Contents

1 Intro & Rings

1.1 Motivation

Definition (Radical): An expression involving only $+, -, *, /, \sqrt[n]{\cdot}$.

After a linear transformation, all cubics can be reduced to $x^3 + px = q$, and there is a formula for solutions to the above. Quartics can also be reduced to a cubic and solved.

The quintic was attempted by Euler, Bezout, Lagrange, etc without success. In 1799, Ruffini gave a 516-page proof on the insolubility of the quintic that was almost right. In 1824, Abel filled in the gap in Ruffini's proof.

The main steps of Galois theory are to:

1. Link a root α of a quintic to $\mathbb{Q}(\alpha)$, the smallest field containing α. It has more structure to be played with. Currently, our knowledge of $\mathbb{Q}(\alpha)$ is lacking. For instance, we don't know how many intermediate fields there are between $\mathbb{Q}\left(\sqrt{2}, \sqrt{3}\right)$ and $\mathbb{Q}$.

$$\begin{tikzcd}[column sep=small] & \mathbb{Q}\left(\sqrt{2},\sqrt{3}\right) \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\ \mathbb{Q}\left(\sqrt{2}\right) \arrow[dr, dash] & \mathbb{Q}\left(\sqrt{3}\right) \arrow[d, dash] & \mathbb{Q}\left(\sqrt{2}+\sqrt{3}\right) \cdots \arrow[dl, dash] \\ & \mathbb{Q} & \end{tikzcd}$$

   We can list infinitely many of these intermediate fields, but how many are actually distinct?

2. To ameliorate the situation, we link the field $\mathbb{Q}(\alpha)$ to a group. Precisely, we associate the field extension $\mathbb{Q}(\alpha)/\mathbb{Q}$ to the group

$$\mathrm{Aut}_{\mathbb{Q}}(\mathbb{Q}(\alpha)) = \left\{\varphi : \mathbb{Q}(\alpha) \to \mathbb{Q}(\alpha) : \varphi|_{\mathbb{Q}} = \mathrm{id}_{\mathbb{Q}}\right\}$$

   i.e. the set of automorphisms that fix the smaller field. It can be shown that if α is "good" then $\mathrm{Aut}_{\mathbb{Q}}(\mathbb{Q}(\alpha))$ is finite. Moreover, there is a bijection between the intermediate fields of $\mathbb{Q}(\alpha)/\mathbb{Q}$ and the subgroups of $\mathrm{Aut}_{\mathbb{Q}}(\mathbb{Q}(\alpha))$. Galois theory is the interplay between fields and groups.

1.2 Review of ring theory

Rings in this course are by and large commutative and unital.

Definition (Integral Domain, Field): A ring R where for all $a, b \in R$ m $ab = 0 \Longrightarrow a = 0$ or $b = 0$ is an **integral domain**. A **field** is a ring R such that $R^* = R \setminus \{0\}$.

Proposition 1.1 (Subrings of fields): Every subring of a field F, including F itself, is an integral domain.

Definition (Ideal): A subset I of a commutative ring such that $0 \in I$, and for $a, b \in I$ and any $r \in R$, $a - b \in I$ and $ra \in I$.

Remark: If $1 \in I$ is an ideal, then $I = R$, since any $r \in R$ satisfies $r1 = r \in I$, so $R \subseteq I$.

The only ideals of a field F are $\{0\}$ and F, since if $a \in I$ with $a \neq 0$, then $aa^{-1} = 1 \in I$, so $I = F$.

Recall that using the division algorithm in $\mathbb{Z}$, we can prove all ideals of $\mathbb{Z}$ are principal ideals.

Remark: The smallest field containing $\mathbb{Z}$ is $\mathbb{Q}$.

Definition ($F[x]$): Define $F[x] = \{a_0 + ... + a_m x^m : a_i \in F\}$.
- If $a_m = 1$, we say f is **monic**.
- If $a_m \neq 0$, the **degree** of f is $\deg(f) = m$. By convention, $\deg(0) = -\infty$.

- For $f, g \in F[x]$, $\deg(fg) = \deg(f) + \deg(g)$.

Notes about $F[x]$:
- $F[x]$ is an integral domain.
- The units of $F[x]$ are $F^* = F \setminus \{0\}$, i.e. the unital constant polynomials.
- The division algorithm works. For f, g with $f \neq 0$, we can write $g(x) = q(x)f(x) + r(x)$ with $\deg(r) < \deg(f)$ (here the $-\infty$ convention is handy).
- Using the DA, we can prove all ideals of $F[x]$ are principal. Moreover, if we impose that generators $f(x)$ are monic, then generators are unique.

Remark: The smallest field containing $F[x]$ is the set of rational functions

$$F(x) := \left\{ \frac{f(x)}{g(x)} : f, g \in F[x] \text{ and } g \neq 0 \right\}$$

Recall when I is an ideal of R, that the additive quotient group R/I is a ring with multiplication $(r + I)(s + I) = rs + I$, and the unit of R/I is $1 + I$.

Theorem 1.2 (First Isomorphism Theorem): Let $\varphi : R \to S$ be a ring homomorphism. Then $\operatorname{Ker}(\varphi)$ is an ideal of R and $R/\operatorname{Ker}(\varphi) \cong \operatorname{Im}(\varphi)$.

Example: Let F be a field, S a ring, and $\varphi : F \to S$ be a ring homomorphism. Then either φ is injective or the zero map, since $\operatorname{Ker}(\varphi)$ is an ideal of F, hence either $\{0\}$ or F.

Definition (Prime, maximal): Let R be a commutative ring. An ideal $P \neq R$ is a **prime** ideal if whenever $rs \in P$, then $r \in P$ or $s \in P$.

An ideal $M \neq R$ of R is **maximal** if whenever A is an ideal such that $M \subseteq A \subseteq R$, then $A = M$ or $A = R$.

Theorem 1.3: Let $I \neq R$ be an ideal of a commutative ring R. Then
(1) I is maximal iff R/I is a field.
(2) I is prime iff R/I is an integral domain.

Proof:
(1) Suppose I is maximal. Note $I \neq R \iff R/I$ is a commutative ring with 1. We show the non-zero elements in R/I have inverses. Let $a \in R$ with $a \notin I$, so $a + I \neq 0 + I$. Since $a \notin I$, we have $I \subsetneq \langle a \rangle + I = \langle I \cup \{a\} \rangle = R$ by maximality, so $\langle a \rangle + I$ contains 1. Notice

$$\langle a \rangle + I = \{ar + m : m \in I, r \in R\}$$

so say $1 = ar + m$ where $r \in R, m \in I$. Then we have our inverse:

$$(a + I)(r + I) = ar + I = (ar + m) + I = 1 + I$$

Conversely if R/I is a field, since $1 + I \neq 0 + I$ we have $1 \notin I$ so $I \neq R$. Let A be an ideal with $I \subseteq A \subseteq R$ and suppose $A \neq I$. Choose $a \in A - I$ so $a + I \neq 0 + I$. Then since R/I is a field, $a + I$ has an inverse, say $b + M$. Then $(a + I)(b + I) = ab + I = 1 + I$. Then $1 - ab \in I \subseteq A$. Since $a \in A$ we have $ab \in A$, so $1 \in A \implies A = R$. Thus I is maximal.

(2) Since $I \neq R$, R/I is a commutative ring with 1. For $a, b \in R$,

$$(a + P)(b + P) = ab + P.$$

and $a + P = 0 + P \iff a \in P$. So $(a + P)(b + P) = 0 + P \iff ab \in P$. The result is immediate.⬠

Corollary 1.4: Every maximal ideal is prime.

2 Domains

2.1 Irreducibles and primes

Definition (Divides): Let R be an integral domain and $a, b \in R$. We say a divides b, denoted $a \mid b$, if $ca = b$ for some $c \in R$.

Notice in $\mathbb{Z}$ that if $n \mid m$ and $m \mid n$, then $n = \pm m$ so $\langle n \rangle = \langle m \rangle$.

Proposition 2.1 (Divisibility characterization): Let R be an integral domain. For $a, b \in R$, TFAE:
(1) $a \mid b$ and $b \mid a$

(2) $a = ub$ for some unit $u \in R$

(3) $\langle a \rangle = \langle b \rangle$

Proof:

($1 \Longrightarrow 2$) Suppose there are $u, v \in R$ so $b = ua$ and $a = vb$. If $a = 0$, then $b = 0$ so $a = 1b$. Otherwise,
$$a = vb = v(ua) = (vu)a \Longrightarrow a(1 - vu) = 0.$$
Since R is an integral domain and $a \neq 0$, $1 - vu = 0 \Longleftrightarrow vu = 1$. Thus v is a unit.

