1 Introduction

Definition 1.1: Syntax

Concerned with form.
Example. $p \vee \neg q$ is well formed. $p\neg \vee q$ is not.

Definition 1.2: Semantics

Concerned with meaning.
Example. The meaning of $p \vee \neg q$ can be defined via truth table.

Definition 1.3: Proposition

A statement that is either true or false.
Example. $4 + 5 = 8$
Statements can be **atomic** or **compound**:

- Compound statements contain connectives: $\neg, \wedge, \vee, \longrightarrow, \longleftrightarrow$

Definition 1.4: Aristotelian Logic

A **syllogism** is a logical argument in which the conclusion is inferred from a set of premises.

Example. $\dfrac{\begin{array}{l}\text{All humans are mortal}\\ \text{Socrates is human}\end{array}}{\text{Socrates is mortal}}$ is a valid argument.

Correctness depends on form/structure, not content.

Definition 1.5: Propositional Logic

Notation. $\mathcal{L}^p$: language of propositional logic that consists of three types of symbols:

- Propositional variables/symbols (atomic formulas)

  Notation. p, q, r, p', p_1

- Connectives

  Notation. $\neg, \wedge, \vee, \longrightarrow, \longleftrightarrow$ listed from highest precedence to lowest

  Example. $\neg p \vee q$ fully expanded becomes $((\neg p) \vee q)$

  $p \longrightarrow q \longrightarrow r$ is ambiguous. Write $p \longrightarrow (q \longrightarrow r)$ or $(p \longrightarrow q) \longrightarrow r$

- Parentheses (for grouping)

  Notation. ()

Definition 1.6: Expression

An expression is a finite string of symbols in $\mathcal{L}^p$.
Example. $a \longrightarrow (b \wedge c)$, $a \longrightarrow b \wedge c$, $\neg$
The second example is shorthand for the first, and the third is ill-formed.

Definition 1.7: Well-formed expressions/Formulas

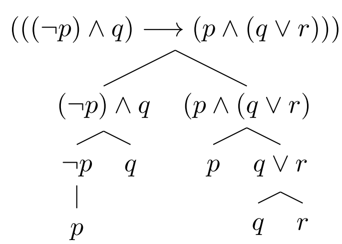
We define $\text{Form}(\mathcal{L}^p)$ as the set of formulas in L^p as follows:

1. **Base case:** A propositional variable p is in $\text{Form}(\mathcal{L}^p)$
2. If $A \in \text{Form}(\mathcal{L}^p)$, then $(\neg A) \in \text{Form}(\mathcal{L}^p)$
3. If $A, B \in \text{Form}(\mathcal{L}^p)$, then:
   - $A \wedge B \in \text{Form}(\mathcal{L}^p)$
   - $A \vee B \in \text{Form}(\mathcal{L}^p)$
   - $A \longrightarrow B \in \text{Form}(\mathcal{L}^p)$
   - $A \longleftrightarrow B \in \text{Form}(\mathcal{L}^p)$

Remark. $\text{Form}(\mathcal{L}^p)$ is a recursively/inductively defined set.

Definition 1.8: Abstract Syntax Tree

There is a bijection between well-formed expressions and abstract syntax trees.
Example. Take the formula $(((\neg p) \wedge q) \longrightarrow (p \wedge (q \vee r)))$:



Proposition 1.9: Properties of Formulas

- Every $A \in \text{Form}(\mathcal{L}^p)$ has balanced parentheses (i.e. equal # of left and right)
- Every $A \in \text{Form}(\mathcal{L}^p)$ has at least one propositional variable
- Any proper prefix of a formula has more left parentheses than right

Theorem 1.10: Translation

The following are equivalent:

- $p \longrightarrow q$
- $\neg q \longrightarrow \neg p$
- p only if q
- q if p
- if p then q
- p implies q
- q is a necessary condition for p
- p is a sufficient condition for q

Proposition 1.11: Rules of Hoare Logic

1. *Assignment:* $\dfrac{}{\{P(e)\}\, x := e\, \{P(x)\}}$

   Example. $\{i + 1 = 43\}\, x := i + 1\, \{x = 43\}$

2. *Consequence:* $\dfrac{\begin{array}{c} P_1 \longrightarrow P_2 \\ \{P_2\}\, S\, \{Q\} \end{array}}{\{P_1\}\, S\, \{Q\}}$ and $\dfrac{\begin{array}{c} Q_1 \longrightarrow Q_2 \\ \{P\}\, S\, \{Q_1\} \end{array}}{\{P\}\, S\, \{Q_2\}}$

   Example. $\dfrac{\begin{array}{c} x > 0 \implies x > 1 \\ \{x > 1\}\, S\, \{Q\} \end{array}}{\{x > 0\}\, S\, \{Q\}}$ and $\dfrac{\begin{array}{c} x > 0 \implies x > 1 \\ \{P\}\, S\, \{x > 0\} \end{array}}{\{P\}\, S\, \{x > 1\}}$

3. *Composition:* $\dfrac{\begin{array}{c} \{P\}\, S\, \{Q\} \\ \{Q\}\, T\, \{R\} \end{array}}{\{P\}\, S; T\, \{R\}}$

   Example. $\dfrac{\begin{array}{c} \{x + 1 = 43\}\, y := x + 1\, \{y = 43\} \\ \{y = 43\}\, z := y\, \{z = 43\} \end{array}}{\{x + 1 = 43\}\, y := x + 1; z := y\, \{z = 43\}}$

4. *Conditional:* $\dfrac{\begin{array}{c} \{P \wedge B\}\, S\, \{Q\} \\ \{P \wedge \neg B\}\, T\, \{Q\} \end{array}}{\{P\}\ \text{if}\ (B)\ S\ \text{else}\ T\, \{Q\}}$

   Example. $\dfrac{\begin{array}{c} \{T \wedge (i < j)\}\, m := i\, \{(m \leq i) \wedge (m \leq j)\} \\ \{T \wedge \neg(i < j)\}\, m := j\, \{(m \leq i) \wedge (m \leq j)\} \end{array}}{\{T\}\ \text{if}\ (i < j)\{m := i\}\,\text{else}\ \{m := j\}\ \{(m \leq i) \wedge (m \leq j)\}}$

5. *While:* (partial correctness) $\dfrac{\{P \wedge B\}\, S\, \{P\}}{\{P\}\ \text{while}\ (B)\ \text{do}\ S\, \{P \wedge \neg B\}}$

   Example. $\dfrac{\{x \leq 10 \wedge x < 10\}\, x := x + 1\, \{x \leq 10\}}{\{x \leq 10\}\ \text{while}\ (x < 10)\,\text{do}\ x := x + 1\, \{x \leq 10 \wedge \neg(x < 10)\}}$