1 Vectors in $\mathbb{R}^n$

Equality of vectors:

$\vec{u} = \vec{v}$ if $u_i = v_i$ for all $i = 1, \ldots, n$.

Properties of Vector Addition:

1. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ (Symmetry)
2. $\vec{u} + \vec{v} + \vec{w} = \vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$ (Associativity)
3. $\vec{0} + \vec{u} = \vec{v} + \vec{0} = \vec{v}$
4. $\vec{u} - \vec{u} = \vec{u} + (-\vec{u}) = (-\vec{u}) + \vec{u} = \vec{0}$ (Additive Inverse)

Properties of Scalar Multiplication:

1. $(c+d)\vec{v} = c\vec{v} + d\vec{v}$
2. $(cd)\vec{v} = c(d\vec{v})$
3. $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$
4. $0\vec{v} = \vec{0}$
5. If $c\vec{v} = \vec{0}$, then $c = 0$ or $\vec{v} = \vec{0}$ (Cancellation Law).

Dot Product in $\mathbb{R}^n$:

$\vec{u} \cdot \vec{v} = u_1 v_1 + \cdots + u_n v_n$

Properties of the Dot Product:

1. $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$
2. $(\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$
3. $(c\vec{u}) \cdot \vec{v} = c(\vec{u} \cdot \vec{v})$
4. $\vec{v} \cdot \vec{v} \geq 0$, with $\vec{v} \cdot \vec{v} = 0 \iff \vec{v} = \vec{0}$

Standard Inner Product in $\mathbb{F}^n$:

$\langle \vec{u}, \vec{v} \rangle = u_1 \overline{v_1} + \cdots + u_n \overline{v_n}$

Properties of the Standard Inner Product:

1. $\langle \vec{u}, \vec{v} \rangle = \overline{\langle \vec{v}, \vec{u} \rangle}$ (Conjugate Symmetry)
2. $\langle \vec{u} + \vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$ (Linearity in the First Argument)
3. $\langle c\vec{u}, \vec{v} \rangle = c\langle \vec{u}, \vec{v} \rangle$ (Linearity in the First Argument)
4. $\langle \vec{u}, \vec{v} + \vec{w} \rangle = \langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, \vec{w} \rangle$ (Conjugate Linearity in the Second Argument)
5. $\langle \vec{u}, c\vec{v} \rangle = \bar{c}\langle \vec{u}, \vec{v} \rangle$ (Conjugate Linearity in the Second Argument)
6. $\langle \vec{v}, \vec{v} \rangle \geq 0$, with $\langle \vec{v}, \vec{v} \rangle = 0 \iff \vec{v} = \vec{0}$

Length (norm/magnitude):

$\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle} = \sqrt{\vec{v} \cdot \vec{v}}$ (in $\mathbb{R}^n$)

Unit vector:

$\vec{v}$ is a unit vector if $\|\vec{v}\| = 1$

We can produce a unit vector in the direction of $\vec{v}$ (normalization) by taking $\hat{v} = \frac{\vec{v}}{\|\vec{v}\|}$

Properties of the Length:

1. $\|c\vec{v}\| = |c|\|\vec{v}\|$ (absolute value for $c \in \mathbb{R}$, modulus for $c \in \mathbb{C}$)
2. $\|\vec{v}\| \geq 0$, with $\|\vec{v}\| = 0 \iff \vec{v} = \vec{0}$

Angle between vectors:

$\vec{v} \cdot \vec{w} = \|\vec{v}\|\|\vec{w}\| \cos\theta$

Cauchy-Schwartz Inequality:

$|\vec{v} \cdot \vec{w}| \leq \|\vec{v}\|\|\vec{w}\|$ for all $\vec{v}, \vec{w} \in \mathbb{R}^n$

Orthogonal:

$\vec{v}, \vec{w}$ are orthogonal if $\langle \vec{v}, \vec{v} \rangle = \vec{v} \cdot \vec{w}$ (in $\mathbb{R}^n$) $= 0$.

Every vector is orthogonal to $\vec{0}$.

Projection:

Let $\vec{v}, \vec{w} \in \mathbb{F}^n$ with $\vec{w} \neq \vec{0}$. The projection of $\vec{v}$ onto $\vec{w}$ is defined as

$$\text{In } \mathbb{R}^n : \operatorname{proj}_{\vec{w}}(\vec{v}) = \frac{(\vec{v} \cdot \vec{w})}{\vec{w} \cdot \vec{w}}\vec{w} = \frac{(\vec{v} \cdot \vec{w})}{\|\vec{w}\|^2}\vec{w} = \frac{\|\vec{v}\|\|\vec{w}\|\cos\theta}{\|\vec{w}\|^2}\vec{w} = (\|\vec{v}\|\cos\theta)\hat{w}$$

$$\text{In } \mathbb{C}^n : \operatorname{proj}_{\vec{w}}(\vec{v}) = \frac{\langle \vec{v}, \vec{w}\rangle}{\langle \vec{w}, \vec{w}\rangle}\vec{w} = \frac{\langle \vec{v}, \vec{w}\rangle}{\|\vec{w}\|^2}\vec{w} = \langle \vec{v}, \hat{w}\rangle\hat{w}$$

Component:

Let $\vec{v}, \vec{w} \in \mathbb{R}^n$ with $\vec{w} \neq \vec{0}$. Then $\|\vec{v}\|\cos\theta = \vec{v} \cdot \hat{w}$ is the scalar component of $\vec{v}$ along $\vec{w}$.

Perpendicular:

Let $\vec{v}, \vec{w} \in \mathbb{R}^n$ with $\vec{w} \neq \vec{0}$. The quantity

$$\operatorname{perp}_{\vec{w}}(\vec{v}) = \vec{v} - \operatorname{proj}_{\vec{w}}(\vec{v})$$

is the perpendicular of $\vec{v}$ onto $\vec{w}$.

1. $\operatorname{perp}_{\vec{w}}(\vec{v})$ is the height of the right triangle whose hypotenuse is $\vec{v}$ and other leg is $\operatorname{proj}_{\vec{w}}(\vec{v})$.
2. The projection and perpendicular are orthogonal: $\operatorname{perp}_{\vec{w}}(\vec{v}) \cdot \operatorname{proj}_{\vec{w}}(\vec{v}) = 0$

Cross Product in $\mathbb{R}^3$:

$$\vec{u} \times \vec{v} = \begin{bmatrix} u_2v_3 - u_3v_2 \\ -(u_1v_3 - u_3v_1) \\ u_1v_2 - u_2v_1 \end{bmatrix}$$

Properties of the Cross Product:

Let $\vec{z} = \vec{u} \times \vec{v}$

1. $\vec{z} \cdot \vec{u} = \vec{z} \cdot \vec{v} = \vec{0}$ (Cross Product is Orthogonal)
2. $\vec{v} \times \vec{u} = -\vec{u} \times \vec{v} = -\vec{z}$ (Skew-symmetry)
3. $\|\vec{u} \times \vec{v}\| = \|\vec{u}\|\|\vec{v}\|\sin\theta$ (Parallelogram Area)
4. $(\vec{u} + \vec{v}) \times \vec{w} = (\vec{u} \times \vec{w}) + (\vec{v} \times \vec{w})$ (Linearity in First)
5. $(c\vec{u}) \times \vec{v} = c(\vec{u} \times \vec{v})$ (Linearity in First)
6. $\vec{u} \times (\vec{v} + \vec{w}) = (\vec{u} \times \vec{v}) + (\vec{u} \times \vec{w})$ (Linearity in Second)
7. $\vec{u} \times (c\vec{v}) = c(\vec{u} \times \vec{v})$ (Linearity in Second)

2 Span, Lines, and Planes

Linear combination:

Let $c_1, \dots, c_k \in \mathbb{F}$ and $\vec{v_1}, \dots, \vec{v_k} \in \mathbb{F}^n$. We refer to any vector of the form $c_1\vec{v_1} + \cdots + c_k\vec{v_k}$ as a linear combination of $\vec{v_1}, \dots, \vec{v_k}$.

Span:

Span $\{\{\}\, \vec{v_1}, \dots, \vec{v_k}\} = \{c_1\vec{v_1} + \cdots + c_k\vec{v_k} : c_1, \dots, c_k \in \mathbb{F}\}$ (i.e. the set of all linear combinations of $\vec{v_1}, \dots, \vec{v_k}$). We call $\{\vec{v_1}, \dots, \vec{v_k}\}$ a **spanning set** of its span. We also say Span $\{\{\}\, \vec{v_1}, \dots, \vec{v_k}\}$ is **spanned by** $\{\vec{v_1}, \dots, \vec{v_k}\}$.

Vector Equations in $\mathbb{R}^n$:

Let $\vec{u}, \vec{v} \in \mathbb{R}^n$ with $\vec{v} \neq \vec{0}$. The **vector equation of a line through $\vec{u}$ with direction $\vec{v}$** is given by

$$\vec{\ell} = \vec{u} + t\vec{v}, \ \ t \in \mathbb{R}$$

If $\vec{\ell_1}$ and $\vec{\ell_2}$ are two lines with direction vectors such that $\vec{v_1} = c\vec{v_2}$ for some $c \neq 0 \in \mathbb{R}$, then they have the **same direction**.