($2 \Longrightarrow 3$) Say $a = ub$. Then $a \in \langle b \rangle$, so $\langle a \rangle \subseteq \langle b \rangle$. Since u is a unit and $b = u^{-1}a$, $\langle b \rangle \subseteq \langle a \rangle$.

($3 \Longrightarrow 1$) If $\langle a \rangle = \langle b \rangle$, then $a \in \langle a \rangle = \langle b \rangle$, so $a = tb$ for some $t \in R$, giving $b \mid a$. Similarly, $a \mid b$. ⬠

Definition (Associated): Let R be an integral domain. For $a, b \in R$, we say a is associated to b, denoted $a \sim b$, if $a \mid b$ and $b \mid a$.

Often this is most useful with $a = ub$ for a unit u. From the previous proposition, we can show $\sim$ is an equivalence relation on R.
- $a = 1a \Longrightarrow a \sim a$
- $a \sim b \Longrightarrow a = ub \Longrightarrow b = u^{-1}a \Longrightarrow b \sim a$
- $a \sim b$ and $b \sim c$ gives $a = ub$ and $b = vc$ so $a = uvc$ where uv is a unit with inverse $v^{-1}u^{-1}$, so $a \sim c$.

Exercise 2.1: Show that for $a, b \in R$ where R is an integral domain,
(a) if $a \sim a'$ and $b \sim b'$ then $ab \sim a'b'$.

(b) if $a \sim a'$ and $b \sim b'$ then $a \mid b \Longleftrightarrow a' \mid b'$.

▶

Example: Let $R = \mathbb{Z}\left[\sqrt{3}\right] = \left\{m + n\sqrt{3} : m, n \in \mathbb{Z}\right\}$. This is an integral domain, where $\left(2 + \sqrt{3}\right)\left(2 - \sqrt{3}\right) = 1$, so $2 + \sqrt{3}$ is a unit in R. Since $3 + 2\sqrt{3} = \left(2 + \sqrt{3}\right)\sqrt{3}$, we have $3 + 2\sqrt{3} \sim \sqrt{3}$.

Definition (Irreducible): Let R be an integral domain. We say $p \in R$ is **irreducible** if $p \neq 0$ and for all $b, c \in R$, if $p = bc$ then one of b, c is a unit.

Example: Let $R := \mathbb{Z}\left[\sqrt{-5}\right]$ and $p := 1 + \sqrt{-5}$. We claim p is irreducible. Suppose it is reducible, so $p = ab$ where a, b are not units.

Definition (Norm in $\mathbb{Z}\left[\sqrt{d}\right]$): Let R be the ring $\mathbb{Z}\left[\sqrt{d}\right]$ where $1 \neq d \in \mathbb{Z}$ is squarefree and non-zero. Define $N : R \to \mathbb{Z}$ by
$$N\left(a + b\sqrt{d}\right) = \left(a + b\sqrt{d}\right)\left(a - b\sqrt{d}\right) = a^2 - db^2.$$

Exercise 2.2: Prove that $N(xy) = N(x)N(y)$ for all $x, y \in R$. ▶

Now $N(p) = 6 = N(a)N(b)$ but $N(a), N(b) \neq 1$ as $N(x) = 1$ iff x is a unit. Therefore $N(a), N(b) \in \{2, 3\}$. Say $a =: u + v\sqrt{-5}$ so $N(a) = u^2 + 5v^2$. Then we see $N(a) = u^2 \bmod 5$, but the possible values of u^2 are

$0, 1, 4$ since $u \in \{0, \pm 1, \pm 2\} \mapsto \{0, 1, 4\}$ mod 5. Therefore $N(a)$ cannot be 2 or 3, a contradiction, so p is irreducible.

Proposition 2.2 (Characterizations of irreducibility): Let R be an integral domain and $p \in R, p \neq 0$ with p not a unit. TFAE:
(1) p is irreducible.

(2) if $d \mid p$, then $d \sim 1$ or $d \sim p$.

(3) if $p \sim ab$, then $p \sim a$ or $p \sim b$.

(4) If $p = ab$, then $p \sim a$ or $p \sim b$.

Proof:

($1 \Longrightarrow 2$) Suppose $p = ad$ so one of a, d is a unit. If a is a unit then $p \sim d$. If d is a unit, $d \sim 1$.

($2 \Longrightarrow 3$) Suppose $p \sim ab$, then $b \mid p$. Then $b \sim 1$ or $b \sim p$. If the latter, we're done, if $b \sim 1$, then $a \sim p$.

($3 \Longrightarrow 4$) Suppose $p = ab$, then $p = 1ab \sim ab$ and by assumption $p \sim a$ or $p \sim b$.

($4 \Longrightarrow 1$) Say $p = ab$. If $p \sim a$ then $a = up$ for a unit u. Since R is commutative, $p = ab = upb = pub$ so $1 = ub$ since R is an integral domain. Thus b is a unit. Similarly, $p \sim b \Longrightarrow a$ is a unit. ⬠

Definition (Prime): Let R be an integral domain and $p \in R$. We say p is **prime** if $p \neq 0$ is not a unit, and if $p \mid ab \in R$ then $p \mid a$ or $p \mid b$.

Remark: If $p \sim q$, then p is prime iff q is prime. Indeed, say p is prime and suppose $q \mid ab \in R$. Then $dq = ab$ for some $d \in R$. Say $p = uq$ for a unit $u \in R$, so $ab = du^{-1}p$ so $p \mid ab$, so $p \mid a$ or $p \mid b$. If $p \mid a$ then $cp = a = cuq$ so $q \mid a$. Similarly $p \mid b \Longrightarrow q \mid b$. The converse is identical.

By induction we can also show if p is prime and $p \mid a_1 \cdots a_n$ then $p \mid a_i$ for some i.

Proposition 2.3 (Primes are irreducible): Let R be an integral domain, $p \in R$. If p is prime, then p is irreducible.

Proof: Say $p = ab \in R$, and wlog $p \mid a$. Write $a = dp, d \in R$, so by commutativity $p = dpb = pdb$ so as $p \neq 0$, we have $db = 1$. Thus b is a unit, so p is irreducible. ⬠

Example: The converse is false. Consider $R = \mathbb{Z}\left[\sqrt{-5}\right]$, where we know $p = 1 + \sqrt{-5}$ is irreducible. Note

$$\left(1 + \sqrt{-5}\right)\left(1 - \sqrt{-5}\right) = 6 = 2 \cdot 3$$

so $p \mid 2 \cdot 3$ but neither of 2 or 3. Indeed, if $p \mid 2$ then $qp = 2$ for some q, then $N(2) = N(q)N(p) \Longleftrightarrow 4 = N(q)6$ but there are no integer solutions to this. The same argument works for 3.

Exercise 2.3: Construct a ring R and an element in R that is irreducible but not prime, different to the above example. ▶

Recall that for a prime $p \in \mathbb{Z}$, $\pm 1 \cdot \pm p$ are the only factorizations of p, so p is irreducible. Also, we can prove Euclid's lemma, showing p is prime. The same things hold for $F[x]$ when F is a field. We want to know the additional property of $\mathbb{Z}$ or $F[x]$ that gives us irreducible implying prime.

2.2 Ascending chains

Definition (ACCP): An integral domain R is said to satisfy the **ascending chain conditions on principal ideals** (ACCP) if for any chain

$$0 \subsetneq \langle a_1 \rangle \subseteq \langle a_2 \rangle \subseteq \dots.$$

of principal ideals in R, there is $n \in \mathbb{N}$ so

$$\langle a_n \rangle = \langle a_{n+1} \rangle = \dots.$$

That is, the chain stabilizes eventually.

Example: $\mathbb{Z}$ satisfies ACCP.

Given a chain, we see $a_2 \mid a_1$ and $a_3 \mid a_2$, and so on. Thus taking absolute values gives

$$|a_1| \geq |a_2| \geq \dots.$$

Since each $|a_n| \geq 0 \in \mathbb{Z}$, we get $|a_n| = |a_{n+1}| = \dots$ for some n, so $a_{n+1} = \pm a_i$ for all $i \geq n$. Thus the chain stabilizes, so $\mathbb{Z}$ satisfies ACCP.

Notice this proof uses the well-ordering principle on $\mathbb{N}$, and so does the proof of unique factorization over $\mathbb{Z}$ (MATH135).

Theorem 2.4 (Product of irreducibles): Let R be an integral domain satisfying ACCP. If $a \in R$ is not zero and not a unit, then a is a product of irreducibles.

