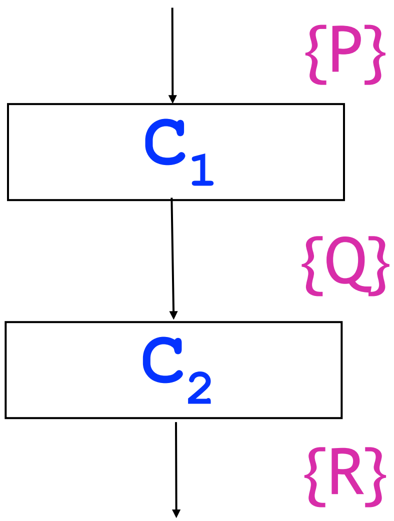

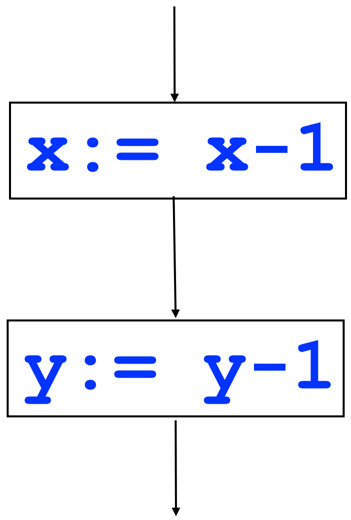

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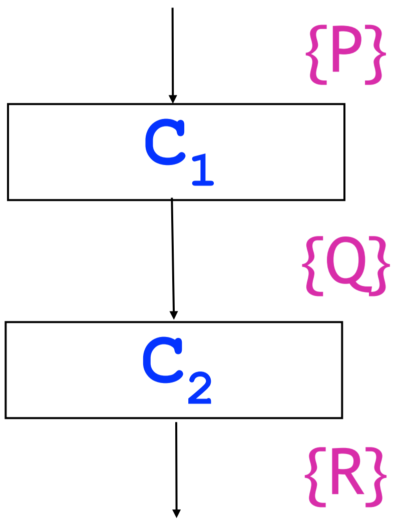

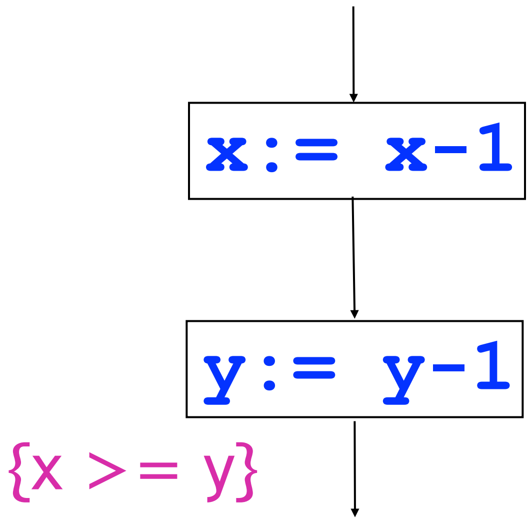

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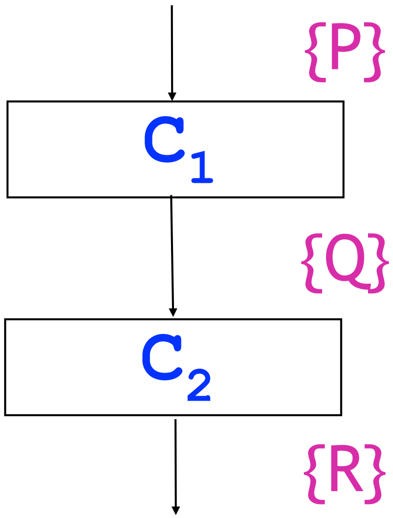
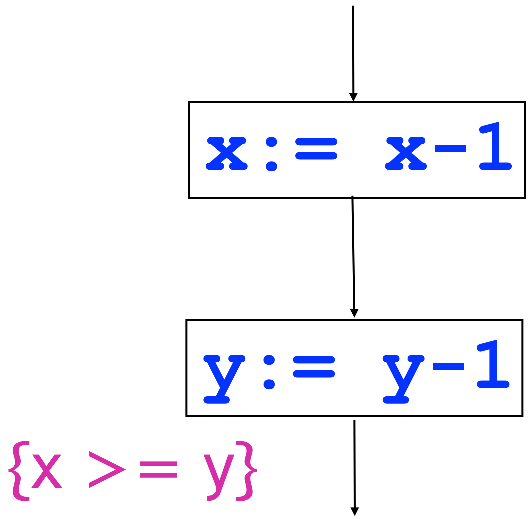
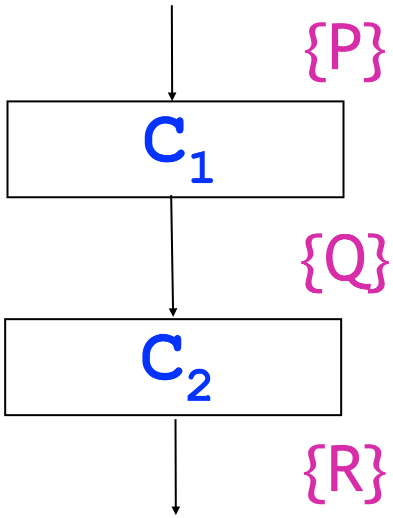
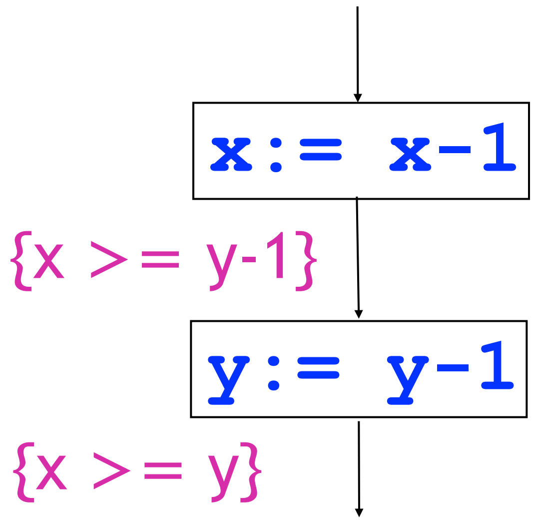




Backwards using weakest preconditions

Sequential Composition





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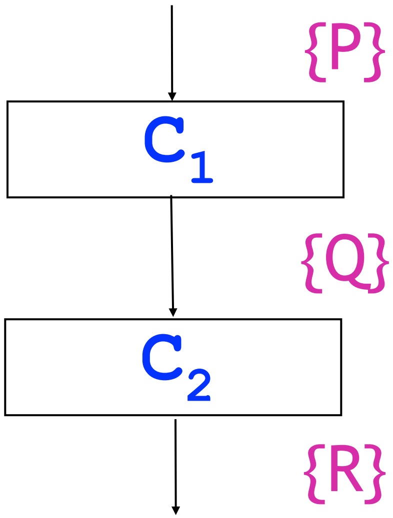

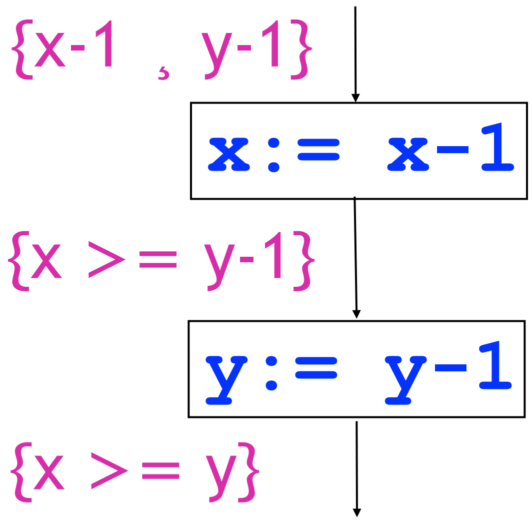

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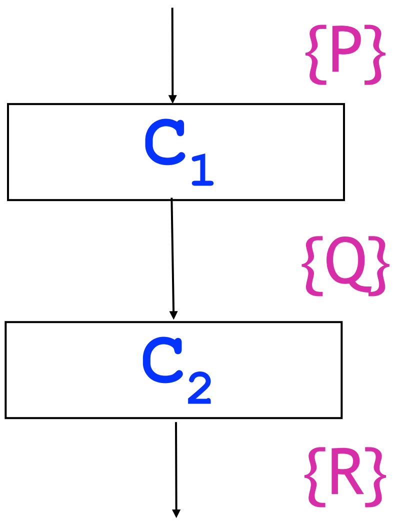

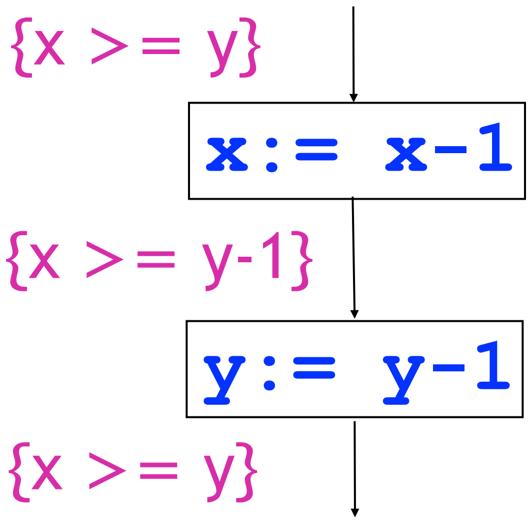

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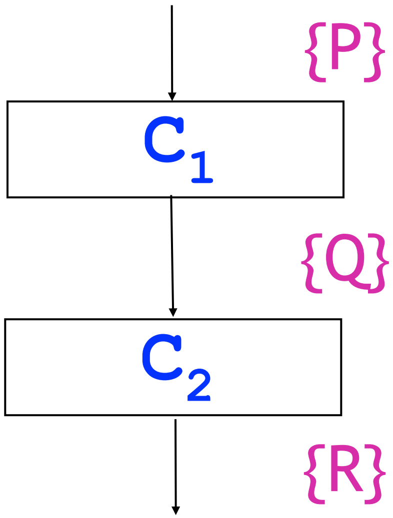
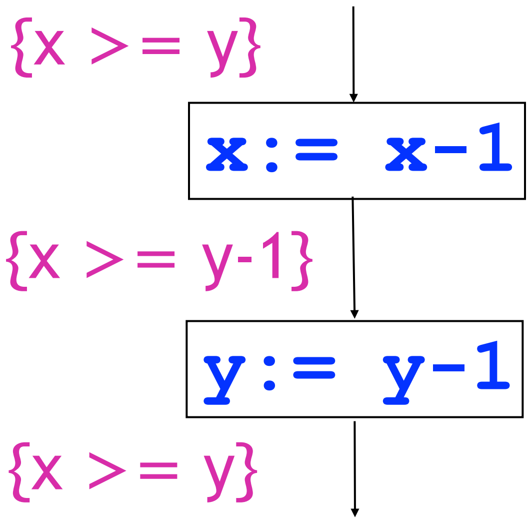




Backwards using weakest preconditions

Forwards using strongest postconditions

Sequential Composition

{P}

C_1

{Q}

C_2

{R}

{x >= y} {x ¸ y}

```
x:= x-1
```

{x >= y-1}

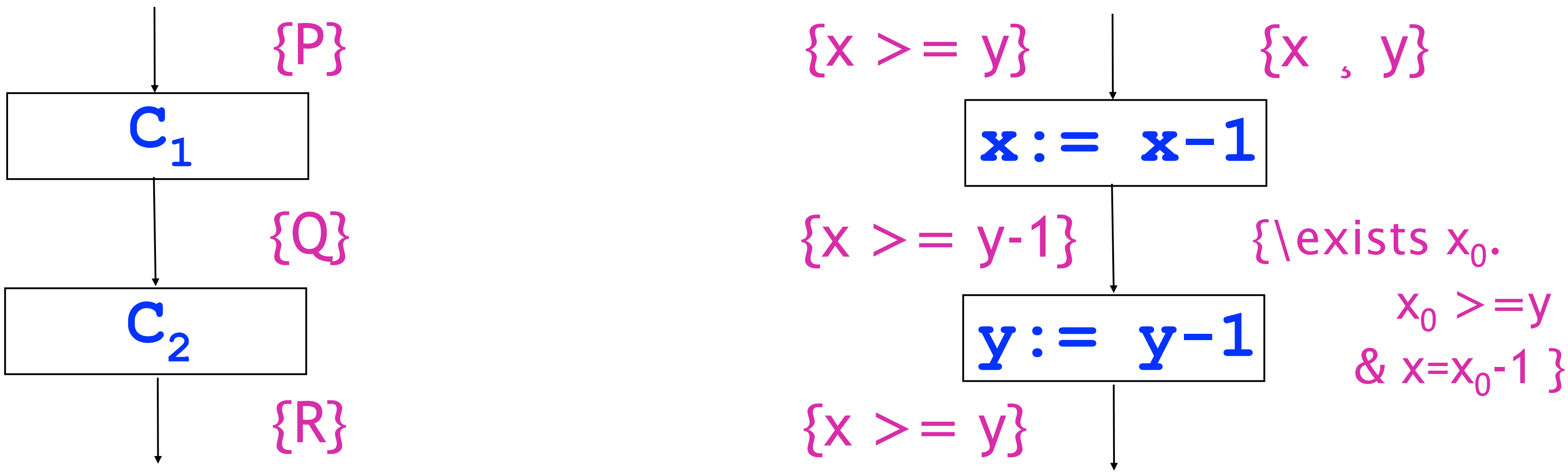
```
y:= y-1
```

{x >= y}

Backwards using weakest preconditions

Forwards using strongest postconditions

Sequential Composition



Backwards using weakest preconditions

Forwards using strongest postconditions

Sequential Composition

{P}

C_1

{Q}

C_2

{R}

{x >= y}

`x:= x-1`

{x >= y-1}

`y:= y-1`

{x >= y}

{x ¸ y}

{\exists x_0.
x_0 >=y
& x=x_0-1 }

{\exists y_0 x_0.
x_0 >= y_0
& x = x_0-1
& y = y_0-1}

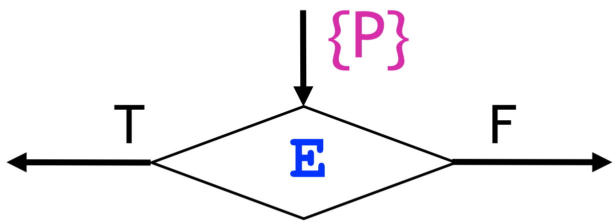
Backwards using weakest preconditions

Forwards using strongest postconditions

Conditionals

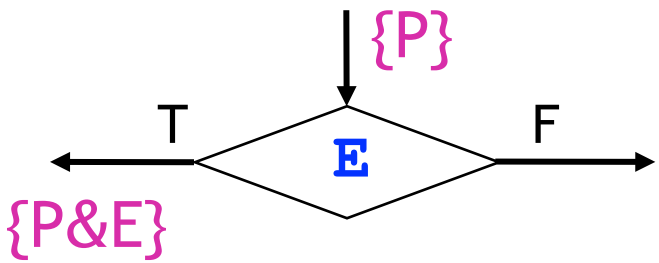
Forwards

Conditionals



Forwards

Conditionals



Forwards

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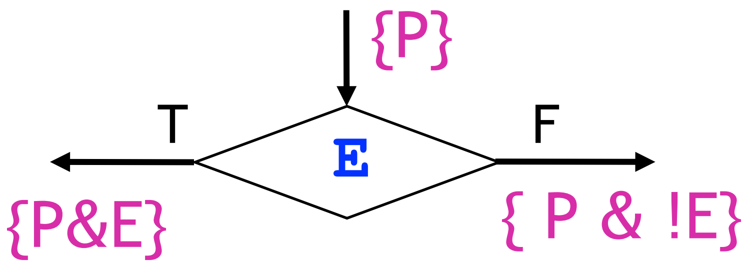
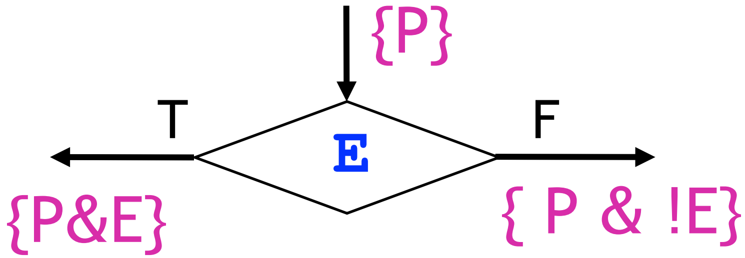
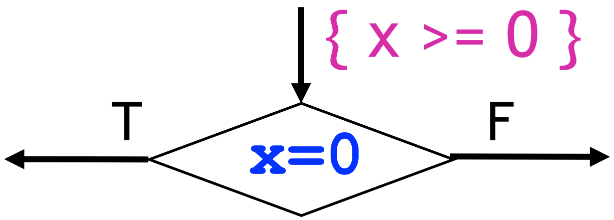


Forwards

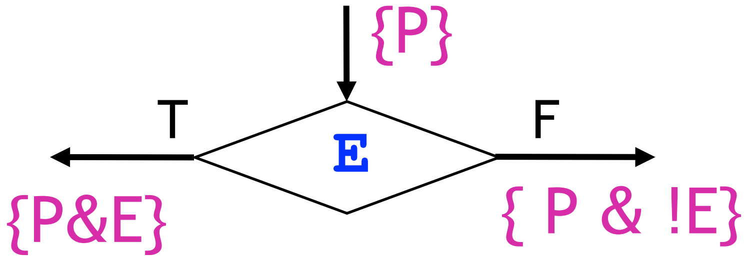
Conditionals

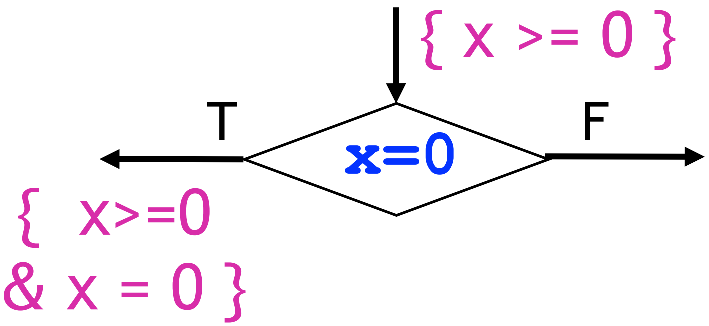




Forwards

Conditionals





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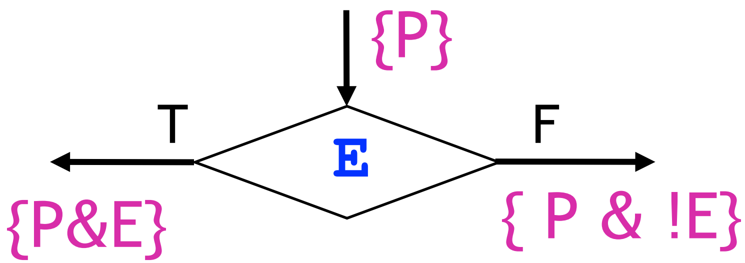
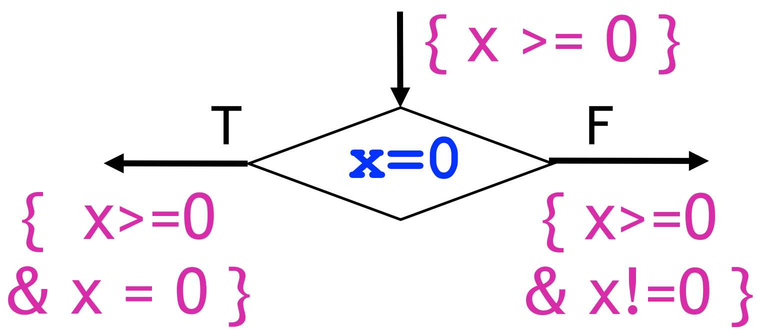
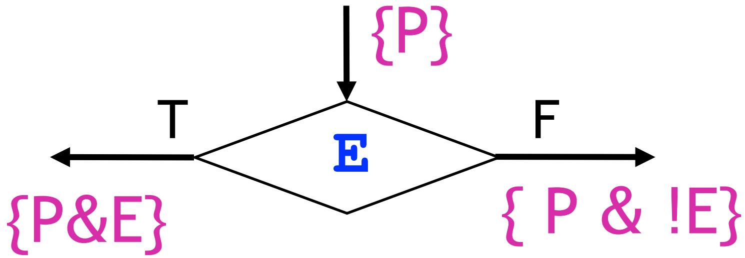
Conditionals

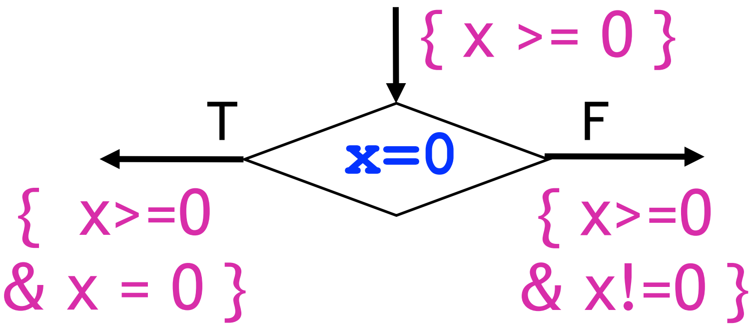




Forwards

Conditionals

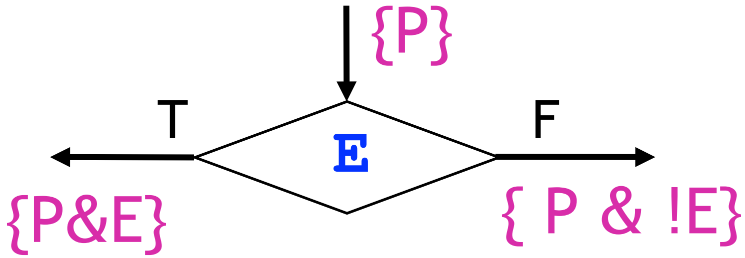


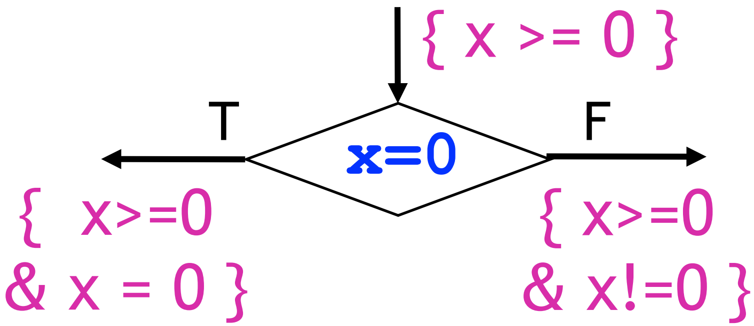


Forwards

Backwards

Conditionals





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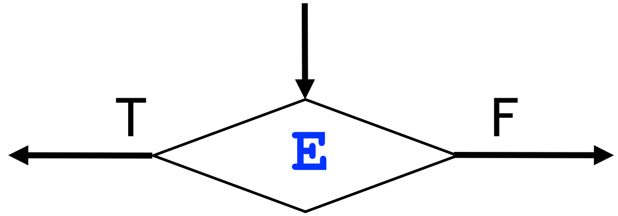
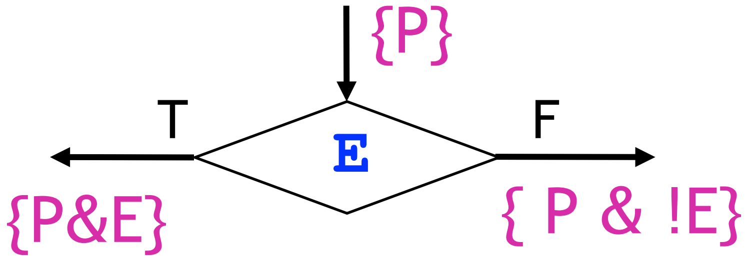
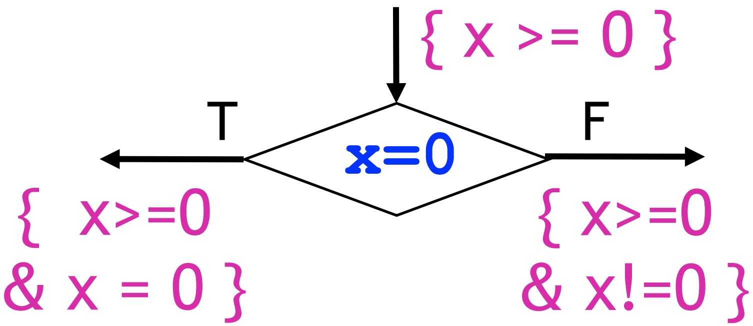


Backwards

Conditionals





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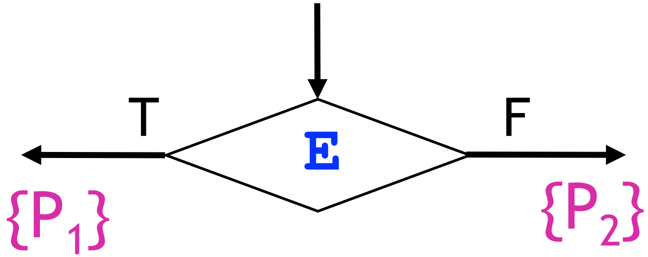

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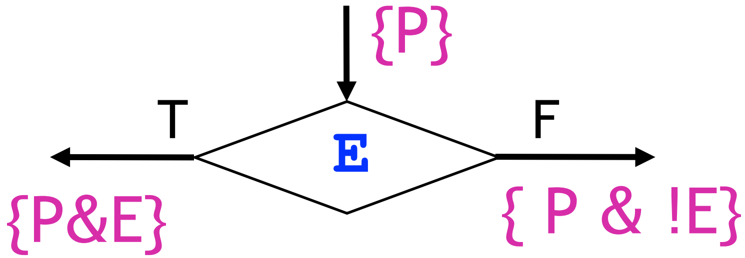
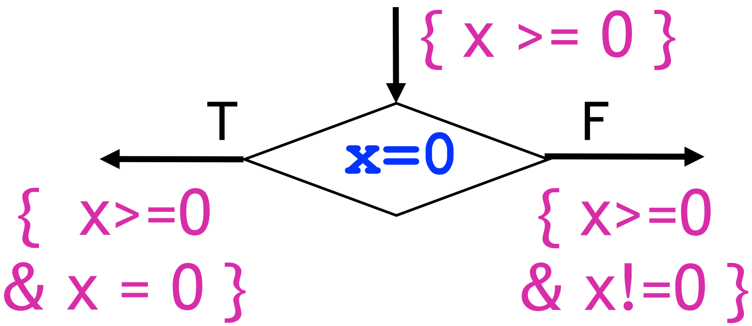
Conditionals





