1 Integration

Theorem 1.1: Integrability Theorem for Continuous Functions:

Let f be continuous on $[a, b]$. Then f is integrable on $[a, b]$. Moreover,

$$\int_a^b f(t)\,dt = \lim_{n\to\infty} S_n$$

where

$$S_n = \sum_{i=1}^{n} f(c_i)\Delta t_i$$

is any Riemann sum associated with the regular n-partitions.

Proposition 1.2: Properties of Integrals:

Assume that f and g are integrable on the interval $[a, b]$. Then:

1. For any $c \in \mathbb{R}$, $\int_a^b cf(t)\,dt = c\int_a^b f(t)\,dt$.
2. $\int_a^b (f + g)(t)\,dt = \int_a^b f(t)\,dt + \int_a^b g(t)\,dt$.
3. If $m \leq f(x) \leq M$ for all $x \in [a, b]$, then $m(b - a) \leq \int_a^b f(t)\,dt \leq M(b - a)$.
4. If $f(x) \geq 0$ for all $x \in [a, b]$, then $\int_a^b f(t)\,dt \geq 0$.
5. If $f(x) \geq g(x)$ for all $x \in [a, b]$, then $\int_a^b f(t)\,dt \geq \int_a^b g(t)\,dt$.
6. The function $|f|$ is integrable on $[a, b]$ and $|\int_a^b f(t)\,dt| \leq \int_a^b |f(t)|\,dt$.

Lemma 1.3: Identical Limits of Integration

Let $f(t)$ be defined at $t = a$. Then we define

$$\int_a^a f(t)\,dt = 0.$$

Lemma 1.4: Switching the Limits of Integration

Let f be integrable on $[a, b]$ where $a < b$. Then we define

$$\int_b^a f(t)\,dt = -\int_a^b f(t)\,dt.$$

Theorem 1.5: Integrals over Subintervals:

Assume that f is integrable on an interval I containing a, b, and c. Then

$$\int_a^b f(t)\,dt = \int_a^c f(t)\,dt + \int_c^b f(t)\,dt.$$

Definition 1.6: Average Value of f:

If f is continuous on $[a,b]$, the average value of f on $[a,b]$ is defined as

$$\frac{1}{b-a}\int_a^b f(t)\,dt.$$

Theorem 1.7: Average Value Theorem (MVT for Integrals):

Assume that f is continuous on $[a,b]$. Then there exists a $c \in [a,b]$ such that

$$f(c) = \frac{1}{b-a}\int_a^b f(t)\,dt.$$

Theorem 1.8: Fundamental Theorem of Calculus (Part 1):

Assume that f is continuous on an open interval I containing a point a. Let

$$G(x) = \int_a^x f(t)\,dt.$$

Then $G(x)$ is differentiable at each $x \in I$ and

$$\forall x \in I, G'(x) = f(x).$$

Lemma 1.9: Extended FTC I

Assume that f is continuous and that g and h are differentiable. Let

$$H(x) = \int_{g(x)}^{h(x)} f(t)\,dt.$$

Then $H(x)$ is differentiable and

$$H'(x) = f(h(x))h'(x) - f(g(x))g'(x)$$

Theorem 1.10: Fundamental Theorem of Calculus (Part 2):

Assume that f is continuous and that F is any antiderivative of f. Then

$$\int_a^b f(t)\,dt = F(b) - F(a).$$

2 Improper Integrals

Theorem 2.1: p-Test for Type I Improper Integrals:

The improper integral

$$\int_1^\infty \frac{1}{x^p}\,dx$$

converges if and only if $p > 1$. If $p > 1$, then

$$\int_1^\infty \frac{1}{x^p}\,dx = \frac{1}{p-1}.$$

Proposition 2.2: Properties of Type I Improper Integrals:

Assume that $\int_a^\infty f(x)\,dx$ and $\int_a^\infty g(x)\,dx$ both converge. Then

1. $\int_a^\infty cf(x)\,dx$ converges for all $c \in \mathbb{R}$ and

$$\int_a^\infty cf(x)\,dx = c\int_a^\infty f(x)\,dx$$

2. $\int_a^\infty f(x) + g(x)\,dx$ converges and

$$\int_a^\infty f(x) + g(x)\,dx = \int_a^\infty f(x)\,dx + \int_a^\infty g(x)\,dx$$