Proof: Suppose bwoc a is not a product of irreducibles. Say $a = x_1 a_1$ where wlog a_1 is not a product of irreducibles, and a is not irreducible so $a \nsim x_1, a_1$. Inductively, construct $a_n = x_{n+1} a_{n+1}$ so $a_n \nsim a_{n+1}$ and a_{n+1} is not a product of irreducibles. Then

$$\langle a \rangle \subsetneq \langle a_1 \rangle \subsetneq \dots$$

which violates ACCP, where the ideal containments are proper since $a_n \nsim a_{n+1}$. ⬠

Theorem 2.5 ($R[x]$ ACCP): If R is an integral domain satisfying ACCP, so is $R[x]$.

Proof: Suppose bwoc there is a chain

$$\{0\} \subsetneq \langle f_1 \rangle \subseteq \langle f_2 \rangle \subsetneq \dots \in R[x].$$

Since $f_{i+1} \mid f_i$, let a_i be the leading coefficient of each f_i to get $a_{i+1} \mid a_i$ for all i. Thus

$$\{0\} \subsetneq \langle a_1 \rangle \subseteq \langle a_2 \rangle \subseteq \dots.$$

Since R has ACCP, there is $n \in N$ so $\langle a_n \rangle = \langle a_{n+1} \rangle = \dots$. For $m \geq n$, we have $f_m = g f_{m+1}$ for some $g(x) \in R[x]$, say g has leading coefficient b. Then $a_m = b a_{m+1}$, so b must be a unit and $\langle a_m \rangle = \langle a_{m+1} \rangle$. Now, if $g = b$ is a constant polynomial, then

$$\langle f_m \rangle = \langle f_{m+1} \rangle,$$

a contradiction, so $\deg(g) \geq 1$. Thus $\deg(f_m) > \deg(f_{m+1})$ for all $m \geq n$, but this is also a contradiction as $\deg(f_i) \geq 0$. ⬠

Example: Since $\mathbb{Z}$ satisfies ACCP, so does $\mathbb{Z}[x]$.

Example: Consider $R = \{n + xf : n \in \mathbb{Z}, f \in \mathbb{Q}[x]\}$, i.e. the set of polynomials in $\mathbb{Q}[x]$ with integer constant term. R is an integral domain, but consider

$$\langle x \rangle = \{x(n + xf)\}, \quad \left\langle \frac{1}{2}x \right\rangle = \left\{\frac{1}{2}x(n + xf)\right\}$$

and so on. This gives

$$\langle x \rangle \subsetneq \left\langle \frac{1}{2}x \right\rangle \subsetneq \left\langle \frac{1}{2^2}x \right\rangle \subsetneq \dots.$$

Thus R is an integral domain that does not satisfy ACCP.

2.3 Unique factorization domains

Definition (UFD): An integral domain R is called a UFD if it satisfies:
- If $a \neq 0 \in R$ is not a unit, then a is a product of irreducibles
- If $p_1 p_2 ... p_n \sim q_1 q_2 ... q_s$ where p_i, q_j are irreducible, then $r = s$ and after possible relabelling, $p_i \sim q_i$ for all $i = 1, ..., r$.

Example: $\mathbb{Z}$ and $F[x]$ are UFDs, and a field F is also a UFD.

Proposition 2.6 (Irreducible implies prime): Let R be a UFD and $p \in R$. If p is irreducible, then p is prime.

Proof: Let $p \in R$ be irreducible. If $p \mid ab \in R$, write $ab = pd$ for $d \in R$. Since R is a UFD, we can factor a, b, d into irreducible elements:

$$\begin{aligned} a &= q_1 ... q_k \\ b &= s_1 ... s_\ell \\ d &= r_1 ... r_m. \end{aligned}$$

We allow k, ℓ, m to be 0 in case a, b, d are units. Now since $pd = ab$,

$$p r_1 ... r_m = q_1 ... q_k s_1 ... s_\ell.$$

Since p is irreducible and R is a UFD, $m + 1 = k + \ell$ and $p \sim q_i$ or $p \sim s_j$ for some i or j. Thus $p \mid a$ or $p \mid b$. ⬠

Example: $\mathbb{Z}$ is a UFD, where we know a prime satisfies Euclid's lemma. A similar statement holds for $F[x]$.

Example: Consider $R = \mathbb{Z}\left[\sqrt{-5}\right]$ and $p = 1 + \sqrt{-5}$. We have seen that p is irreducible but not prime, so R is not a UFD. Claim: R satisfies ACCP. Say

$$\{0\} \subsetneq \langle a_1 \rangle \subsetneq \langle a_2 \rangle \subsetneq ....$$

Then $a_{i+1} \mid a_i$ for all i, and as the norm is non-negative, and multiplicative,

$$N(a_{i+1}) \leq N(a_i).$$

Therefore,

$$N(a_1) \geq N(a_2) \geq ...,$$

but each $N(a_n) \geq 0$, so we must have $N(a_n) = N(a_{n+1}) = ...$ for some $n \in \mathbb{N}$.

The takeaway here is UFD implies ACCP, but ACCP does not imply UFD. We would like to know exactly how much stronger a UFD is than an integral domain with ACCP.

Definition (GCD): Let R be an integral domain and $a, b \in R$. We say $d \in R$ is a **greatest common divisor** of a and b, denoted $\gcd(a, b)$ if:
- $d \mid a, b$.
- If $e \in R$ with $e \mid a, b$ then $e \mid d$.

Remark: One can show if R is a UFD and a, b are non-zero and $p_1, ..., p_k$ are non-associated primes dividing a, b, say

$$\begin{aligned} a &\sim p_1^{\alpha_1} ... p_k^{\alpha_k} \\ b &\sim p_1^{\beta_1} ... p_k^{\beta_k} \end{aligned}$$

Then $\gcd(a, b) = p_1^{\min\{\alpha_1, \beta_1\}} ... p_k^{\min\{\alpha_k, \beta_k\}}$.

Furthermore, if R is a UFD and $d, a_1, ..., a_m \in R$, we have

$$\gcd(da_1, ..., da_m) = d \gcd(a_1, ..., a_m).$$

Exercise 2.4: Prove the above remark.

Theorem 2.7 (UFD characterization): Let R be an integral domain. TFAE:

(1) R is a UFD

(2) R satisfies ACCP and $\gcd(a, b)$ exists for all $a, b \neq 0 \in R$

(3) R satisfies ACCP and every irreducible element is prime.

Proof:

($1 \Longrightarrow 2$) By the previous remark, $\gcd(a, b)$ exists for all $a, b \neq 0$. Also, suppose

$$\{0\} \subsetneq \langle a_1 \rangle \subsetneq \langle a_2 \rangle \subsetneq \dots.$$

Since $\langle a_1 \rangle \neq \{0\}$ and a_1 is not a unit, we can write $a \sim p_1^{k_1} \cdots p_r^{k_r}$ where p_i are non-associated primes and $k_i \in \mathbb{N}$. Since $a_i \mid a_1$ for all i we have

$$a_i \sim p_1^{d_{i,1}} \cdots p_r^{d_{i,r}}$$

for $0 \leq d_{i,j} \leq k_j$ $(1 \leq j \leq r)$. Thus there are only finitely many non-associated choices for a_i, and so there exist $m \neq n$ with $a_m \sim a_n \Longrightarrow \langle a_m \rangle = \langle a_n \rangle$, a contradiction. Hence R satisfies ACCP.

($2 \Longrightarrow 3$) Let r be irreducible and suppose $p \mid ab \in R$. Then let $d = \gcd(a, p)$. Since $d \mid p$ which is irreducible, $d \sim 1$ or $d \sim p$. If $d \sim p$ then $d \mid a \Longrightarrow p \mid a$. Otherwise, $d \sim 1$ so $1 \sim \gcd(a, p) \Longrightarrow b \sim \gcd(ab, pb)$, where $p \mid ab$ and $p \mid pb$, so $p \mid b$.

($3 \Longrightarrow 1$) R satisfies ACCP, so for $a \neq 0 \in R$ not a unit, a is a product of irreducibles, so it suffices to prove such factorizations are unique. Suppose we have

$$p_1 \cdots p_r \sim q_1 \dots q_s$$

where each p_i, q_j is irreducible. Since p_1 is prime by assumption, we have $p_1 \mid q_j$ for some j, say wlog $p_1 \mid q_1$. Thus $p_1 \sim q_1$. Since $p_1 \sim q_1$ we can divide out and repeat inductively to get $p_1 \cdots p_r \sim q_1 \dots q_s$ has $r = s$ and $p_i \sim q_i$ $(1 \leq i \leq r)$. Thus the factorization is unique. ⬠

2.4 Principal ideal domains

Definition (PID): An integral domain R is a **principal ideal domain** (PID) if every ideal in R is principal (singly-generated).

Example: $\mathbb{Z}$ and $F[x]$ are PIDs, as are fields. Note that although all ideals in $\mathbb{Z}_n$ are principal, $\mathbb{Z}_n$ is not an integral domain, so is not a PID.