Parametric Equations in $\mathbb{R}^n$:

The parametric equations of the line $\vec{\ell} = \vec{u} + t\vec{v}$ are

$$\begin{aligned} \vec{\ell_1} &= u_1 + tv_1 \\ &\vdots \\ \vec{\ell_n} &= u_n + tv_n, \ \ t \in \mathbb{R} \end{aligned}$$

Line in $\mathbb{R}^n$:

Let $\vec{u}, \vec{v} \in \mathbb{R}^n$ with $\vec{v} \neq \vec{0}$. The set of vectors

$$\mathcal{L} = \{\vec{u} + t\vec{v}, \ \ t \in \mathbb{R}\}$$

is the line $\mathcal{L}$ through $\vec{u}$ with direction $\vec{v}$.

1. Letting $t = 0$ gives us that the vector $\vec{u}$ is on the line.
2. The line passes through the terminal point U associated with $\vec{u}$. The other points on the line move from U in the $\vec{v}$ direction by scalar multiples of $\vec{v}$
3. We say that $\vec{v}$ is **parallel** to the line and that $\vec{v}$ is a **direction vector** to the line.
4. The vector $\vec{v}$ is parallel to the line. However, the terminal point V associated with $\vec{v}$ is not usually a point on the line; in fact, V is a point on the line if and only if the vector $\vec{v}$ is a scalar multiple of the vector $\vec{u}$

Plane in $\mathbb{R}^n$ Through the Origin:

Let $\vec{u}, \vec{v} \in \mathbb{R}^n$ with $\vec{u}, \vec{v} \neq \vec{0}$ and $\vec{u} \neq c\vec{v}$ for all $c \in \mathbb{R}$. Then

$$\mathcal{P} = \text{Span}\,\{\{\}\,\vec{u}, \vec{v}\} = \{s\vec{u} + t\vec{v} : s, t \in \mathbb{R}\}$$

is the plane $\mathcal{P}$ through the origin with direction vectors $\vec{u}$ and $\vec{v}$.

1. $\mathcal{P}$ contains U and V, which are the terminal points of the vectors $\vec{u}$ and $\vec{v}$, respectively.
2. If a point P with associated vector $\vec{p}$ lies on the plane, then $\vec{p} \in \mathcal{P}$.
3. Any plane defined by the span of two vectors passes through the origin.
4. If two vectors are parallel or one is zero, their span is not a plane but a line. If both are zero, their span is simply $\{\vec{0}\}$, the origin.

Vector Equation of a Plane in $\mathbb{R}^n$ Through the Origin:

Let $\vec{u}, \vec{v} \in \mathbb{R}^n$ with $\vec{u}, \vec{v} \neq \vec{0}$ and $\vec{u} \neq c\vec{v}$ for all $c \in \mathbb{R}$. Then

$$\vec{p} = s\vec{u} + t\vec{v}$$

is a **vector equation** of the plane $\mathcal{P}$ through the origin with direction vectors $\vec{u}$ and $\vec{v}$.

Plane in $\mathbb{R}^n$:

Let $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$ with $\vec{v}, \vec{w} \neq \vec{0}$ and $\vec{v} \neq c\vec{w}$ for all $c \in \mathbb{R}$. Then

$$\mathcal{P} = \{\vec{u} + s\vec{v} + t\vec{w} : s, t \in \mathbb{R}\}$$

is the plane $\mathcal{P}$ through $\vec{u}$ with direction vectors $\vec{v}$ and $\vec{w}$. We say that $\vec{v}$ and $\vec{w}$ are **parallel** to $\mathcal{P}$

1. $s = t = 0$ gives us that $\vec{u}$ lies on the plane.
2. The plane passes through U. The other points on the plane move from U as linear combinations of $\vec{v}$ and $\vec{w}$.
3. V and W are usually not on the plane. V is on the plane iff $\vec{u}$ is parallel to $\vec{v}$ and likewise for W.

Vector Equation of a Plane in $\mathbb{R}^n$:

Let $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$ with $\vec{v}, \vec{w} \neq \vec{0}$ and $\vec{u} \neq c\vec{v}$ for all $c \in \mathbb{R}$. Then

$$\vec{p} = \vec{u} + s\vec{v} + t\vec{w}$$

is a **vector equation** of the plane $\mathcal{P}$ through $\vec{u}$ with direction vectors $\vec{v}$ and $\vec{w}$.

Normal Form of a Plane:

Let $\mathcal{P}$ be a plane in $\mathbb{R}^3$ with direction vectors $\vec{v}$ and $\vec{w}$ and a normal vector $\vec{n} \neq \vec{0}$. Let $\vec{u}, \vec{p} \in \mathcal{P}$ with $\vec{u} \neq \vec{p}$. A normal form of $\mathcal{P}$ is given by

$$\vec{n} \cdot (\vec{p} - \vec{u}) = 0$$

Scalar Equation of a Plane:

Expanding above, we get

$$ax + by + cz = d$$

where $d = \vec{n} \cdot \vec{u}$.

$\mathcal{P}$ goes through the origin if and only if

1. $\vec{0}$ satisfies the scalar equation
2. $(\vec{v} \times \vec{w}) \cdot (\vec{u}) = 0$
3. $\vec{u} = a\vec{v} + b\vec{v}$ for some $a, b \in \mathbb{R}$
4. Both V and W lie on the plane

3 Systems of Linear Equations

Solve, Solution:

The scalars $y_1, \ldots, y_n \in \mathbb{F}$ **solve** the system if when we set $x_i = y_i$ for all $i = 1, \ldots, n$ each equation is satisfied. We also say that the vector $\vec{y} = \begin{bmatrix} y_1 & \ldots & y_n \end{bmatrix}^T$ is a **solution** to the system. The **solution set** is all solutions to a system.

A system either has no solutions, a unique solution, or infinite solutions.

Inconsistent, Consistent:

A system is inconsistent if its solution set is empty, and consistent otherwise.

Equivalent systems:

Two linear systems are equivalent if they have the same solution set.

Elementary Operations:

1. Swap: interchange two equations
2. Scale: multiply one equation by a non-zero scalar
3. Add: add a multiple of one equation to another

Performing a finite number of elementary operations on a system yields an equivalent system.

Coefficient, augmented matrix:

Let a system be

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\ &\vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= b_m. \end{aligned}$$

The **coefficient matrix,** A, of the system is given by

$$A = \begin{bmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \ldots & a_{mn} \end{bmatrix}$$

The **augmented matrix,** $[A|\vec{b}\,]$, of the system is given by

$$\left[\begin{array}{ccc|c} a_{11} & \ldots & a_{1n} & b_1 \\ \vdots & \ddots & \vdots & \vdots \\ a_{m1} & \ldots & a_{mn} & b_m \end{array}\right]$$

The entry in the i^{th} row and j^{th} column of a matrix is the $(i,j)^{th}$ entry, denoted a_{ij} or $(A)_{ij}$

Trivial equation:

$0 = 0$. Any other equation is **non-trivial**.

Elementary Row Operations:

These are analogous to the Elementary Operations, and a matrix B obtained from A by a finite number of EROs is **row equivalent** to A.

Zero Row:

In a matrix, a row whose entries are all zero.

Row Echelon Form:

1. All zero rows occur as the final rows in the matrix
2. The leading entry in a non-zero row appears in a column to the right of any leading entries of a row above it

Pivots:

The leading entries of a matrix in REF are called **pivots** and their positions are called **pivot positions**. Any row/column containing a pivot position is called a **pivot row/column**.

Reduced Row Echelon Form:

1. The matrix is in REF
2. All pivots are 1
3. The only entry in a pivot column is the pivot itself

The RREF of a matrix is unique.

Basic, free variables:

If the i^{th} column is a pivot column, x_i is a basic variable. Otherwise, it is a free variable.

Rank:

If $\text{RREF}(A)$ has r pivots, then $\text{rank}(A) = r$.

Rank bounds:

Let $A \in M_{m\times n}(\mathbb{F})$. $\text{rank}(A) \leq \min\{m, n\}$.

Consistent System Test:

A system with augmented matrix $[A|\vec{b}\,]$ is consistent if and only if $\text{rank}(A) = \text{rank}([A|\vec{b}\,])$.

System Rank Theorem:

Let $A \in M_{m\times n}(\mathbb{F})$ with $\operatorname{rank}(A) = r$.

1. Let $\vec{b} \in \mathbb{F}^m$. If the system with augmented matrix $[A|\vec{b}]$ is consistent, then its solution set contains $n - r$ parameters.
2. The system with augmented matrix $[A|\vec{b}]$ is consistent for all $\vec{b} \in \mathbb{F}^m$ if and only if $r = m$.

Nullity:

The nullity of an $m \times n$ matrix A, denoted $\operatorname{nullity}(A)$, is $n - \operatorname{rank}(A)$.

Homogeneous system:

A system is **homogeneous** if all the constant terms on the right-hand side are zero (i.e. a system of the form $A\vec{x} = \vec{0}$), and **non-homogeneous** otherwise.