3. If $f(x) \geq g(x)$ for all $x \geq a$, then

$$\int_a^\infty f(x)\,dx \geq \int_a^\infty g(x)\,dx$$

4. If $a < c < \infty$, then $\int_c^\infty f(x)\,dx$ converges and

$$\int_a^\infty f(x)\,dx = \int_a^c f(x)\,dx + \int_c^\infty f(x)\,dx.$$

Theorem 2.3: Comparison Test for Type I Improper Integrals:

Assume that $0 \leq g(x) \leq f(x)$ for all $x \geq a$ and that f and g are both continuous for all $x \geq a$. Then

1. If $\int_a^\infty f(x)\,dx$ converges, then $\int_a^\infty g(x)\,dx$ converges as well.
2. If $\int_a^\infty g(x)\,dx$ diverges, then $\int_a^\infty f(x)\,dx$ diverges as well.

Theorem 2.4: Absolute Convergence Theorem for Improper Integrals:

Let f be integrable on $[a, b]$ for all $a < b$. Then $|f|$ is also integrable (Properties of Integrals). Moreover, if we assume that

$$\int_a^\infty |f(x)|\,dx$$

converges, then

$$\int_a^\infty f(x)\,dx$$

converges. In particular, if both f and g are integrable, $0 \le |f(x)| \le g(x)$ for all $x \ge a$, and $\int_a^\infty g(x)\,dx$ converges, then so does

$$\int_a^\infty f(x)\,dx.$$

Theorem 2.5: p-Test for Type II Improper Integrals:

The improper integral

$$\int_0^1 \frac{1}{x^p}\,dx$$

converges if and only if $p < 1$. If $p < 1$, then

$$\int_0^1 \frac{1}{x^p}\,dx = \frac{1}{1-p}.$$

3 Applications of Integrals

Theorem 3.1: Volumes of Revolution:

Definition 3.2: Disks/Washers Method:

Suppose the axis of rotation is horizontal. Let r_{out} and r_{in} be continuous on $[a, b]$ with $r_{out}(x)$ further from the axis of rotation than $r_{in}(x)$ for all $x \in [a, b]$. Let W be the region bounded by the graphs of r_{out} and r_{in}, and the lines $x = a$ and $x = b$. Then the volume V of the solid of revolution obtained by rotating the region W around the axis of rotation is given by

$$dV = \pi(r_{out}^2 - r_{in}^2)\,dx \implies V = \pi \int_a^b (r_{out}^2 - r_{in}^2)\,dx.$$

If the axis of rotation is vertical, interchange x with y and we have

$$dV = \pi(r_{out}^2 - r_{in}^2)\,dy \implies V = \pi \int_a^b (r_{out}^2 - r_{in}^2)\,dy.$$

Definition 3.3: Shells Method:

Let r and $h = f - g$ be continuous on $[a, b]$ where $r(x)$ represents the lateral distance to the axis of rotation for all $x \in [a, b]$ and $h(x)$ represents the height of the region W bounded by the graphs of f and g, and the lines $x = a$ and $x = b$. Then the volume V of the solid of revolution obtained by rotating the region W around the axis of rotation is given by

$$dV = 2\pi r h\,dx \implies V = 2\pi \int_a^b r h\,dx.$$

If the axis of rotation is horizontal, interchange x with y and we have

$$dV = 2\pi r h\,dy \implies V = 2\pi \int_a^b r h\,dy.$$

Lemma 3.4: Summary

Method \ Axis	Horizontal	Vertical
Disks/Washers	$dV = \pi(r_{out}^2 - r_{in}^2)\,dx$	$dV = \pi(r_{out}^2 - r_{in}^2)\,dy$
Shells	$dV = 2\pi r h\,dy$	$dV = 2\pi r h\,dx$

Theorem 3.5: Arc Length:

Let f be continuously differentiable on $[a, b]$. Then the arc length S of the graph of f over the interval $[a, b]$ is given by

$$S = \int_a^b \sqrt{1 + (f'(x))^2}\,dx.$$

4 Differential Equations

Definition 4.1: Separable Differential Equation:

A first-order differential equation is said to be *separable* if there exist functions $f = f(x)$ and $g = g(y)$ such that

$$y' = f(x)g(y).$$

Definition 4.2: First-Order Linear Differential Equation (FOLDE):

A first-order differential equation is said to be *linear* if it can be written in the form

$$y' = f(x)y + g(x).$$

Theorem 4.3: Existence and Uniqueness Theorem for FOLDEs:

Assume that f and g are continuous on an interval I. Then for each $x_0 \in I$ and for all $y_0 \in \mathbb{R}$, the initial value problem

$$\begin{aligned} y' &= f(x)y + g(x) \\ y(x_0) &= y_0 \end{aligned}$$

has exactly *one* solution on the interval I.

