Question 1.

(a)

$$S = \left\{ \begin{array}{ccc} (P_1, P_1) & \cdots & (P_1, P_4) \\ \vdots & \ddots & \vdots \\ (P_4, P_1) & \cdots & (P_4, P_4) \end{array} \right\}$$

(b) $P(\text{Repeat prof}) = \frac{4}{16} = \frac{1}{4}$.

Question 2.

(a) $S = \{TTT, TTH, THT, THH, HTT, HTH, HHT, HHH\}$

(b) $\frac{3}{8}$

(c) $\frac{1}{4}$

Question 3.

(a)

$$S = \left\{ \begin{array}{ccc} (1,2) & \cdots & (1,5) \\ \vdots & \ddots & \vdots \\ (5,1) & \cdots & (5,4) \end{array} \right\}$$

(b) $\frac{6}{20}$

(c) $\frac{8}{20}$

Question 4.

Let the envelopes be ordered, labelled as W, X, Y, Z such that the arrangement $WXYZ$ represents the letter addressed to W placed in envelope W, and so on. Then

$$S = \left\{ \begin{array}{c} WXYZ, WXZY, WYXZ, WYZX, WZXY, WZYX, \\ XWYZ, XWZY, XYWZ, XYZW, XZWY, XZYW, \\ YWXZ, YWZX, YXWZ, YXZW, YZWX, YZXW, \\ ZWXY, ZWYX, ZXWY, ZXYW, ZYWX, ZYXW \end{array} \right\}$$

(b)

$$\begin{aligned} A &= \{WXYZ, WXZY, WYXZ, WYZX, WZXY, WZYX\} \\ B &= \{XWZY, XYZW, XZWY, YWZX, YZWX, YZXW, ZWXY, ZYWX, ZYXW\} \\ C &= \{WXZY, WYXZ, WZYX, XWYZ, YXWZ, ZXYW\} \\ D &= \varnothing \end{aligned}$$

(c)

$$\begin{aligned} A &= \frac{6}{24} \\ B &= \frac{9}{24} \\ C &= \frac{6}{24} \\ D &= 0 \end{aligned}$$

Question 5.

(a) Let xyz denote ball 1 in box x, ball 2 in box y, and ball 3 in box z. For example, 123 denotes ball 1 in box 1, ball 2 in box 2, and ball 3 in box 3.

$$S = \left\{ \begin{array}{c} 111, 112, 113, 121, 122, 123, 131, 132, 133, \\ 211, 212, 213, 221, 222, 223, 231, 232, 233, \\ 311, 312, 313, 321, 322, 323, 331, 332, 333 \end{array} \right\}$$

(b)

$$A = \frac{8}{27}$$
$$B = \frac{1}{27}$$
$$C = \frac{6}{27}$$

(c) For A, we simply need to avoid box 1 when placing the three balls. There are $n-1$ other boxes, so the probability that box 1 is empty is

$$A = \left(\frac{n-1}{n}\right)^3.$$

For B, we avoid the first two boxes.

$$B = \left(\frac{n-2}{n}\right)^3.$$

For C, we want each ball to be in a different box. The first ball can go anywhere, then there are $n-1$ and $n-2$ places the next two balls can go, respectively.

$$C = \frac{(n-1)(n-2)}{n^2}$$

(d) Following similar strategies,

$$A = \left(\frac{n-1}{n}\right)^k$$
$$B = \left(\frac{n-2}{n}\right)^k$$
$$C = \frac{(n-1)(n-2)\ldots(n-k+1)}{n^{k-1}}$$

Question 6.

(a) $\frac{18}{1000}$

(b) $\frac{20}{1000}$

(c) $\frac{18}{78}$

Question 7.

(a)

$$S = \left\{ \begin{array}{ccc} (0,0) & \cdots & (0,9) \\ \vdots & \ddots & \vdots \\ (9,0) & \cdots & (9,9) \end{array} \right\}$$

(b)

	0	1	2	3	4	5	6	7	8	9
0	0.096						0.004			
1		0.100								
2			0.100							
3				0.100						
4					0.098			0.002		
5						0.095				0.005
6	0.004						0.096			
7					0.002			0.098		
8									0.100	
9						0.005				0.095

Fraction correctly identified is given by sum along the trace, 0.978.

Question 8.

4 and 5 work with classical probability and are fine. 6 is based off relative frequency and could be inaccurate, especially given the small sample size. For 7 it is unclear how the probabilities were arrived at, but was most likely done through relative frequency analysis. These are all appropriate choices since for 6 and 7 we require data from experimentation. For 4 and 5 we can work out probabilities strictly theoretically.

Question 9.

Let (x, y) represent the situation where x is the professor's number and y is Allen's guess.

$$S = \left\{ \begin{array}{ccc} (1,1) & \cdots & (1,9) \\ \vdots & \ddots & \vdots \\ (9,1) & \cdots & (9,9) \end{array} \right\}$$

Note that Beth can always copy Allen, and so her lowest possible mark is 85. We want to have an expected value above 85 to deviate from this strategy. We work case-wise. Suppose Allen's guess is 1. Then Beth can guess 2, and will be closer $\frac{8}{9}$ times. Her expected score, then, is $90 \cdot \frac{8}{9} + 80 \cdot \frac{1}{9} = 88.89$. Similarly, if Allen's guess is 2, Beth can guess 3 and will be closer $\frac{7}{9}$ times. Her expected score is $90 \cdot \frac{7}{9} + 80 \cdot \frac{2}{9} = 87.78$. So long as Beth is closer for at least $\frac{5}{9}$ possibilities, she should guess on the other side of what Allen guessed.

Continuing, we get

Allen's guess	Optimal guess	Proportion closer	Expected score
1	2	$\frac{8}{9}$	88.89
2	3	$\frac{7}{9}$	87.78
3	4	$\frac{6}{9}$	86.67
4	5	$\frac{5}{9}$	85.56
5	5	N/A	85
6	5	$\frac{5}{9}$	85.56
7	6	$\frac{6}{9}$	86.67
8	7	$\frac{7}{9}$	87.78
9	8	$\frac{8}{9}$	88.89