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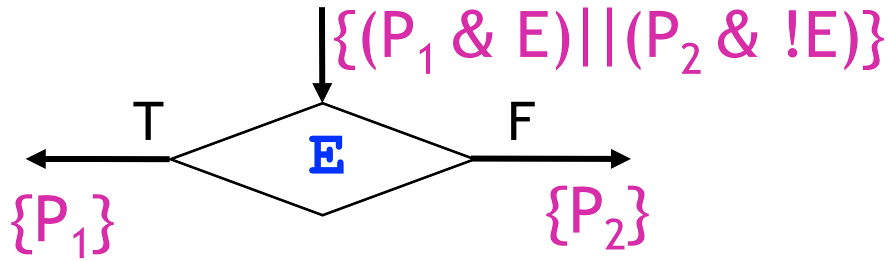
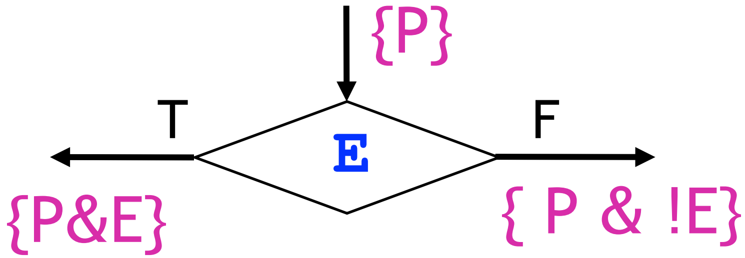
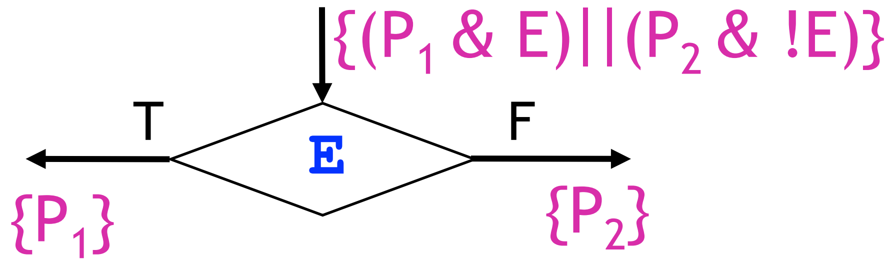
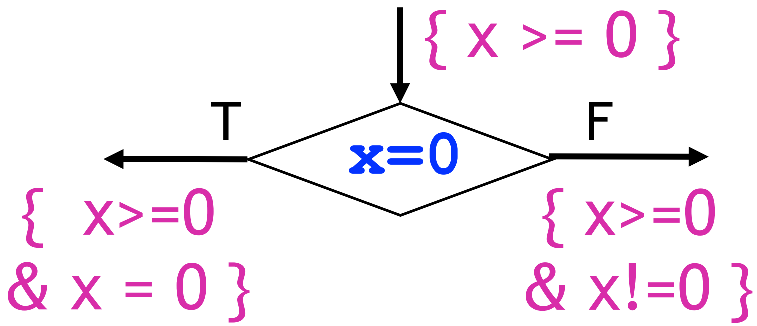


Backwards

Conditionals







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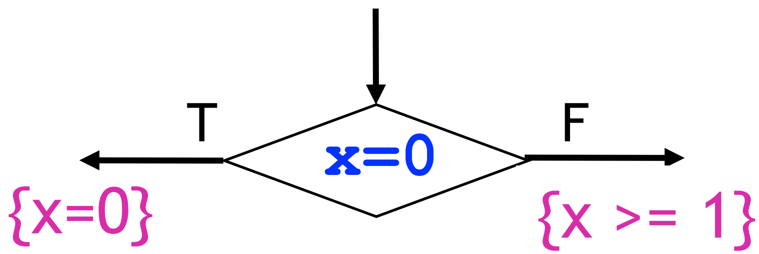
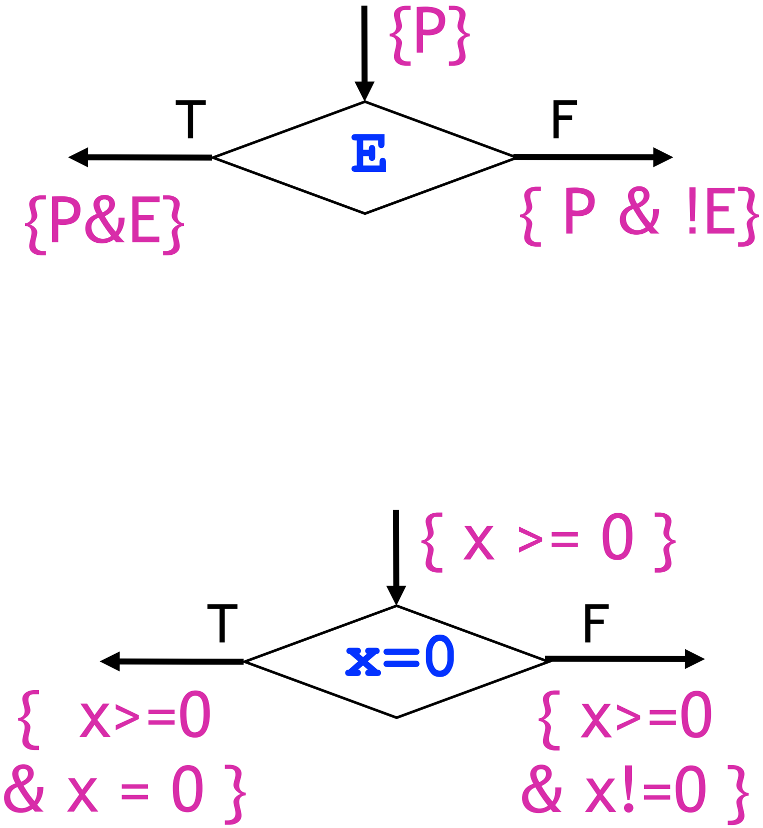


Backwards

Conditionals



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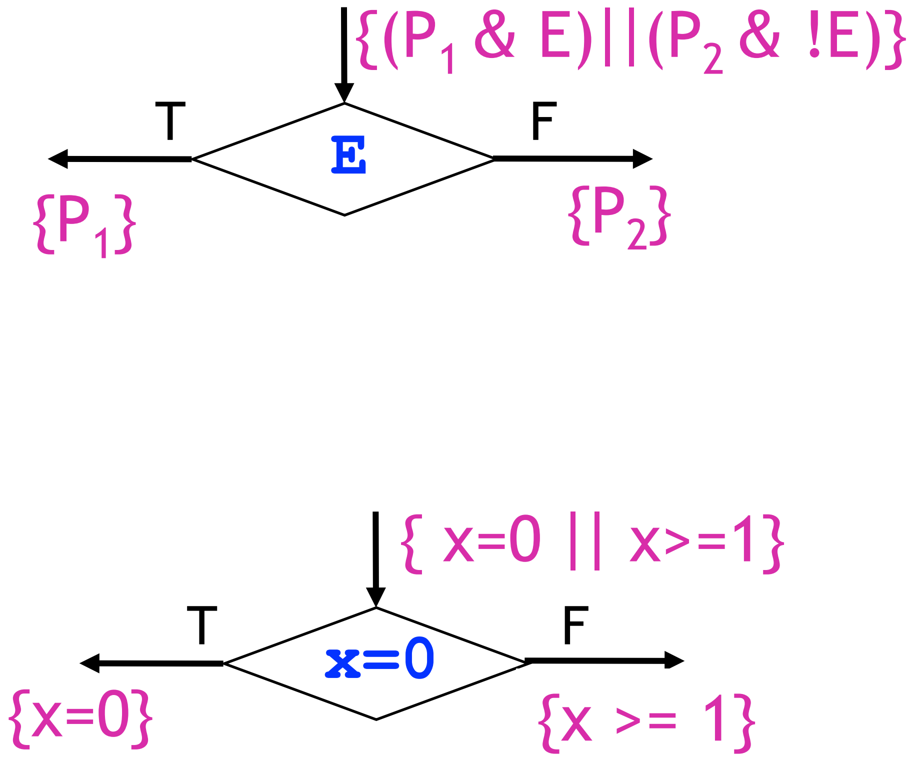
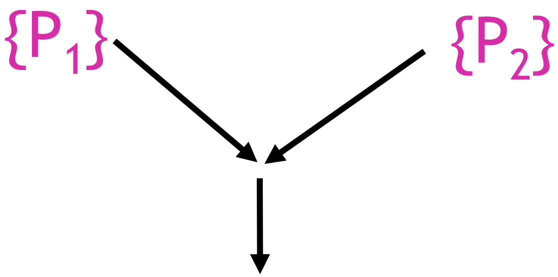


Backwards

Joins



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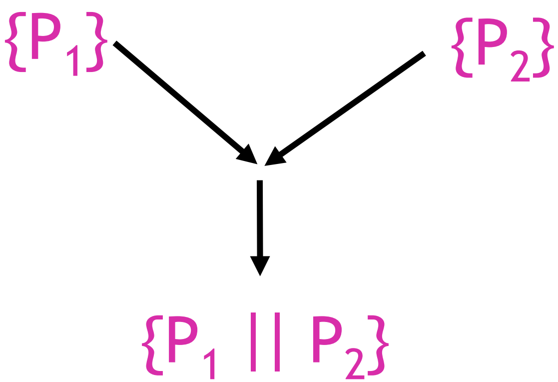


Forwards

Joins

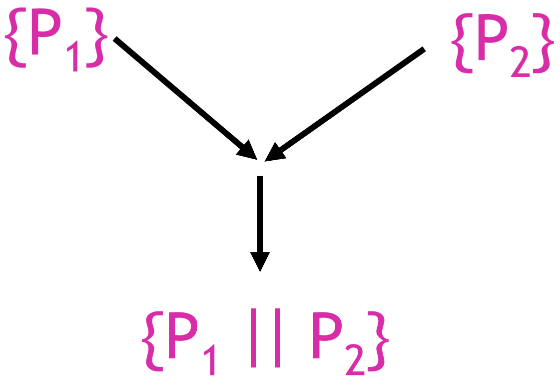
$\{P_1\}$ $\{P_2\}$

$\{P_1 \mid\mid P_2\}$

Forwards

Backwards

Joins



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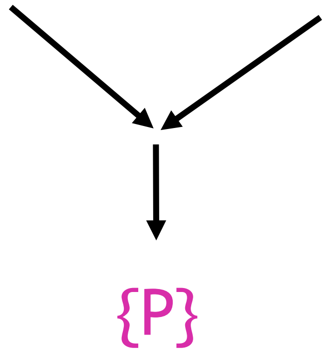

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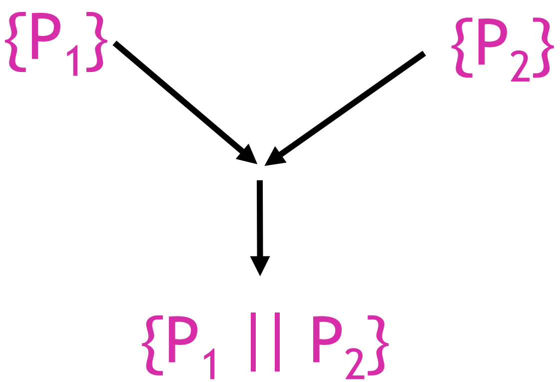
Joins



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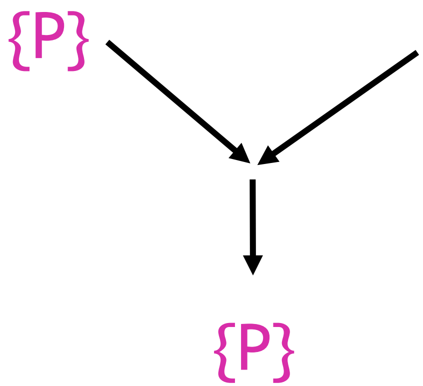
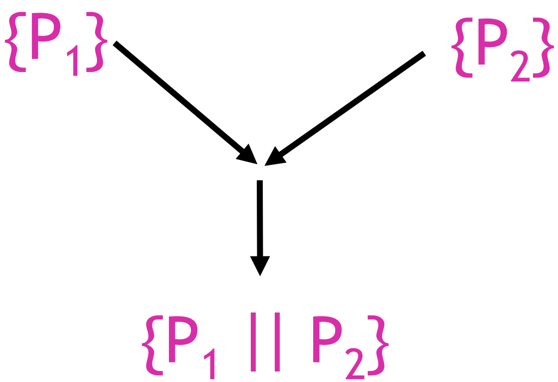


Backwards

Joins



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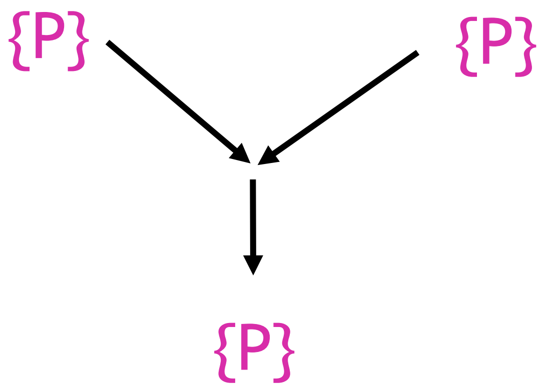
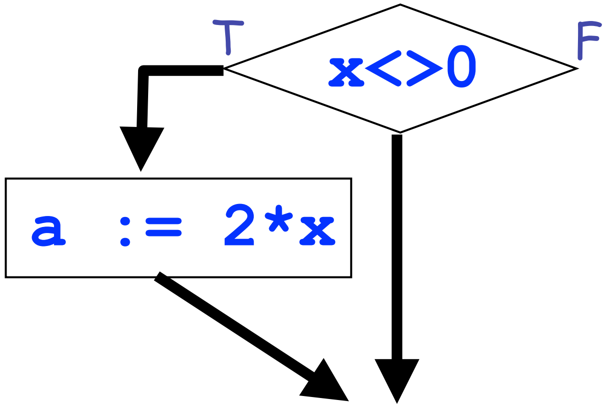
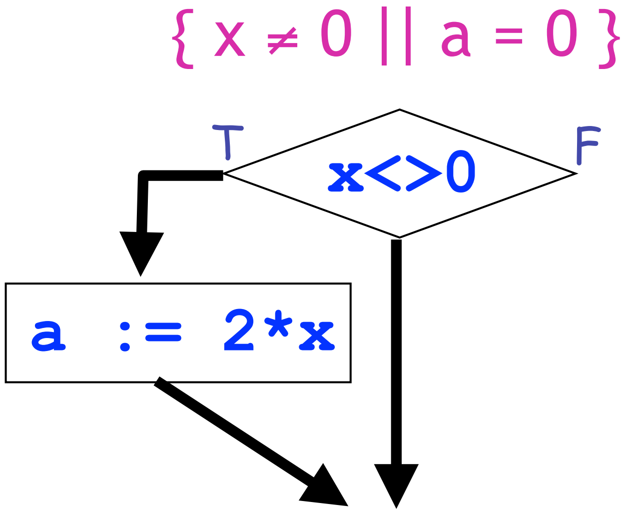


Backwards

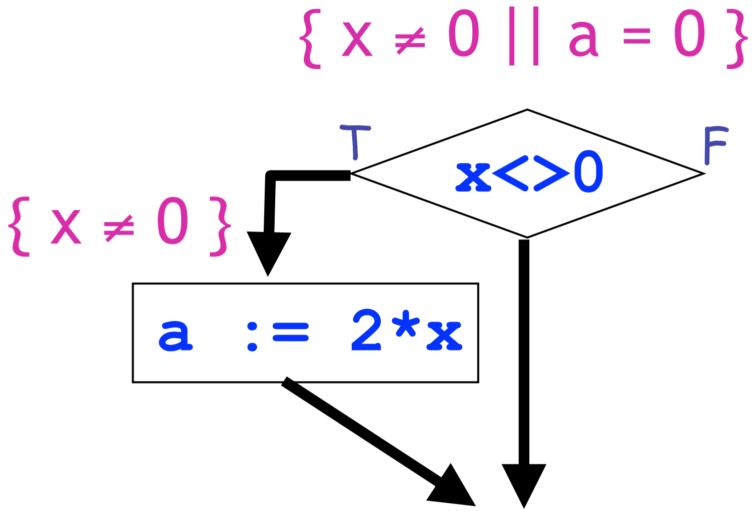
Conditional+Join: Forward



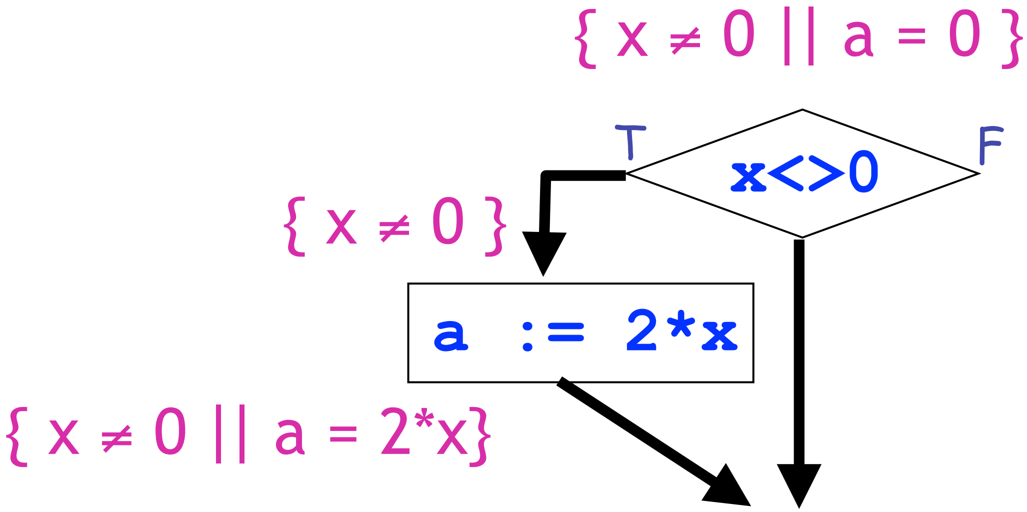
Conditional+Join: Forward



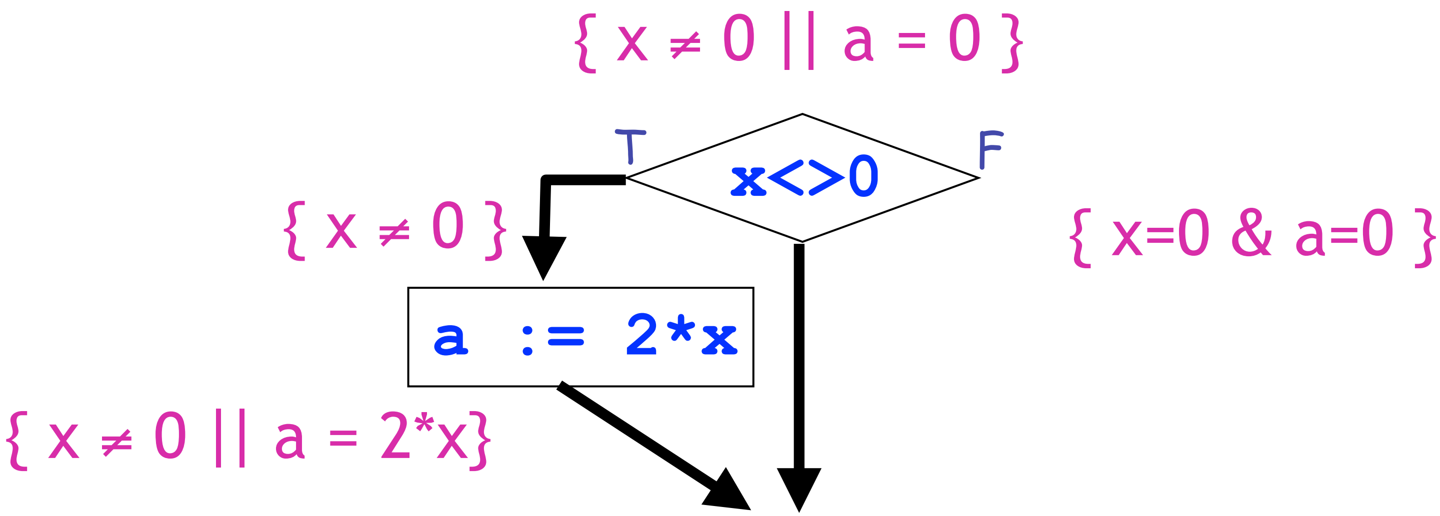
Conditional+Join: Forward



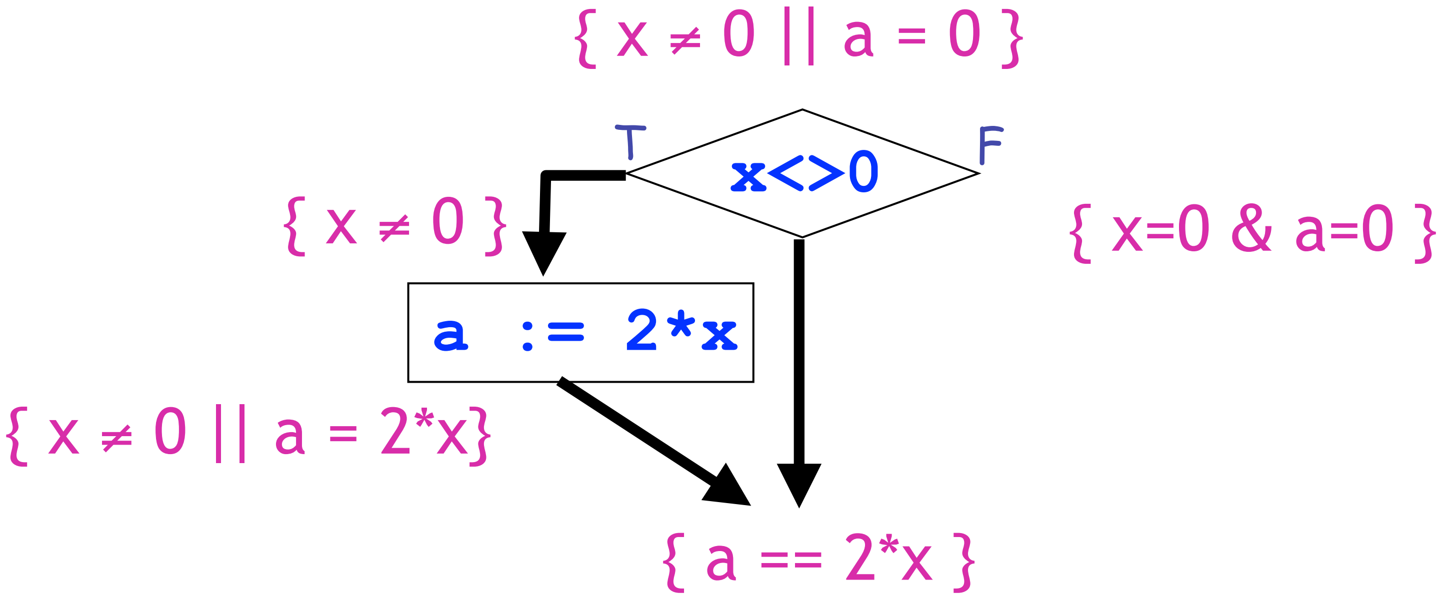
Conditional+Join: Forward



Conditional+Join: Forward

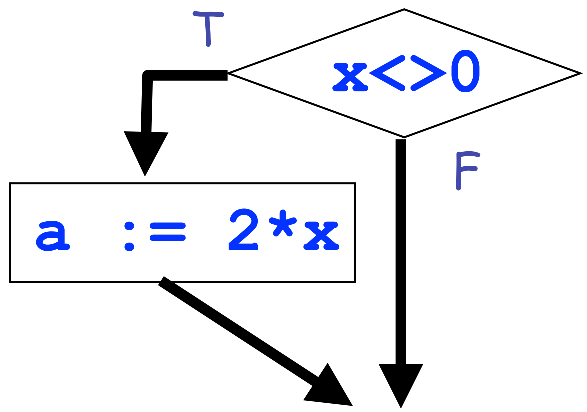


Conditional+Join: Forward

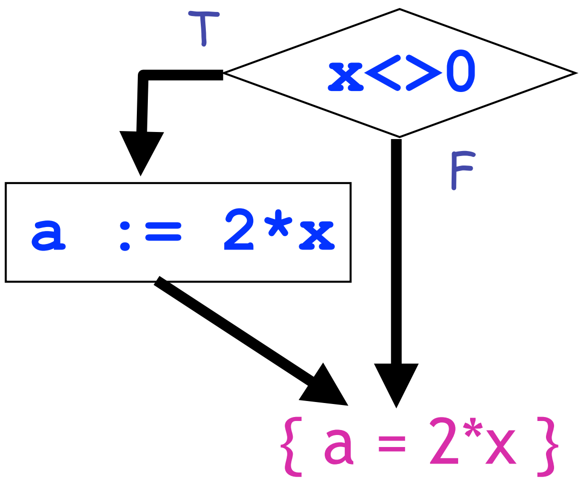


- Check the implications (simplifications)

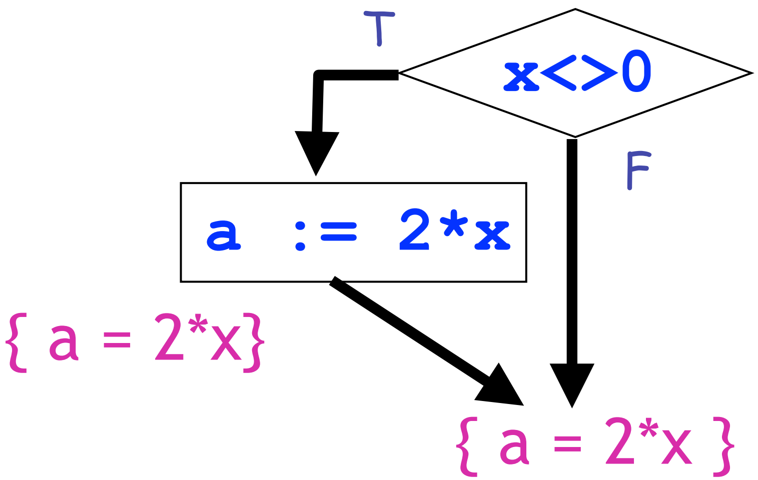
Conditionals+Joins: Backward



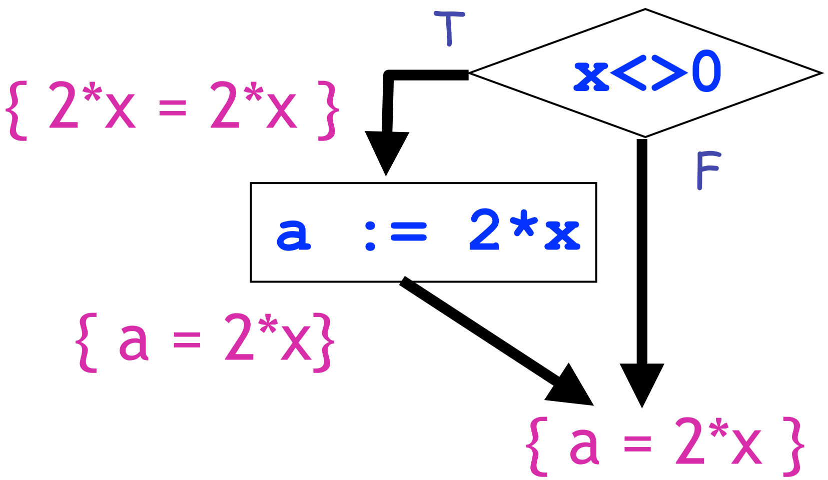
Conditionals+Joins: Backward



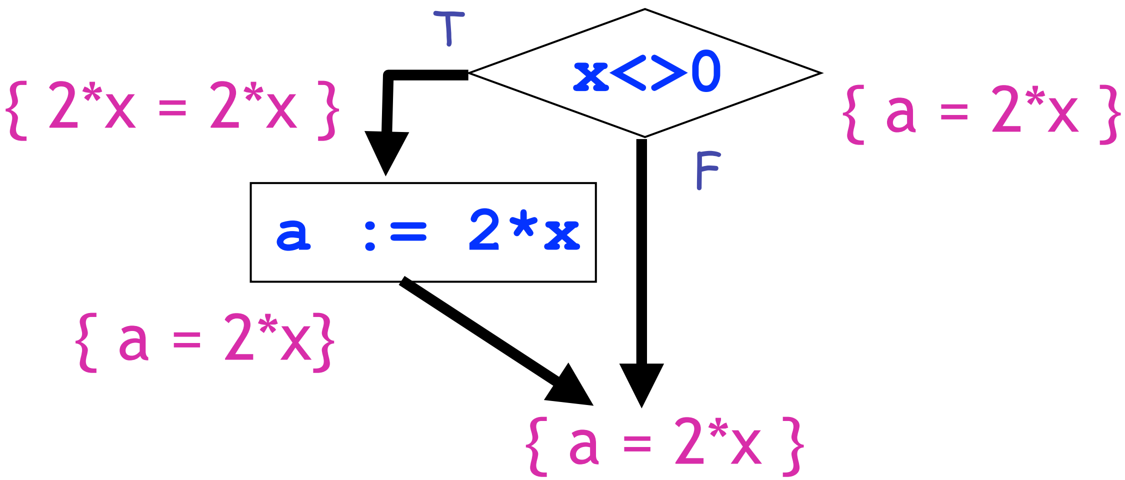
Conditionals+Joins: Backward



Conditionals+Joins: Backward

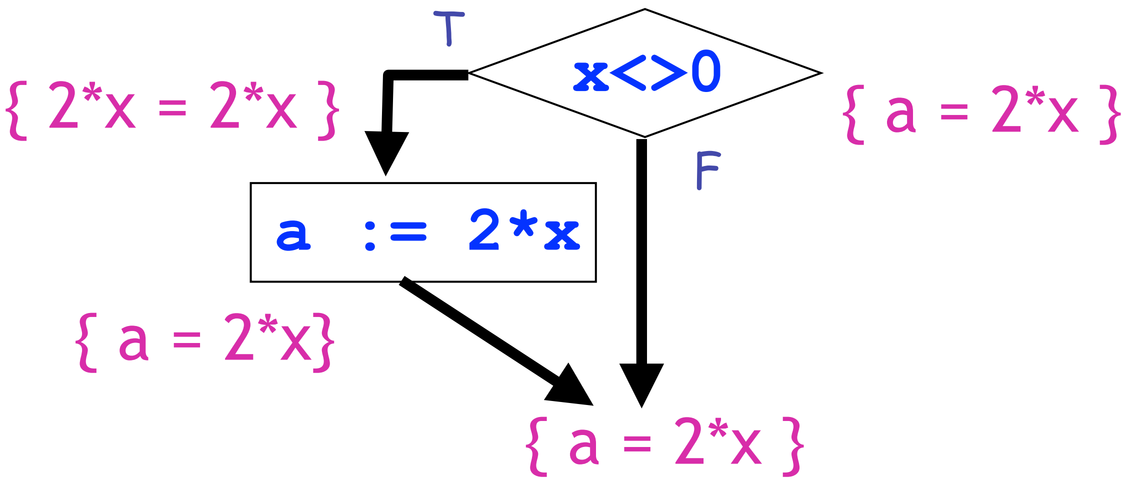


Conditionals+Joins: Backward



Conditionals+Joins: Backward

$\{ (x \neq 0 \;\&\; \text{true}) \;||\; (x = 0 \;\&\; a = 2{*}x) \}$

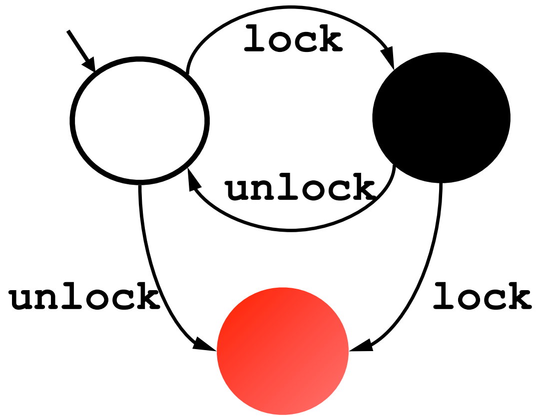


Forward or Backward ?