Definition 4.4: Newton's Law of Cooling:

Let T_a be the ambient temperature of an object's surroundings with $T(t)$ denoting the temperature of the object at time t. Then, there is a constant $k < 0$ such that

$$T' = k(T - T_a).$$

Definition 4.5: Logistic Growth:

Suppose there is a carrying capacity M and let $P(t)$ denote population at time t. Then there exists a constant k such that

$$P' = kP(M - P).$$

The differential equation

$$y' = ky(M - y)$$

is called the *logistic equation*.

5 Series

Theorem 5.1: Divergence Test:

If $\lim_{n\to\infty} a_n \neq 0$ or diverges/does not exist, then $\sum_{n=1}^{\infty} a_n$ diverges.

Remark. If $\lim_{n\to\infty} a_n = 0$, we cannot conclude anything.

Theorem 5.2: Geometric Series Test:

The geometric series $\sum_{n=0}^{\infty} r^n$ converges if and only if $|r| < 1$ and diverges otherwise. If $|r| < 1$, then

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}.$$

Theorem 5.3: Arithmetic for Series I:

Assume that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge. Then

1. The series $\sum_{n=1}^{\infty} ca_n$ converges for all $c \in \mathbb{R}$ and

$$\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n.$$

2. The series $\sum_{n=1}^{\infty} (a_n + b_n)$ converges and

$$\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n.$$

Theorem 5.4: Arithmetic for Series II:

1. If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=j}^{\infty} a_n$ also converges for all $j \in \mathbb{F}^n N$.

2. If $\sum_{n=j}^{\infty} a_n$ converges for some $j \in \mathbb{F}^n N$, then $\sum_{n=1}^{\infty} a_n$ converges.

In either case,

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_{j-i} + \sum_{n=j}^{\infty} a_n.$$

Theorem 5.5: Comparison Test for Series:

Assume that $0 \leq a_n \leq b_n$ for all $n > K$, where $K \in \mathbb{F}^n N$. We have:

1. If $\sum\limits_{n=1}^{\infty} b_n$ converges, then $\sum\limits_{n=1}^{\infty} a_n$ converges as well.
2. If $\sum\limits_{n=1}^{\infty} a_n$ diverges, then $\sum\limits_{n=1}^{\infty} b_n$ diverges as well.

Theorem 5.6: Limit Comparison Test (LCT):

Assume that $a_n, b_n > 0$ for all $n \in \mathbb{F}^n N$ and that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L$$

where we have three possibilities:

1. <u>Case 1: $0 < L < \infty$.</u> Then $\sum\limits_{n=1}^{\infty} a_n$ converges if and only if $\sum\limits_{n=1}^{\infty} b_n$ converges.
2. <u>Case 2: $L = 0$.</u> If $\sum\limits_{n=1}^{\infty} a_n$ diverges, then $\sum\limits_{n=1}^{\infty} b_n$ diverges as well.
3. <u>Case 3: $L = \infty$.</u> If $\sum\limits_{n=1}^{\infty} a_n$ converges, then $\sum\limits_{n=1}^{\infty} b_n$ converges as well.

Theorem 5.7: Integral Test:

Assume that

1. f is continuous on $[t, \infty)$,
2. $f(x) > 0$ on $[t, \infty)$,
3. f is decreasing on $[t, \infty)$, and
4. $a_n = f(n)$ for all $n \in \mathbb{F}^n N$.

For each $k \in \mathbb{F}^n N$, let $S_k = \sum\limits_{n=1}^{k} a_n$. Then

1. $\sum\limits_{n=t}^{\infty} a_n$ converges if and only if $\int_t^{\infty} f(x)\,dx$ converges.
2. Let $t = 1$. For all $k \in \mathbb{F}^n N$,
$$\int_k^{k+1} f(x)\,dx \le S_k \le a_1 + \int_1^k f(x)\,dx.$$
3. In the case that $\sum\limits_{n=1}^{\infty} a_n$ converges to some $S \in \mathbb{R}$, then
$$\int_1^{\infty} f(x)\,dx \le \sum_{n=1}^{\infty} a_n \le a_1 + \int_1^{\infty} f(x)\,dx, \text{ and}$$
$$\int_{k+1}^{\infty} f(x)\,dx \le S - S_k \le \int_k^{\infty} f(x)\,dx.$$

Theorem 5.8: p-Series Test:

The series $\sum\limits_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$.