A homogeneous system always has the **trivial solution** $\vec{x} = \vec{0}$.

Nullspace:

The solution set of a *homogeneous* system with coefficient matrix A is the **nullspace** of A, denoted $\operatorname{Null}(A)$.

Matrix-Vector Multiplication:

Let $A \in M_{m\times n}(\mathbb{F})$. M-V M is only defined for $\vec{x} \in \mathbb{F}^n$ as

$$A\vec{x} = x_1\vec{a_1} + \cdots + x_n\vec{a_n}.$$

Linearity of Matrix-Vector Multiplication:

Let $A \in M_{m\times n}(\mathbb{F})$, let $\vec{x}, \vec{y} \in \mathbb{F}^n$, and let $c \in \mathbb{F}$.

1. $A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y}$
2. $A(c\vec{x}) = cA\vec{x}$

Matrix-Vector Representation of a System:

We can represent the system with augmented matrix $[A|\vec{b}]$ as $A\vec{x} = \vec{b}$.

If $A\vec{x} = \vec{e_i}$ is consistent for all $i = 1, \ldots, m$, then $\operatorname{rank}(A) = m$.

Let $A\vec{x} = \vec{0}$ be a homogeneous system with solution set S. If $\vec{v}, \vec{w} \in S$, then $\vec{v} + \vec{w}, c\vec{v} \in S$ for all $c \in \mathbb{F}$. We combine these results to state that $a\vec{v} + b\vec{w} \in S$ for all $a, b \in \mathbb{F}$.

Associated Homogeneous System:

Let $A\vec{x} = \vec{b}$, $\vec{b} \neq \vec{0}$ be a non-homogeneous system. The **associated homogeneous system** is the system $A\vec{x} = \vec{0}$.

Particular Solution:

Let $A\vec{x} = \vec{b}$ be a consistent system. We refer to any vector $\vec{x_p}$ that satisfies this system as a particular solution to the system.

Solutions to $A\vec{x} = \vec{0}$ and $A\vec{x} = \vec{b}$:

Let S_b be the solution set to $A\vec{x} = \vec{b}$. If S is the solution set to the associated homogeneous system $A\vec{x} = \vec{0}$ and $\vec{x_p}$ is any element of S_b, then

$$S_b = \{\vec{x_p} + \vec{x} : \vec{x} \in S\}$$

Solutions to $A\vec{x} = \vec{b}$ and $A\vec{x} = \vec{c}$:

Let S_b, S_c be the solution sets to $A\vec{x} = \vec{b}$ and $A\vec{x} = \vec{c}$ with particular solutions $\vec{x_b}$ and $\vec{x_c}$, respectively. Then,

$$S_c = \{(\vec{x_c} - \vec{x_b}) + \vec{z} : \vec{z} \in S_b\}$$

4 Matrices

Column Space:

We define the column space of A, denoted $\text{Col}(A)$ to be the span of the columns of A.

$$\text{Col}(A) = \text{Span}\{\vec{a_1}, \dots, \vec{a_n}\}$$

Consistent System and Column Space:

The system $A\vec{x} = \vec{b}$ is consistent if and only if $b \in \text{Col}(A)$.

Transpose:

The transpose of A, denoted A^T, is defined as $(A^T)_{ij} = (A)_{ji}$.

Row Space:

The span of the transposed rows of A, i.e. $\text{Row}(A) = \text{Col}\left(A^T\right)$.

Row Spaces of Row Equivalent Matrices:

If A and B are row equivalent, then $\text{Row}(A) = \text{Row}(B)$.

This property is not true for the column space.

Matrix Equality:

$A = B$ if they are the same size and each entry is equal.

Column Extraction:

$A\vec{e_i} = \vec{a_i}$ for all $i = 1, \dots, n$. That is to say, the matrix-vector product of A with the i^{th} standard basis vector yields the i^{th} column of A.

Equality of Matrices:

$A = B \iff A\vec{x} = B\vec{x}$ for all $\vec{x} \in \mathbb{F}^n$.

Matrix Addition:

Let $A, B \in M_{m\times n}(\mathbb{F})$. We define the matrix sum $A + B = C$ to be the matrix whose entries are $c_{ij} = a_{ij} + b_{ij}$ for all $i = 1, \dots, m$ and $j = 1, \dots n$.

Additive Inverse:

The additive inverse of A, denoted $-A$, is the matrix whose entries are all $-a_{ij}$.

Zero Matrix:

The zero matrix $\mathcal{O}$ is the matrix whose entries are all 0.

Properties of Matrix Addition:

1. $A + B = B + A$
2. $A + B + C = (A + B) + C = A + (B + C)$
3. $A + (-A) = (-A) + A = \mathcal{O}$
4. $A + \mathcal{O} = \mathcal{O} + A = A$

Matrix Multiplication:

The matrix product of $A \in M_{m \times n}(\mathbb{F})$ and $B \in M_{n \times p}(\mathbb{F})$, is defined as the matrix $AB = C, C \in M_{m \times p}(\mathbb{F})$

$$C = AB = A\left[\vec{b_1}\ \vec{b_2}\ \dots\ \vec{b_n}\right] = \left[A\vec{b_1}\ A\vec{b_2}\ \dots\ A\vec{b_n}\right].$$

That is, $\vec{c_j} = A\vec{b_j}$ for all $j = 1, \dots, p$.

1. The number of columns of A must equal the number of rows of B for the product AB to be defined.
2. Matrix multiplication is non-commutative: $AB = BA$ does not hold for all A, B.
3. The j^{th} column of C is by definition in $\mathrm{Col}(A)$.

We may also construct the product entry-by-entry:

$$(C)_{ij} = (\vec{c_j})_i = (A\vec{b_j})_i = \sum_{k=1}^{n} a_{ik} b_{kj}$$

Note that the sum is the dot product between the (transposed) i^{th} row of A and the j^{th} column of B.

Properties of Matrix Multiplication:

1. $(A + B)C = AC + BC$
2. $A(C + D) = AC + AD$
3. $ACE = A(CE) = (AC)E$

Matrix-Scalar Multiplication:

The product of $c \in \mathbb{F}$ and A is the matrix whose entries are all ca_{ij}.

Properties of Matrix-Scalar Multiplication:

1. $s(A+B) = sA + sB$
2. $(r+s)A = rA + sA$
3. $r(sA) = (rs)A$
4. $s(AC) = (sA)C = A(sC)$

Properties of Transpose:

1. $(A+B)^T = A^T + B^T$
2. $(sA)^T = sA^T$
3. $(AC)^T = C^T A^T$
4. $(A^T)^T = A$

Square Matrix:

A matrix is square if it is $n \times n$.

Upper Triangular:

A square matrix is upper triangular if $a_{ij} = 0$ for all $i > j$ with $i = 1, \ldots m, j = 1, \ldots, n$.

Lower Triangular:

A square matrix is lower triangular if $a_{ij} = 0$ for all $i < j$ with $i = 1, \ldots m, j = 1, \ldots, n$.

1. The transpose of an upper (lower) triangular matrix is a lower (upper) triangular matrix, respectively.
2. The product of upper (lower) triangular matrices is upper (lower) triangular, respectively.

Diagonal:

A matrix is diagonal if it is both upper and lower triangular. That is, $a_{ij} = 0$ for all $i \neq j$ with $i = 1, \ldots, m, j = 1, \ldots, n$. We denote A in shorthand by $A = \text{diag}(a_{11}, \ldots, a_{nn})$.

Identity Matrix:

$I = \text{diag}(1, \ldots, 1)$. We specify size by I_n.

I behaves as a multiplicative identity: $I_m A = A I_n = A$, and $I_n \vec{x} = \vec{x}$ for all $\vec{x} \in \mathbb{F}^n$.

Elementary Matrix:

A matrix that can be obtained by performing a single ERO on the identity matrix is called an elementary matrix.

EROs by Elementary Matrices:

Suppose a single ERO is performed on A to yield B. Suppose we perform the same ERO on I_m to produce the elementary matrix E. Then,

$$EA = B.$$

We can also carry out a sequence of k EROs on A, which gives $C = E_k \dots E_2 E_1 A$.

Invertible:

An $n \times n$ matrix is **invertible** if there exist $n \times n$ matrices B and C such that $AB = CA = I_n$. We denote the unique inverse of A by A^{-1} such that $AA^{-1} = A^{-1}A = I_n$

The inverse of an invertible matrix is unique.

Invertibility Criteria:

For a square matrix A,

$$A \text{ is invertible} \iff \operatorname{rank}(A) = n \iff \operatorname{RREF}(A) = I_n$$

Algorithm for Checking Invertibility and Finding the Inverse:

1. Construct the super-augmented matrix $[A \mid I_n]$.
2. Find the RREF, $[R \mid B]$ of $[A \mid I_n]$.
3. If $R = I_n$, A is invertible and $A^{-1} = B$. Otherwise, A is not invertible.