- Forward reasoning
  - Know the precondition
  - Want to know what postcond the code guarantees

- Backward reasoning
  - Know what we want to code to establish
  - Want to know under what preconditions this happens

Another Example: Double Locking



*"An attempt to re-acquire an acquired lock or release a released lock will cause a **deadlock**."*

Calls to lock and unlock must alternate.

Locking Rules

Locking Rules

Boolean variable `locked` states if lock is held or not

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- {!locked & P[true/locked] } `lock` { P }

  `lock` behaves as `assert(!locked);locked:=true`

Locking Rules

Boolean variable `locked` states if lock is held or not

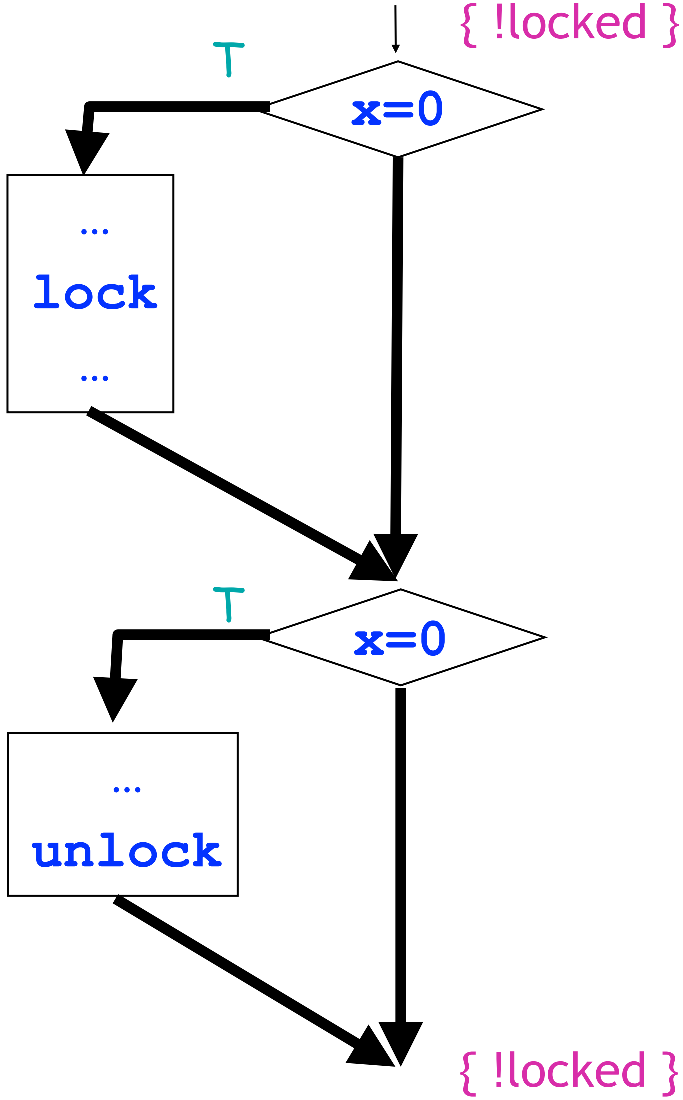
- {!locked & P[true/locked] } `lock` { P }

  `lock` behaves as `assert(!locked);locked:=true`

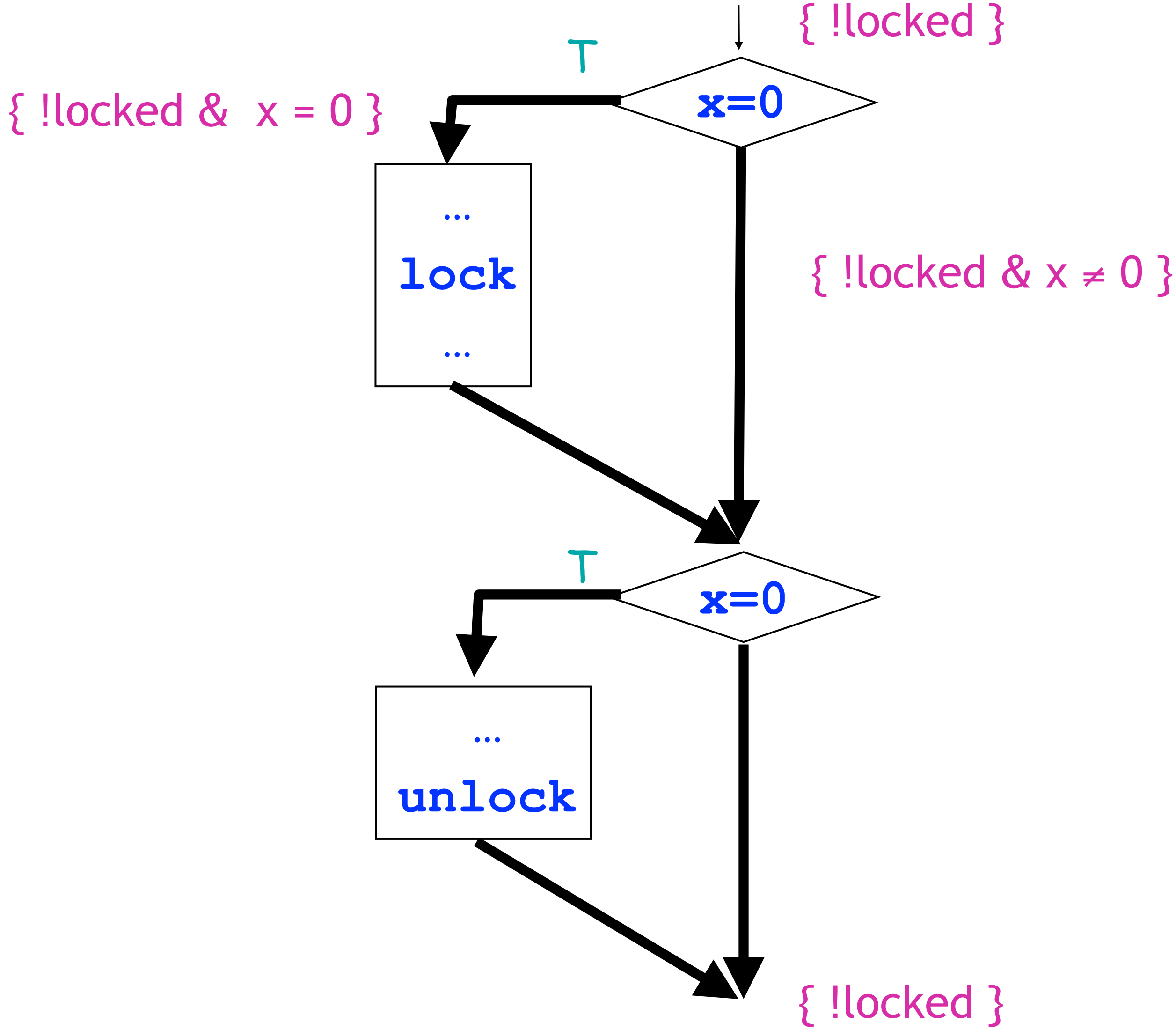
- { locked & P[false/locked] } `unlock` { P }

  `unlock` behaves as `assert(locked);locked:=false`

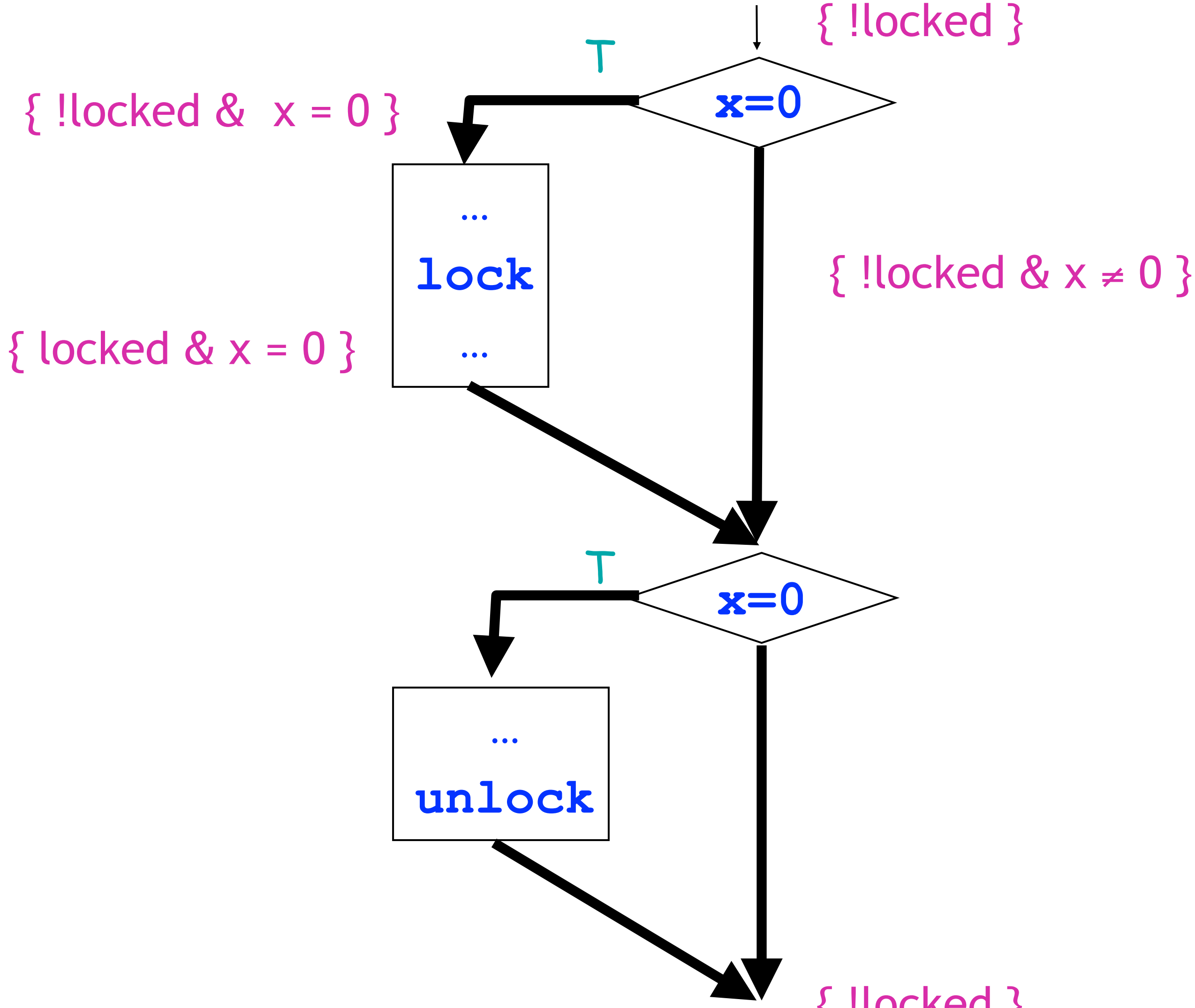
Locking Example



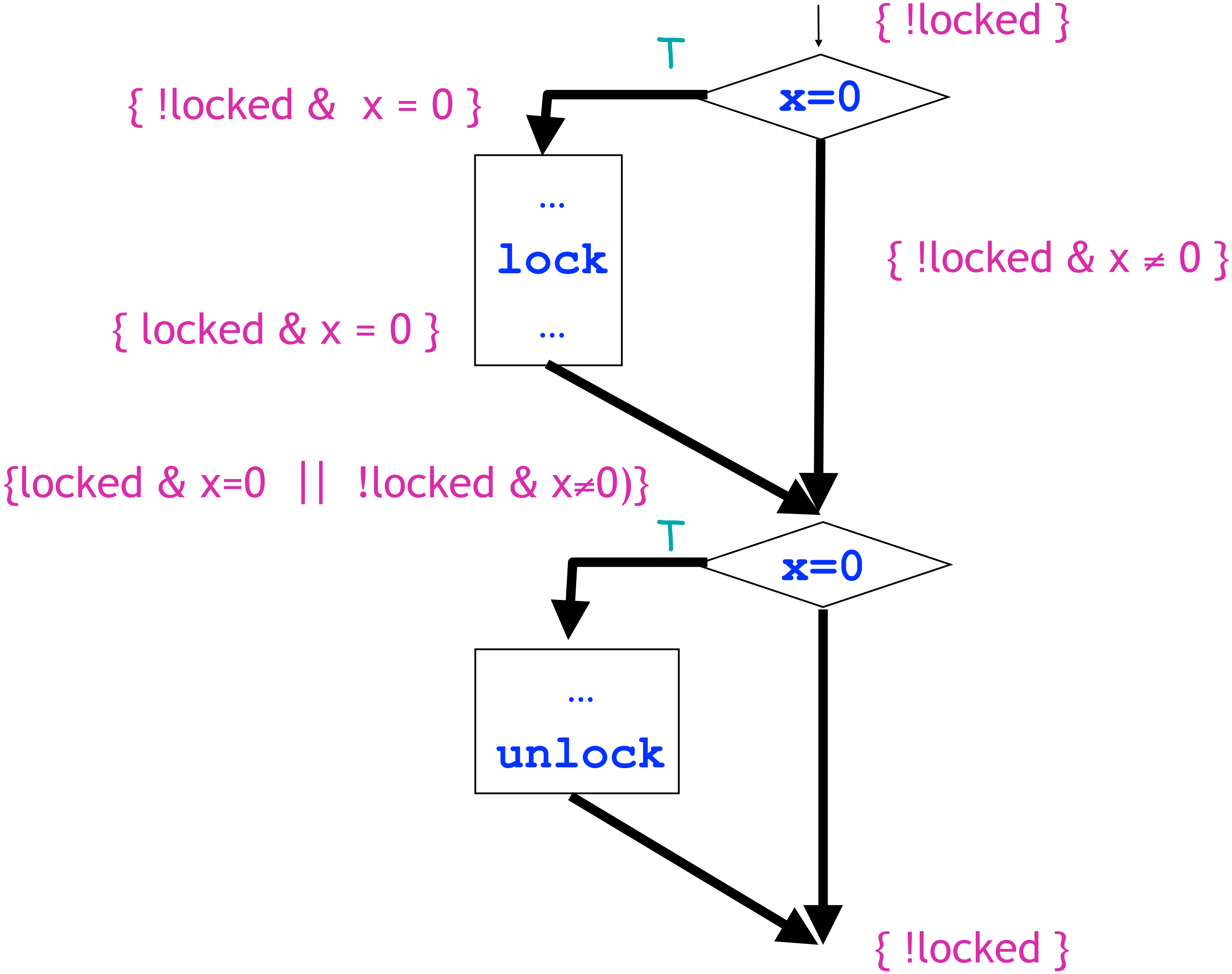
Locking Example



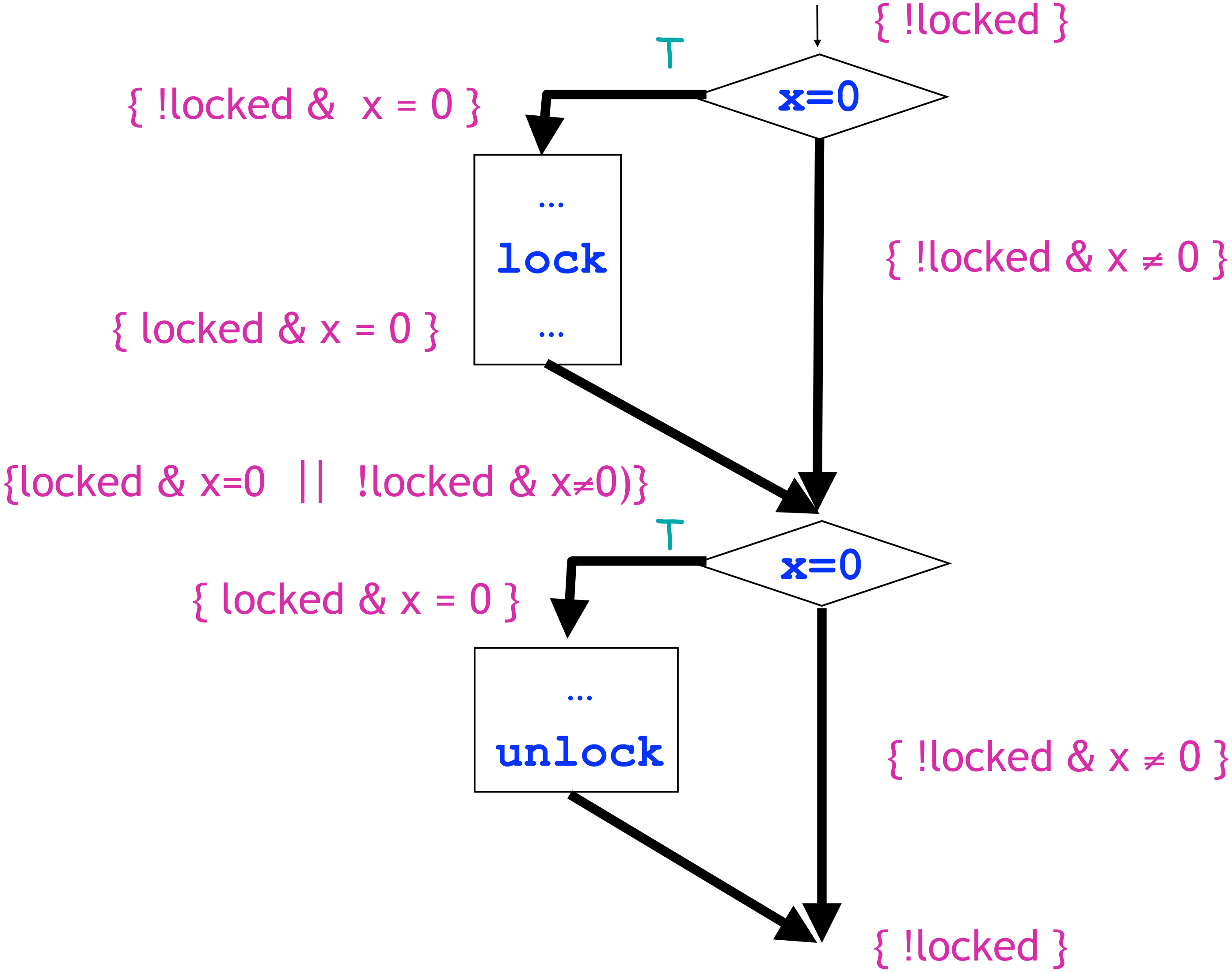
Locking Example



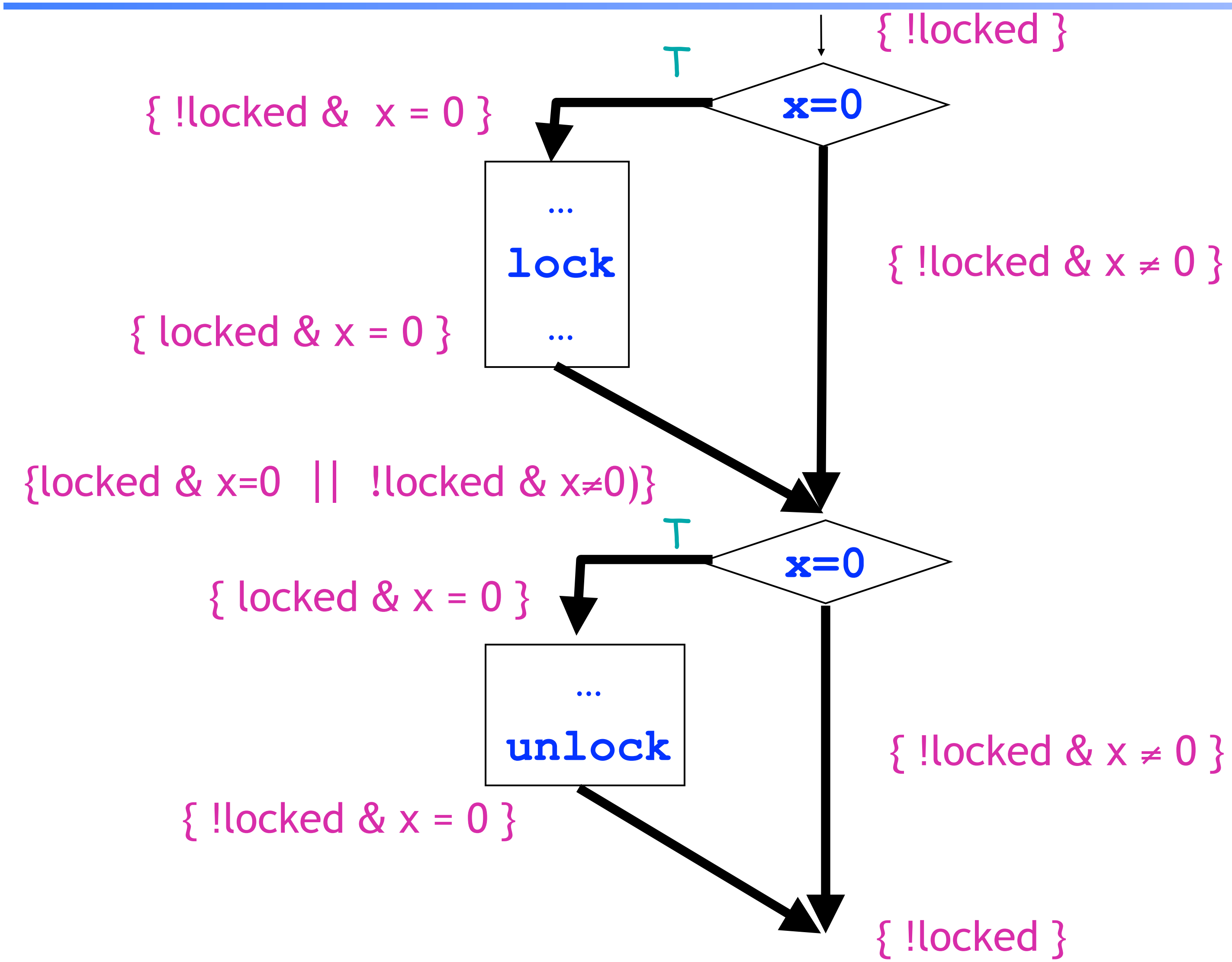
Locking Example



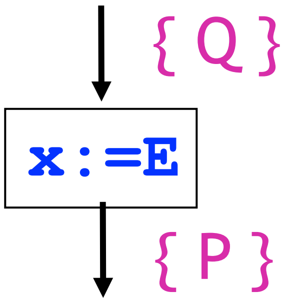
Locking Example



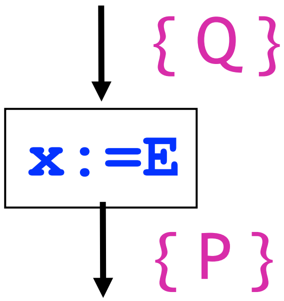
Locking Example



Review

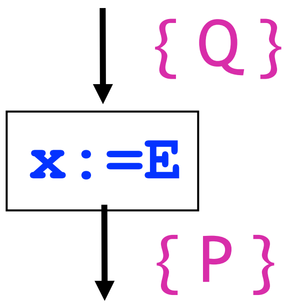


Review

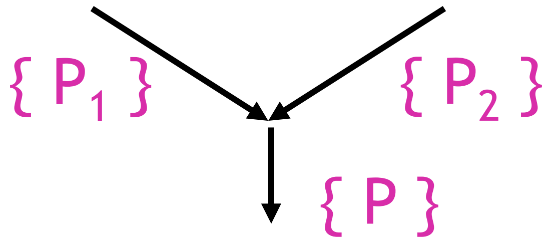


if Q => P[E\x] }

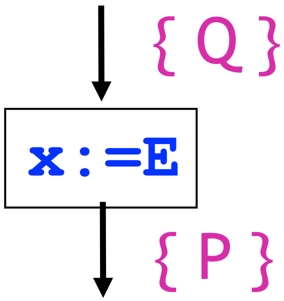
Review



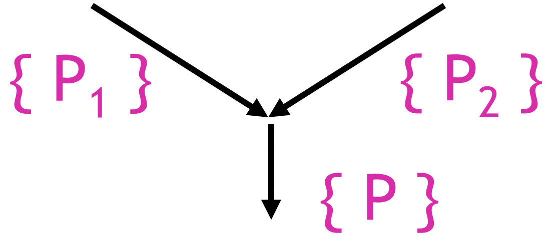
if Q => P[E\x] }



Review

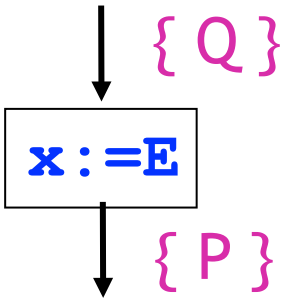


if $Q \Rightarrow P[E\backslash x]$ }



if $P_1 \Rightarrow P$ and $P_2 \Rightarrow P$

Review



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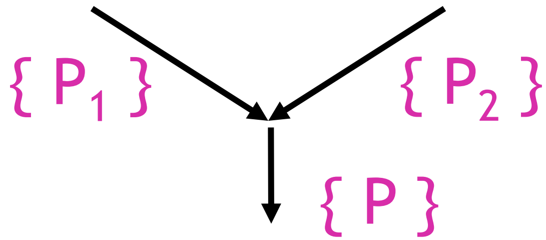