Theorem 5.9: Alternating Series Test:

Assume that there exists some $N \in \mathbb{F}^n N$ such that

1. $a_n > 0$ for all $n \geq N$ (a_n is eventually positive)
2. $a_{n+1} < a_n$ for all $n \geq N$ (a_n is eventually decreasing)
3. $\lim\limits_{n\to\infty} a_n = 0$.

Then the alternating series

$$\sum_{n=1}^{\infty}(-1)^{n-1}a_n$$

converges to some $S \in \mathbb{R}$. If $S_k = \sum\limits_{n=1}^{k}(-1)^{n-1}a_n$, then the partial sums approximate the sum with an error that is at most a_{k+1} for all $n \geq N$. That is, for all $k \geq N$,

$$|S - S_k| \leq a_{k+1}$$

Remark. If the first term of the alternating series is positive, the even partial sums are underestimates while the odd partial sums are overestimates and vice-versa if the first term is negative.

Theorem 5.10: Absolute Convergence Theorem:

If $\sum\limits_{n=1}^{\infty} |a_n|$ converges, then so does $\sum\limits_{n=1}^{\infty} a_n$.

Remark. The two series have different sums unless $|a_n| = a_n \iff a_n \geq 0$ for all n.

Theorem 5.11: Ratio/Root Test:

Given a series $\sum\limits_{n=1}^{\infty} a_n$, assume that

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L_1 \qquad\qquad \lim_{n\to\infty}\sqrt[n]{|a_n|} = L_2$$

where $L_1 \in \mathbb{R}$ or $L_1 = \infty$ and $L_2 \in \mathbb{R}$ or $L_2 = \infty$.

1. If $0 \leq L_1 < 1$ or $0 \leq L_2 < 1$, the series converges absolutely.
2. If $L_1 > 1$ or $L_1 = \infty$ or $L_2 > 1$ or $L_2 = \infty$, the series diverges.
3. If $L_1 = L_2 = 1$, we know nothing.

6 Power Series

Definition 6.1: Power Series:

A power series centered at $x = a$ is a series of the form

$$\sum_{n=0}^{\infty} a_n(x-a)^n$$

where x is considered a variable and a_n is called the coefficient of the term $(x-a)^n$.

Theorem 6.2: Test for the Radius of Convergence:

Let $\sum\limits_{n=0}^{\infty} a_n(x-a)^n$ be a power series for which

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L$$

where $0 \leq L < \infty$ or $L = \infty$. Then the radius of convergence, R, is $\frac{1}{L}$.

Theorem 6.3: Abel's Theorem: Continuity of Power Series:

Assume the power series $\sum\limits_{n=0}^{\infty} a_n(x-a)^n$ has interval of convergence I. Let

$$f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$$

for all $x \in I$. Then $f(x)$ is continuous on I.

Theorem 6.4: Addition of Power Series:

Assume that f and g are represented by power series both centered at $x = a$ with

$$f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n, \qquad g(x) = \sum_{n=0}^{\infty} b_n(x-a)^n,$$

respectively. Assume also that the radii of convergence of these series are R_f and R_g with intervals of convergence I_f and I_g. Then

$$(f+g)(x) = \sum_{n=0}^{\infty}(a_n+b_n)(x-a)^n,$$

with radius of convergence $R = \min\{R_f, R_g\}$ and interval of convergence $I = I_f \cap I_g$.

Theorem 6.5: Multiplication of a Power Series by $(x-a)^n$:

Assume that f is represented by $f(x) = \sum\limits_{n=0}^{\infty} a_n(x-a)^n$ with radius of convergence R_f and interval of convergence I_f. Assume that $h(x) = (x-a)^m f(x)$ where $m \in \mathbb{F}^n N$. Then $h(x)$ can also be represented by a power series by

$$h(x) = \sum_{n=0}^{\infty} a_n (x-a)^{n+m}$$

with radius of convergence R_f and interval of convergence I_f.

