1 Sample Spaces and Probability

Definition 1.1: Probability

- **Classical definition:** The probability of some event is

$$\frac{\text{Number of ways event can occur}}{\text{Number of total possible outcomes}}$$

  Example. Probability of rolling 6 is $\frac{1}{6}$ on a die

- **Relative frequency:** Limiting proportion of occurrence in a series of independent trials.

  Example. After many rolls, $P(6) \to \frac{1}{6}$.

- **Subjective probability:** Best guess at the chances of an event.

  Example. Weather forecast predicts 30% chance of rain.

Definition 1.2: Sample space

A sample space S is a set of distinct outcomes of an experiment with the property that in a single trial of the experiment only one of these outcomes occurs.

Example. $S = \{1, 2, 3, 4, \dots\}$

Definition 1.3: Discrete

A sample space is said to be discrete if it is finite or countably infinite. Otherwise, it is non-discrete.

Definition 1.4: Event

An event is a subset of a sample space S.

Notation. We write $A \subseteq S$ to mean A is an event from S.

Definition 1.5: Types of events

A **simple event** contains only one sample point.

A **compound event** contains two or more sample points. It occurs if any of the simple events occur.

Definition 1.6: Disjoint

We say two events A and B are disjoint (mutually exclusive) to mean that $A \cap B = \varnothing$.

Definition 1.7: Axioms of Probability

1. *Scale.* If A is an event, $0 \leq P(A) \leq 1$.
2. *Something happens.* $P(S) = 1$.
3. *Additivity (infinite).* If $A_1, A_2, \dots$ is a sequence of mutually disjoint events, then

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

Definition 1.8: Probability Distribution

Let $\mathcal{S}$ denote the set of all events of a sample space S.

Example. $S = \{1,2,3\} \implies \mathcal{S} = \{\varnothing, \{1\}, \{2\}, \{3\}\}$.

A **probability** defined on $\mathcal{S}$ is a function

$$P : \mathcal{S} \to \mathbb{R},$$

that satisfies the axioms of probability. If S is discrete, then $S = \{A_1, A_2, \dots\}$. In this case, if probabilities $P(A_i), i = 1, 2, \dots$ may be assigned to each outcome in the sample space such that for all $i = 1, 2, \dots$

1. $0 \leq P(A_i) \leq 1$
2. $\sum_{i=1}^{\infty} P(A_i) = 1$,

then the set of probabilities $\{P(A_1), P(A_2), \dots\}$ is called a **probability distribution** on S, and satisfies the probability axioms so long as additivity holds.

Proposition 1.9: Properties of the Probability Function

- $P(\varnothing) = 0$
- *Additivity (finite).* If $A_1, A_2, \dots, A_n$ is a sequence of mutually disjoint events, then

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i)$$

- $P(\bar{A}) = 1 - P(A)$

Definition 1.10: Random Experiment

A random experiment is a trial under controlled conditions of some phenomena

- Consistent possible outcomes
- Repeatable

Definition 1.11: Probability Model

A probability model contains a sample space, set of events, and a way of assigning probabilities. It models a random experiment.

2 Counting Techniques

Definition 2.1: Odds

The odds of an event A with respect to another event B is given by

$$\frac{P(A)}{P(B)}.$$

The **odds** of A is given by

$$\frac{P(A)}{1-P(A)}$$

and the **odds against** A is given by

$$\frac{1-P(A)}{P(A)}.$$

Definition 2.2: Equally likely sample space

The probability of every individual outcome in a finite sample space S is the same.

Definition 2.3: Cardinality

The size of a set. Commonly, the number of outcomes in an event A or sample space S.

Notation. $|A|$ denotes the cardinality of A.

Example. In the case of an equally likely sample space,

$$1 = P(S) = \sum_{i=1}^{|S|} P(A_i) = |S|P(A_i).$$

So

$$P(A_i) = \frac{1}{|S|}$$

and therefore

$$P(A) = \frac{|A|}{|S|}$$

Theorem 2.4: The Addition Rule (OR)

If E and F are disjoint events, then

$$|E \cup F| = |E| + |F|.$$

Theorem 2.5: The Multiplication Rule (AND)

An ordered k-tuple is an ordered set of values $(a_1, \dots a_k)$. If the outcomes in an event A can be written as an ordered k-tuple where there are n_i choices for a_i for $i = 1, \dots, k$, then

$$|A| = n_1 \cdots n_k = \prod_{i=1}^{k} n_i$$

Definition 2.6: Factorial

The number of arrangements of length n of n symbols using each once and only once:

$$n! = n(n-1)\cdots(2)(1)$$

Definition 2.7: Permutation

Given n distinct objects, a **permutation** of size k is an *ordered* subset of k objects. The number of permutations of size k taken from n objects is

$$n^{(k)} = n(n-1)\cdots(n-k+1) = \frac{n!}{(n-k)!}$$

Definition 2.8: Combination

Given n distinct objects, a **combination** of size k is an *unordered* subset of k objects. The number of combinations of size k taken from n objects is given by

$$\binom{n}{k} = \frac{n^{(k)}}{k!} = \frac{n!}{k!(n-k)!}$$

Remark. There are $k!$ ways to rearrange k objects, so we divide the number of permutations of length k by the number of rearrangements to count each subset once and only once.

Remark. We assume $0! = 1$ and hence $\binom{n}{0} = 1$.

Lemma 2.9: Non-unique permutations

When counting permutations of non-unique symbols, count the total permutations and divide by the number of permutations double-counted.

Example. The unique permutations of MISSISSIPPI are $\frac{11!}{1!4!4!2!}$

Theorem 2.10: Geometric Series

For $t \neq 1$,

$$\sum_{i=0}^{n-1} t^i = 1 + t + \cdots + t^{n-1} = \frac{1-t^n}{1-t}.$$

If $|t| < 1$, then

$$\sum_{n=0}^{\infty} t^n = \frac{1}{1-t} \implies \sum_{n=1}^{\infty} n t^{n-1} = \frac{1}{(1-t)^2}.$$

Theorem 2.11: Binomial Theorem

If $\alpha \in \mathbb{R}$ and $t \in \mathbb{R}$, then for $|t| < 1$,

$$(1+t)^\alpha = \sum_{i=0}^{\infty} \binom{\alpha}{i} t^i$$

Theorem 2.12: Multinomial Theorem

Theorem 2.13: Hypergeometric Identity

Let $a, b, x \in \mathbb{R}$. If $|x| < 1$, then

$$\sum_{x=0}^{\infty} \binom{a}{x}\binom{b}{n-x} = \binom{a+b}{n}$$

Theorem 2.14: Exponential Series

For all $t \in \mathbb{R}$,

$$e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!} = \lim_{n\to\infty} \left(1 + \frac{t}{n}\right)^n$$

Proposition 2.15: Special sums

For $n \in \mathbb{N}$,

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$
$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$
$$\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$