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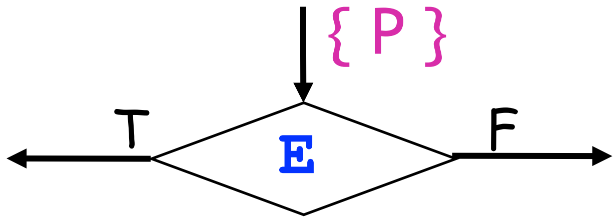
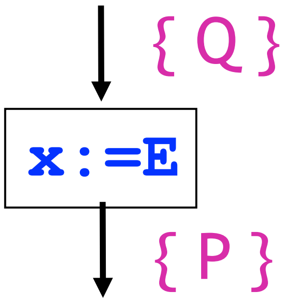
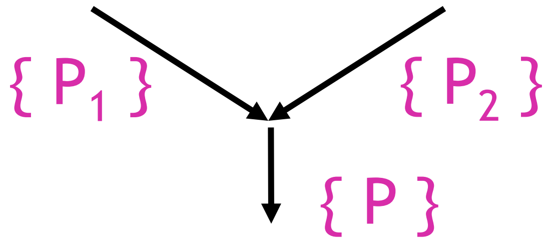


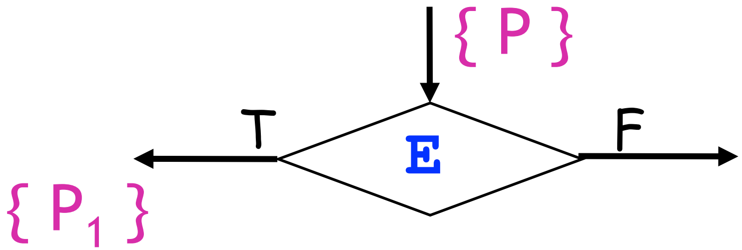
Review



if Q => P[E\x] }

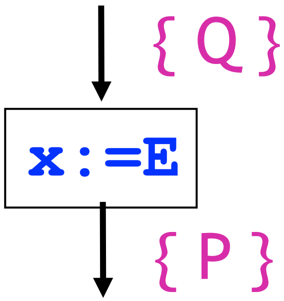


if P_1 => P and P_2 => P

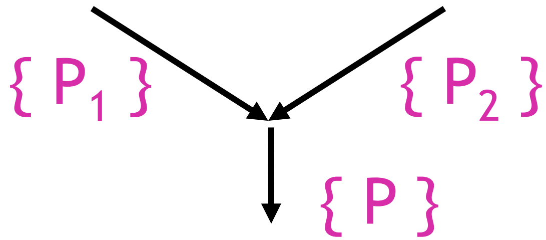


if P&E => P_1

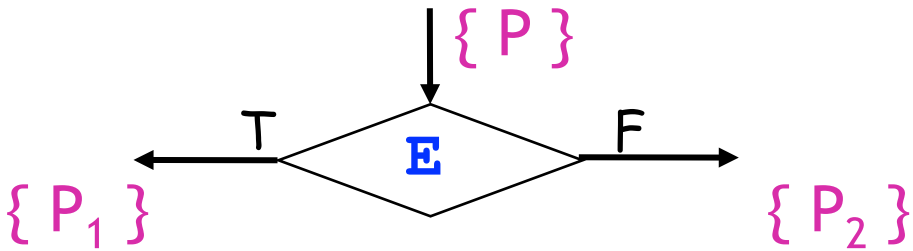
Review



if $Q => P[E\backslash x]$ }



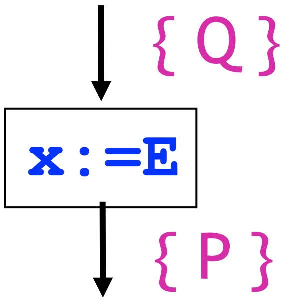
if $P_1 => P$ and $P_2 => P$



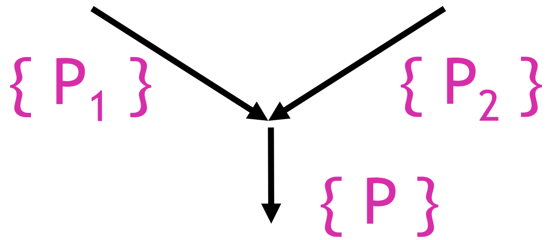
if $P\&E => P_1$

if $P\& !E => P_2$

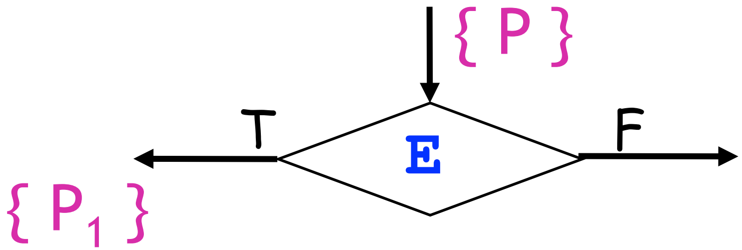
Review



if $Q => P[E\backslash x]$ }



if $P_1 => P$ and $P_2 => P$



if $P\&E => P_1$

if $P\& !E => P_2$

Implication is always in the direction of the control flow

What about real languages ?

- Loops
- Function calls
- Pointers

Reasoning about loops: Rules

Rewrite A with I : Loop Invariant

Rule of Consequence

Reasoning about loops: Rules

$$\frac{\vdash \{A \,\&\, b\}\ \mathtt{c}\ \{A\}}{\vdash \{A\}\ \mathtt{while\ b\ do\ c}\ \{A \,\&\, !b\}}$$

Rewrite A with I : Loop Invariant

Rule of Consequence

Reasoning about loops: Rules

$$\frac{\vdash \{A \;\&\; b\}\ \texttt{c}\ \{A\}}{\vdash \{A\}\ \texttt{while b do c}\ \{A \;\&\; !b\}}$$

Rewrite A with I : Loop Invariant

$$\frac{\vdash \{I \;\&\; b\}\ \texttt{c}\ \{I\}}{\vdash \{I\}\ \texttt{while b do c}\ \{I \;\&\; !b\}}$$

Rule of Consequence

Reasoning about loops: Rules

$$\frac{\vdash \{A \mathbin{\&} b\}\ \mathtt{c}\ \{A\}}{\vdash \{A\}\ \mathtt{while\ b\ do\ c}\ \{A \mathbin{\&} !b\}}$$

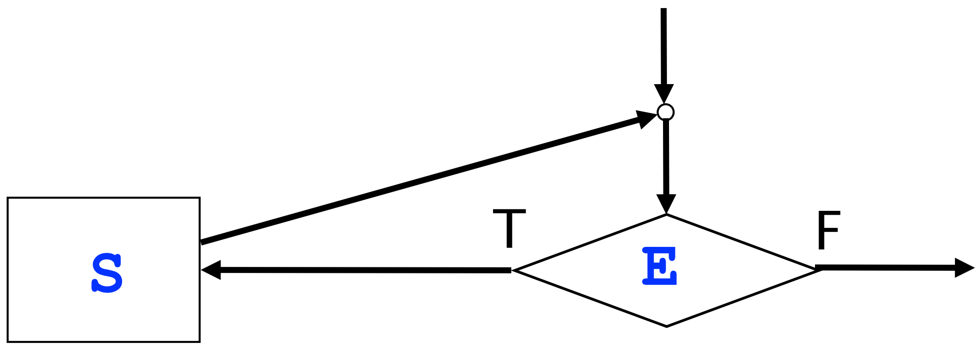
Rewrite A with I : Loop Invariant

$$\frac{P \Rightarrow I \qquad \dfrac{\vdash \{I \mathbin{\&} b\}\ \mathtt{c}\ \{I\}}{\vdash \{I\}\ \mathtt{while\ b\ do\ c}\ \{I \mathbin{\&} !b\}} \qquad I \mathbin{\&} !b \Rightarrow Q}{\vdash \{P\}\ \mathtt{while\ b\ do\ c}\ \{Q\}}$$

Rule of Consequence

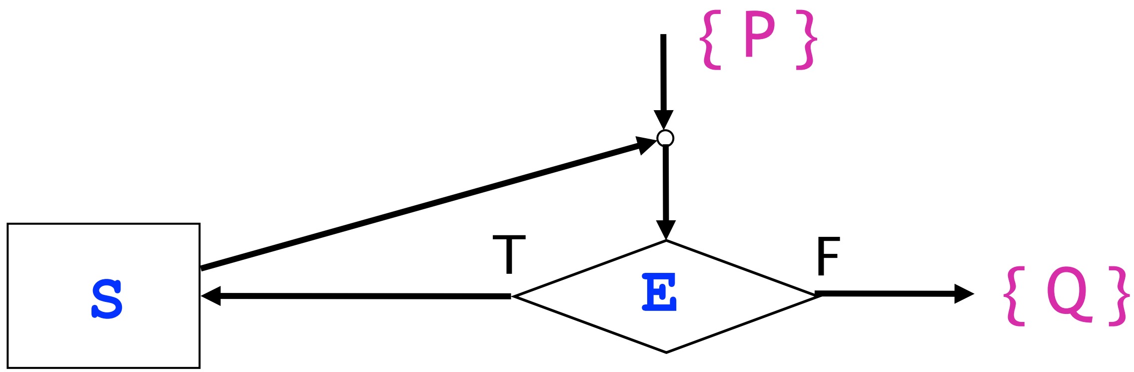
Reasoning about loops: Flow Graphs

- Loops can be handled using conditionals and joins
- Consider the `while b do S` statement



Reasoning about loops: Flow Graphs

- Loops can be handled using conditionals and joins
- Consider the **`while b do S`** statement



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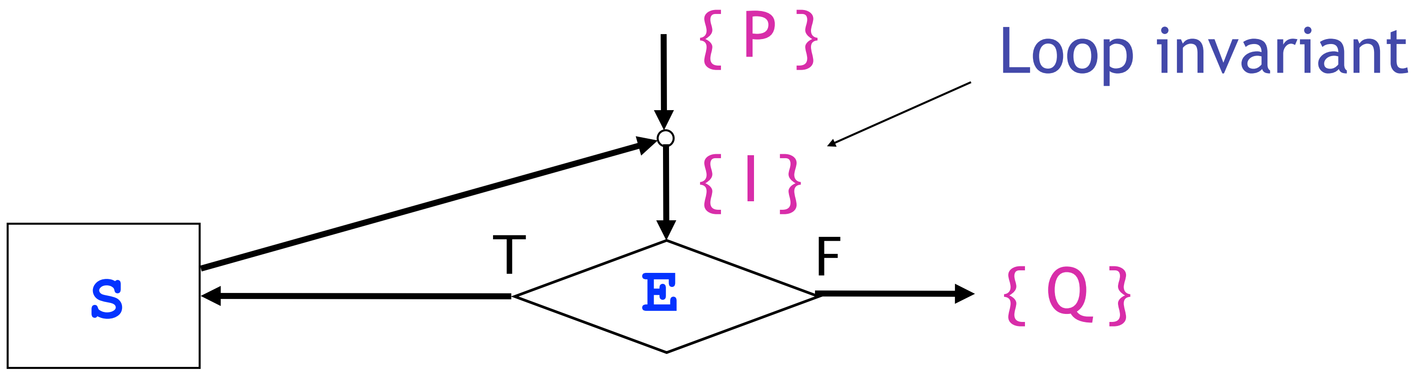

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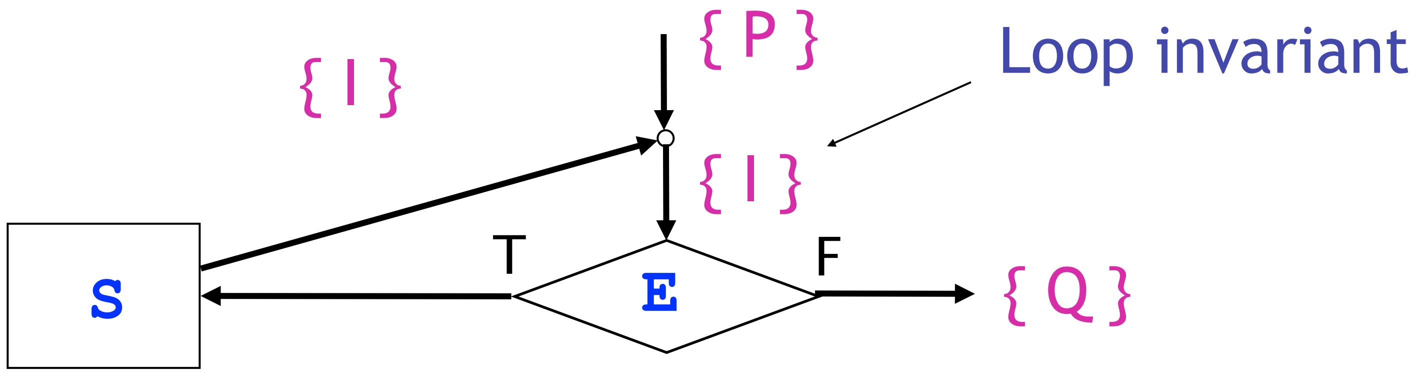

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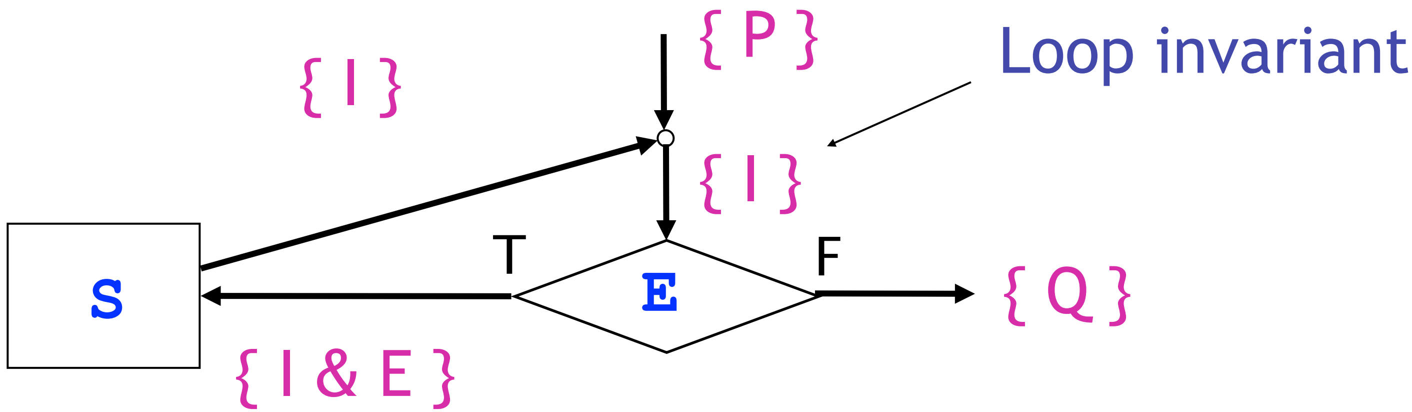

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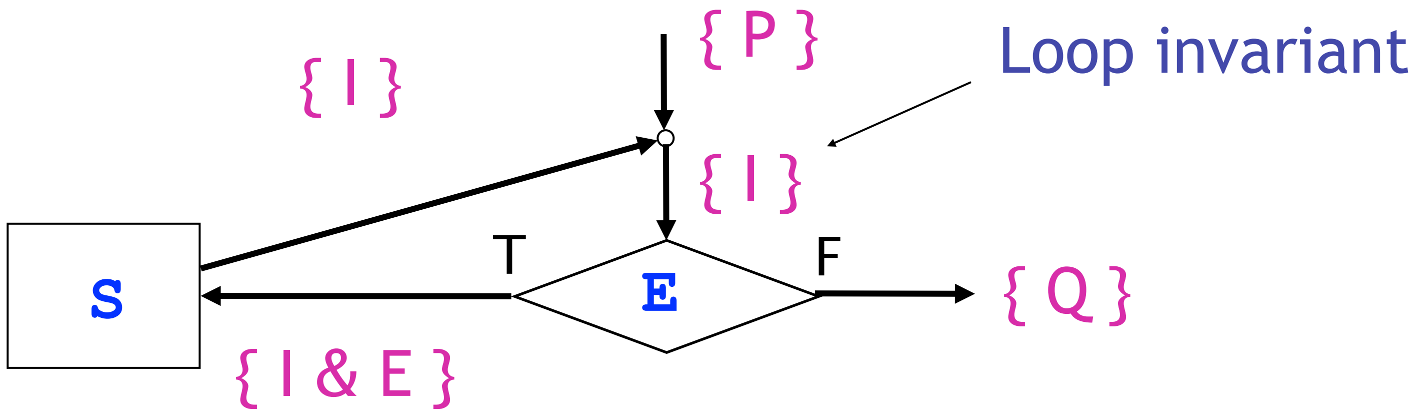

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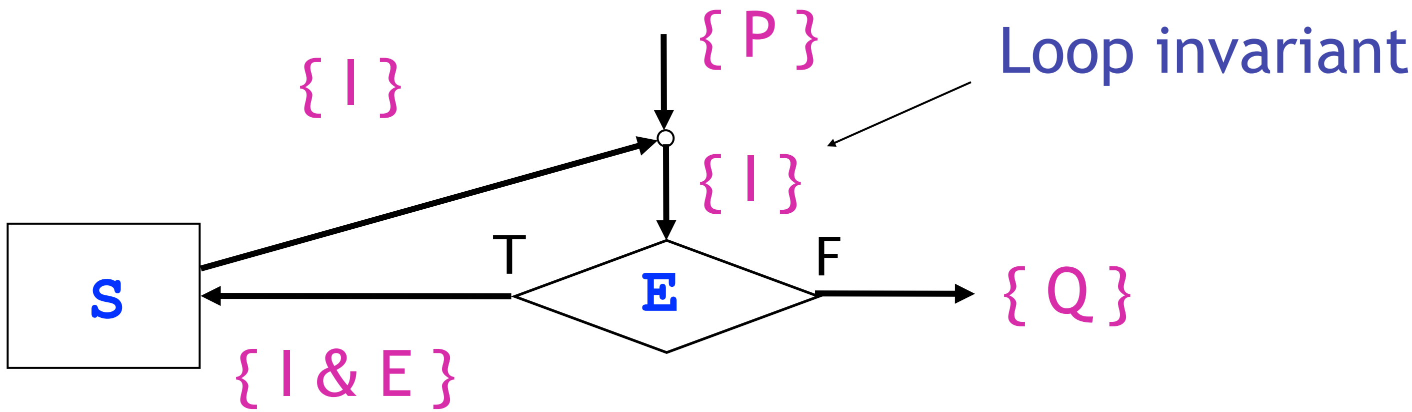

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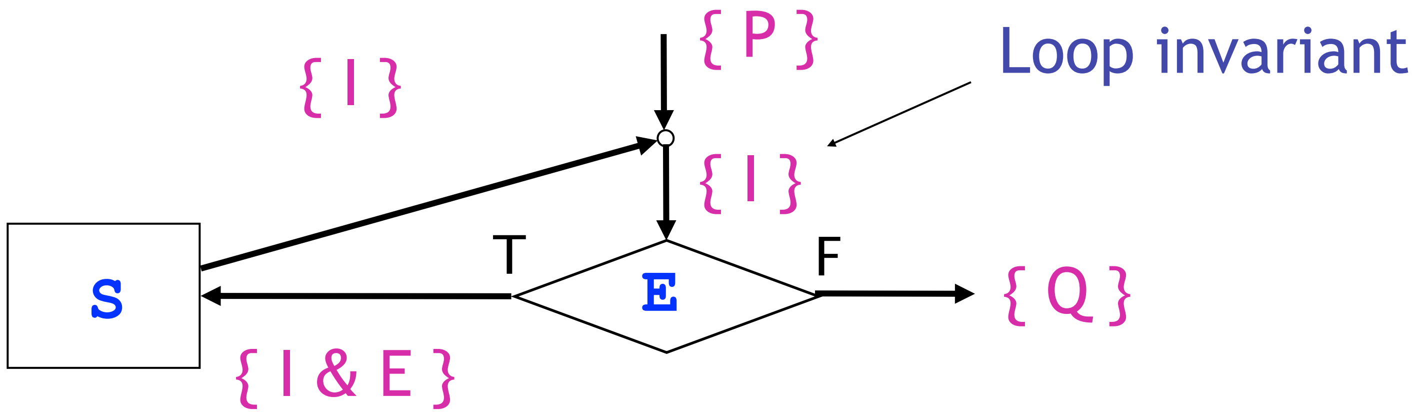

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and	I & !b => Q	(loop establishes the postcondition)
and	{ I & b } **S** { I }	(loop invariant is preserved)

Loop Example

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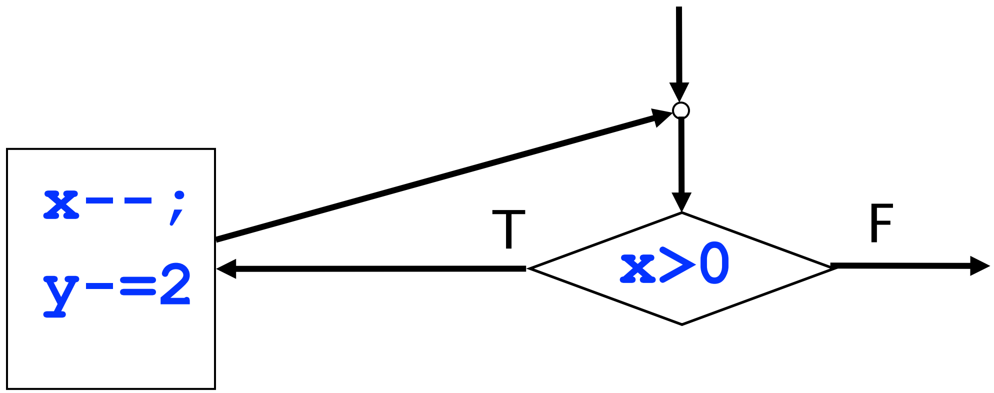

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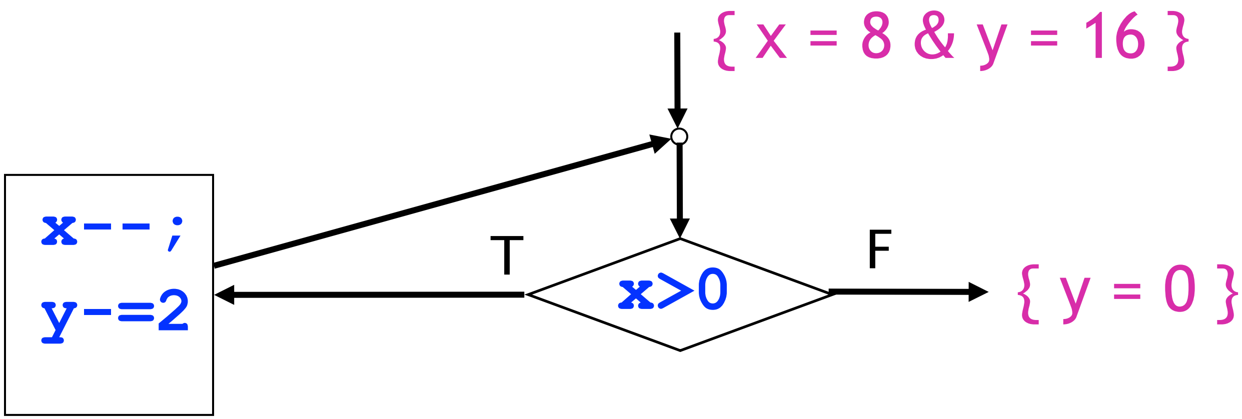

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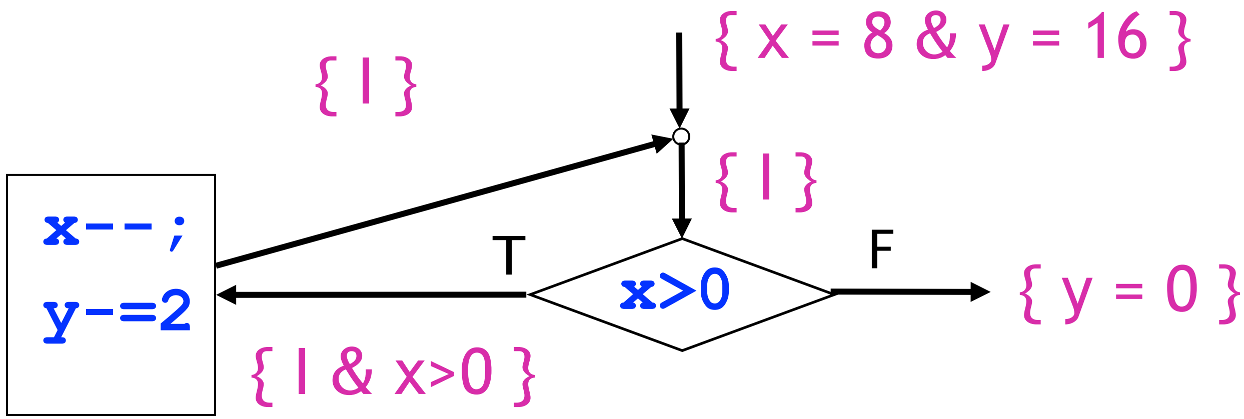
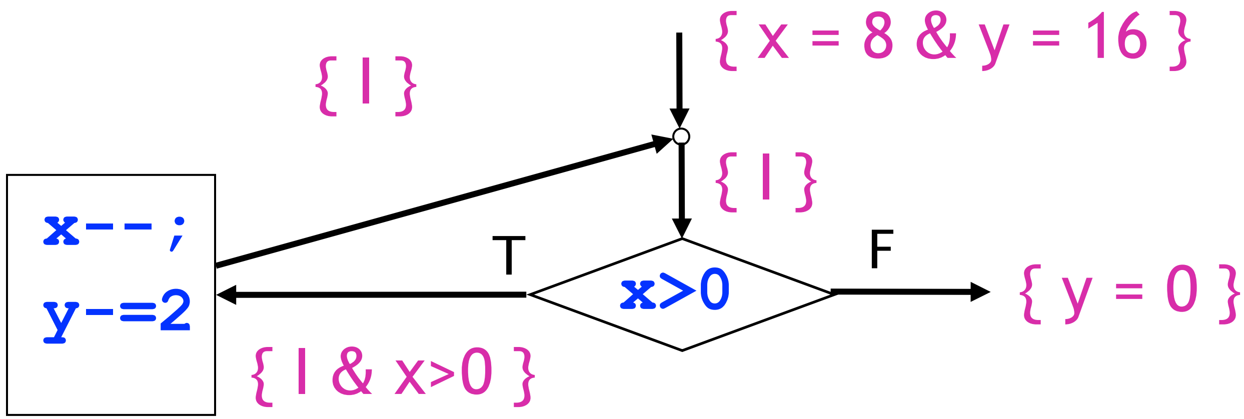


Loop Example

Verify:

{x=8 & y=16} `while(x>0){x--; y-=2;}` {y = 0}



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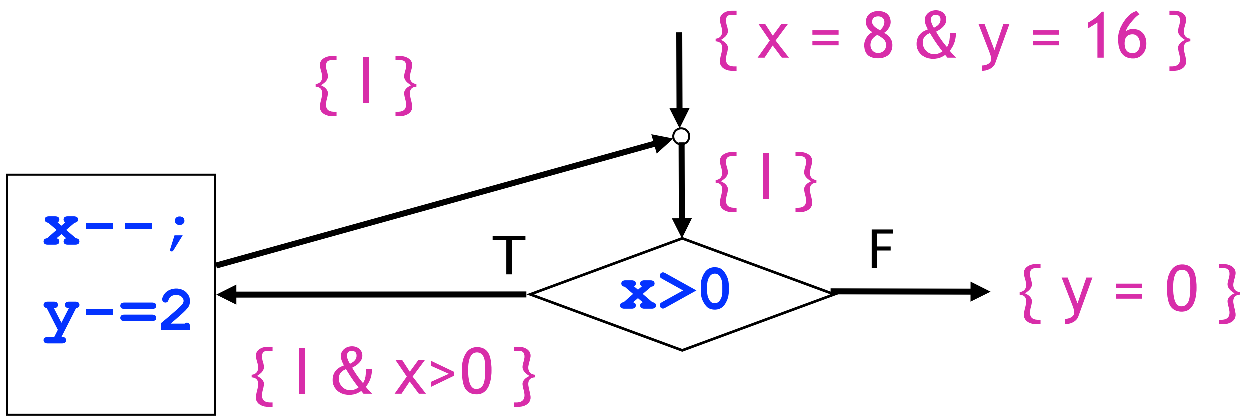

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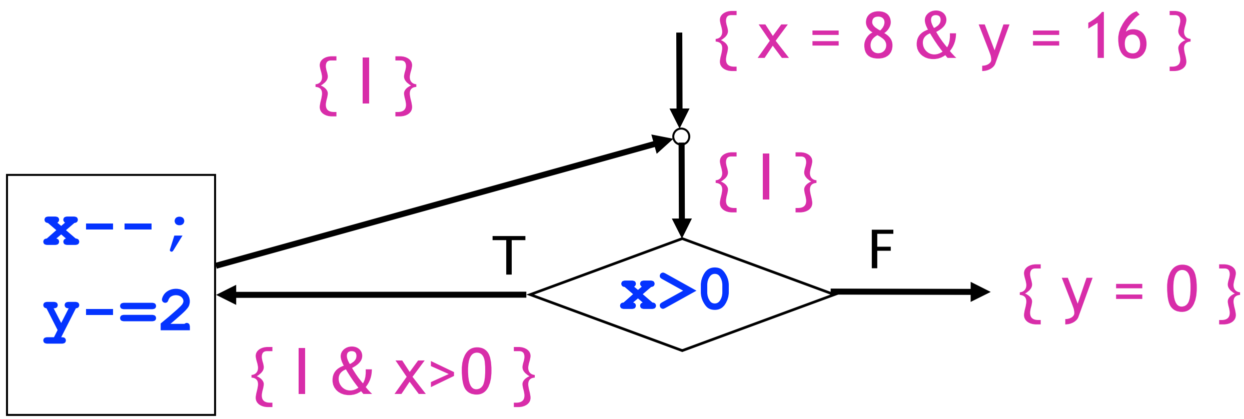
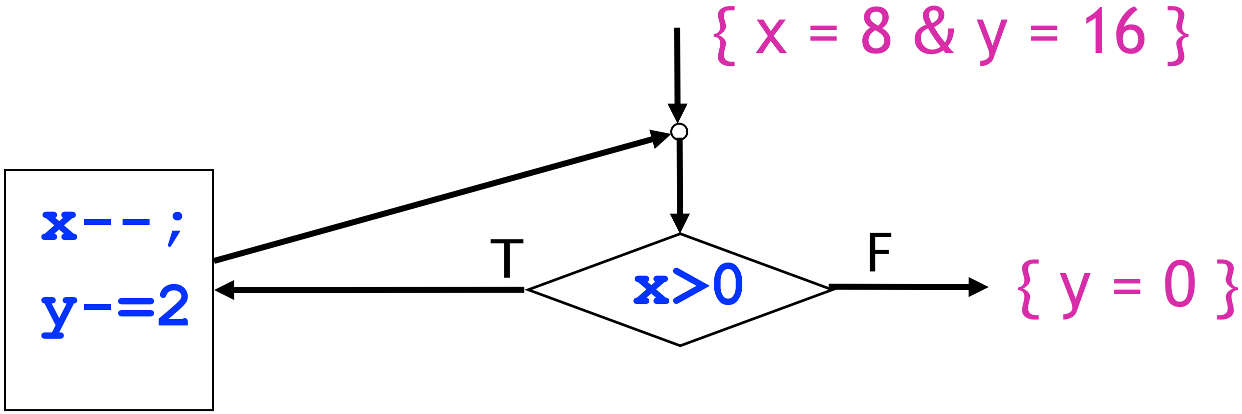


Find an appropriate invariant I

- Holds initially x = 8 & y = 16
- Holds at end y == 0

Loop Example (II)



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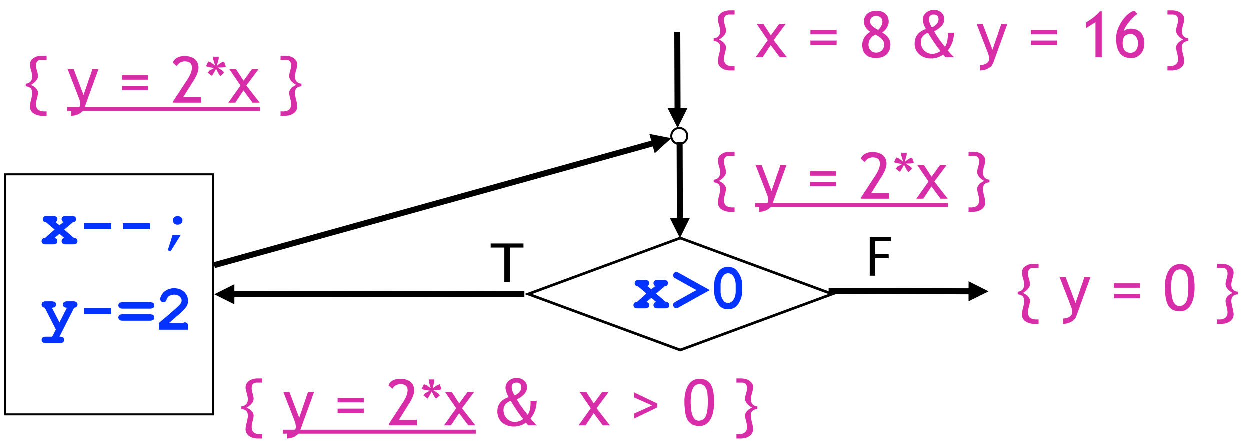

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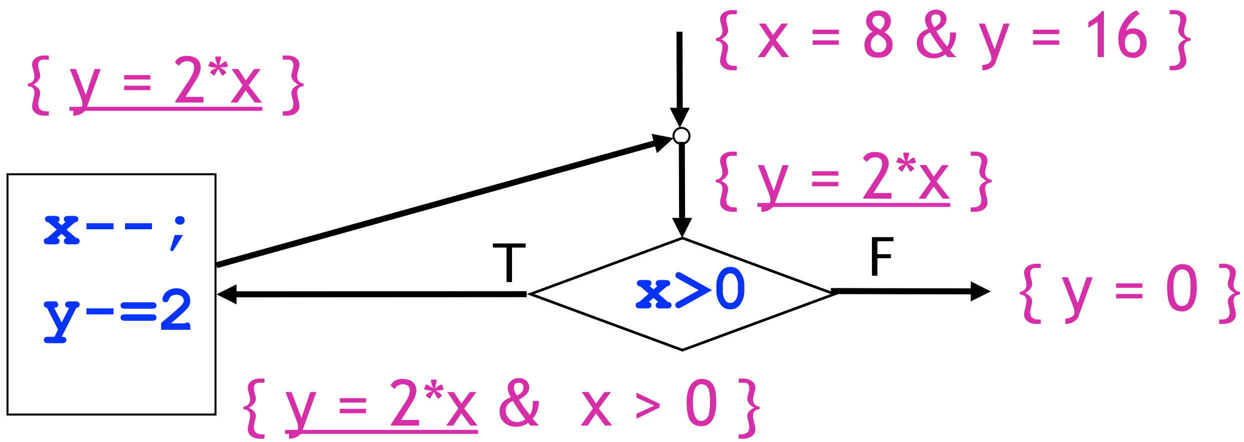
Guess invariant y = 2*x



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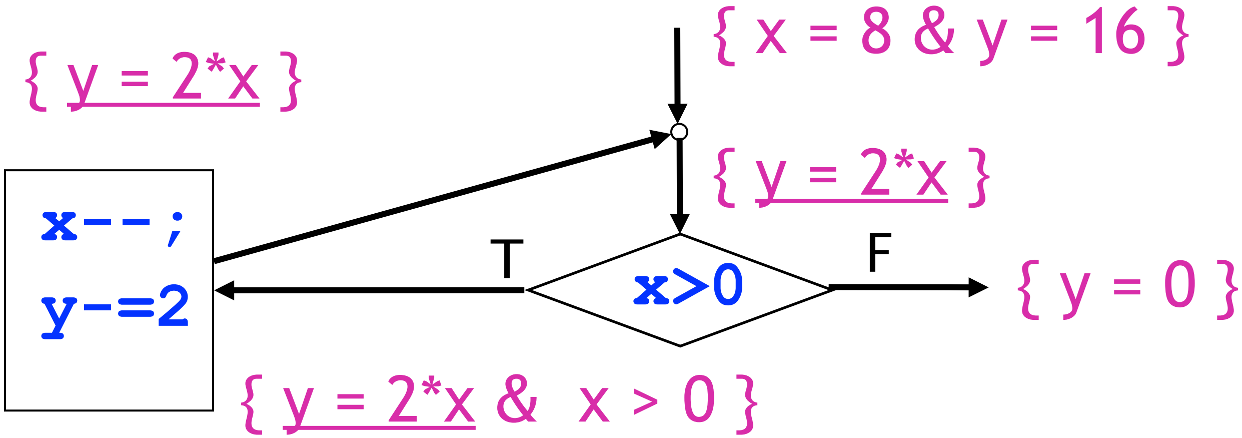

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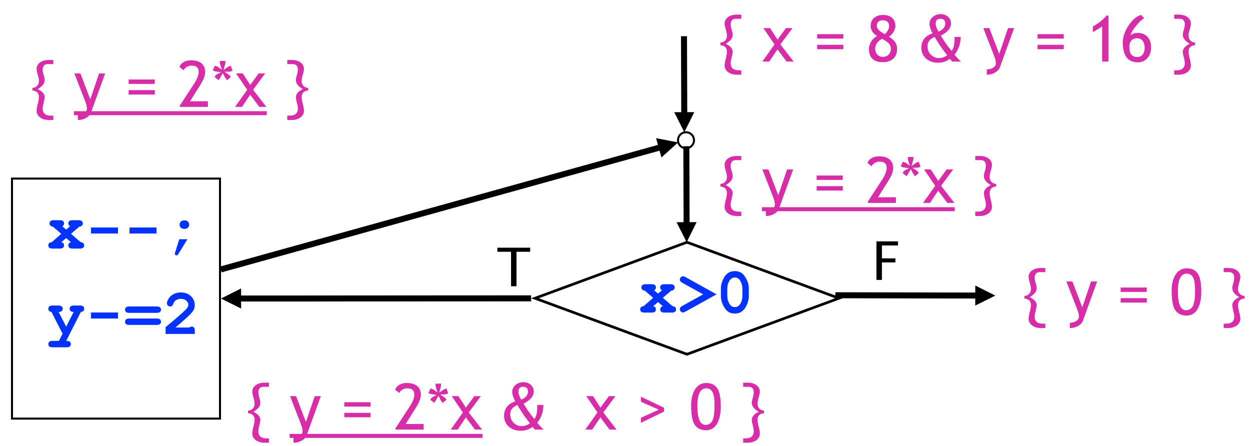

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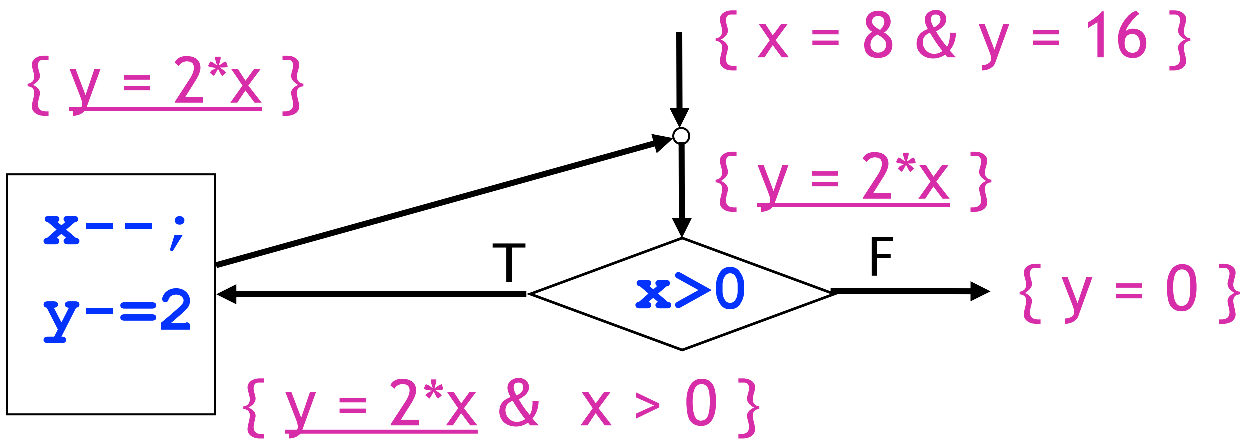

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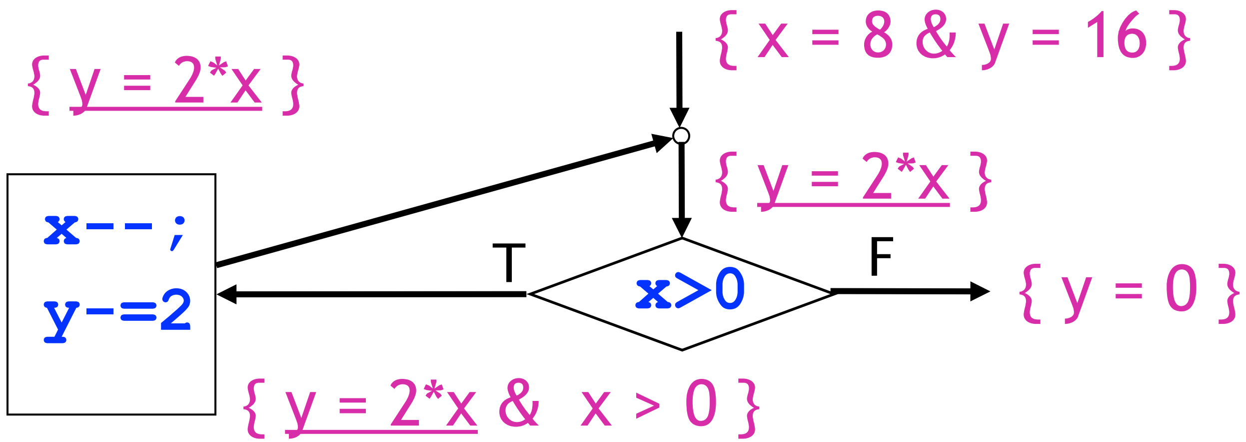

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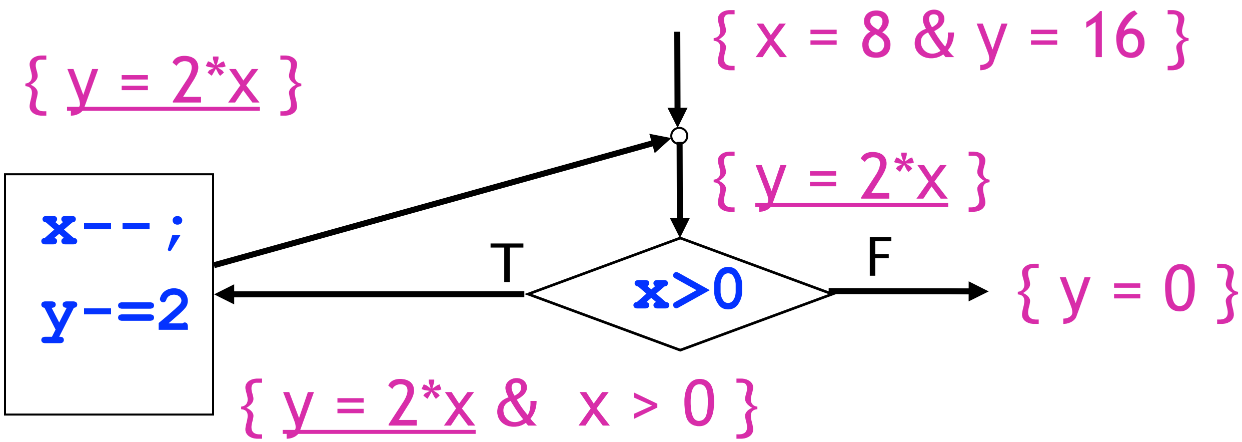

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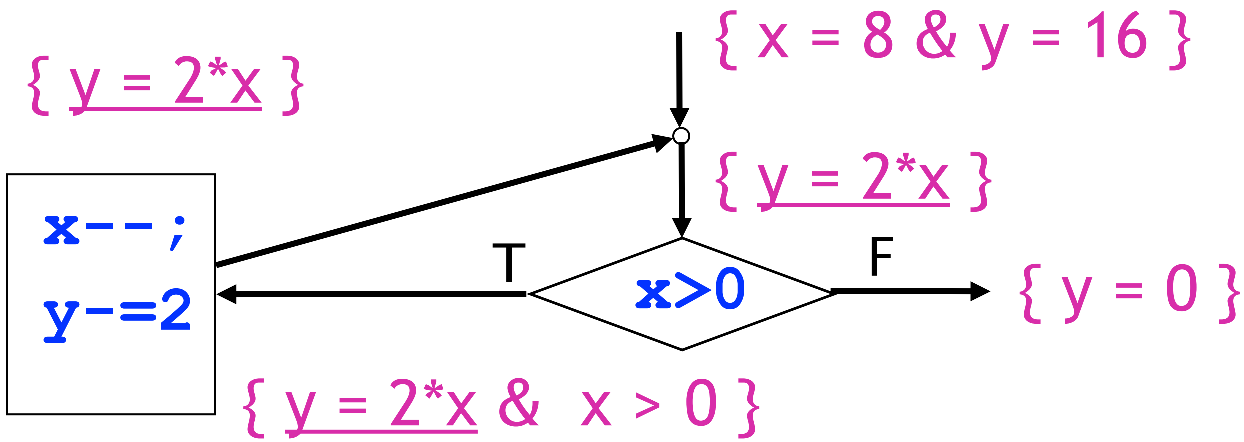

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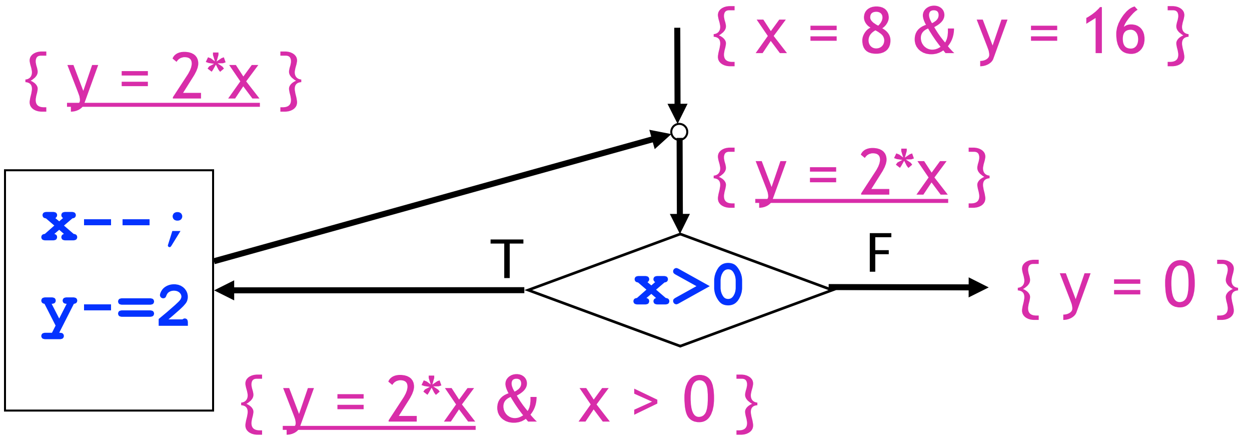

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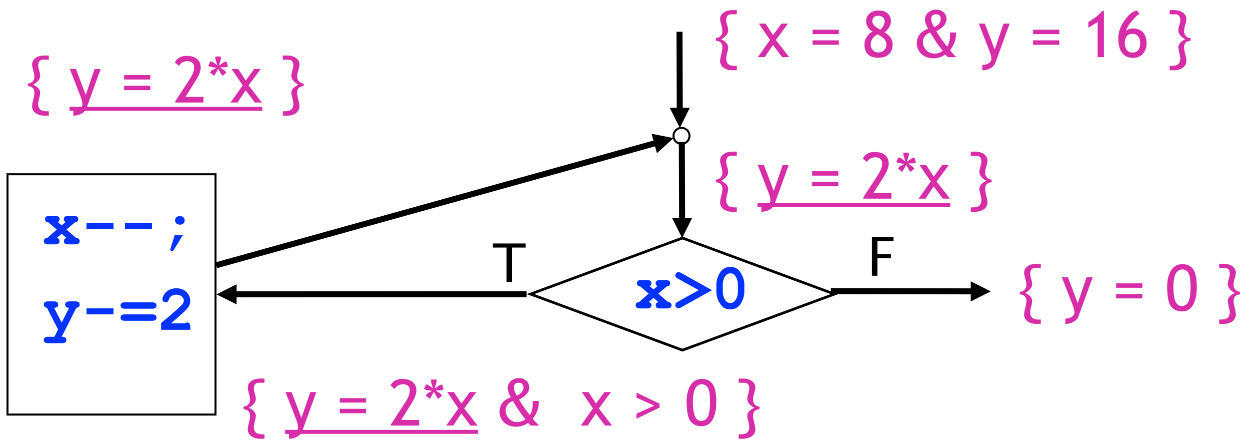

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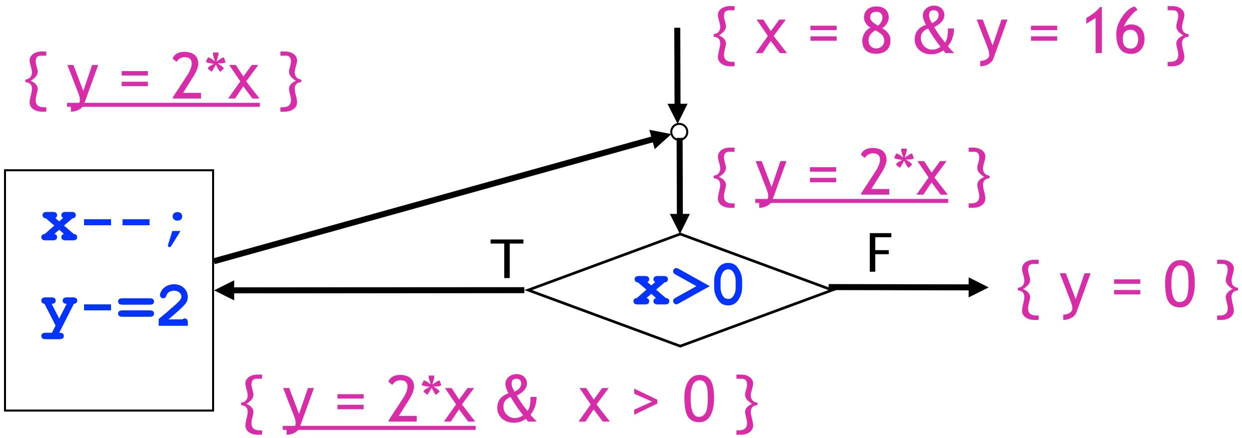

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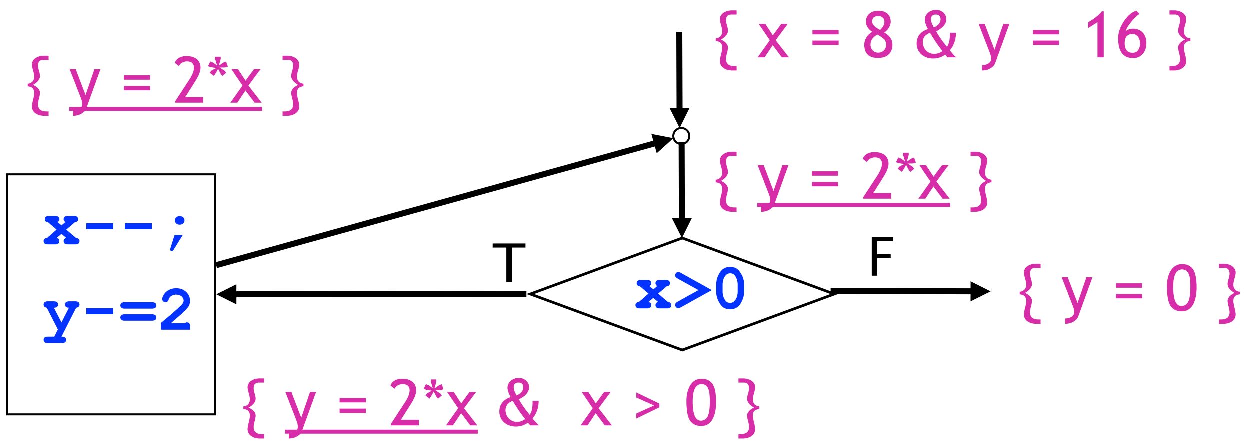

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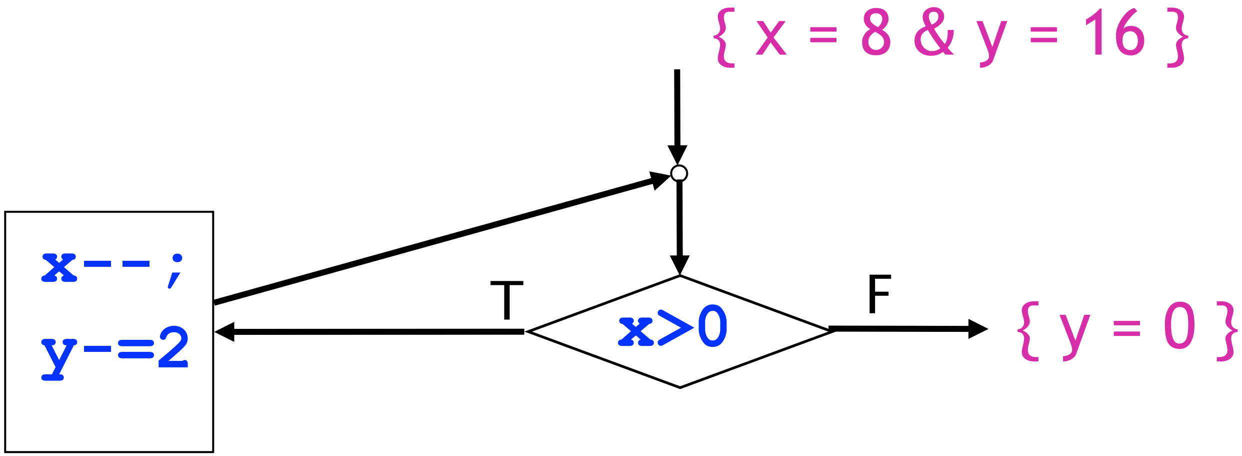