Proposition 2.8: Let R be a PID and $a_1, \dots, a_n \neq 0$. Then $d \sim \gcd(a_1, \dots, a_n)$ exists, and there exist $r_1, \dots, r_n \in R$ so that

$$\gcd(a_1, \dots, a_n) \sim r_1 a_1 + \dots + r_n a_n.$$

Proof: Let $A = \langle a_1, \dots, a_n \rangle = \{r_1 a_1 + \dots + r_n a_n : r_i \in R\}$ so A is an ideal, hence principal i.e. there is $d \in R$ so $A = \langle d \rangle$. In particular,

$$d = r_1 a_1 + \dots + r_n a_n$$

for some $r_i \in R$ as $d \in A$. We claim $d \sim \gcd(a_1, \dots, a_n)$. For each $i \in [n]$, $a_i \in \langle d \rangle$ so $a_i = qd$ for some q, hence $d \mid a_i$. Also, if $r \mid a_i$ for all i, then $r \mid (r_1 a_1 + \dots + r_n a_n) \Longleftrightarrow r \mid d$, so $d \sim \gcd(a_1, \dots, a_n) \sim r_1 a_1 + \dots + r_n a_n$ by definition. ⬠

Theorem 2.9 (PIDs are UFDs): Every PID is a UFD.

Proof: If R is a PID, by Theorem 2.7 and Proposition 2.8 it suffices to show R satisfies ACCP. Suppose

$$\{0\} \subsetneq \langle a_1 \rangle \subseteq \langle a_2 \rangle \subseteq ....$$

Let $A = \bigcup_{i \in \mathbb{N}} \langle a_i \rangle$, which is an ideal, so $\langle a \rangle = A$ for some $a \in R$. Then as $a \in A$, there is $n \in \mathbb{N}$ so $a \in \langle a_n \rangle$. Thus $a \in \langle a_m \rangle$ for all $m \geq n$, so $\langle a \rangle \subseteq \langle a_m \rangle \subseteq \langle a \rangle \Longrightarrow \langle a \rangle = \langle a_m \rangle$, so the chain stabilizes. Thus R satisfies ACCP, so is a UFD. ⬠

Example: We claim $\mathbb{Z}[x]$ is not a PID. Consider

$$A := \{2n + xf(x) : n \in \mathbb{Z}, f(x) \in \mathbb{Z}[x]\}$$

i.e. those polynomials with even constant term. Suppose $A = \langle g(x) \rangle$ for some $g(x) \in \mathbb{Z}[x]$. Since $2 \in A$, $g(x) \mid 2$, so $g(x) \sim 1$ or $g(x) \sim 2$. In the former case, $1 \in A \Longrightarrow A = \mathbb{Z}[x]$ is a contradiction, and in the latter case, $A = \{2f(x) : f(x) \in \mathbb{Z}[x]\}$ which is also a contradiction, since e.g. $x \in A$. Therefore there exist ideals that are not principal.

Theorem 2.10: Let R be a PID. If $0 \neq p \in R$ is not a unit, TFAE:

(1) p is prime

(2) $R/\langle p \rangle$ is a field (iff $\langle p \rangle$ is a maximal ideal)

(3) $R/\langle p \rangle$ is an integral domain (iff $\langle p \rangle$ is a prime ideal)

Proof:

($1 \Longrightarrow 2$) Let p be prime and let $0 + \langle p \rangle \neq a + \langle p \rangle \in R/\langle p \rangle$ for some $a \in R$ such that $p \nmid a$. We wish to show $(a + \langle p \rangle)^{-1}$ exists. Consider the ideal

$$A = \langle a, p \rangle = \{ra + sp : r, s \in R\}.$$

Since R is a PID, $A = \langle d \rangle$ for some $d \in R$. Since $p \in A$ we have $d \mid p$, but as p is prime hence irreducible, $d \sim 1$ or $d \sim p$. Notice if $d \sim p$ then $\langle p \rangle = \langle d \rangle = A$ where $a \in A$, so then $p \mid a$, a contradiction.

Thus we have $d \sim 1$, so $A = \langle d \rangle = \langle 1 \rangle = R$. Hence $1 = ba + cp$ for some $b, c \in R$, giving

$$\begin{aligned}(a + \langle p \rangle)(b + \langle p \rangle) &= ab + \langle p \rangle \\ &= (1 - cp) + \langle p \rangle \\ &= 1 + \langle p \rangle.\end{aligned}$$

Therefore $(a + \langle p \rangle)^{-1}$ exists, so $R/\langle p \rangle$ is a field.

($2 \Longrightarrow 3$) Every field is an integral domain.

($3 \Longrightarrow 1$) Suppose $p \mid ab \in R$. Then

$$(a + \langle p \rangle)(b + \langle p \rangle) = ab + \langle p \rangle = 0 + \langle p \rangle$$

because $p \mid ab \Longrightarrow ab \in \langle p \rangle$. Since $R/\langle p \rangle$ is an integral domain, one of $a + \langle p \rangle, b + \langle p \rangle$ is $0 + \langle p \rangle$, so one of $a, b \in \langle p \rangle$ i.e. $p \mid a$ or $p \mid b$, so p is prime. ⬠

Remark: The proofs for (2) $\Longrightarrow$ (3) and (3) $\Longrightarrow$ (1) work for integral domains, only (1) $\Longrightarrow$ (2) leverages that R is a PID.

Note: We have the following relations between algebraic structures:

$$\begin{array}{ccccccccccccc} \text{Field} & \subsetneq & \text{PID} & \subseteq & \text{UFD} & \subsetneq & \text{ACCP} & \subsetneq & \text{ID} & \subsetneq & \text{Comm Ring} & \subsetneq & \text{Ring} \\ \mathbb{Q} & & \mathbb{Z} & & \mathbb{Z}[x] & & \mathbb{Z}\left[\sqrt{-5}\right] & & A & & \mathbb{Z}_n & & M_n(\mathbb{R}). \end{array}$$

where $A = \{n + xf : n \in \mathbb{Z}, f \in \mathbb{Q}[x]\}$.

We don't yet know if the PID $\subseteq$ UFD containment is proper, but we will show $\mathbb{Z}[x]$ is a UFD eventually.

Remark: Theorem 2.10 fails for UFDs. Consider $\langle x \rangle \in \mathbb{Z}[x]$, then $\mathbb{Z}[x]/\langle x \rangle \cong \mathbb{Z}$ which is an integral domain but not a field, i.e. $\langle x \rangle$ is a prime ideal but not a maximal ideal.

In a PID, non-zero proper ideals are prime iff they are maximal. In general, only maximal implies prime.

In a UFD, non-zero non-units are prime iff they are irreducible. In general, only prime implies irreducible.

2.5 Polynomials

Consider $2x + 4$, which is irreducible in $\mathbb{Q}[x]$, but factors as $2(x + 4)$ in $\mathbb{Z}[x]$ where 2 is not a unit, so it is reducible in $\mathbb{Z}[x]$. This motivates the following definition:

Definition (Content, primitive): If R is a UFD and $0 \neq f(x) \in R[x]$, a greatest common divisor of all coefficients of f is called a **content** of f, denoted $c(f)$. If $c(f) \sim 1$, we say f is a **primitive** polynomial.

Example: In $\mathbb{Z}[x]$, $c(6 + 10x^2 + 15x^3) \sim \gcd(6, 10, 15) \sim 1$ so this is primitive. However, $c(6 + 9x^2 + 15x^3) \sim \gcd(6, 9, 15) \sim 3$, so this is not primitive.

Lemma 2.11: Let R be a UFD and $0 \neq f(x) \in \mathbb{R}[x]$.
- $f(x)$ can be written as $f(x) = c(f)f_1(x)$ for some primitive $f_1(x) \in \mathbb{R}[x]$
- if $0 \neq b \in R$, then $c(bf) \sim bc(f)$.

Proof: Let $f(x) = a_m x^m + ... + a_0$. Let $c(f) \sim \gcd(a_m, ..., a_0)$ and write $a_i = c(f)b_i$ for all i, so

$$f(x) = c(f)f_1(x), \text{where } f_1(x) = b_m x^m + ... + b_0.$$

We show f_1 is primitive. Indeed,

$$c(f) \sim \gcd(a_m, ..., a_0) \sim \gcd(c(f)b_m, ..., c(f)b_0) \sim c(f) \gcd(b_m, ..., b_0).$$

Hence $1 \sim \gcd(b_m, ..., b_0) \Longleftrightarrow c(f_1) \sim 1$, so f_1 is primitive. Furthermore, the coefficients of bf for $b \neq 0$ are $ba_m, ..., ba_0$, so

$$c(bf) \sim \gcd(ba_m, ..., ba_0) \sim b \gcd(a_m, ..., a_0) \sim bc(f).$$

Thus $c(bf) \sim bc(f)$. ⬠

Lemma 2.12: Let R be a UFD and $\ell(x) \in R[x]$ be irreducible with $\deg(\ell) \geq 1$. Then $c(\ell) \sim 1$.