Inverse of a 2×2 Matrix:

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. A is invertible iff $ad - bc \neq 0$, and its inverse is given by

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

5 Linear Transformations

Function Determined by a Matrix:

Let $A \in M_{m\times n}(\mathbb{F})$. The function determined by the matrix A is the function

$$T_A : \mathbb{F}^n \to \mathbb{F}^m$$

defined by

$$T_A(\vec{x}) = A\vec{x}$$

Linear Transformation:

We say that the function $T : \mathbb{F}^n \to \mathbb{F}^m$ is a linear transformation (or linear mapping) if T satisfies the following for all $\vec{x}, \vec{y} \in \mathbb{F}^n$ and $c \in \mathbb{F}$:

1. $T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y})$ (linearity over addition)
2. $T(c\vec{x}) = cT(\vec{x})$ (linearity over scalar multiplication)

The function determined by any matrix is linear.

More succinctly, T is a linear transformation if and only if for all $\vec{x}, \vec{y} \in \mathbb{F}^n$ and $c \in \mathbb{F}$,

$$T(c\vec{x} + \vec{y}) = cT(\vec{x}) + T(\vec{y})$$

If T is linear, $T(\vec{0}_{\mathbb{F}^n}) = \vec{0}_{\mathbb{F}^m}$ (Zero Maps to Zero).

Range:

Let $T : \mathbb{F}^n \to \mathbb{F}^m$ be a transformation. The range of T, denoted $\operatorname{Range}(T)$ is defined as

$$\operatorname{Range}(T) = \{T(\vec{x}) : \vec{x} \in \mathbb{F}^n\}$$

That is, the set of all outputs of T. Note that $\operatorname{Range}(T) \subseteq \mathbb{F}^m$.

The range of a linear transformation, T_A, determined by a matrix A is given by

$$\operatorname{Range}(T_A) = \operatorname{Col}(A)$$

The system $A\vec{x} = \vec{b}$ is consistent if and only if $\vec{b} \in \operatorname{Range}(T_A)$.

Onto/Surjective:

We say the transformation $T : \mathbb{F}^n \to \mathbb{F}^m$ is surjective if $\operatorname{Range}(T) = \mathbb{F}^m$.

Onto Criteria:

Let T_A be the linear transformation determined by $A \in M_{m\times n}(\mathbb{F})$. The following are equivalent:

1. T_A is surjective.
2. $\text{Col}\,(A) = \mathbb{F}^m$.
3. $\text{rank}(A) = m$.

Kernel:

Let $T : \mathbb{F}^n \to \mathbb{F}^m$ be a transformation. The kernel of T, denoted $\text{Ker}\,(T)$ is defined as

$$\text{Ker}\,(T) = \{\vec{x} \in \mathbb{F}^n : T(\vec{x}) = \vec{0}_{\mathbb{F}^m}\}$$

That is, the set of all inputs that get mapped to zero. Note $\text{Ker}\,(T) \subseteq \mathbb{F}^n$.

The kernel of a linear transformation, T_A, determined by a matrix A is given by

$$\text{Ker}\,(T_A) = \text{Null}\,(A)$$

The kernel of T_A is equal to the solution set of the homogeneous system $A\vec{x} = \vec{0}$.

One-to-One/Injective:

We say the transformation $T : \mathbb{F}^n \to \mathbb{F}^m$ is injective if whenever $T(\vec{x}) = T(\vec{y})$ then $\vec{x} = \vec{y}$.

Taking the contrapositive, we have that

$$\forall \vec{x}, \vec{y} \in \mathbb{F}^n, \vec{x} \neq \vec{y} \implies T(\vec{x}) \neq T(\vec{y})$$

That is, T maps distinct elements from the domain to distinct elements in the codomain.

One-to-One Test:

Let $T : \mathbb{F}^n \to \mathbb{F}^m$ be a linear transformation. Then T is injective if and only if $\text{Ker}\,(T) = \{\vec{0}_{\mathbb{F}^n}\}$.

One-to-One Criteria:

1. T_A is one-to-one.
2. $\text{Null}\,(A) = \{\vec{0}_{\mathbb{F}^m}\}$.
3. $\text{nullity}\,(A) = 0$.
4. $\text{rank}(A) = n$.

Invertibility Criteria:

Let T_A be the linear transformation determined by $A \in M_{n\times n}(\mathbb{F})$. The following are equivalent:

1. A is invertible.
2. T_A is injective.
3. T_A is surjective.
4. $\operatorname{Null}(A) = \{\vec{0}\}$.
5. $\operatorname{Col}(A) = \mathbb{F}^n$.
6. $\operatorname{nullity}(A) = 0$.
7. $\operatorname{rank}(A) = n$.
8. $\operatorname{RREF}(A) = I_n$.

Standard Matrix:

Let $T : \mathbb{F}^n \to \mathbb{F}^m$ be a linear transformation. We define the standard matrix of T, denoted $[T]_{\mathcal{E}}$, as

$$[T]_{\mathcal{E}} = \begin{bmatrix} T(\vec{e_1}) & T(\vec{e_2}) & \cdots & T(\vec{e_n}) \end{bmatrix}$$

Every Linear Transformation is Determined by a Matrix:

Let $T : \mathbb{F}^n \to \mathbb{F}^m$ be a linear transform with standard matrix $[T]_{\mathcal{E}}$. Then for all $\vec{x} \in \mathbb{F}^n$,

$$T(\vec{x}) = [T]_{\mathcal{E}}\,\vec{x}$$

That is, $T = T_{[T]_{\mathcal{E}}}$ is the linear transformation determined by $[T]_{\mathcal{E}}$.

If $T : \mathbb{R} \to \mathbb{R}$ is a linear transformation, then there exists an $m \in \mathbb{R}$ such that $T(x) = mx$ for all $x \in \mathbb{R}$.

Properties of a Standard Matrix:

Let $A \in M_{m\times n}(\mathbb{F})$, let T_A be the linear transformation determined by A, and let $T : \mathbb{F}^n \to \mathbb{F}^m$ be a linear transformation. Then

1. $T_{[T]_{\mathcal{E}}} = T$.
2. $[T_A]_{\mathcal{E}} = A$.
3. T is onto if and only if $\operatorname{rank}([T]_{\mathcal{E}}) = m$.
4. T is one-to-one if and only if $\operatorname{rank}([T]_{\mathcal{E}}) = n$.

Identity Transformation:

The linear transformation $\operatorname{id}_n : \mathbb{F}^n \to \mathbb{F}^n$ such that $\operatorname{id}_n(\vec{x}) = \vec{x}$ for all $\vec{x} \in \mathbb{F}^n$. Note that $[\operatorname{id}_n]_{\mathcal{E}} = I_n$.

Special Linear Transformations:

The standard matrix for a counter-clockwise rotation by θ in $\mathbb{R}^2$ is given by

$$[R_\theta]_\mathcal{E} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Let $\vec{w} = \begin{bmatrix} w_1 & w_2 \end{bmatrix}^T$ be non-zero in $\mathbb{R}^2$. The standard matrix for projection onto $\vec{w}$ is given by

$$[\text{proj}_{\vec{w}}]_\mathcal{E} = \frac{1}{w_1^2 + w_2^2} \begin{bmatrix} w_1^2 & w_1 w_2 \\ w_1 w_2 & w_2^2 \end{bmatrix}$$

The standard matrix for reflection along the line Span $\{()\ \vec{w})$ is given by

$$[\text{refl}_{\vec{w}}]_\mathcal{E} = \frac{1}{w_1^2 + w_2^2} \begin{bmatrix} w_1^2 - w_2^2 & 2w_1 w_2 \\ 2w_1 w_2 & w_2^2 - w_1^2 \end{bmatrix}$$

The projection of $\vec{v}$ onto a plane through the origin, $\mathcal{P}$, can be defined using the normal vector, $\vec{n}$, and some nice geometric intuition as

$$\text{proj}_\mathcal{P}(\vec{v}) = \text{perp}_{\vec{n}}(\vec{v})$$

Composition of Linear Transformations:

Let $T_1 : \mathbb{F}^n \to \mathbb{F}^m$ and $T_2 : \mathbb{F}^m \to \mathbb{F}^p$ be linear transformations. We define the function $T_2 \circ T_1 : \mathbb{F}^n \to \mathbb{F}^p$ for all $\vec{x} \in \mathbb{F}^n$ by

$$(T_2 \circ T_1)(\vec{x}) = T_2(T_1(\vec{x}))$$

The function $T_2 \circ T_1$ is the **composite function** of T_2 and T_1.

The composite function of two linear transformations is a linear transformation.

The standard matrix of a composite function is given by

$$[T_2 \circ T_1]_\mathcal{E} = [T_2]_\mathcal{E} [T_1]_\mathcal{E}$$

T^p:

Let $T : \mathbb{F}^n \to \mathbb{F}^n$ and let $p > 1$ be an integer. We define the p^{th} power of T inductively by

$$T^p = T \circ T^{p-1}.$$

We also define $T^0 = \text{id}_n$.