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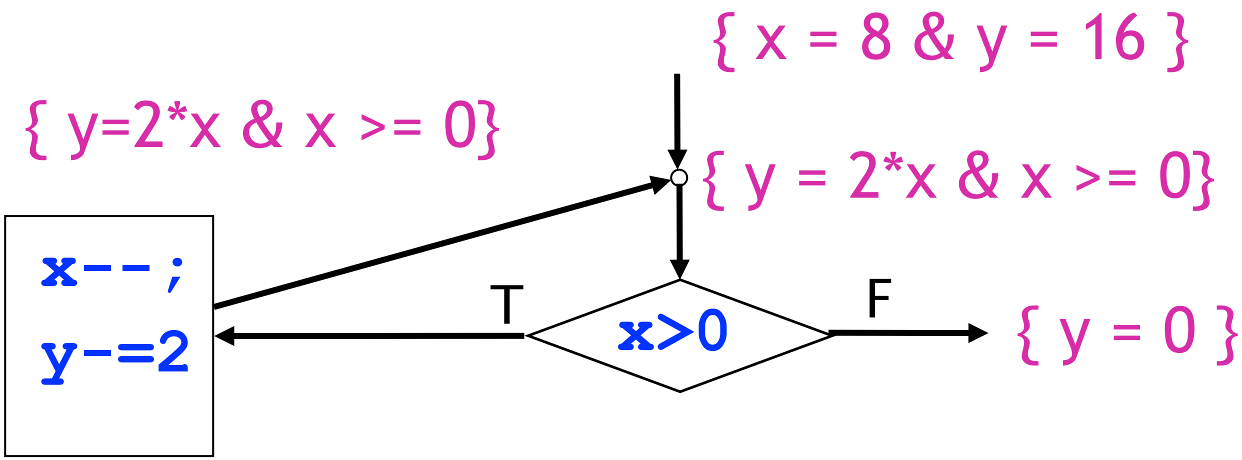

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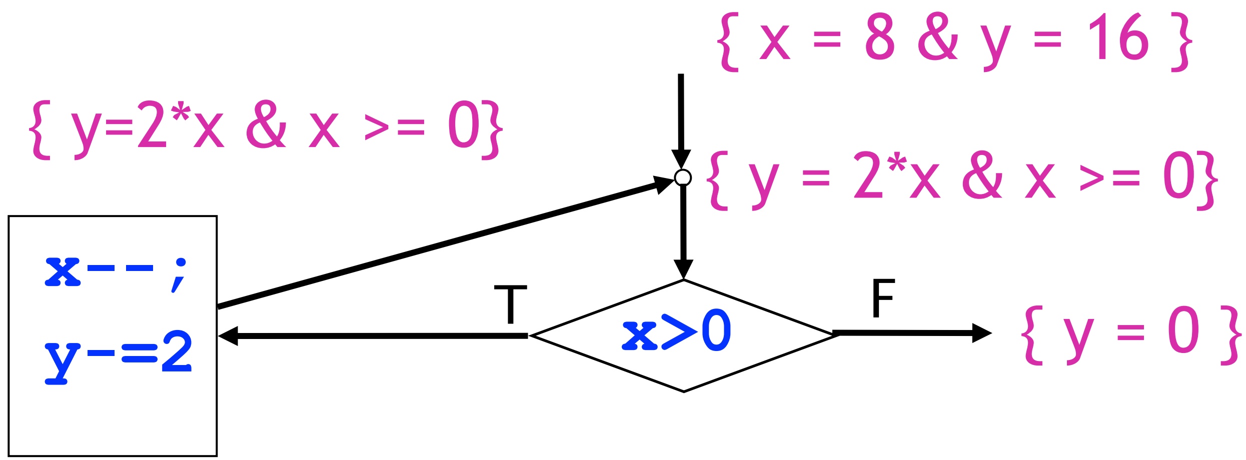

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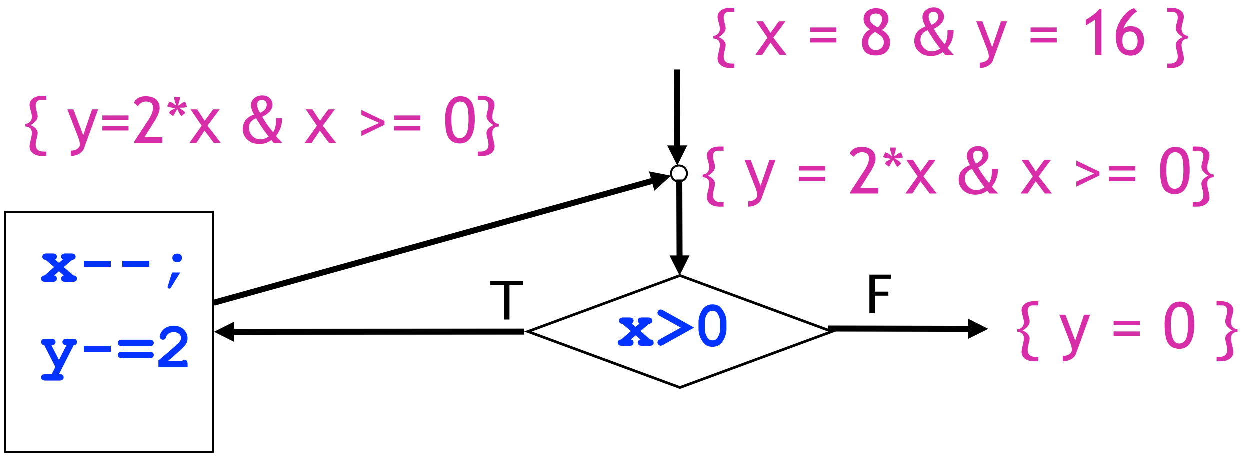

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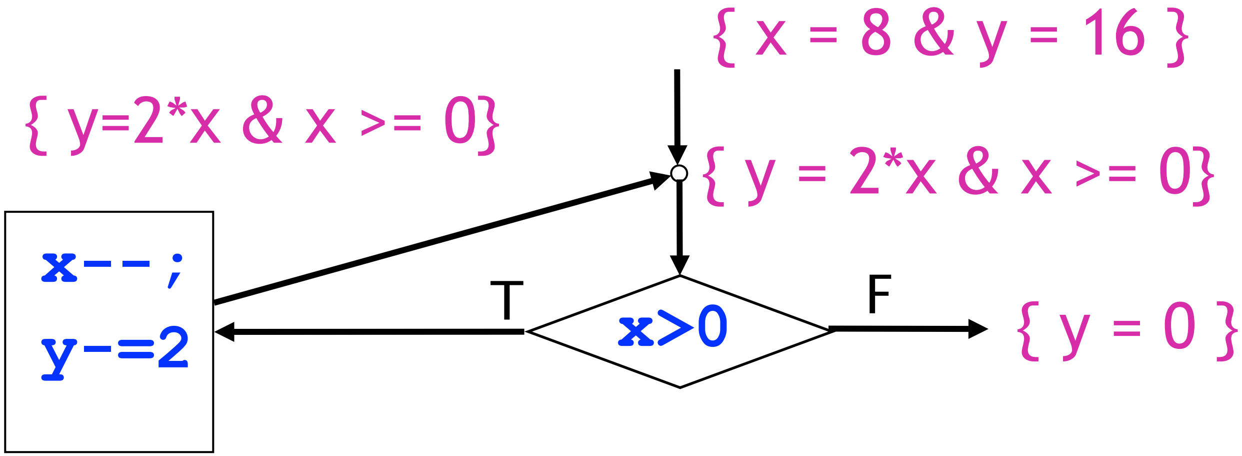

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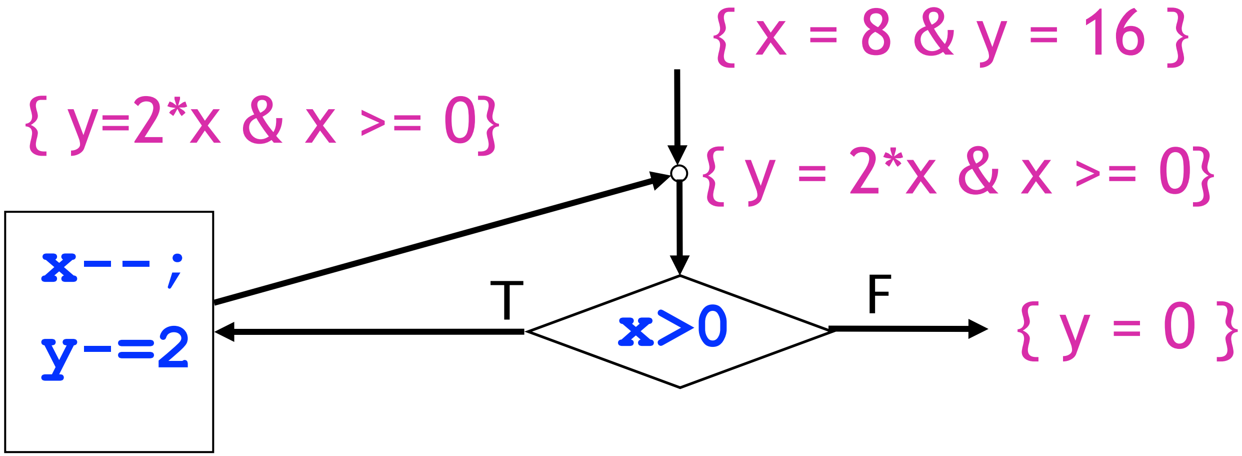

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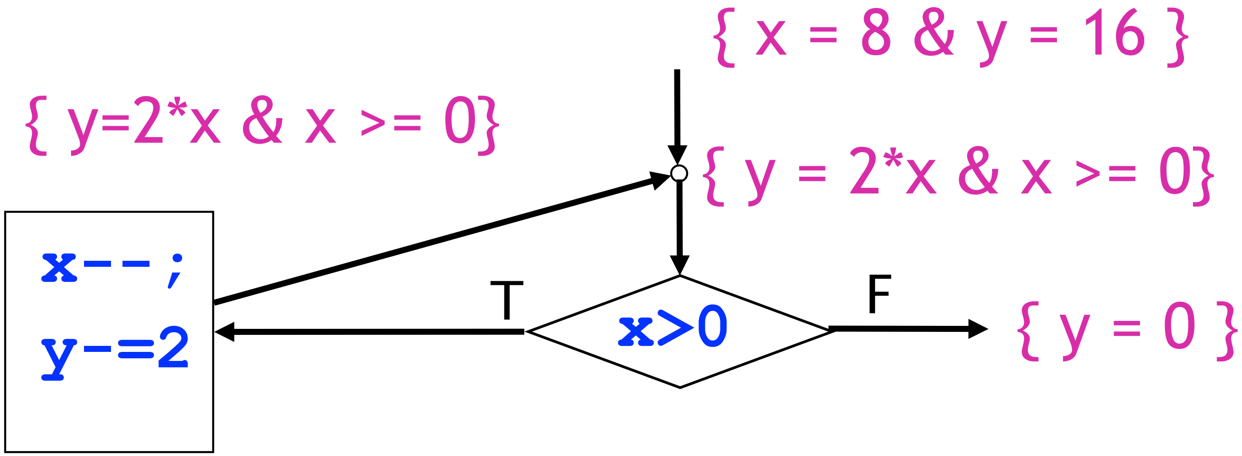

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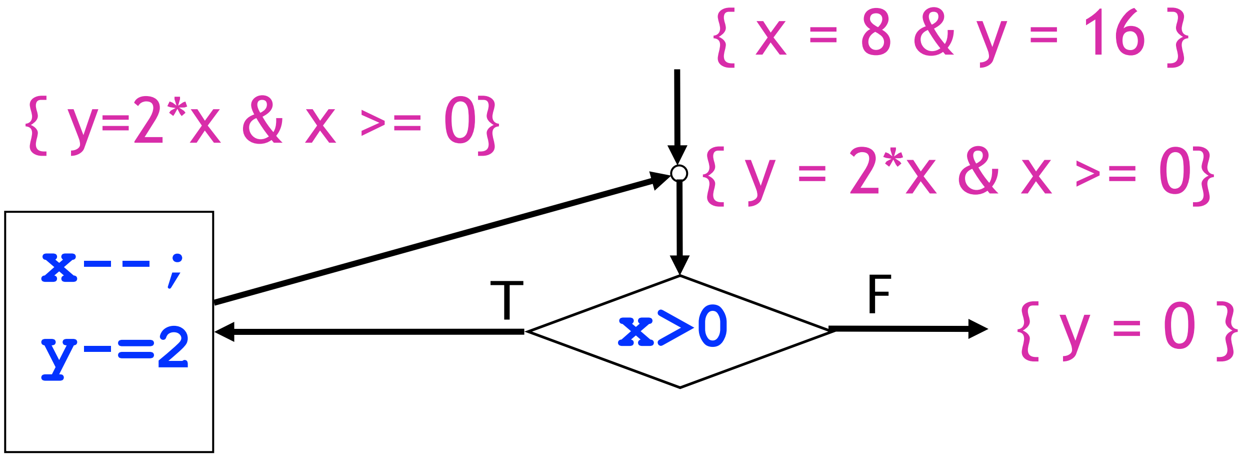

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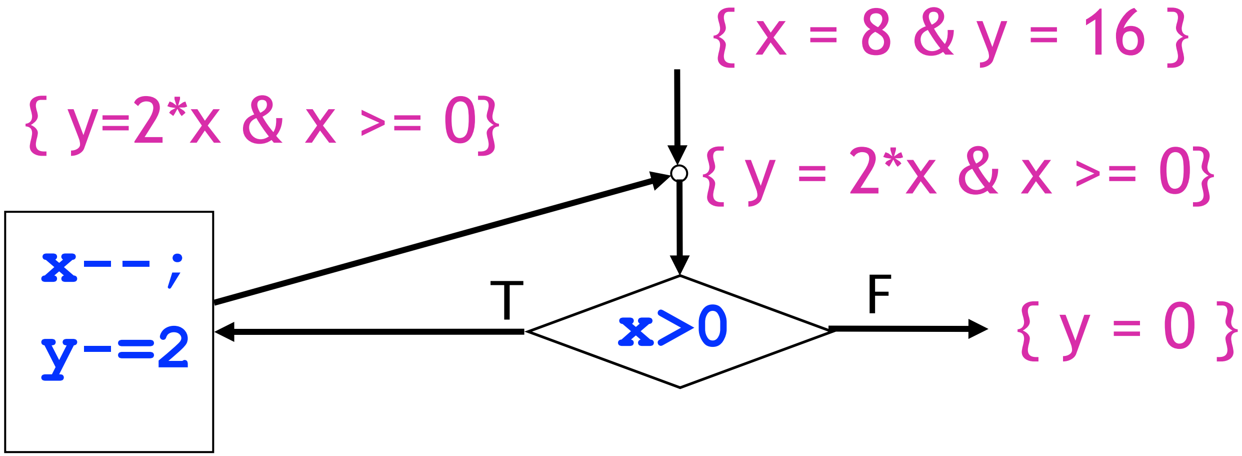

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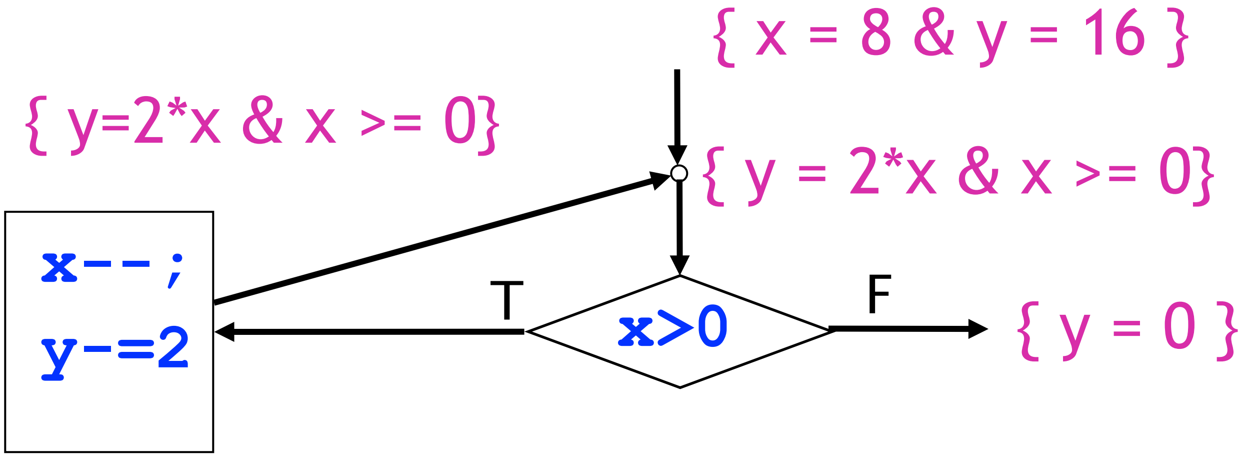

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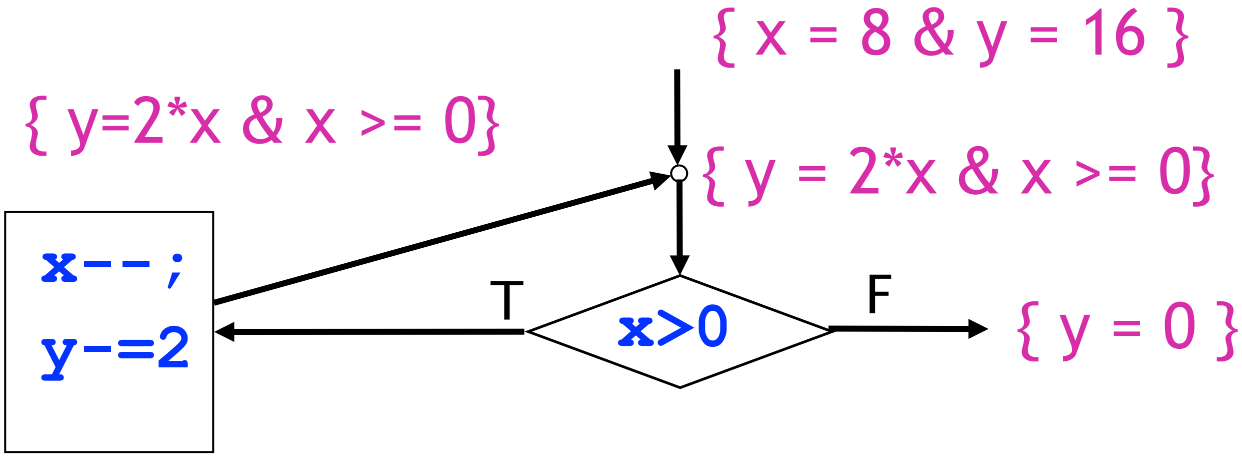

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Note: Invariant depends on your proof goal!