Proof: Write $\ell(x) = c(\ell)\ell_1(x)$ with ℓ_1 primitive and $\deg(\ell_1) = \deg(\ell) = 1$. Since ℓ is irreducible one of $c(\ell), \ell_1$ must be a unit but clearly ℓ_1 cannot be, so $c(\ell) \sim 1$. ⬠

Theorem 2.13 (Gauss' Lemma): Let R be a UFD. If $f, g \neq 0 \in R[x]$ then $c(fg) \sim c(f)c(g)$. In particular, the product of primitive polynomials is again primitive.

Proof: Let $f = c(f)f_1$ and $g = c(g)g_1$ with f_1, g_1 primitive. Then

$$c(fg) \sim c(c(f)f_1 c(g)g_1) \sim c(f)c(g)c(f_1 g_1).$$

It suffices then to prove a product of primitives is primitive. Suppose bwoc f, g are primitive but fg is not. Write

$$f(x) = a_0 + ... + a_m x^m$$
$$g(x) = b_0 + ... + b_n x^n.$$

Since R is a UFD, there is a prime p dividing each coefficient of fg. Since f, g are primitive, there is some k, s so $p \nmid a_k, b_s$. Let k and s be the minimum such values. Then
- $p \nmid a_k$ but $p \mid a_i$ for $i = 0, ..., k - 1$
- $p \nmid b_s$ but $p \mid b_j$ for $j = 0, ..., s - 1$

Now the coefficient c_{k+s} of x^{k+s} in fg is

$$\begin{aligned} c_{k+s} &= \sum_{i+j=k+s} a_i b_j \\ &= a_0 b_{k+s} + \dots + a_{k-1} b_{s+1} + a_k b_s + a_{k+1} b_{s-1} + \dots + a_{k+s} b_0. \end{aligned}$$

Now p divides every term on the left of $a_k b_s$ and every term on the right of it. However, it does not divide $a_k b_s$, hence cannot divide the sum, i.e. $p \nmid c_{k+s}$, a contradiction. Thus fg is primitive. ⬠

Theorem 2.14: Let R be a UFD whose field of fractions F is

$$F = \left\{ \frac{a}{b} : a, b \in R, b \neq 0 \right\}.$$

Regard R as a subring of F. If $\ell(x) \in R[x]$ is irreducible in $R[x]$, then $\ell(x)$ is irreducible in $F[x]$.

Proof: Let $\ell(x) \in R[x]$ be irreducible. Suppose $\ell(x) = g(x)h(x) \in F[x]$. If a, b are the products of the denominators of the coefficients of $g(x)$ and $h(x)$, then $g_1(x) = ag(x) \in R[x]$ and $h_1(x) = bh(x) \in R[x]$. Notice that $ab\ell(x) = g_1(x)h_1(x)$ is a factorization in $R[x]$. Since $\ell(x)$ is irreducible, $c(\ell) \sim 1$. Also, by Gauss' lemma, we have

$$ab \sim abc(\ell) \sim c(ab\ell) \sim c(g_1 h_1) \sim c(g_1)c(h_1). \quad (\star)$$

Now, write $g_1(x) = c(g_1)g_2(x)$ and $h_1(x) = c(h_1)h_2(x)$ where $g_2(x), h_2(x)$ are primitive in $R[x]$. Then

$$ab\ell(x) = g_1(x)h_1(x) = c(g_1)c(h_1)g_2(x)h_2(x).$$

By $(\star)$ we have $\ell(x) \sim g_2(x)h_2(x)$ in $R[x]$. Since $\ell(x)$ is irreducible, it follows that $h_2(x) \sim 1$ or $g_2(x) \sim 1$.

If $g_2(x) \sim 1$, then $ag(x) = g_1(x) = c(g_1)g_2(x)$. Thus $g(x) = a^{-1}c(g_1)g_2(x)$ with $g_2(x) \sim 1$ is a unit in $F[x]$. Similarly if $h_2(x) \sim 1$, we can show $h(x)$ is a unit in $F[x]$. Thus $\ell(x) = g(x)h(x)$ in $F[x]$ implies that either $g(x)$ or $h(x)$ is a unit in $F[x]$, so $\ell(x)$ is irreducible in $F[x]$. ⬠

Recall the converse is false: $2x + 4$ is irreducible in $\mathbb{Q}[x]$ but reducible in $\mathbb{Z}[x]$. What's notable about this example is the content of $2x + 4$ is not a unit. One might wonder if this is the only such restriction preventing an iff statement. Indeed it is.

Proposition 2.15: Let F be a UFD whose field of fractions is F. Let $f(x) \in R[x]$ with $\deg(x) \geq 1$. TFAE:

(1) $f(x)$ is irreducible in $R[x]$.

(2) $f(x)$ is primitive and irreducible in $F[x]$.

Proof:

($1 \implies 2$) Follows from Lemma 2.12 and Theorem 2.14.

($2 \implies 1$) Suppose $f(x)$ is primitive and irreducible in $F[x]$ but reducible in $R[x]$. Then a nontrivial factorization of $f(x)$ in $R[x]$ must be of the form $f(x) = dg(x)$ with $d \in R$ and $d \nsim 1$ (if both factors have degree ≥ 1, then it would be a nontrivial factorization in $F[x]$). Since $d \mid f(x)$, $d \nsim 1$ divides each coefficient of $f(x)$, contradicting the fact that $f(x)$ is primitive. Thus $f(x)$ is irreducible in $R[x]$. ⬠

Notice that primitive guarantees irreducibility in $R[x]$ iff $F[x]$. Only the $R[x] \implies F[x]$ direction holds for general polynomials.

Theorem 2.16: If R is a UFD, then so is $R[x]$.

Let R be a UFD and $x_1, \dots, x_n$ be n commutative variables and define the ring $R[x_1, \dots, x_n]$ of polynomials in n variables inductively by

$$R[x_1, \dots, x_n] = (R[x_1, \dots, x_{n-1}])[x_n].$$

Corollary 2.17: If R is a UFD, then for all $n \in \mathbb{Z}^+$, $R[x_1, ..., x_n]$ is a UFD.

Since $\mathbb{Z}$ is a UFD, $\mathbb{Z}[x]$ and $\mathbb{Z}[x_1, ..., x_n]$ are UFDs. With this, we can say that PID $\subsetneq$ UFD because $\mathbb{Z}[x]$ is a UFD but not a PID.

Theorem 2.18 (Eisenstein's criterion): Let R be a UFD with field of fractions F. Let $h(x) = c_n x^n + ... + c_1 x + c_0 \in R[x]$ with $n \geq 1$. Let $\ell \in R$ be irreducible. If:

- $\ell \nmid c_n$
- $\ell \mid c_i$ for all $i = 0, ..., n-1$
- $\ell^2 \nmid c_0$

Then h is irreducible in $F[x]$.

Proof: By contradiction. If $h(x)$ is reducible in $F[x]$, by Gauss' lemma there are $r(x), s(x) \in R[x]$ of degree at least 1 so $h(x) = s(x)r(x)$. Write

$$s(x) = a_0 + ... + a_m x^m$$
$$r(x) = b_0 + ... + b_k x^k.$$

where $1 \leq m, k < n$. Since $h(x) = s(x)r(x)$ we have

$$c_0 = a_0 b_0, ..., c_{k+s} = \sum_{i+j=k+s} a_i b_j.$$

Consider the constant term. Since $\ell \mid c_0$, we have $\ell \mid a_0 b_0$. Since ℓ is irreducible and R is a UFD, ℓ is prime, hence $\ell \mid a_0$ or $\ell \mid b_0$. Wlog, suppose $\ell \mid a_0$. Since $l^2 \nmid c_0$, we have $\ell \nmid b_0$.

If we consider the coefficient of x, since $\ell \mid c_1$ we have $\ell \mid (a_0 b_1 + a_1 b_0)$ where $\ell \mid a_0$ but $\ell \nmid b_0$, hence $\ell \mid a_1 b_0 \Longrightarrow \ell \mid a_1$.