If T is linear, the standard matrix of T^p is given by

$$[T^p]_\mathcal{E} = ([T]_\mathcal{E})^p.$$

6 The Determinant

Determinant of 1×1 and 2×2 Matrices:

The determinant of a 1×1 matrix is

$$\det(A) = a_{11}.$$

The determinant of a 2×2 matrix is

$$\det(A) = a_{11}a_{22} - a_{12}a_{21}.$$

Submatrix, Minor:

Let $A \in M_{n\times n}(\mathbb{F})$. The $(i,j)^{\text{th}}$ **submatrix** of A, denoted $M_{ij}(A)$, is obtained by removing the i^{th} row and j^{th} column of A:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{12} & a_{33} \end{bmatrix} \implies M_{21}(A) = \begin{bmatrix} a_{12} & a_{13} \\ a_{12} & a_{33} \end{bmatrix}$$

The determinant of $M_{ij}(A)$ is called the $(i,j)^{\text{th}}$ **minor** of A.

Determinant Function:

Let $A \in M_{n\times n}(\mathbb{F})$ for $n \geq 2$. We define $\det : M_{n\times n}\mathbb{F} \to \mathbb{F}$ as

$$\det(A) = \sum_{j=1}^{n} a_{1j}(-1)^{1+j}\det(M_{1j}(A)).$$

We can also expand along any row. Let $i \in \{1,\dots,n\}$. Then

$$\det(A) = \sum_{j=1}^{n} a_{ij}(-1)^{i+j}\det(M_{ij}(A)).$$

Furthermore, we can expand along any column. Let $j \in \{1,\dots,n\}$. Then

$$\det(A) = \sum_{i=1}^{n} a_{ij}(-1)^{i+j}\det(M_{ij}(A)).$$

Easy Determinants:

Let $A \in M_{n\times n}(\mathbb{F})$. Then

1. If A has a zero row, then $\det(A) = 0$.
2. If A has a zero column, then $\det(A) = 0$.
3. If A is upper or lower triangular, then $\det(A) = a_{11}\dots a_{nn}$.

$$\det(I_n) = 1$$

Properties of the Determinant:

Let $A, B \in M_{n\times n}(\mathbb{F})$. Then

1. $\det(A) = \det\left(A^T\right)$.
2. $\det(AB) = \det(A)\det(B) = \det(BA)$.
3. If A is invertible, $\det\left(A^{-1}\right) = \frac{1}{\det(A)}$.

It is not true in general that $\det(A+B) = \det(A) + \det(B)$.

Effect of EROs on the Determinant:

Let $A \in M_{n\times n}(\mathbb{F})$.

1. If B is obtained from A by row swap, then $\det(B) = -\det(A)$.
2. If B is obtained from A by multiplying a row by $m \neq 0$, then $\det(B) = m\det(A)$.
3. If B is obtained from A by a non-zero row addition, then $\det(B) = \det(A)$.

Since $\det(A) = \det\left(A^T\right)$, the above holds for column operations as well.

If A has two rows or columns that are scalar multiples of each other, then $\det(A) = 0$.

If E is an elementary matrix, then

1. (Row swap) $\det(E) = -1$.
2. (Row scale) $\det(E) = m$.
3. (Row addition) $\det(E) = 1$.

Suppose we perform k EROs on A, each with elementary matrix E_i such that $B = E_k \dots E_1 A$. Then

$$\det(B) = \det(E_k \dots E_1 A) = \det(E_k)\dots\det(E_1)\det(A),$$

Invertible Iff Non-Zero Determinant:

Let $A \in M_{n\times n}(\mathbb{F})$. Then A is invertible if and only if $\det(A) \neq 0$.

Cofactor:

Let $A \in M_{n\times n}(\mathbb{F})$. The $(i,j)^{\text{th}}$ **cofactor** of A, denoted $C_{ij}(A)$, is defined as

$$C_{ij}(A) = (-1)^{i+j}\det(M_{ij}(A)).$$

Adjugate:

Let $A \in M_{n\times n}(\mathbb{F})$. The **adjugate** of A, denoted $\operatorname{adj}(A)$, is the $n \times n$ matrix whose $(i,j)^{\text{th}}$ entry is

$$(\operatorname{adj}(A))_{ij} = C_{ji}(A).$$

That is, the adjugate of A is the *transpose* of the matrix of cofactors of A.

$$A \operatorname{adj}(A) = \operatorname{adj}(A)\, A = \det(A) I_n$$

Inverse by Adjugate:

Let $A \in M_{n\times n}(\mathbb{F})$. If $\det(A) \neq 0$, then

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A).$$

Cramer's Rule:

Let $A \in M_{n\times n}(\mathbb{F})$ and suppose the system $A\vec{x} = \vec{b}$ has a unique solution $\vec{x}$. If we construct B_j by replacing the j^{th} column of A with $\vec{b}$, then for all $j = 1, \dots, n$:

$$x_j = \frac{\det(B_j)}{\det(A)}.$$

Area of Parallelogram:

Let $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ be vectors in $\mathbb{R}^2$. The area of the parallelogram with sides $\vec{v}$ and $\vec{w}$ is

$$\left| \det\left(\begin{bmatrix} v_1 & v_2 \\ w_1 & w_2 \end{bmatrix} \right) \right|$$

7 Eigenvalues and Diagonalization:

Eigenvector, Eigenvalue, Eigenpair:

Let $A \in M_{n\times n}(\mathbb{F})$. A *non-zero* vector $\vec{x}$ is called an **eigenvector** of A over $\mathbb{F}$ if there exists some $\lambda \in \mathbb{F}$ such that

$$A\vec{x} = \lambda\vec{x}.$$

The scalar λ is called an **eigenvalue** of A over $\mathbb{F}$, and $(\lambda, \vec{x})$ is called an **eigenpair** of A over $\mathbb{F}$.

Characteristic Polynomial:

Let $A \in M_{n\times n}(\mathbb{F})$ and $\lambda \in \mathbb{F}$. The **characteristic polynomial** of A is

$$C_A(\lambda) = \det(A - \lambda I_n).$$

The **characteristic equation** of A is

$$C_A(\lambda) = 0.$$

Features of the Characteristic Polynomial:

Let $A \in M_{n\times n}(\mathbb{F})$ have characteristic polynomial $C_A(\lambda) = \det(A - \lambda I_n)$. Then $C_A(\lambda)$ is an n degree polynomial of the form

$$C_A(\lambda) = c_n\lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_0$$

where:

1. $c_n = (-1)^n$.
2. $c_{n-1} = (-1)^{n-1}\operatorname{tr}(A)$.
3. $c_0 = \det(A)$.

A is invertible if and only if $\lambda = 0$ is not an eigenvalue of A.

Let A have n (possibly repeated) eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ over $\mathbb{C}$. Then

1. $c_{n-1} = (-1)^{n-1}\sum_{i=1}^{n}\lambda_i = (-1)^{n-1}\operatorname{tr}(A)$, and
2. $c_0 = \prod_{i=1}^{n}\lambda_i = \det(A)$

Eigenspace:

Let $A \in M_{n\times n}(\mathbb{F})$ and let $\lambda \in \mathbb{F}$. The eigenspace of A associated with λ, denoted $E_\lambda(A)$, is the solution set to the system

$$E_\lambda(A) = \mathrm{Null}\,(A - \lambda I)\,.$$

Note that $\vec{0}$ is always in E_λ. However, $\vec{0}$ is not an eigenvector. Thus E_λ consists of all the eigenvectors that have eigenvalue λ *together with the zero vector.*

λ is an eigenvalue for A if and only if $E_\lambda \neq \{\vec{0}\}$.

$E_0(A) = \mathrm{Null}\,(A - 0I) = \mathrm{Null}\,(A)$.

Similar Matrices:

Let $A, B \in M_{n\times n}(\mathbb{F})$. We say that A is similar to B over $\mathbb{F}$ if there exists an invertible matrix $P \in M_{n\times n}(\mathbb{F})$ such that $A = P^{-1}BP$.

If A and B are similar over $\mathbb{F}$, then they have the same characteristic polynomial and the same eigenvalues over $\mathbb{F}$. Furthermore,

1. $\det(A) = \det(B)$.
2. $\mathrm{tr}(A) = \mathrm{tr}(B)$.

Diagonalizable:

Let $A \in M_{n\times n}(\mathbb{F})$. We say that A is **diagonalizable** over $\mathbb{F}$ if it is similar over $\mathbb{F}$ to a diagonal matrix $D \in M_{n\times n}(\mathbb{F})$; that is, is there exists an invertible matrix $P \in M_{n\times n}(\mathbb{F})$ such that $P^{-1}AP = D$. We say that P diagonalizes A.

It is possible for a real matrix to be diagonalizable over $\mathbb{C}$ and not over $\mathbb{R}$ (it will have at least one complex eigenvalue).

Diagonalizable $\implies$ n Eigenvalues:

Let $A \in M_{n\times n}(\mathbb{F})$. If A is diagonalizable over $\mathbb{F}$, then $C_A(\lambda)$ has n roots (possibly repeated) over $\mathbb{F}$.