Verification Example

```
        k = k + 1;
    }
    return r;
}
```

Verification Example

```
    }
    return r;
}
```

Verification Example

```
    return r;
  }
```

Verification Example

```
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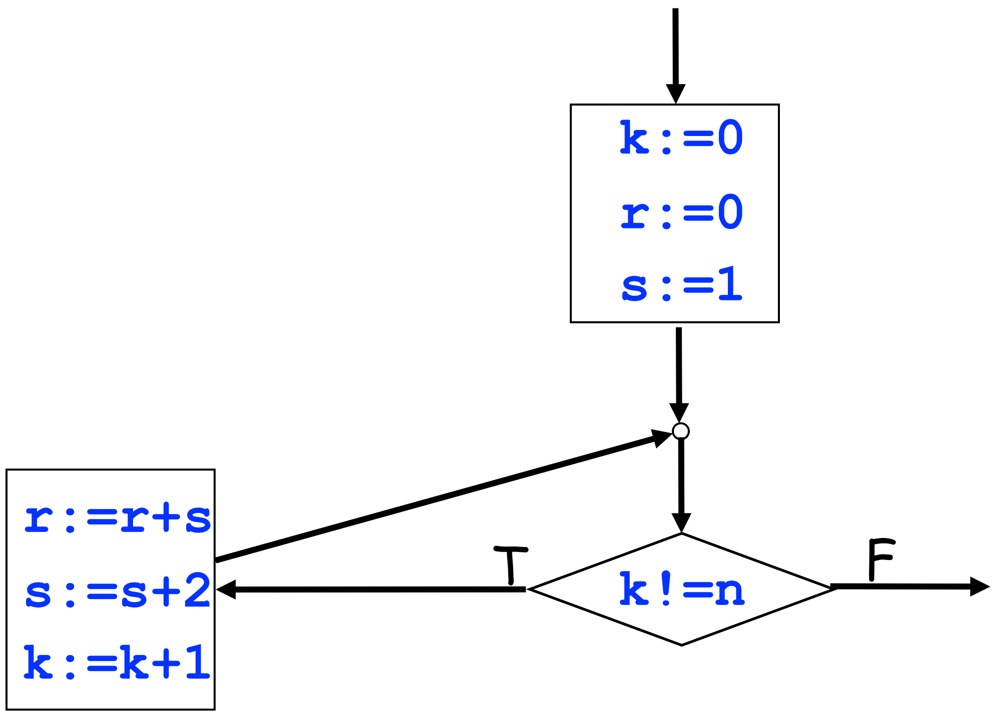

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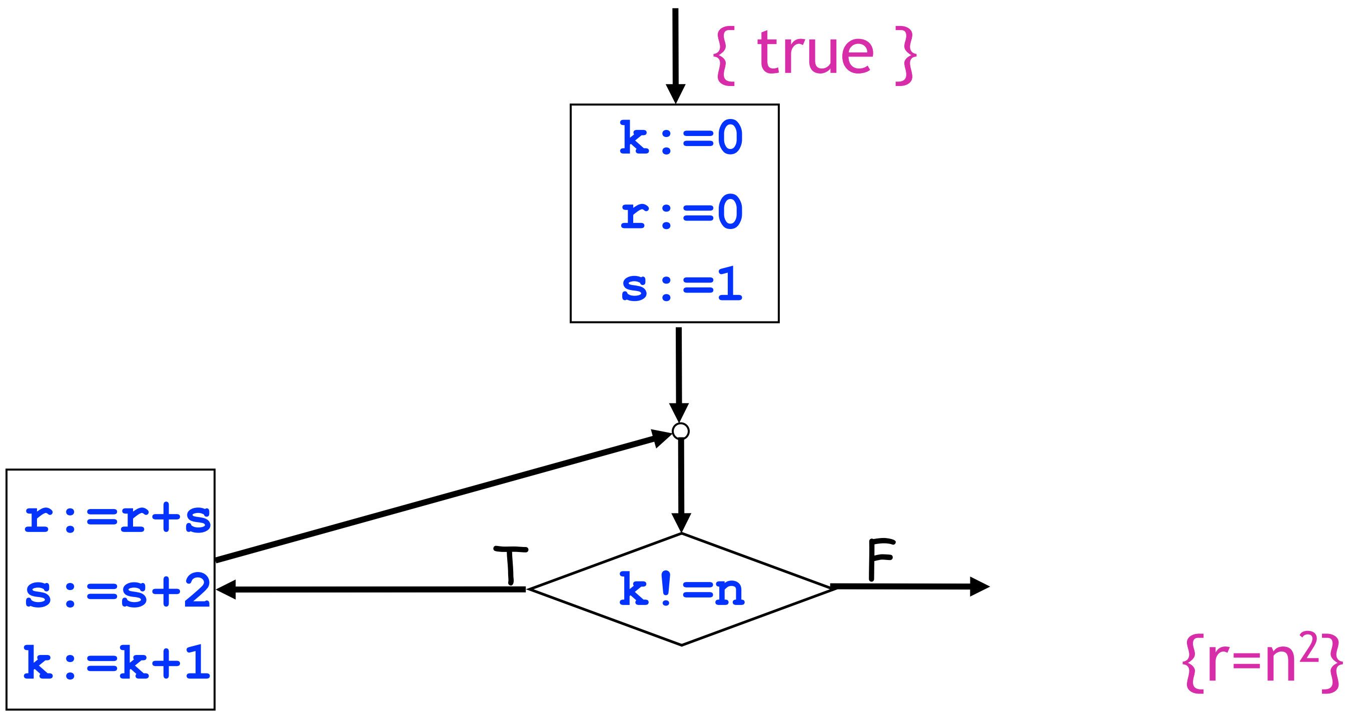

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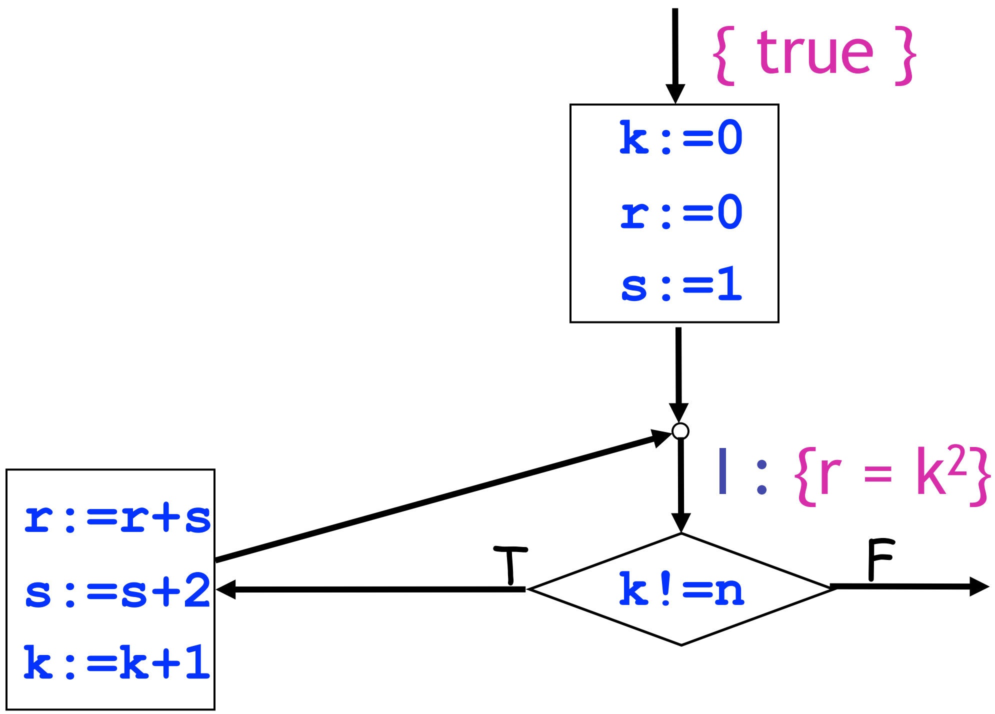

Pick I: $r = k^2$

$\{r=n^2\}$

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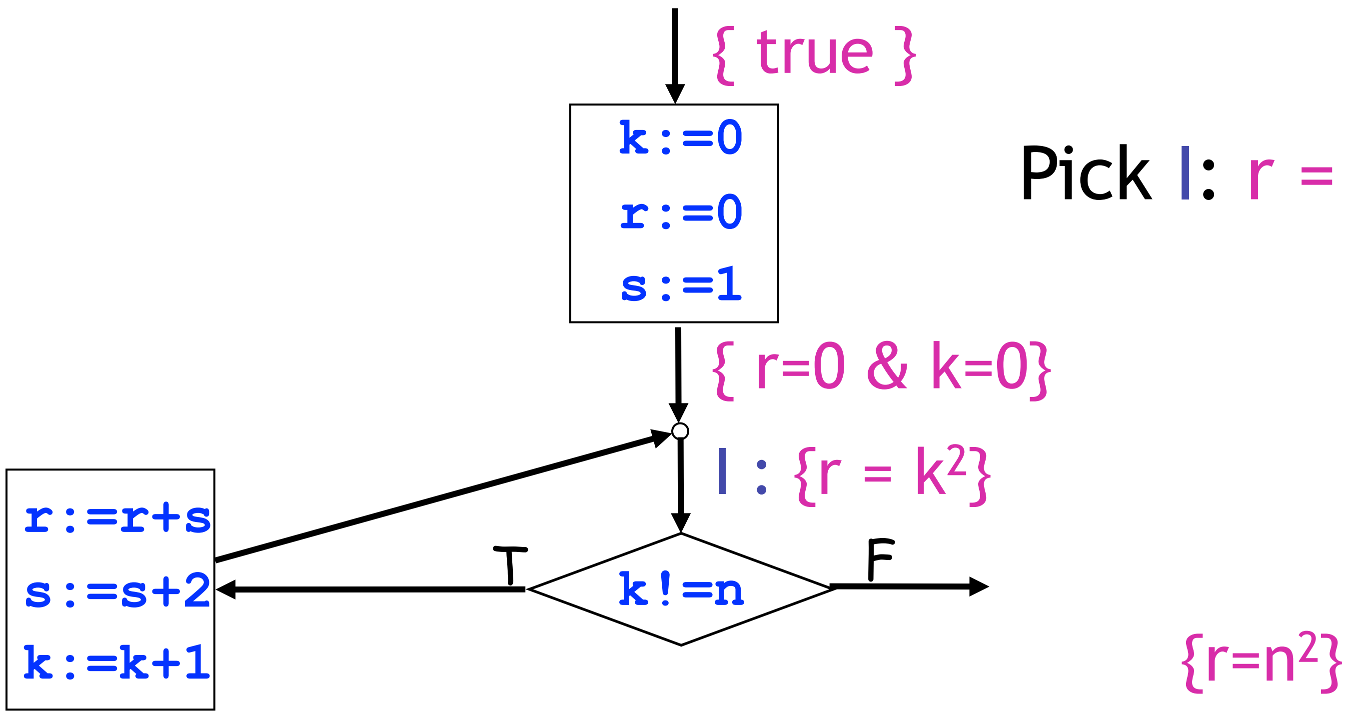

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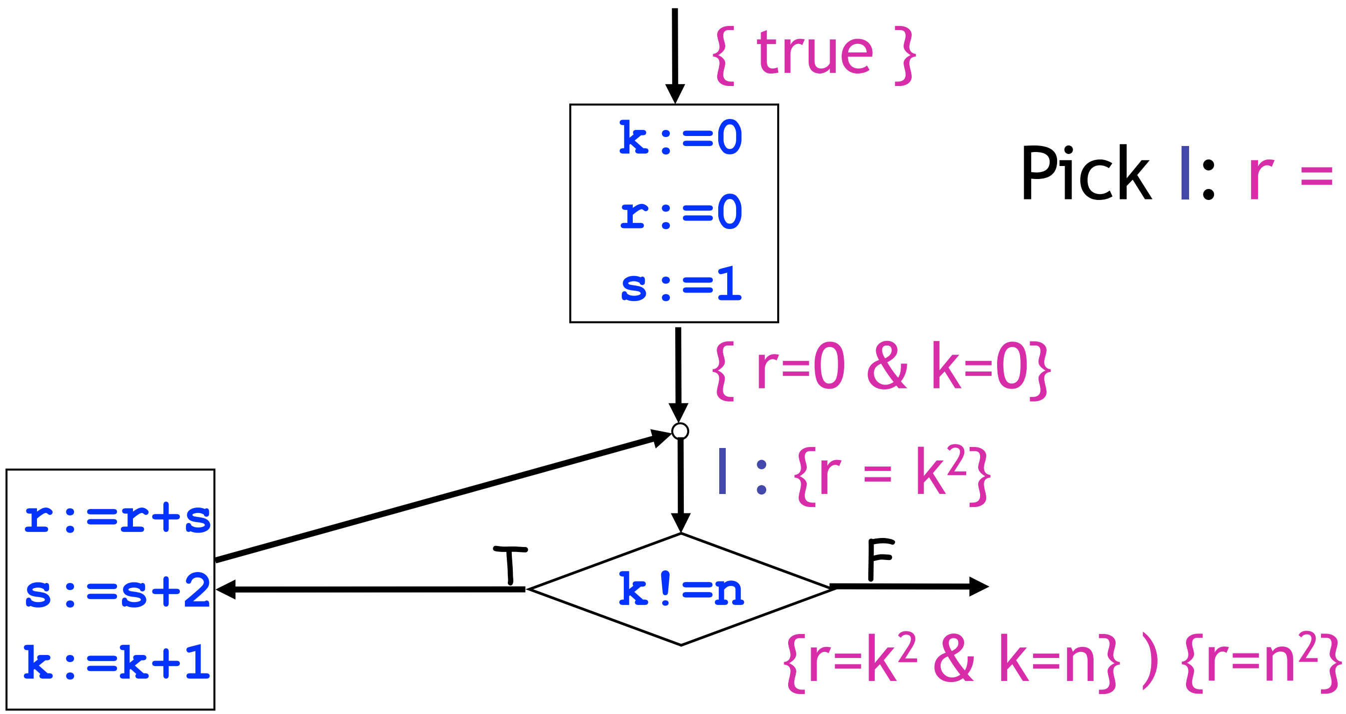

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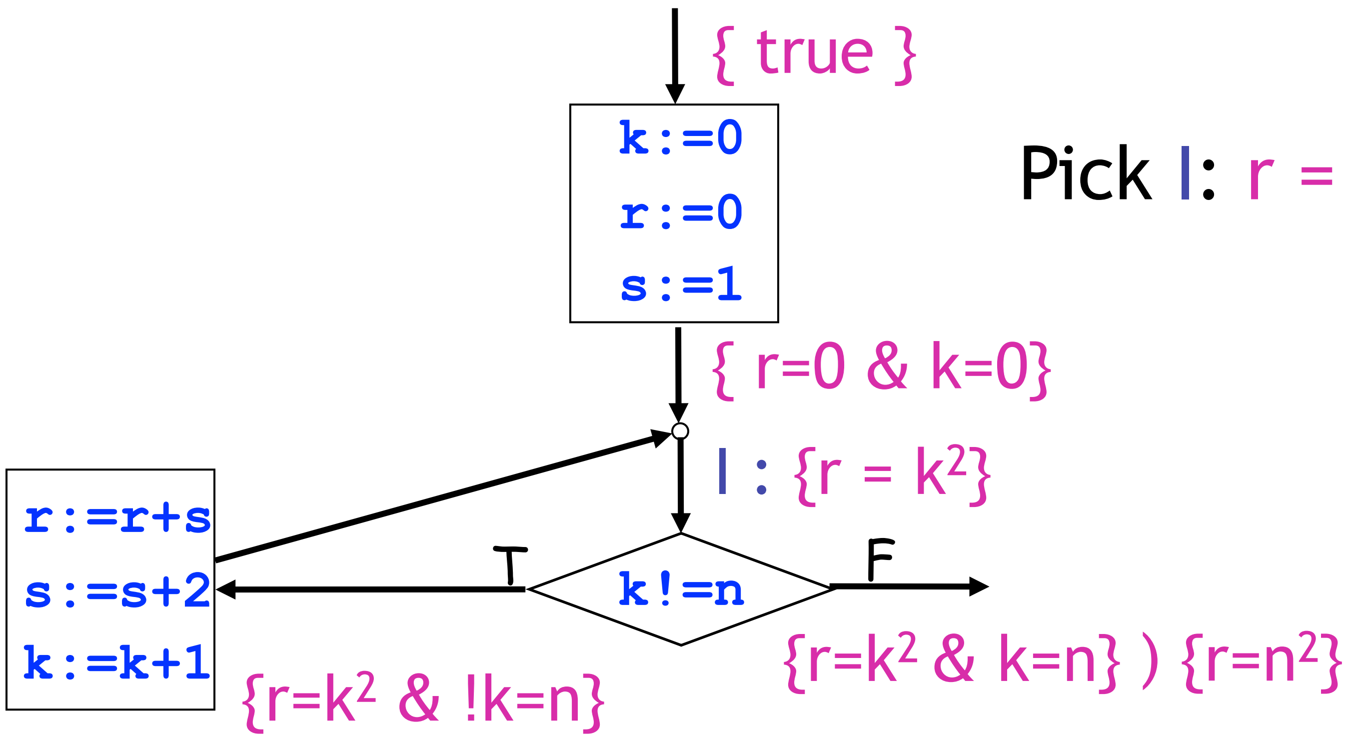

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Need: $\{r=k^2 \ \& \ !k=n\}$ **c** $\{r=k^2\}$

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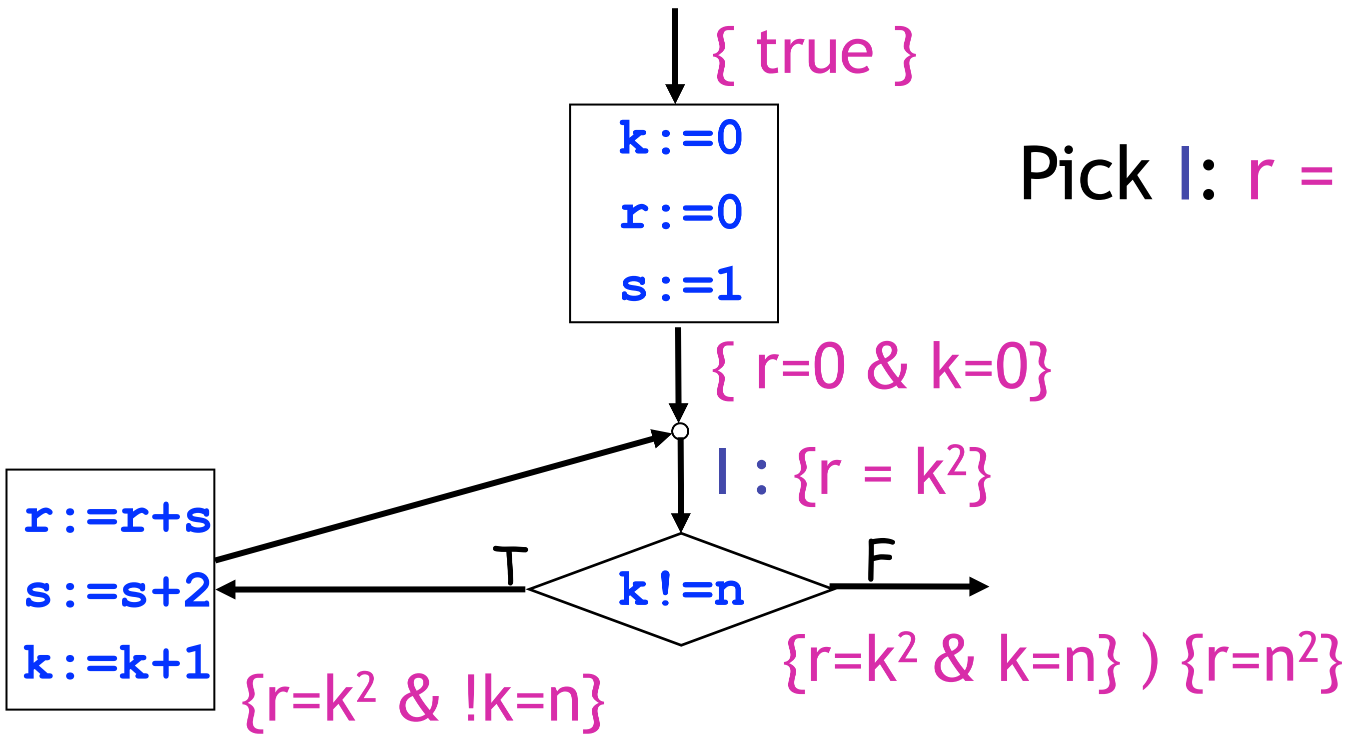

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Need: $\{r=k^2 \ \& \ !k=n\}$ **c** $\{r=k^2\}$

i.e. $\{r=k^2 \ \& \ !k=n\}$ => **WP(c,**$\{r=k^2\}$**)**

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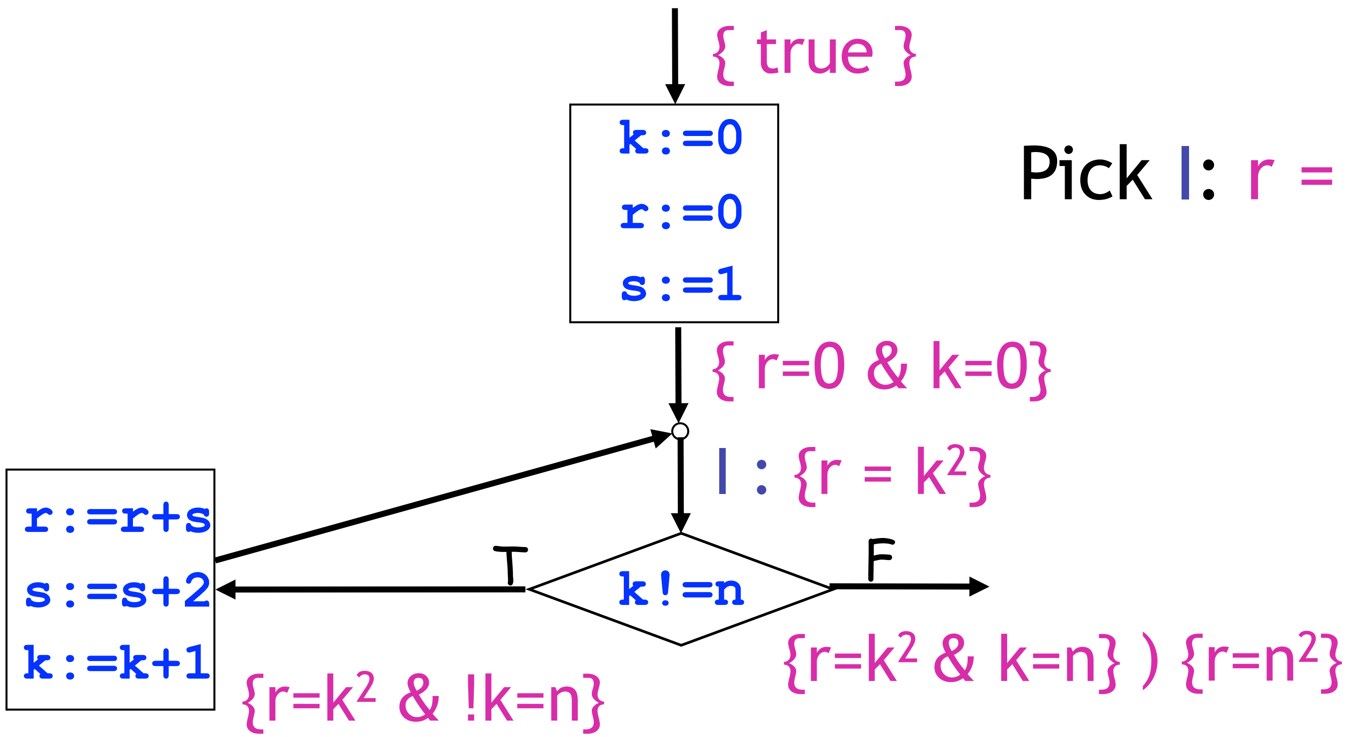

Pick I: r = k^2

Need: {r=k^2 & !k=n} **c** {r=k^2}

i.e. {r=k^2 & !k=n} => WP(**c**,{r=k^2})

i.e. {r=k^2 & !k=n} => {r+s=(k+1)^2}

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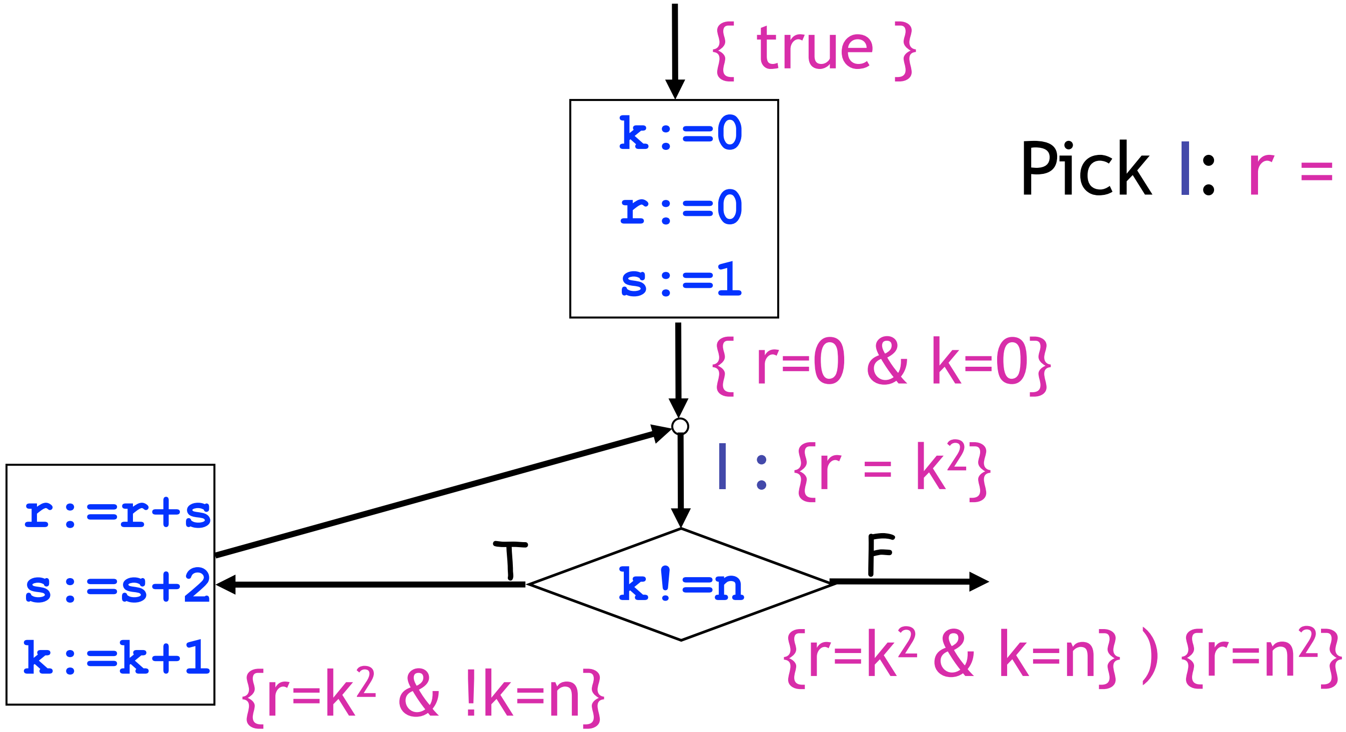

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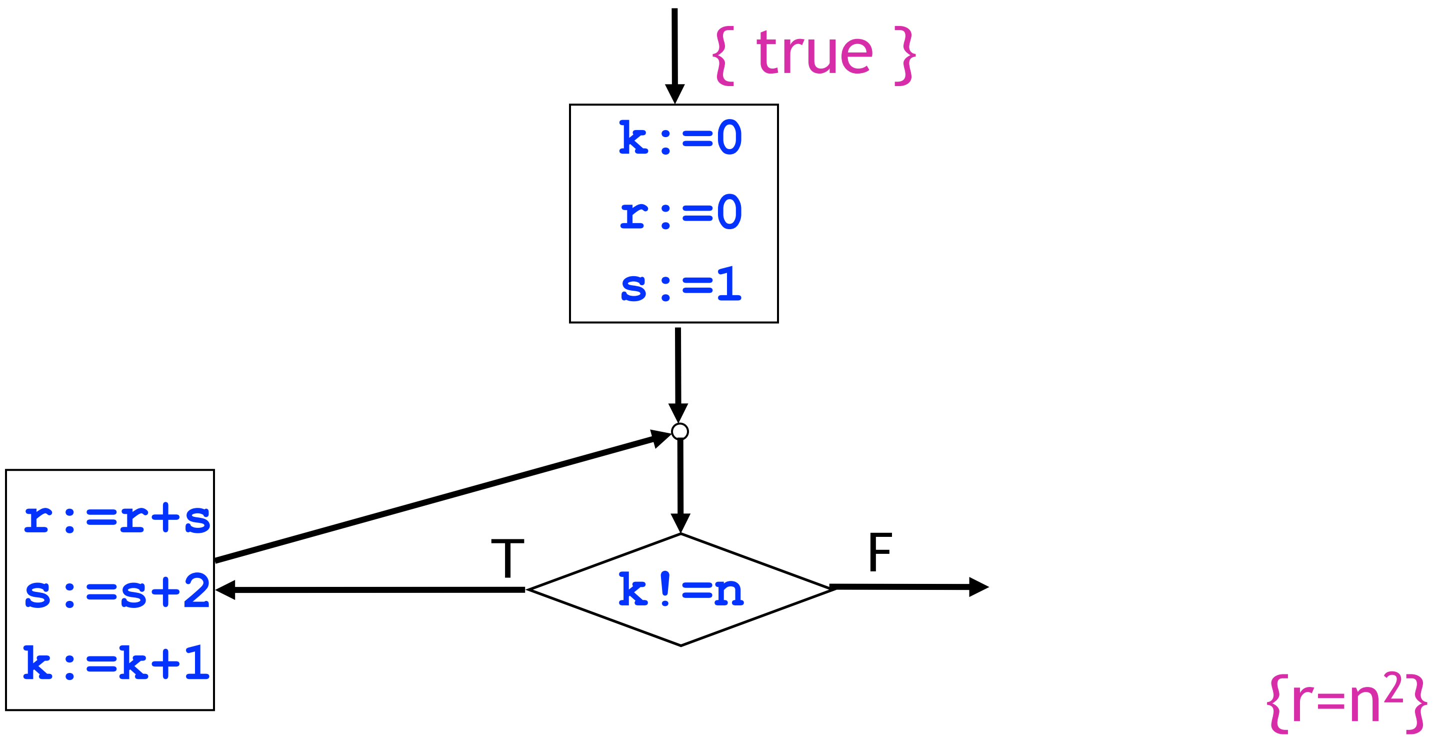
Invalid

Verification Example



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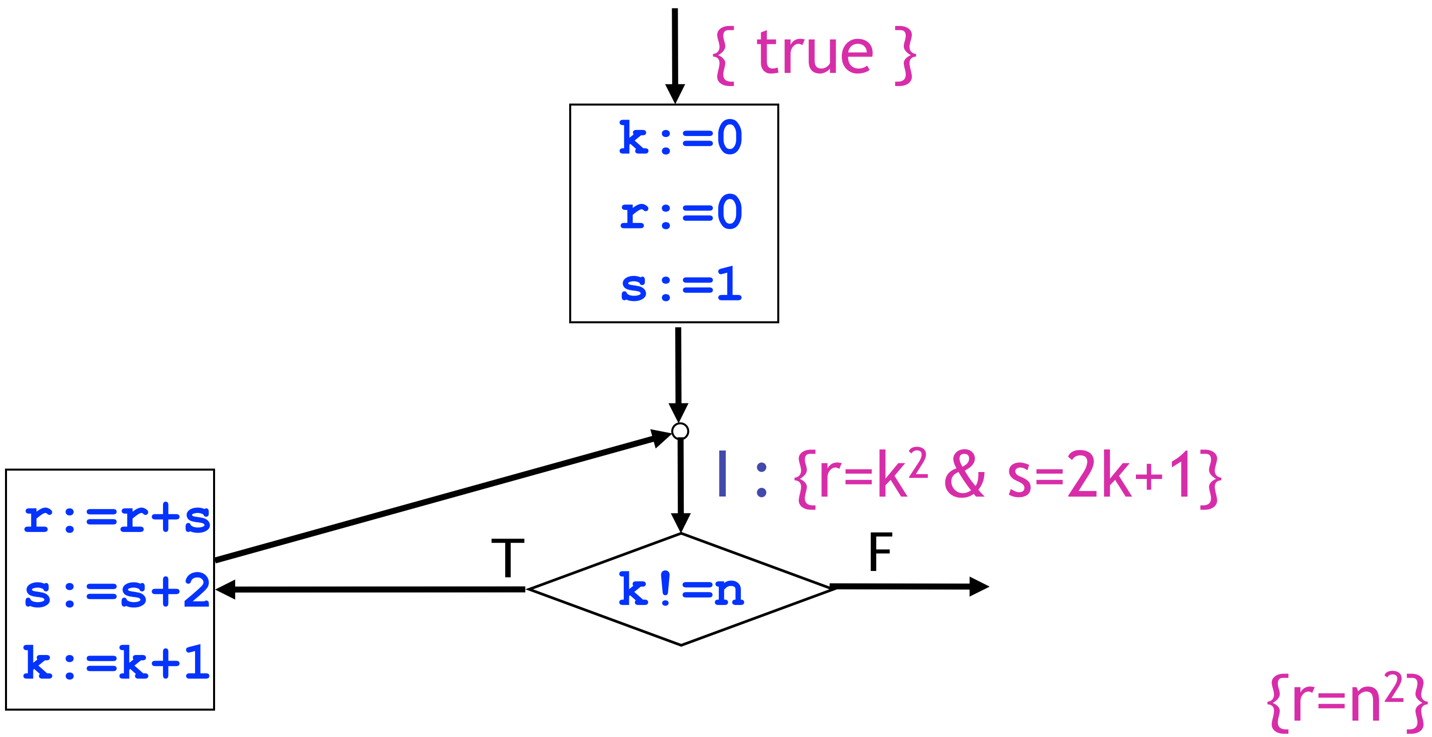

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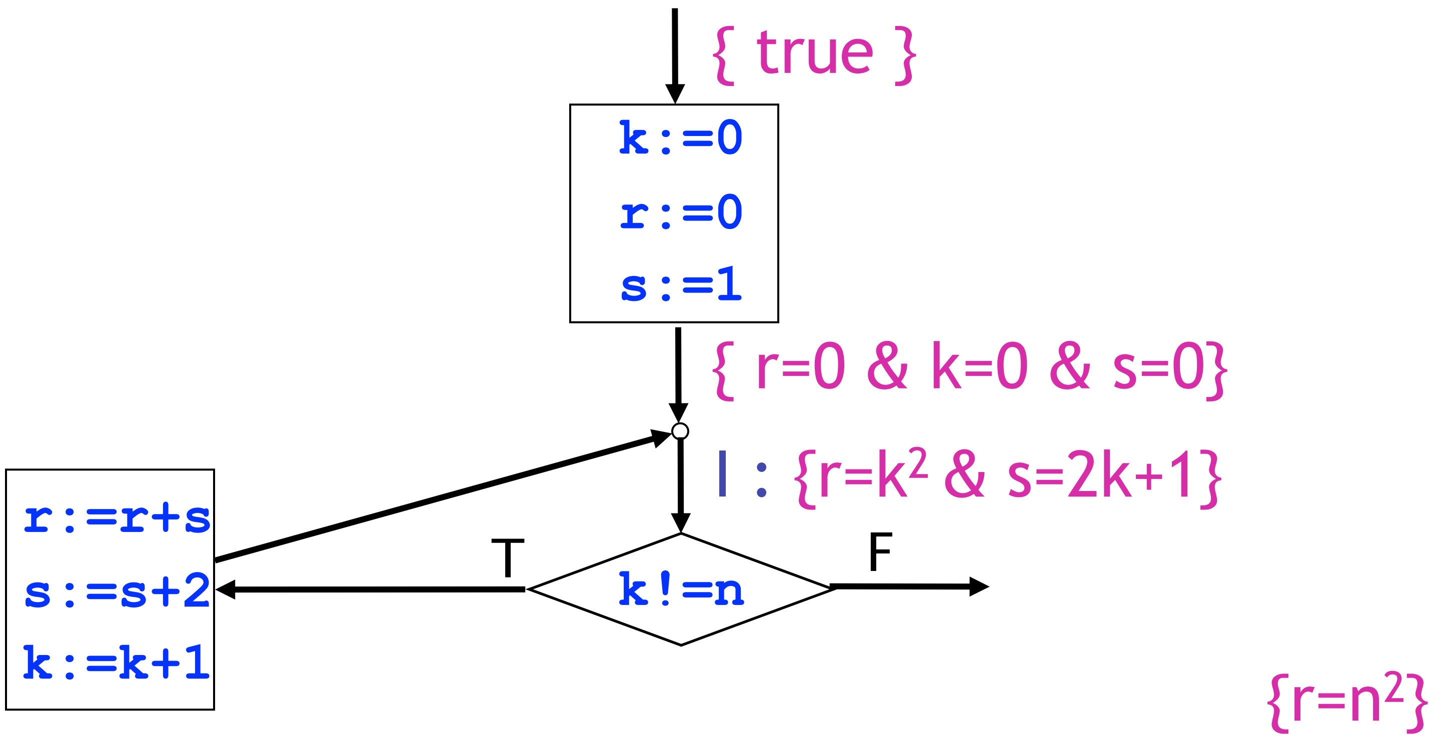

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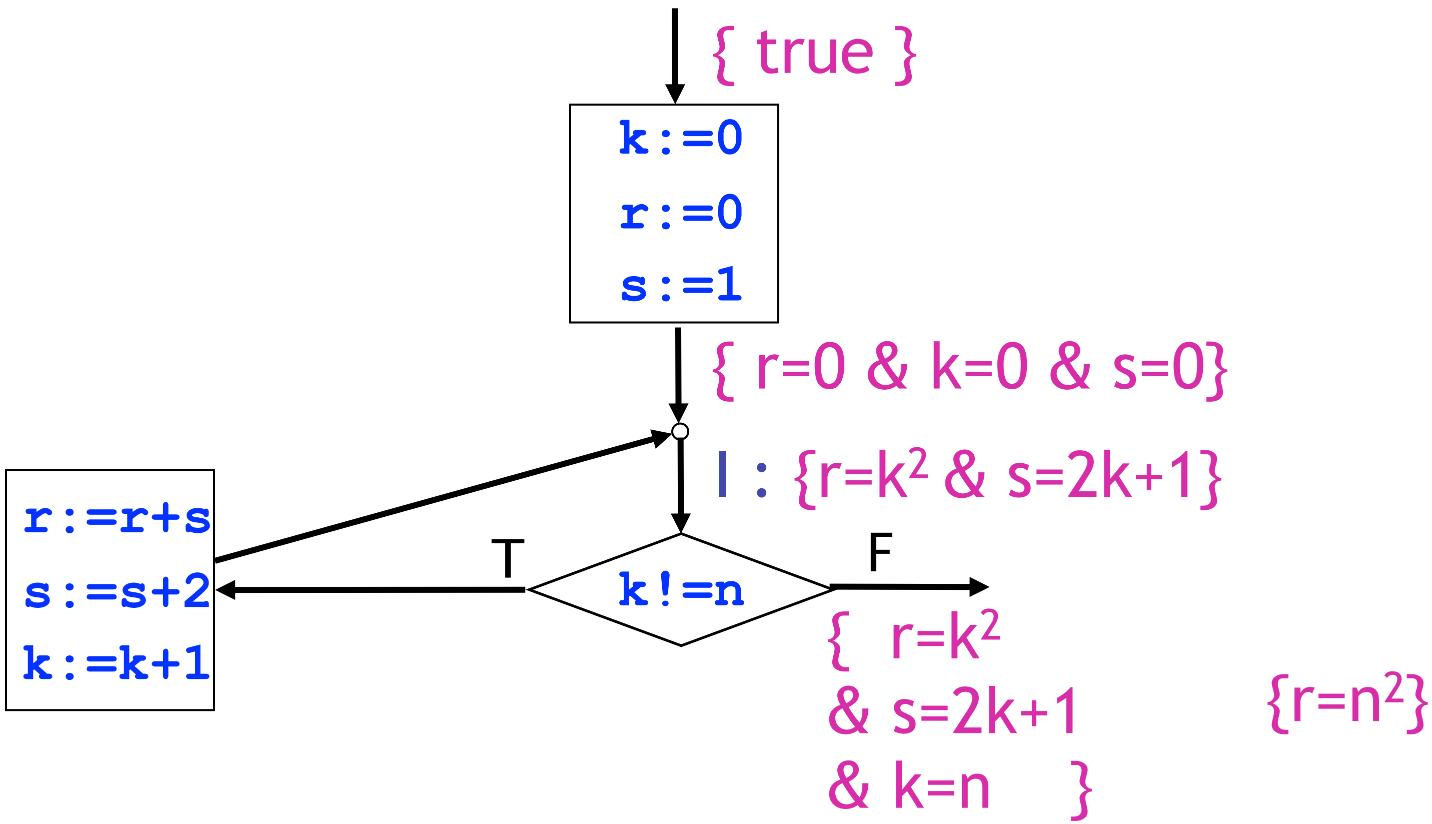
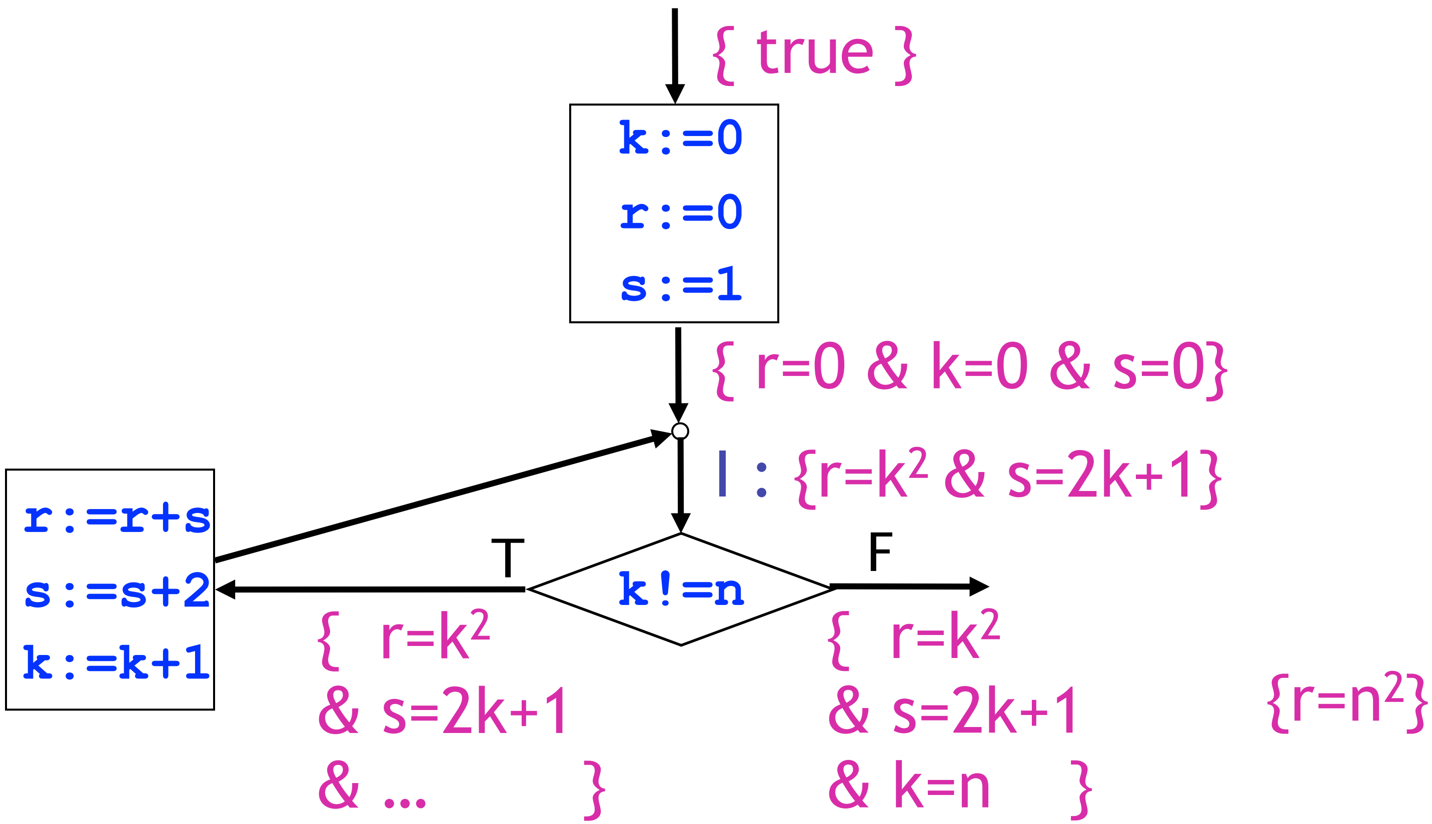


Verification Example

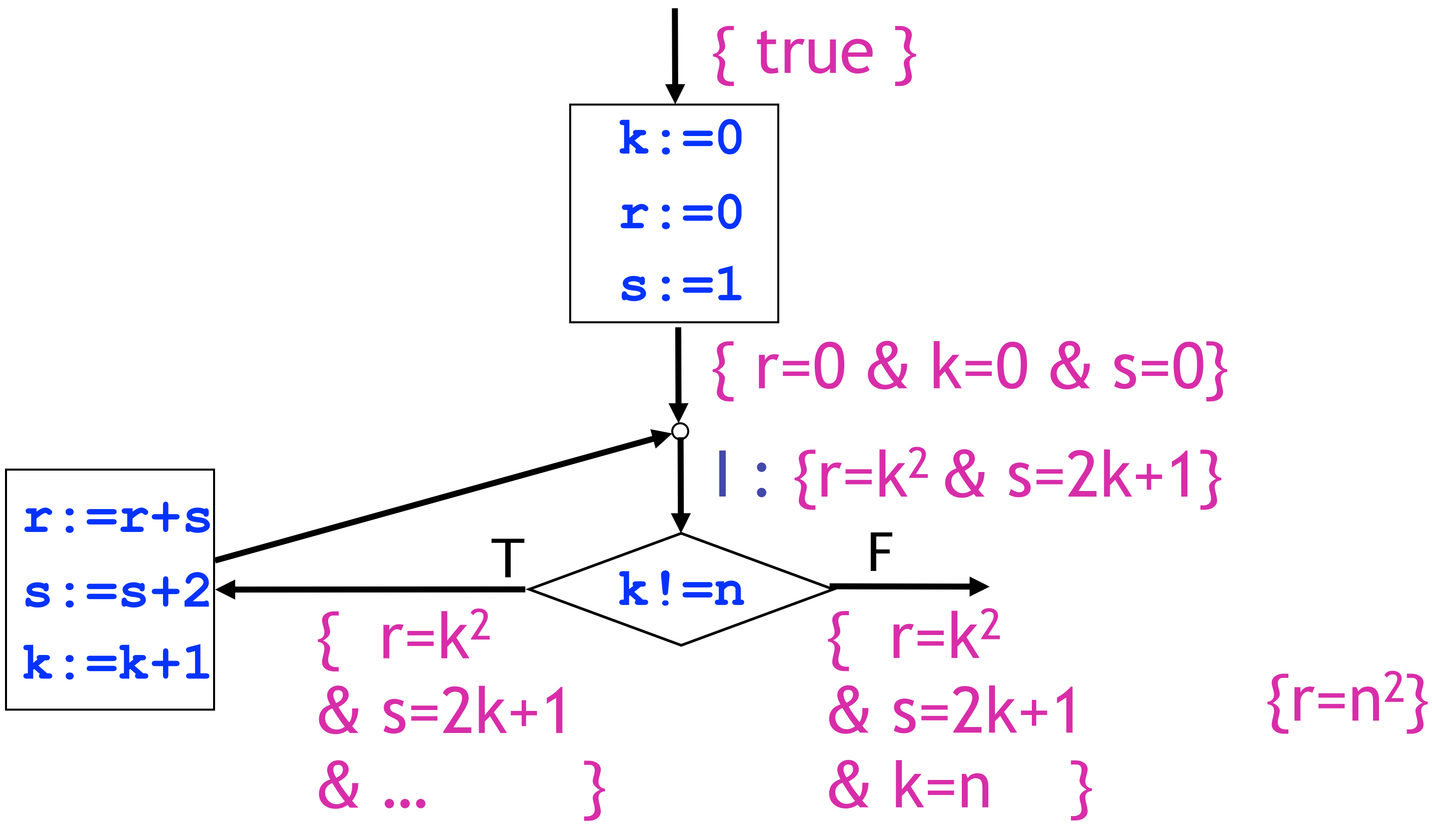
Need: {$r=k^2$ & $s=2k+1$ & …} **c** {$r=k^2$ & $s=2k+1$}



Verification Example

Need: $\{r=k^2 \ \& \ s=2k+1 \ \& \ \ldots\}$ **c** $\{r=k^2 \ \& \ s=2k+1\}$

i.e. $\{r=k^2 \ \& \ s=2k+1 \ \ldots\} \Rightarrow$ WP(**c**, $\{r=k^2 \ \& \ s=2k+1\}$)



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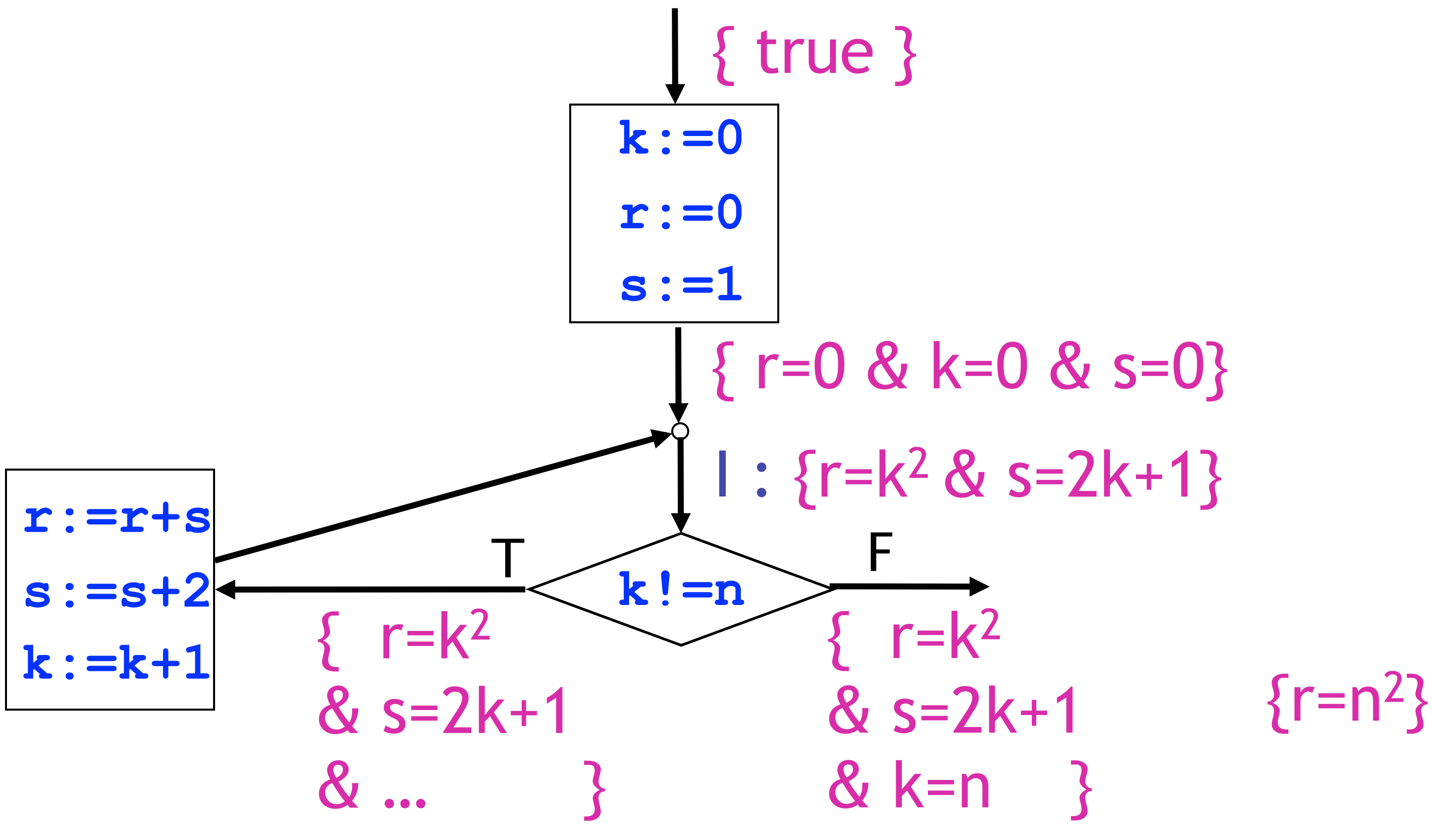

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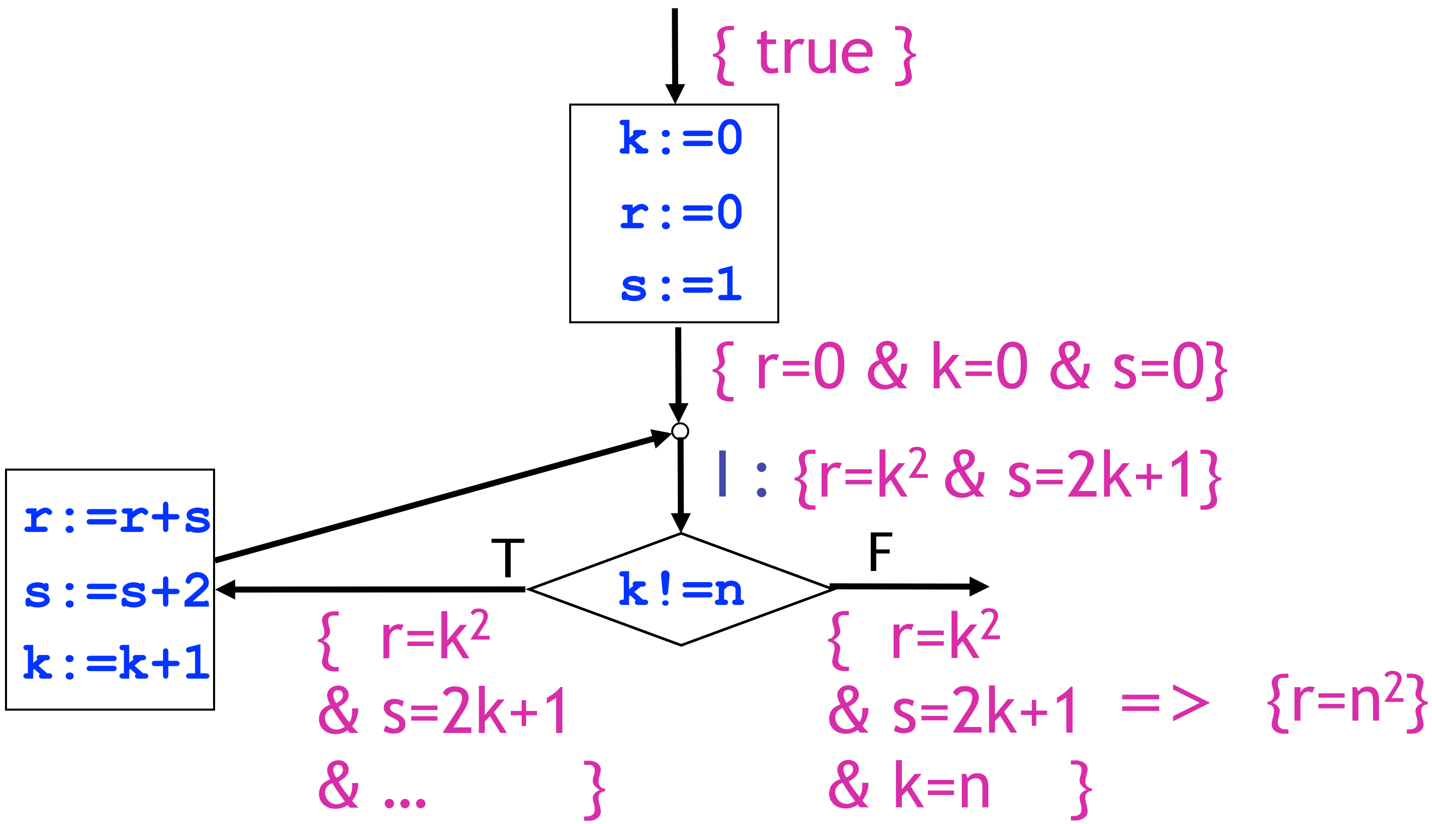
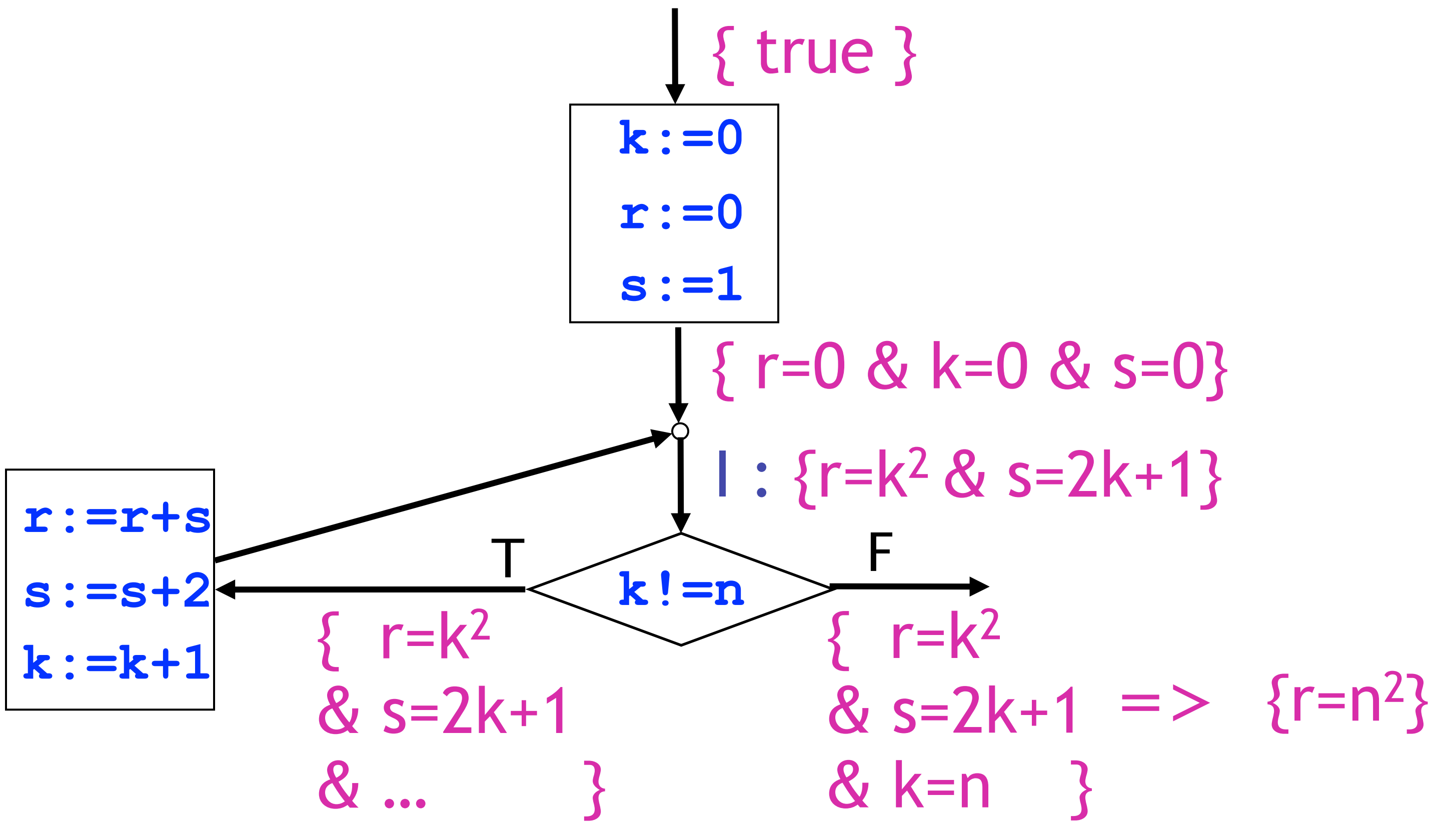


Verification Example

Need: {r=k² & s=2k+1 & …} c {r=k² & s=2k+1}

i.e. {r=k² & s=2k+1 …} => WP(c, {r=k² & s=2k+1})

i.e. {r=k² & s=2k+1 …} => {r+s=(k+1)² Æ (s+2) = 2(k+1)+1} Valid



What about real languages ?

- Loops
- Function calls
- Pointers

Functions are big instructions

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Suppose we have verified `bsearch`

Postcondition
"Ensures"

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- Function spec = precondition + postconditon
- Also called a contract

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- Consider a call to function `y:= f(e)`
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```
  return r;
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```

- sorted[array] => Pre[a := arr]

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$\{arr[y]=5\}$

- sorted[array] => Pre[a := arr]
- Post[y/r, arr/a, 5/p] => (y=-1 || arr[y]=5)

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Assignment and Aliasing

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Does assignment rule work with aliasing ?

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If *`x` and *`y` are aliased then:

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If *`x` and *`y` are aliased then:

{x=y} *`x:=5` {*x + *y=10}

Hoare Rules: Assignment and References