By repeating the above argument, conditions on coefficients of $h(x)$ imply that $\ell \mid a_i$ for all $1 \leq \ell \leq m-1$. However, $\ell \nmid a_m$ since $\ell \nmid c_m$. Consider the reduction $\overline{h}(x) = \overline{s}(x)\overline{r}(x)) \in (R/\langle \ell \rangle)[x]$. By the assumption on the coefficients of h, we have $\overline{h}(x) = \overline{c_n} x^n$. However, since $\overline{s}(x) = \overline{a_m} x^m$ and $\ell \nmid b_0$, $\overline{s}(x)\overline{r}(x)$ contains the term $\overline{a_m b_0} x^m$, which is a contradiction. Thus $h(x)$ is irreducible in $F[x]$. ⬠

Example: Consider $2x^7 + 3x^4 + 6x^2 + 12$, where for $p = 3$ by Eisenstein's criterion this is irreducible in $\mathbb{Q}[x]$.

Example: Let p be prime and $\zeta_p = \exp\left(\frac{2\pi i}{p}\right)$ be a p^{th} root of unity. Now ζ_p is a root of the p^{th} cyclotomic polynomial

$$\begin{aligned}\Phi_p(x) &= \frac{x^p - 1}{x - 1} \\ &= x^{p-1} + x^{p-2} + ... + x + 1.\end{aligned}$$

Eisenstein's does not work directly here, but $\Phi_p(x+1)$ is irreducible iff $\Phi_p(x)$ is, so

$$\begin{aligned}\Phi_p(x+1) &= \frac{(x+1)^p - 1}{x} \\ &= x^{p-1} + \binom{p}{1} x^{p-2} + \binom{p}{2} x^{p-3} + ... + \binom{p}{p-2} x + \binom{p}{p-1} \in \mathbb{Z}[x].\end{aligned}$$

Then $p \mid \binom{p}{i}$ for $i = 1, ..., p-1$, but $p \nmid 1$ and $p^2 \nmid \binom{p}{p-1} = p$. Thus by Eisenstein's criterion $\Phi_p(x+1)$ is irreducible iff $\Phi_p(x)$ is irreducible in $\mathbb{Q}[x]$. Furthermore, observe $\Phi_p(x)$ is primitive, so by Proposition 2.15 it is irreducible in $\mathbb{Z}[x]$ as well.

3 Field Extensions

3.1 Basics

Definition (Field extension): If E is a field containing another field F, we say E is a **field extension** of F, denoted E/F.

Remark: E/F does *not* mean a quotient ring, as the only ideals are $\{0\}$ and E.

If E/F is a field extension, we can view E as a vector space over F with the obvious addition and scaling.

Definition (Degree): The dimension of E over F is called the **degree** of E over F, denoted $[E : F]$. If $[E : F] < \infty$ we say E/F is a finite extension, and otherwise it is an infinite extension.

Example: $[\mathbb{C} : \mathbb{R}] = 2$ is a finite extension.

Example: Let F be a field and let $F(x) = \left\{\frac{f(x)}{g(x)} : f(x), g(x) \in F[x], g(x) \neq 0\right\}$. Then $[F(x) : F]$ is an infinite extension since $\{1, x, x^2, ...\}$ is linearly independent over F.

Theorem 3.1 (Intermediate field extensions): If E/K and K/F are finite field extensions then E/F is a finite field extension with

$$[E : F] = [E : K][K : F].$$

In particular, if K is an intermediate field of a finite extension F, then $[K : F] \mid [E : F]$.

Proof: Suppose $[E : K] = m$ and $[K : F] = n$. Let $\{a_1, ..., a_m\}$ and $\{b_1, ..., b_n\}$ be bases for E/K and K/F respectively. It suffices to show $\{a_i b_j\}$ is a basis for E/F.

For $e \in E$ we have

$$e = \sum_{i=1}^{m} k_i a_i$$

for some $k_i \in K$, and for each k_i we have

$$k_i = \sum_{j=1}^{n} c_{i,j} b_j$$

with each $c_{i,j} \in F$. Hence

$$e = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{i,j} b_j a_i \in \operatorname{Span}_F\{a_i b_j\}.$$

Next, we have

$$\begin{aligned} &\sum_{i=1}^{m} \sum_{j=1}^{n} c_{i,j} a_i b_j = 0 \\ \Longrightarrow &\sum_{i=1}^{m} a_i \sum_{j=1}^{n} c_{i,j} b_j = 0. \end{aligned}$$

Since the a_i are LI in E/K with each sum term in K, by linearly independence of the a_i over K we have

$$\sum_{j=1}^{n} c_{i,j} b_j = 0$$

for each i. Then by the linear independence of the b_j over K/F, we have each $c_{i,j} = 0$, so the $\{a_i b_j\}$ are LI. ⬠

Definition (Algebraic, transcendental): Let E/F be a field extension and $\alpha \in E$. We say α is **algebraic over F** if there is $f(x) \in F[x] \setminus \{0\}$ such that $f(\alpha) = 0$. Otherwise, α is **transcendental over F**.

Example: $q \in \mathbb{Q}$ and $\sqrt{2}$ are algebraic over $\mathbb{Q}$, but e and π are transcendental over $\mathbb{Q}$.

Example: Claim: $\alpha = \sqrt{2} + \sqrt{3}$ is algebraic over $\mathbb{Q}$.

$$\begin{aligned}\left(\alpha - \sqrt{2}\right)^2 &= 3 \\ \alpha^2 - 1 &= 2\sqrt{2}\alpha \\ \alpha^4 - 10\alpha^2 + 1 &= 0\end{aligned}$$

So α is a root of $x^4 - 10x^2 + 1 \in \mathbb{Q}[x]$, so is algebraic over $\mathbb{Q}$.

Notation: Let E/F be a field extension and $\alpha \in E$. Then $F[\alpha]$ denotes the smallest subring of E containing F and α, and $F(\alpha)$ denotes the smallest subfield of E containing F and α. For $\alpha, \beta \in E$ we define $F[\alpha, \beta]$ and $F(\alpha, \beta)$ similarly.

3.2 Simple extensions

Definition (Simple extension): If $E = F(\alpha)$ for some $\alpha \in E$, we say E is a **simple** extension of F.

We would like to know what $[F(\alpha) : F]$ is.

Definition (F-homomorphism): Let R, R_1 be two rings containing a field F. A ring hom $\varphi : R \to R_1$ is called an F-homomorphism if $\varphi|_F = \mathrm{id}$.

Theorem 3.2: Let E/F be a field extension and $\alpha \in E$. If α is transcendental over F, then $F[\alpha] \cong F[x]$ and $F(\alpha) \cong F(x)$. In particular, $F[\alpha] \neq F(\alpha)$.

Proof: Define $\psi : F(x) \to F(\alpha)$ as the unique F-hom mapping $x \mapsto \alpha$. Then for $f(x), g(x) \in F[x]$ with $g(x) \neq 0$,

$$\psi\left(\frac{f(x)}{g(x)}\right) = \frac{f(\alpha)}{g(\alpha)}.$$

Notice this is well-defined as α is transcendental, so $g(\alpha) \neq 0$. Now $\operatorname{Ker}\psi$ is an ideal of $F(x)$, so ψ is injective as $x \notin \operatorname{Ker}(\psi)$. Also, since $F(x)$ is a field, so too is $\operatorname{Im}(\psi)$, which contains F and α, so $F(\alpha) \subseteq \operatorname{Im}(\psi)$. Thus $F(\alpha) = \operatorname{Im}(\psi)$ and by the first isomorphism theorem, $F(x)/\operatorname{Ker}(\psi) \cong F(x) \cong \operatorname{Im}(\psi) = F(\alpha)$. As $F[x]$ and $F[\alpha]$ are subrings of these fields, they too are isomorphic. ⬠

Theorem 3.3: Let E/F be a field extension with $\alpha \in E$. If α is algebraic over F, there is a unique monic irreducible $p(x) \in F[x]$, called the **minimal polynomial of α over F**, such that there is an F-isomorphism $\varphi : F[x]/\langle p(x) \rangle \to F[\alpha]$ with $\varphi(x) = \alpha$ from which we conclude $F[\alpha] = F(\alpha)$.

Remark: Since α is algebraic, the map in the proof of Theorem 3.2 is not well-defined.

Proof: Consider the unique F-homomorphism $\varphi : F[x] \to F[\alpha]$ sending $x \mapsto \alpha$. Since $F[x]$ is a ring, $\operatorname{Im}(\varphi)$ is a ring containing F and α, so $F[\alpha] \subseteq \operatorname{Im}(\varphi)$ gives $\operatorname{Im}(\varphi) = F[\alpha]$.

Let $I = \operatorname{Ker}(\varphi) = \{f(x) \in F[x] : f(\alpha) = 0\}$. Since α is algebraic, $I \neq \{0\}$, where I is an ideal of $F[x]$. Since $F[x]/I \cong \operatorname{Im}\varphi = F[\alpha]$ is an integral domain, I is a prime ideal. As $F[x]$ is a PID, there is a unique monic irreducible $p(x)$ so that $I = \langle p(x) \rangle$. Since I is a prime ideal and therefore a maximal ideal, $F[x]/\langle p(x) \rangle$ is a field by Theorem 2.10.