Moreover, if P diagonalizes A, then the entries of $D = P^{-1}AP$ are the eigenvalues of A.

n Distinct Eigenvalues $\implies$ Diagonalizable:

Let $A \in M_{n\times n}(\mathbb{F})$ have n *distinct* eigenvalues over $\lambda_1, \lambda_2, \ldots, \lambda_n$ over $\mathbb{F}$, let $(\lambda_1, \vec{v_1}), \ldots, (\lambda_n, \vec{v_n})$ be corresponding eigenpairs over $\mathbb{F}$, and let $P = [\vec{v_1} \ldots \vec{v_n}]$. Then,

1. P is invertible, diagonalizes A, and
2. $P^{-1}AP = D = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$.

8 Subspaces and Bases

Subspace:

A subset V of $\mathbb{F}^n$ is called a **subspace** of $\mathbb{F}^n$ if the following properties are satisfied:

1. $\vec{0} \in V$.
2. For all $\vec{x}, \vec{y} \in V, \vec{x} + \vec{y} \in V$ (**closure under addition**).
3. For all $\vec{x} \in V$ and $c \in \mathbb{F}, c\vec{x} \in V$ (**closure under scalar multiplication**).

It is important to note that $c \in \mathbb{F}$, and so is not necessarily an element of V.

Examples of Subspaces:

1. $\{\vec{0}\}$ and $\mathbb{F}^n$ are subspaces of $\mathbb{F}^n$.
2. If $\{\vec{v_1}, \dots, \vec{v_k}\}$ is a subset of $\mathbb{F}^n$, then $\text{Span}\{\vec{v_1}, \dots, \vec{v_k}\}$ is a subspace of $\mathbb{F}^n$.
3. If $A \in M_{m\times n}(\mathbb{F})$, then
   (a) $\text{Null}(A)$ is a subspace of $\mathbb{F}^n$.
   (b) $\text{Col}(A)$ is a subspace of $\mathbb{F}^m$.
4. If $T : \mathbb{F}^n \to \mathbb{F}^m$ is a linear transformation, then
   (a) $\text{Ker}(T)$ is a subspace of $\mathbb{F}^n$.
   (b) $\text{Range}(T)$ is a subspace of $\mathbb{F}^m$.
5. If $A \in M_{n\times n}(\mathbb{F})$ and if $\lambda \in \mathbb{F}$, then the eigenspace E_λ is a subspace of $\mathbb{F}^n$.

Subspace Test:

Let V be a subset of $\mathbb{F}^n$. Then V is a subspace of $\mathbb{F}^n$ if and only if:

1. V is non-empty, and
2. for all $\vec{x}, \vec{y} \in V$ and $c \in \mathbb{F}, c\vec{x} + \vec{y} \in V$.

Linear Dependence and Independence:

We say that the vectors $\vec{v_1}, \dots, \vec{v_k} \in \mathbb{F}^n$ are **linearly dependent** if there exist scalars $c_1, \dots, c_k \in \mathbb{F}$, not all zero, such that

$$c_1\vec{v_1} + \cdots + c_k\vec{v_k} = \vec{0}.$$

If $U = \{\vec{v_1}, \dots, \vec{v_k}\}$, then we say that U is **linearly independent** to mean the vectors $\vec{v_1}, \dots, \vec{v_k}$ are linearly dependent.

If the only solution to $c_1\vec{v_1} + \cdots + c_k\vec{v_k} = \vec{0}$ is the **trivial solution** $c_1 = \cdots = c_k = 0$, then we say that $\vec{v_1}, \dots, \vec{v_k}$ are **linearly independent**, and the set $U = \{\vec{v_1}, \dots, \vec{v_k}\}$ is linearly independent.

$\varnothing$ is by convention linearly **independent** as it vacuously satisfies the definition.

Linear Dependence Check:

1. The vectors $\vec{v_1}, \ldots, \vec{v_k}$ are linearly dependent if and only if one of the vectors can be written as a linear combination of some of the other vectors.
2. The vectors $\vec{v_1}, \ldots, \vec{v_k}$ are linearly independent if and only if

$$c_1\vec{v_1} + \cdots + c_k\vec{v_k} = \vec{0} \ (c_i \in \mathbb{F}) \implies c_1 = \cdots = c_k = 0.$$

Let $S \subseteq \mathbb{F}^n$.

1. If $\vec{0} \in S$, then S is linearly dependent.
2. If $S = \{\vec{x}\}$ contains only one vector, then S is linearly dependent if and only if $\vec{x} = \vec{0}$.
3. If $S = \{\vec{x}, \vec{y}\}$ contains only two vectors, then S is linearly dependent if and only if $\vec{x}$ is parallel to $\vec{y}$.

Pivots and Linear Dependence:

Let $S = \{\vec{v_1}, \ldots, \vec{v_k}\}$ be a set of k vectors in $\mathbb{F}^n$. Let $A = \begin{bmatrix}\vec{v_1} & \cdots & \vec{v_k}\end{bmatrix}^T$ be the $n \times k$ matrix whose columns are the vectors in S. Suppose that $\mathrm{rank}(A) = r$ and that A has pivot columns $q_1, \ldots, q_r$. Let $U = \{\vec{v_{q1}}, \ldots, \vec{v_{qk}}\}$ be the set of pivot columns labelled above. Then

1. S is linearly independent if and only if $r = k$.
2. U is linearly independent.
3. if $\vec{v} \in S$ but $\vec{v} \notin U$, then the set $\{\vec{v_{q1}}, \ldots, \vec{v_{qk}}, \vec{v}\}$ is linearly dependent.
4. $\mathrm{Span}(U) = \mathrm{Span}(S)$.

Bound on Number of Linearly Independent Vectors:

Let $S = \{\vec{v_1}, \ldots, \vec{v_k}\}$ be a set of k vectors in $\mathbb{F}^n$. If $n < k$, then S is linearly dependent.

Every Subspace Has a Spanning Set:

Let V be a subspace of $\mathbb{F}^n$. Then there exist vectors $\vec{v_1}, \ldots, \vec{v_k} \in V$ such that

$$V = \mathrm{Span}\{\vec{v_1}, \ldots, \vec{v_k}\}.$$

Span of a Subset:

Let V be a subspace of $\mathbb{F}^n$ and let $S = \{\vec{v_1}, \ldots, \vec{v_k}\} \subseteq V$. Then $\mathrm{Span}(S) \subseteq V$.

Spans $\mathbb{F}^n$ iff rank is n:

Let $S = \{\vec{v_1}, \ldots, \vec{v_k}\}$ be a set of k vectors in $\mathbb{F}^n$ and let $A = \begin{bmatrix} \vec{v_1} & \ldots & \vec{v_k} \end{bmatrix}$ be the matrix whose columns are the vectors in S. Then

$$\mathrm{Span}(S) = \mathbb{F}^n \iff \mathrm{rank}(A) = n.$$

Basis:

Let V be a subspace of $\mathbb{F}^n$ and let $\mathcal{B} = \{\vec{v_1}, \ldots, \vec{v_k}\}$ be a finite set of vectors contained in V. We say that $\mathcal{B}$ is a **basis** for V if

1. $\mathcal{B}$ is linearly independent, and
2. $V = \mathrm{Span}(\mathcal{B})$.

Ordered Basis:

Let V be a subspace of $\mathbb{F}^n$. An ordered basis for V is a basis $\mathcal{B} = \{\vec{v_1}, \ldots, \vec{v_k}\}$ for V with a fixed ordering.

By convention, $\varnothing$ is a basis for the zero subspace $V = \{\vec{0}\}$.

Standard Basis:

In $\mathbb{F}^n$, let $\vec{e_i}$ be the vector whose i^{th} component is 1 with all other components 0. The set

$$\mathcal{E} = \{\vec{e_1}, \ldots, \vec{e_n}\}$$

is called the standard basis for $\mathbb{F}^n$.

Every Subspace Has a Basis:

Let V be a subspace of $\mathbb{F}^n$. Then V has a basis.

Size of Basis for $\mathbb{F}^n$:

Let $S = \{\vec{v_1}, \ldots, \vec{v_k}\}$ be a set of k vectors in $\mathbb{F}^n$. If S is a basis for $\mathbb{F}^n$, then $k = n$.

n Vectors in $\mathbb{F}^n$ Span iff Independent:

Let $S = \{\vec{v_1}, \ldots, \vec{v_n}\}$, $\vec{v_1}, \ldots, \vec{v_n} \in \mathbb{F}^n$. Then S is linearly independent iff $\mathrm{Span}(S) = \mathbb{F}^n$.

Basis for $\mathrm{Col}(A)$:

Let $A = \begin{bmatrix} \vec{a_1} & \ldots & \vec{a_n} \end{bmatrix} \in M_{m\times n}(\mathbb{F})$ and suppose that $\mathrm{RREF}(A)$ has pivots in columns $q_1, \ldots, q_r$ where $r = \mathrm{rank}(A)$. Then $\{\vec{a_{q_1}}, \ldots, \vec{a_{q_r}}\}$ is a basis for $\mathrm{Col}(A)$.

Note it is the columns of A, not $\mathrm{RREF}(A)$, that form the basis.