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Hoare Rules: Assignment and References

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- A should be " *y = 5 or x = y "

- but Hoare rule for assignment gives:

  [5/*x](*x + *y = 10)

  = 5 + *y = 10

  = *y = 5

  (uh oh! we lost one case! What happened?)

Hoare Rules: Assignment and References

Modeling writes with memory expressions

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Reason about memory expressions with McCarthy's rule

$$sel(upd(M, E_1, E_2), E_3) = \begin{cases} E_2 & \text{if } E_1 = E_3 \\ sel(M, E_3) & \text{if } E_1 \neq E_3 \end{cases}$$

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Assignment (update) changes the value of memory

$$\frac{}{\{B[upd(M, E_1, E_2)/M]\}\ \texttt{*}\mathtt{E_1}\texttt{:=}\mathtt{E_2}\ \{B\}}$$

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= (x = y & 5 = 5) || (x != y & sel(M, y) = 5)

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= sel(upd(M, x, 5), y) = 5
= (x = y & 5 = 5) || (x != y & sel(M, y) = 5)
= x=y || *y = 5

Program Verification Tools

- Semi-automated
  - You write some invariants and specifications
  - Tool tries to fill in the other invariants
  - And to prove all implications
  - Explains when implication is invalid: counterexample for your specification

- ESC/Java is one of the best tools
- ... Spec#, Verifast, VCC

Algorithmic Program Verification

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…or how does ESC/Java work ?

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Q: How to algorithmically prove {P} `c` {Q} ?

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1. Compute: WP(`c`,Q)

Algorithmic Program Verification

…or how does ESC/Java work ?

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Algorithmic Program Verification

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Verification Condition

Algorithmic Program Verification

…or how does ESC/Java work ?

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Verification Condition

Proved By SMT Solver

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VC Generation for Loops

Suppose all loops annotated with Invariant

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Compute VC:

SMTValid(VC) **implies** |- {P} `c` {Q}

Q: Why not iff ?

1. Loop invariants may be bogus…
2. SMT solver may not handle logic…

VCGen

We will write a function

vcgen :: Pred -> Com -> (Pred, [Pred])

Suppose (Q',L') = VCG(c,(Q,L;))

Then VC for {P} c {Q} is: $P \Rightarrow Q' \ \&\&_{\{f \text{ in } L'\}} f$

- L' : the set of conditions that must be true
  - From loops (init, preservation, final)
- Q' : "precondition" modulo invariants…

VCGen

```
-----------------------------------------------------
verify        :: Pred -> Com -> Pred -> Bool
-----------------------------------------------------

-- | The top level verifier, takes:
--   in : pre `p`, command `c` and post `q`
--   out: True iff {p} c {q} is a valid Hoare-Triple

verify              :: Pred -> Com -> Pred -> Bool
verify p c q        = all smtValid queries
  where
    (q', conds) = runState (vcgen q c) []
    queries     = p `implies` q' : conds
```

VCGen

```
vcgen :: Pred -> Com -> VC Pred

vcgen (Skip) q
  = return q

vcgen (Asgn x e) q
  = return $ q `subst` (x, e)

vcgen (If b c1 c2) q
  = do q1     <- vcgen q c1
       q2     <- vcgen q c2
       return $ (b `And` q1) `Or` (Not b `And` q2)

vcgen (While i b c) q
  = do q'     <- vcgen i c
       valid  $ (i `And` Not b) `implies` q'
       valid  $ (i `And` b)     `implies` q
       return $ i
```

ESC/Java

Semi-automated “Deductive Verification”

- You write the invariants
- ESC/Java:
  - VCGen
  - Simplify: SMT used to prove VC
- Explains when implication is invalid: counterexample for your specification