Then, $F[x]/\langle p(x) \rangle \cong F[\alpha]$ is a field containing F and α, so $F(\alpha) \subseteq F[\alpha]$. The reverse containment is obvious, so $F[\alpha] = F(\alpha)$. ⬠

Remark: If $p(x)$ is the minimal polynomial of α over F, we have $\langle p(x) \rangle = \{f(x) \in F[x] : f(\alpha) = 0\}$. In particular, if $f(x) \in F[x]$ satisfies $f(\alpha) = 0$, then $p(x) \mid f(x)$.

As a direct consequence of these theorems, we have the following result:

Theorem 3.4 (Degree of a simple extension): Let E/F be a field extension, $\alpha \in E$.

(1) α is transcendental over F iff $[F(\alpha) : F]$ is infinite.

(2) α is algebraic over F iff $[F(\alpha) : F]$ is finite. Moreover, if $p(x)$ is the minimal polynomial of α over F, $[F(\alpha) : F] = \deg(p)$ and $\left\{1, \alpha, \alpha^2, ..., \alpha^{\deg(p)-1}\right\}$ is a basis for $F(\alpha)/F$.

Proof: Notice (1) and (2) are equivalent contrapositives, so it suffices to just prove the forwards direction of each.

(1) ($\Longrightarrow$) By Theorem 3.2 we have $F(x) \cong F(\alpha)$. In $F(x)$, the elements $\left\{1, x, x^2, ...\right\}$ are linearly independent over F, so $[F(\alpha) : F] = \infty$.

(2) ($\Longrightarrow$) By Theorem 3.3, $F(\alpha) \cong F[x]/\langle p(x) \rangle$. Note that

$$F[x]/\langle p(x) \rangle = \{r(x) \in F[x] : \deg(r) < \deg(p)\}$$

so $\left\{1, x, x^2, ..., x^{\deg(p)-1}\right\}$ is a basis of $F[x]/\langle p(x) \rangle$. ⬠

Example: Let p be a prime and $\zeta_p = \exp\left(\frac{2\pi i}{p}\right)$ be a root of $\Phi_p(x) = \frac{x^p - 1}{x-1} = x^{p-1} + ... + x + 1$. We know $\Phi_p(x)$ is irreducible, so it is the minimal polynomial of ζ_p. Thus by Theorem 3.4, $\left[\mathbb{Q}\left(\zeta_p\right) : \mathbb{Q}\right] = p - 1$.

Example: $\alpha = \sqrt{2} + \sqrt{3}$ is algebraic, as a root of $x^4 - 10x^2 + 1$. We would like to show that this is the minimal polynomial of α over $\mathbb{Q}$ by showing $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 4$. Notice

$$\left(\alpha - \sqrt{2}\right)^2 = \sqrt{3}^2 \Longrightarrow \sqrt{2} = \frac{\alpha^2 - 1}{2\alpha},$$

so $\sqrt{2} \in \mathbb{Q}(\alpha)$. We have the following diagram:

$$\begin{array}{c} \mathbb{Q}(\alpha) \\ | \\ \mathbb{Q}\left(\sqrt{2}\right) \\ | \\ \mathbb{Q} \end{array}$$

Since $\sqrt{2}$ is a root of $x^2 - 2$, which is irreducible, we have $\left[\mathbb{Q}\left(\sqrt{2}\right) : \mathbb{Q}\right] = 2$. Also, $\sqrt{3} \notin \mathbb{Q}\left(\sqrt{2}\right)$, giving $\left[\mathbb{Q}(\alpha) : \mathbb{Q}\left(\sqrt{2}\right)\right] \geq 2$. Since $\alpha \in \mathbb{Q}(\alpha) \setminus \mathbb{Q}\left(\sqrt{2}\right)$, it follows that $[\mathbb{Q}(\alpha) : \mathbb{Q}] \geq 4$. However, α is a root of a degree 4 polynomial, so $[\mathbb{Q}(\alpha) : \mathbb{Q}] \leq 4$, so we have equality, and thus $x^4 - 10x^2 + 1$ is the minimal polynomial of α over $\mathbb{Q}$.

Exercise 3.1: Show that $\sqrt{3} \notin \mathbb{Q}\left(\sqrt{2}\right)$. ▶

Exercise 3.2: Can we show that $x^4 - 10x^2 + 1$ is irreducible in $\mathbb{Q}[x]$ by Eisenstein's? ▶

3.3 Finite extensions and the algebraic closure

It turns out that to understand finite field extensions, it suffices to understand simple ones.

Theorem 3.5: Let E/F be a field extension. If $[E : F] < \infty$, there exist $\alpha_1, ..., \alpha_n \in E$ such that

$$F \subsetneq F(\alpha_1) \subsetneq ... \subsetneq F(\alpha_1, ..., \alpha_n) = E.$$

Proof: By induction on $[E : F]$. If $[E : F] = 1$, then $E = F$ and we are done. Suppose $[E : F] > 1$ and the statement holds for all field extensions E_1/F_1 with $[E_1 : F_1] < [E : F]$. Let $\alpha_1 \in E \setminus F$ so by Theorem 3.1 we have

$$[E : F] = [E : F(\alpha_1)][F(\alpha_1) : F].$$

Since $[F(\alpha_1) : F] > 1$, we have $[E : F(\alpha_1)] < [E : F]$ so by the IH, there are $a_2, ..., a_n \in E$ such that

$$F(\alpha_1) \subsetneq F(a_1, \alpha_2) \subsetneq ... \subsetneq F(\alpha_1, \alpha_2, ..., \alpha_n) = E.$$

Therefore placing $F \subsetneq F(\alpha_1)$ at the start of this chain gives the desired result. ⬠

Definition (Algebraic field extension): A field extension E/F is **algebraic** if every $\alpha \in E$ is algebraic over F. Otherwise, it is **transcendental**.

Theorem 3.6: Let E/F be a field extension. If $[E : F] < \infty$, then E/F is algebraic.

Proof: Suppose $[E : F] = n$. For $\alpha \in E$, the elements $\{1, \alpha, \alpha^2, ..., \alpha^n\}$ are not linearly independent over F, so there exist $c_i \in F$ not all zero such that

$$\sum_{i=0}^{n} c_i \alpha^i = 0$$

i.e. that α is a root of $c_0 + ... + c_n x^n \in F[x]$, so α is algebraic over F. ⬠

Theorem 3.7 (Algebraic closure): Let E/F be a field extension. Define

$$L := \{\alpha \in E : [F(\alpha) : F] < \infty\}.$$

Then L, called the **algebraic closure of F in E**, is an intermediate field of E/F.

Proof: Certainly $F \subseteq L$, so if $\alpha, \beta \in L$ we need to show $\alpha \pm \beta, \alpha\beta$, and $\frac{\alpha}{\beta}$ for $\beta \neq 0$ are all in L. By definition, $[F(\alpha) : F], [F(\beta) : F] < \infty$.

Consider the field $F(\alpha, \beta)$. Notice the minimal polynomial of α over F, say $p(x) \in F[x]$, is also an element of $F(\beta)[x]$ with $p(\alpha) = 0$. Therefore the minimal polynomial of α over $F(\beta)$ divides the minimal polynomial of α over F, so the former has at most the degree of the latter. It follows by Theorem 3.1 that

$$[F(\alpha, \beta) : F(\beta)] \leq [F(\alpha) : F]$$
$$[F(\alpha, \beta) : F] = [F(\alpha, \beta) : F(\beta)][F(\beta) : F] \leq [F(\alpha) : F][F(\beta)F] < \infty.$$

Now since $\alpha \pm \beta \in F(\alpha, \beta)$, we have $[F(\alpha + \beta) : F] \leq [F(\alpha, \beta) : F] < \infty$, so $\alpha \pm \beta \in L$. Similarly, we can show $\alpha\beta, \frac{\alpha}{\beta} \in L$, so L is a field. ⬠

Definition (Algebraically closed): A field F is **algebraically closed** if for any algebraic extension E/F, we have $E = F$.

Example: By the fundamental theorem of algebra, $\mathbb{C}$ is algebraically closed. Moreover, $\mathbb{C}$ is the algebraic closure of $\mathbb{R}$ in $\mathbb{C}$.

Example: Let $\overline{\mathbb{Q}}$ be the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$, i.e.

$$\overline{\mathbb{Q}} = \{\alpha \in \mathbb{C} : \alpha \text{ is algebraic over } \mathbb{Q}\}.$$

For a prime p, since $\zeta_p \in \overline{\mathbb{Q}}$ as ζ_p is a root of its minimal polynomial $\Phi_p(x)$, we have

$$\left[\overline{\mathbb{Q}} : \mathbb{Q}\right] \geq [\mathbb{Q}(\zeta_p) : \mathbb{Q}] = p - 1.$$

As there are infinitely many primes, $\left[\overline{\mathbb{Q}} : \mathbb{Q}\right] = \infty$. In particular, this example shows that an algebraic extension need not be finite, i.e. the converse of Theorem 3.6 is false.