Basis for $\mathrm{Null}(A)$:

Let $A \in M_{m\times n}(\mathbb{F})$ and consider the homogeneous system $A\vec{x} = \vec{0}$. Suppose after applying the Gauss-Jordan algorithm that we obtain k free parameters so that the solution set is given by

$$\mathrm{Span}\{\vec{x_1}, \dots, \vec{x_k}\}.$$

Then, $\{\vec{x_1}, \dots, \vec{x_k}\}$ is a basis for $\mathrm{Null}(A)$.

Here, $k = \mathrm{nullity}(A) = n - \mathrm{rank}(A)$.

Dimension:

The number of elements in a basis for a subspace V of $\mathbb{F}^n$ is called the **dimension** of V. We denote this number by $\dim(V)$.

Dimension is Well-Defined:

Let V be a subspace of $\mathbb{F}^n$. If $\mathcal{B} = \{\vec{v_1}, \dots, \vec{v_k}\}$ and $\mathcal{C} = \{\vec{w_1}, \dots, \vec{w_\ell}\}$ are bases for V, then $k = \ell$.

Bound on Dimension of Subspace:

Let V be a subspace of $\mathbb{F}^n$. Then $\dim(V) \le n$.

Rank-Nullity Theorem:

Let $A \in M_{m\times n}(\mathbb{F})$. Then

$$n = \mathrm{rank}(A) + \mathrm{nullity}(A) = \dim(\mathrm{Col}(A)) + \dim(\mathrm{Null}(A)).$$

Unique Representation Theorem:

Let V be a subspace of $\mathbb{F}^n$ and let $\mathcal{B} = \{\vec{v_1}, \dots, \vec{v_k}\}$ be a basis for V. Then, for every $\vec{v} \in V$, there exist *unique* scalars $c_1, \dots, c_k \in \mathbb{F}$ such that

$$c_1\vec{v_1} + \cdots + c_k\vec{v_k} = \vec{v}.$$

Coordinates and Components:

Let V be a subspace of $\mathbb{F}^n$ and let $\mathcal{B} = \{\vec{v_1}, \dots, \vec{v_k}\}$ be a basis for V. Let the vector $\vec{v} \in V$ have the representation

$$\vec{v} = c_1\vec{v_1} + \cdots + c_k\vec{v_k} = \sum_{i=1}^{k} c_i\vec{v_i}, \quad (c_i \in \mathbb{F}).$$

We call the scalars $c_1, \dots, c_k$ the **coordinates** (or **components**) of $\vec{v}$ with respect to $\mathcal{B}$, or the $\mathcal{B}$-coordinates of $\vec{v}$.

Coordinate Vector:

Let $\mathcal{B} = \{\vec{v_1}, \ldots, \vec{v_k}\}$ be an ordered basis for the subspace V of $\mathbb{F}^n$. Let $\vec{v} \in V$ have $\mathcal{B}$-coordinates $c_1, \ldots, c_k$ with matching ordering to $\mathcal{B}$. Then, the coordinate vector of $\vec{v}$ with respect to $\mathcal{B}$ (or the $\mathcal{B}$-coordinate vector of $\vec{v}$) is the column vector in $\mathbb{F}^n$:

$$[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_k \end{bmatrix}.$$

Linearity of Taking Coordinates:

Let $\mathcal{B} = \{\vec{v_1}, \ldots, \vec{v_k}\}$ be an ordered basis for V. Then the function $[\,]_{\mathcal{B}} : V \to \mathbb{F}^k$ given by $\vec{x} \mapsto [\vec{x}]_{\mathcal{B}}$ is a linear transformation.

Change of Basis Matrix:

Let $\mathcal{B} = \{\vec{v_1}, \ldots, \vec{v_k}\}$ and $\mathcal{C} = \{\vec{w_1}, \ldots, \vec{w_\ell}\}$ be ordered bases for a subspace V of $\mathbb{F}^n$.

The change of basis matrix from $\mathcal{B}$-coordinates to $\mathcal{C}$-coordinates is the $k \times k$ matrix

$${}_{\mathcal{C}}[I]_{\mathcal{B}} = [[\vec{v_1}]_{\mathcal{C}} \;\cdots\; [\vec{v_k}]_{\mathcal{C}}]$$

whose columns are the $\mathcal{C}$-coordinates of the vectors in $\mathcal{B}$. Similarly, the change of basis matrix from $\mathcal{C}$-coordinates to $\mathcal{B}$-coordinates is the $k \times k$ matrix

$${}_{\mathcal{B}}[I]_{\mathcal{C}} = [[\vec{w_1}]_{\mathcal{B}} \;\cdots\; [\vec{w_\ell}]_{\mathcal{B}}\,.]$$

Changing a Basis:

$[\vec{x}]_{\mathcal{C}} = {}_{\mathcal{C}}[I]_{\mathcal{B}}\,[\vec{x}]_{\mathcal{B}}$ and $[\vec{x}]_{\mathcal{B}} = {}_{\mathcal{B}}[I]_{\mathcal{C}}\,[\vec{x}]_{\mathcal{C}}$ for all $\vec{x} \in V$.

Let $\vec{x} = [\vec{x}]_{\mathcal{E}} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ be a vector in $\mathbb{F}^n$, where $\mathcal{E}$ is the standard ordered basis. If $\mathcal{C}$ is any ordered basis for $\mathbb{F}^n$, then $[\vec{x}]_{\mathcal{C}} = {}_{\mathcal{C}}[I]_{\mathcal{E}}\,[\vec{x}]_{\mathcal{E}}$.

Inverse of Change of Basis Matrix:

Let $\mathcal{B}$ and $\mathcal{C}$ be two ordered bases of $\mathbb{F}^n$. Then

$${}_{\mathcal{C}}[I]_{\mathcal{B}}\;{}_{\mathcal{B}}[I]_{\mathcal{C}} = I_n \quad \text{and} \quad {}_{\mathcal{B}}[I]_{\mathcal{C}}\;{}_{\mathcal{C}}[I]_{\mathcal{B}} = I_n.$$

That is, ${}_{\mathcal{C}}[I]_{\mathcal{B}} = ({}_{\mathcal{B}}[I]_{\mathcal{C}})^{-1}$ and ${}_{\mathcal{B}}[I]_{\mathcal{C}} = ({}_{\mathcal{C}}[I]_{\mathcal{B}})^{-1}$.

9 Diagonalization

$\mathcal{B}$-Matrix of T:

Let $T : \mathbb{F}^n \to \mathbb{F}^n$ be a linear operator and let $\mathcal{B} = \{\vec{v_1}, \dots, \vec{v_n}\}$ be an ordered basis for $\mathbb{F}^n$. We define the $\mathcal{B}$-matrix of T as follows:

$$[T]_{\mathcal{B}} = \left[[T(\vec{v_1})]_{\mathcal{B}} \ \dots \ [T(\vec{v_n})]_{\mathcal{B}} \right]$$

That is, after applying T to each vector in $\mathcal{B}$, we construct $[T]_{\mathcal{B}}$ from the $\mathcal{B}$-coordinate vectors of these images.

If $\vec{v} \in \mathbb{F}^n$, then

$$[T(\vec{v})]_{\mathcal{B}} = [T]_{\mathcal{B}} [\vec{v}]_{\mathcal{B}}$$

Similarity of Matrix Representations:

Let $T : \mathbb{F}^n \to \mathbb{F}^n$ be a linear operator. Let $\mathcal{B}$ and $\mathcal{C}$ be ordered bases for $\mathbb{F}^n$. Then

$$[T]_{\mathcal{C}} = {}_{\mathcal{C}}[I]_{\mathcal{B}} [T]_{\mathcal{B}} \, {}_{\mathcal{B}}[I]_{\mathcal{C}} = \left({}_{\mathcal{B}}[I]_{\mathcal{C}} \right)^{-1} [T]_{\mathcal{B}} \, {}_{\mathcal{B}}[I]_{\mathcal{C}}$$

and

$$[T]_{\mathcal{B}} = {}_{\mathcal{B}}[I]_{\mathcal{C}} [T]_{\mathcal{C}} \, {}_{\mathcal{C}}[I]_{\mathcal{B}} = \left({}_{\mathcal{C}}[I]_{\mathcal{B}} \right)^{-1} [T]_{\mathcal{C}} \, {}_{\mathcal{C}}[I]_{\mathcal{B}}.$$

That is, $[T]_{\mathcal{B}}$ and $[T]_{\mathcal{C}}$ are similar over $\mathbb{F}$.

Finding the Standard Matrix:

$$\begin{aligned} [T]_{\mathcal{E}} &= {}_{\mathcal{E}}[I]_{\mathcal{B}} [T]_{\mathcal{B}} \, {}_{\mathcal{B}}[I]_{\mathcal{E}} \\ &= ({}_{\mathcal{B}}[I]_{\mathcal{E}})^{-1} [T]_{\mathcal{B}} \, {}_{\mathcal{B}}[I]_{\mathcal{E}} \\ [T]_{\mathcal{B}} &= {}_{\mathcal{B}}[I]_{\mathcal{E}} [T]_{\mathcal{E}} \, {}_{\mathcal{E}}[I]_{\mathcal{B}} \\ &= ({}_{\mathcal{E}}[I]_{\mathcal{B}})^{-1} [T]_{\mathcal{E}} \, {}_{\mathcal{E}}[I]_{\mathcal{B}}. \end{aligned}$$

Eigenthings of a Linear Operator:

Let $T : \mathbb{F}^n \to \mathbb{F}^n$ be a linear operator. We say that the *non-zero* vector $\vec{x} \in \mathbb{F}^n$ is an **eigenvector** of T to mean that there exists a scalar $\lambda \in \mathbb{F}$ such that

$$T(\vec{x}) = \lambda \vec{x}.$$

The scalar λ is an **eigenvalue** of T and $(\lambda, \vec{x})$ is called an **eigenpair** of T.