Theorem 6.6: Power Series of Composite Functions:

Assume that $f(u) = \sum_{n=0}^{\infty} a_n u^n$ centered at $u = 0$ with RoC R_f and IoC I_f. Let $h(x) = f(c \cdot x^m)$ where c is a non-zero constant. Then h has a power series representation centered at $x = 0$ given by

$$h(x) = f(c \cdot x^m) = \sum_{n=0}^{\infty} a_n (cx^m)^n = \sum_{n=0}^{\infty} a_n c^n x^{mn}$$

with RoC $|c \cdot x^m| < 1$ or equivalently

$$R_h = \sqrt[m]{\frac{R_f}{|c|}}.$$

Definition 6.7: Formal Derivative:

Given a power series $\sum_{n=0}^{\infty} a_n (x-a)^n$, its formal derivative is the series

$$\sum_{n=0}^{\infty} n a_n (x-a)^{n-1} = \sum_{n=1}^{\infty} n a_n (x-a)^{n-1}.$$

Theorem 6.8: Term-by-term Differentation of Power Series:

Let $f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$ with RoC $R > 0$ for all $x \in (a-R, a+R)$. Then f is differentiable on $(a-R, a+R)$ and for all $x \in (a-R, a+R)$,

$$f'(x) = \sum_{n=1}^{\infty} n a_n (x-a)^{n-1}.$$

Theorem 6.9: Uniqueness of Power Series Representations:

Suppose that

$$f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$$

for all $x \in (a-R, a+R)$ where $R > 0$. Then

$$a_n = \frac{f^{(n)}(a)}{n!},$$

for all $n \geq 0$. In particular,

$$f(x) = \sum_{n=0}^{\infty} b_n (x-a)^n \implies b_n = a_n$$

for all $n \geq 0$.

Definition 6.10: Formal Antiderivative:

The formal antiderivative of the power series $\sum\limits_{n=0}^{\infty} a_n(x-a)^n$ is defined as

$$\sum_{n=0}^{\infty} \int a_n (x-a)^n \, dx = C + \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x-a)^{n+1}$$

where C is an arbitrary constant.

Theorem 6.11: Term-by-term Integration of Power Series:

Let $f(x) = \sum\limits_{n=0}^{\infty} a_n(x-a)^n$ with RoC $R > 0$ for all $x \in (a-R, a+R)$. Then the series

$$\sum_{n=0}^{\infty} \int a_n (x-a)^n \, dx = C + \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x-a)^{n+1}$$

also has RoC R and if

$$F(x) = C + \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x-a)^{n+1},$$

then $F'(x) = f(x)$. Furthermore, if $[c, b] \subset (a-R, a+R)$, then

$$\begin{aligned} \int_c^b f(x)\, dx &= \int_c^b \sum_{n=0}^{\infty} a_n (x-a)^n \, dx \\ &= \sum_{n=0}^{\infty} \int_c^b a_n (x-a)^n \, dx \\ &= \sum_{n=0}^{\infty} \frac{a_n}{n+1} \cdot \left((b-a)^{n+1} - (c-a)^{n+1}\right). \end{aligned}$$

Theorem 6.12: Taylor's Theorem:

Assume that f is $n+1$ times differentiable on an interval I containing $x = a$. Let $x \in I$. Then there exists a point c between x and a such that

$$f(x) - T_{n,a}(x) = R_{n,a}(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}.$$

Theorem 6.13: Taylor's Approximation Theorem I:

Assume that $f^{(k+1)}$ is continuous on $[-1,1]$. Then there exists a constant $M > 0$ such that

$$|f(x) - T_{k,0}(x)| \leq M|x|^{k+1}$$

for each $x \in [-1,1]$.

Theorem 6.14: Convergence Theorem for Taylor Series:

Assume that $f(x)$ has derivatives of all orders on an interval I containing $x = a$ and that there exists an M such that

$$|f^{(k)}(x)| \leq M$$

for all $k \in \mathbb{F}^n N$ and $x \in I$. Then

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

for all $x \in I$.

Theorem 6.15: Binomial Theorem:

Let $a \in \mathbb{R}$ and $n \in \mathbb{N}$. Then for each $x \in \mathbb{R}$ we have that

$$(a+x)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} x^k.$$

Theorem 6.16: Generalized Binomial Theorem:

Let $\alpha \in \mathbb{R}$. Then for each $x \in (-1,1)$ we have that

$$(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{k} x^k.$$

Remark. For all $\alpha \in \mathbb{R}$,

$$\binom{\alpha}{0} = 1$$