Coinduction demystified

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There will also be *inference rules*.

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I hope I can make it feel less inscrutable!

Let's talk about *induction*

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  $P\ n$ implies $P\ (\mathsf{S}\ n)$

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and learned to prove it by induction:

- Prove P $\mathtt{Nil}$
- Prove that for any head h, and for any tail t,

  $P\ t$ **implies** $P\ (\mathtt{Cons}\ h\ t)$

Why is this sane?

Let's put on our proof theory hats!

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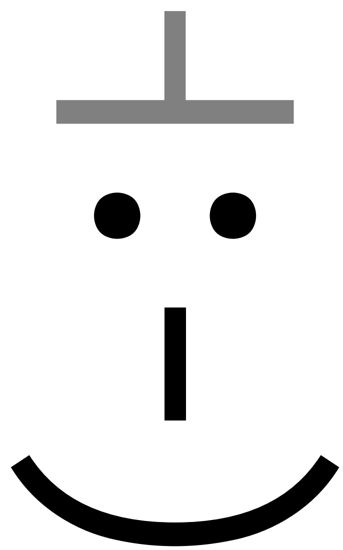

Inductive List T :=

$$\frac{}{\cdot \vdash x \in \texttt{List}\ T}$$

```
Inductive List (T : Type) :=
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Inductive List (T : Type) ≔
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Inductive List (T : Type) :=
| Nil  : List T
| Cons : (h : T) → (t : List T) → List T
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```

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I will give you an intuition for all three definitions now.

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2. the *smallest* set closed under those rules

Let's try to compute `List Nat` starting with `{}`,
assuming `· ⊢ Nat ∈ Type` and `Nat = {0, 1, 2, ...}`

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We will shortly see in what sense it could be larger!

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Set-theoretic argument

Because T is the *smallest* set closed under its rules, $T \subseteq P$ and therefore $T = P$

Let's now consider *coinduction*

"Coinductive types model infinite structures unfolded on demand, like politicians' excuses: for each attack, there is a defence but no likelihood of resolution."

- Conor McBride

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Read backward!?

$$\frac{x \equiv \mathtt{CoZero}}{\cdot \vdash x \in \mathtt{CoNat}}$$

$$\frac{\cdot \vdash n \in \mathtt{CoNat} \qquad x \equiv \mathtt{CoSucc}\ n}{\cdot \vdash x \in \mathtt{CoNat}}$$

Coinductive lists

$$\cfrac{\cdot \vdash T \in \mathtt{Type} \qquad x \equiv \mathtt{CoNil}}{\cdot \vdash x \in \mathtt{CoList}\ T}$$

$$\cfrac{\cdot \vdash T \in \mathtt{Type} \qquad \cdot \vdash h \in T \qquad \cdot \vdash t \in \mathtt{CoList}\ T \qquad x \equiv \mathtt{CoCons}\ h\ t}{\cdot \vdash x \in \mathtt{CoList}\ T}$$

Coinductive streams

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The lack of power of equality

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Huh!?

Let us prove:

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by *induction* on `Nat`

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$$\dfrac{\dfrac{\dfrac{\dfrac{}{1 + n = n + 1 \Rightarrow S\ (S\ n) = S\ (S\ n)}}{1 + n = n + 1 \Rightarrow S\ (S\ n) = S\ (1 + n)}}{1 + n = n + 1 \Rightarrow S\ (S\ n) = S\ (n + 1)}}{1 + n = n + 1 \Rightarrow 1 + S\ n = S\ n + 1}$$

Let us prove:

$$\forall\ (n \in \mathrm{Nat}),\ 1 + n = n + 1$$

by ***induction*** **on** Nat

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$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{1 + n = n + 1 \Rightarrow \mathrm{S}\ (\mathrm{S}\ n) \equiv \mathrm{S}\ (\mathrm{S}\ n)}}{1 + n = n + 1 \Rightarrow \mathrm{S}\ (\mathrm{S}\ n) = \mathrm{S}\ (\mathrm{S}\ n)}}{1 + n = n + 1 \Rightarrow \mathrm{S}\ (\mathrm{S}\ n) = \mathrm{S}\ (1 + n)}}{1 + n = n + 1 \Rightarrow \mathrm{S}\ (\mathrm{S}\ n) = \mathrm{S}\ (n + 1)}}{1 + n = n + 1 \Rightarrow 1 + \mathrm{S}\ n = \mathrm{S}\ n + 1}$$

More values, more problems!

Co-fixpoints are not judgmentally equal either! :-(

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comap (+ 1) zeroes ≢ ones
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Solutions

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- **Build an *inductive* argument for finite observations**

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Solutions

- Build an *inductive* argument for finite observations
- Build a *coinductive* argument for infinite observations

Stream "equality"

$$\frac{h_1 \equiv h_2 \qquad t_1 \approx t_2}{\mathsf{CoCons}\ h_1\ t_1 \approx \mathsf{CoCons}\ h_2\ t_2}$$

In practice, the coinduction principle I presented
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Bisimulation techniques demonstrate how some binary relations are closed under the *destructors* of a type.

There is a dual proof technique, *congruence*,
capturing binary relations closed under the *constructors* of a type.

Finally, because this is a Galois talk:

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There is yet another, categorical, interpretation of

inductive and *coinductive* types.

Inductive types are *initial* algebras:

$$\mathtt{T\ A} \longrightarrow \mathtt{A}$$

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of functors like:

$\mathsf{T}\ \mathsf{X} = 1 + \mathsf{X}$ (natural numbers)

$\mathsf{T}\ \mathsf{X} = 1 + \mathsf{T} \times \mathsf{X}$ (lists of Ts)

Coinductive types are *final co*algebras:

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of functors like:

$T\ X = 1 + X$ (predecessor function)

$T\ X = 1 + T \times X$ (possibly-finite streams of Ts)

Great things about the categorical intuition

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- Gives a great intuition for the constructor / destructor roles
- The duality story is very crisp
- Justifies case analysis by the existence of a homomorphism
- Justifies *induction* and *coinduction* by the uniqueness of said homomorphism

Learn more!

Jacobs, B. and Rutten, J., 1997.
A tutorial on (co) algebras and (co) induction.
Bulletin-European Association for Theoretical Computer Science, 62, pp.222-259.