Eigenpairs of T and $[T]_{\mathcal{B}}$:

Let $T : \mathbb{F}^n \to \mathbb{F}^n$ be a linear operator and let $\mathcal{B}$ be an ordered basis of $\mathbb{F}^n$. Then $(\lambda, \vec{x})$ is an eigenpair of T if and only if $(\lambda, [\vec{x}]_{\mathcal{B}})$ is an eigenpair of the matrix $[T]_{\mathcal{B}}$.

Diagonalizable:

Let $T : \mathbb{F}^n \to \mathbb{F}^n$ be a linear operator. We say that T is diagonalizable over $\mathbb{F}$ to mean that there exists an ordered basis $\mathcal{B}$ of $\mathbb{F}^n$ such that $[T]_{\mathcal{B}}$ is a diagonal matrix.

Eigenvector Basis Criterion for Diagonalizability:

Let $T : \mathbb{F}^n \to \mathbb{F}^n$ be a linear operator. Then T is diagonalizable over $\mathbb{F}$ if and only if there exists an ordered basis $\mathcal{B} = \{\vec{v_1}, \dots, \vec{v_n}\}$ of $\mathbb{F}^n$ consisting of eigenvectors of T

Matrix Version:

Let $A \in M_{n\times n}(\mathbb{F})$. Then A is diagonalizable over $\mathbb{F}$ if and only if there exists a basis of $\mathbb{F}^n$ consisting of eigenvectors of A.

T Diagonalizable iff $[T]_{\mathcal{B}}$ Diagonalizable:

Let $T : \mathbb{F}^n \to \mathbb{F}^n$ be a linear operator and let $\mathcal{B}$ be an ordered basis of $\mathbb{F}^n$. Then T is diagonalizable over $\mathbb{F}$ if and only if $[T]_{\mathcal{B}}$ is diagonalizable over $\mathbb{F}$.

Eigenvectors Corresponding to Distinct Eigenvalues are Linearly Independent:

Let $A \in M_{n\times n}(\mathbb{F})$ have eigenpairs $(\lambda_1, \vec{v_1}), \dots, (\lambda_k, \overrightarrow{v_k s})$ for $1 \le k \le n$.
If the eigenvalues $\lambda_1, \dots, \lambda_k$ are all distinct, then $\{\vec{v_1}, \dots, \vec{v_k}\}$ is linearly independent.

Let $A \in M_{n\times n}(\mathbb{F})$ with distinct eigenvalues $\lambda_1, \dots, \lambda_k$. If their corresponding eigenspaces, $E_{\lambda_1}, \dots, E_{\lambda_k}$ have bases $\mathcal{B}_1, \dots, \mathcal{B}_k$, then

$$\mathcal{B} = \mathcal{B}_1 \cup \cdots \cup \mathcal{B}_k$$

is linearly independent.

Characteristic Polynomial:

Let $T : \mathbb{F}^n \to \mathbb{F}^n$ be a linear operator and let $\mathcal{B}$ be a basis for $\mathbb{F}^n$. Then

$$C_T(\lambda) = C_{[T]_{\mathcal{B}}}(\lambda)$$

This definition is unambiguous because the matrices $[T]_{\mathcal{B}}$ and $[T]_{\mathcal{C}}$ are similar for any bases $\mathcal{B}$ and $\mathcal{C}$, and similar matrices have identical characteristic polynomials.

Geometric and Algebraic Multiplicities:

Let λ_i be an eigenvalue of $A \in M_{n\times n}(\mathbb{F})$. The **algebraic multiplicity** of λ_i, denoted a_{λ_i}, is the largest positive integer such that $(\lambda - \lambda_i)^{a_{\lambda_i}}$ divides $C_A(\lambda)$.

The **geometric multiplicity** of λ_i, denoted g_{λ_i} is the dimension of the eigenspace E_{λ_i}. That is, $g_{\lambda_i} = \dim(E_{\lambda_i})$.

Geometric and Algebraic Multiplicities:

Let λ_i be an eigenvalue of $A \in M_{n\times n}(\mathbb{F})$. Then

$$1 \le g_{\lambda_i} \le a_{\lambda_i}.$$

Diagonalizability Test:

Let $A \in M_{n\times n}(\mathbb{F})$ with characteristic polynomial

$$C_A(\lambda) = h(\lambda)(\lambda - \lambda_1)^{a_{\lambda_1}} \cdots (\lambda - \lambda_k)^{a_{\lambda_k}}$$

where $\lambda_1, \ldots, \lambda_k$ are all distinct eigenvalues of A over $\mathbb{F}$ with corresponding algebraic multiplicities $a_{\lambda_1}, \ldots, a_{\lambda_k}$ and $h(\lambda)$ is a polynomial irreducible over $\mathbb{F}$. Then A is diagonalizable if and only if $\deg(h(\lambda)) = 0$ and $a_{\lambda_i} = g_{\lambda_i}$ for all $i = 1, \ldots, k$.

That is, A is diagonalizable if and only if $C_A(\lambda)$ is reducible to linear terms over $\mathbb{F}$ (enough eigenvalues) and each algebraic and geometric multiplicities match (enough linearly independent eigenvectors).

Powers of Similar Matrices:

Let $A, B \in M_{n\times n}(\mathbb{F})$ such that $B = P^{-1}AP$ for some invertible $P \in M_{n\times n}(\mathbb{F})$. Then for all $k \in \mathbb{N}$,

$$B^k = P^{-1}A^kP.$$

10 Vector Spaces

Vector Space:

A non-empty set, $\mathbb{V}$, is a vector space over a field, $\mathbb{F}$, under the operations of addition, $\oplus$, and scalar multiplication, $\odot$, provided the following ten axioms are met:

1. Closure under $\oplus$ and $\odot$.
2. $\oplus$ and $\odot$ are associative and have identity elements.
3. $\oplus$ is commutative and every vector in $\mathbb{V}$ has an additive inverse.
4. $\odot$ distributes over $\oplus$.
5. Field addition distributes over $\odot$.

Vector:

A **vector** is an element of a vector space.

$L(\mathbb{F}^n, \mathbb{F}^m)$:

We use $L(\mathbb{F}^n, \mathbb{F}^m)$ to denote the vector space over $\mathbb{F}$ comprised of all linear transformations $T : \mathbb{F}^n \to \mathbb{F}^m$, with the following operations for all $\vec{x} \in \mathbb{F}^n$ and $c \in \mathbb{F}$ defined as follows:

$$(T_1 + T_2)(\vec{x}) = T_1(\vec{x}) + T_2(\vec{x}),$$
$$(cT_1)(\vec{x}) = cT_1(\vec{x}).$$

$P_n(\mathbb{F})$:

We use $P_n(\mathbb{F})$ to denote the vector space over $\mathbb{F}$ comprised of the set of all polynomials of degree at most n with coefficients in $\mathbb{F}$, with addition and scalar multiplication defined obviously.

Properties of Vector Spaces:

Let $\mathbb{V}$ be a vector space over $\mathbb{F}$ and let $\vec{x} \in \mathbb{V}$.

1. The zero vector in $\mathbb{V}$ is unique.
2. The additive inverse of $\vec{x}$ is unique.
3. $0 \odot \vec{x} = \vec{0}$.
4. For all $a \in \mathbb{F}, a \odot \vec{0} = \vec{0}$.
5. $-\vec{x} = (-1) \odot \vec{x}$.
6. If $a \odot \vec{x} = \vec{0}$, then $a = 0$ or $\vec{x} = \vec{0}$ (Cancellation Law).

Subspace Test:

Let $\mathbb{V}$ be a subspace over $\mathbb{F}$ and let $\mathbb{U} \subseteq \mathbb{V}$. Then $\mathbb{U}$ is a subspace of $\mathbb{V}$ if and only if:

1. $\mathbb{U}$ is non-empty, and
2. $\mathbb{U}$ is closed under addition and scalar multiplication.

Vector Space Ideas:

The following ideas carry over verbatim from $\mathbb{F}^n$:

1. Linear combinations
2. Span
3. Subspaces
4. Linear Independence/Dependence
5. Bases

   The dimension of the zero space $\{\vec{0}\}$ is 0.

   If $\mathbb{V}$ does not have a basis with a finite number of vectors in it, then $\mathbb{V}$ is said to be **infinite-dimensional**.

6. Dimension
7. Unique Representation Theorem
8. $\mathcal{B}$-coordinates
9. Change of Basis Matrix