4 Splitting Fields

4.1 Existence

Definition (Splits over): Let E/F be a field extension. We say $f(x) \in F[x]$ **splits over E** if E contains all roots of $f(x)$, i.e. f can be written as a product of linear factors in $E[x]$.

Definition (Splitting field): Let $\tilde{E}/F$ be a field extension, $f(x) \in F[x]$, and $F \subseteq E \subseteq \tilde{E}$. If
- $f(x)$ splits over E and
- $f(x)$ does not split over any proper subfield of E

we say that E is a splitting field of $f(x)$ in $\tilde{E}$.

Theorem 4.1: Let $p(x) \in F[x]$ be irreducible. The quotient ring $F[x]/\langle p(x)\rangle$ is a field containing F and a root of $p(x)$.

Proof: Since $p(x)$ is irreducible, $I := \langle p(x)\rangle$ is a prime ideal. Since $F[x]$ is a PID, I is maximal iff $E := F[x]/I$ is a field. Consider the map

$$\begin{aligned} \varphi : F &\to E \\ a &\mapsto a + I. \end{aligned}$$

Since F is a field and $\varphi \neq 0$, φ is injective. Thus by identifying F with $\varphi(F)$, we view F as a subfield of E. We claim $\alpha := x + I$ is a root of $p(x)$. Write

$$\begin{aligned} p(x) &= a_0 + a_1 x + ... + a_n x^n \in F[x] \\ &= (a_0 + I) + (a_1 + I)x + ... + (a_n + I)x^n \in E[x]. \end{aligned}$$

We have

$$\begin{aligned} p(\alpha) &= a_0 + I + (a_1 + I)\alpha + ... + (a_n + I)\alpha^n \\ &= (a_0 + I) + (a_1 + I)(x + I) + ... + (a_n + I)(x + I)^n \\ &= (a_0 + a_1 x + ... + a_n x^n) + I \\ &= p(x) + I = 0 + I = I. \end{aligned}$$

Thus $\alpha = x + I \in E$ is a root of $p(x)$. ⬠

Theorem 4.2 (Kronecker's theorem): Let $f(x) \in F[x]$. There exists a field E containing F such that $f(x)$ splits over E.

Proof: By induction on $\deg(f)$ with any field. If $\deg(f) = 1$, we let $E = F$ and are done. If $\deg(f) > 1$, write $f(x) = p(x)h(x)$ with $p(x)$ irreducible in $F[x]$. By Theorem 4.1, there is a field K with $F \subseteq K$ containing a root of $p(x)$, say α. Thus

$$\begin{aligned} p(x) &= (x - \alpha)q(x) \\ \implies f(x) &= (x - \alpha)q(x)h(x) \end{aligned}$$

where $q(x) \in K[x]$. Since $\deg(qh) < \deg(f)$, by induction there is a field E containing K over which $q(x)h(x)$ splits. It follows that $f(x)$ splits over E. ⬠

Theorem 4.3 (Splitting fields are finite extensions): Every $f(x) \in F[x]$ has a splitting field which is a finite extension of F.

Proof: For $f(x) \in F[x]$, by Theorem 4.2 there is a field extension E/F over which $f(x)$ splits. Say $\alpha_1, ..., \alpha_n$ are the roots of $f(x)$ in E. Consider $L := F(\alpha_1, ..., \alpha_n)$, which is the smallest subfield of E containing all roots of $f(x)$, so $f(x)$ does not split over any proper subfield of L. Thus L/F is a splitting field of $f(x)$ in E. In addition, since the α_i are all algebraic in L, $[L : F]$ is finite. ⬠

Example: Consider $x^3 - 2 \in \mathbb{Q}[x]$. We know $x^3 - 2 = \left(x - \sqrt[3]{2}\right)\left(x - \sqrt[3]{2}\zeta_3\right)\left(x - \sqrt[3]{2}\zeta_3^2\right)$ where $\zeta_3 = \exp\left(\frac{2\pi i}{3}\right)$. Hence the splitting field of $x^3 - 2$ over $\mathbb{Q}$ is

$$\mathbb{Q}\left(\sqrt[3]{2}, \sqrt[3]{2}\zeta_3, \sqrt[3]{2}\zeta_3^2\right) = \mathbb{Q}\left(\sqrt[3]{2}, \sqrt[3]{2}\zeta_3\right).$$

4.2 Uniqueness

Question: If we have two field extensions E/F and E_1/F, what is the relation between the splitting field of $f(x)$ in E and in E_1?

Definition (Homomorphism extension): Let $\varphi : R \to R_1$ be a ring homomorphism, and $\Phi : R[x] \to R_1[x]$ be the unique ring homomorphism satisfying $\Phi|_R = \varphi$ and $\Phi(x) = x$. We say Φ **extends** φ.

More generally, if $R \subseteq S$ and $R_1 \subseteq S_1$ are all rings and $\Phi : S \to S_1$ is a ring homomorphism with $\Phi|_R = \varphi$, we say Φ extends φ.

Theorem 4.4: Let $\varphi : F \to F_1$ be a field isomorphism and $f(x) \in F[x]$. Let $\Phi : F[x] \to F_1[x]$ extend φ. Let $f_1(x) = \Phi(f(x))$ and $E/F, E_1/F_1$ be splitting fields of $f(x)$ and $f_1(x)$ respectively. Then there is an isomorphism $\psi : E \to E_1$ which extends φ.

Proof: By induction on $[E : F]$. If $[E : F] = 1$, then $f(x)$ is a product of linear factors in $F[x]$, and so is $f_1(x)$ in $F_1[x]$. Thus $E = F$, $E_1 = F_1$, so let $\psi = \varphi$ and we are done.

Suppose $[E : F] > 1$ and the statement holds for all $\tilde{E}/\tilde{F}$ with $\left[\tilde{E} : \tilde{F}\right] < [E : F]$. Let $p(x) \in F[x]$ be an irreducible factor of $f(x)$ with $\deg(p) \geq 2$. Such a p exists, as otherwise all the irreducible factors of f are degree 1, giving $[E : F] = 1$. Define $p_1(x) := \Phi(p(x))$.

Let $\alpha \in E$ and $\alpha_1 \in E_1$ be roots of $p(x)$ and $p_1(x)$ respectively. From Theorem 3.3, we have the F and F_1-isomorphisms

$$\begin{aligned} F(\alpha) &\cong F[x]/\langle p(x)\rangle, \quad \alpha \mapsto x + \langle p(x)\rangle \\ F_1(\alpha_1) &\cong F_1[x]/\langle p_1(x)\rangle, \quad \alpha_1 \mapsto x + \langle p_1(x)\rangle. \end{aligned}$$

Consider the isomorphism $\Phi : F[x] \to F_1[x]$ extending φ. Since $p_1(x) = \Phi(p(x))$, there is a field isomorphism $\tilde{\Phi}$ given by

$$\begin{aligned} \tilde{\Phi} : F[x]/\langle p(x)\rangle &\to F_1[x]/\langle p_1(x)\rangle \\ x + \langle p(x)\rangle &\mapsto x + \langle p_1(x)\rangle \end{aligned}$$

which extends φ. It follows from the commutative diagram below that there exists a field isomorphism $\tilde{\varphi} : F(\alpha) \to F_1(\alpha_1), \alpha \mapsto \alpha_1$ extending φ.

$$\begin{tikzcd}
E \arrow[rrr, "\psi"] \arrow[dd, dash] & & & E_1 \arrow[dd, dash] \\
& F[x]/\langle p(x)\rangle \arrow[r, "\tilde{\Phi}"] & F_1[x]/\langle p_1(x)\rangle \arrow[dr, "\sim"] & \\
F(\alpha) \arrow[ur, "\sim"] \arrow[rrr, bend right, "\tilde{\varphi}"] \arrow[d, dash] & & & F_1(\alpha_1) \arrow[d, dash] \\
F \arrow[rrr, "\varphi"] & & & F_1
\end{tikzcd}$$

Notice since $\deg(p) \geq 2$, we have $[E : F(\alpha)] < [E : F]$. Since E (resp. E_1) is the splitting field of $f(x) \in F(\alpha)[x]$ (resp. $f_1(x) \in F_1(\alpha_1)[x]$) over $F(\alpha)$ (resp. $F_1(\alpha_1)$), by induction there is an isomorphism $\psi : E \to E_1$ which extends $\tilde{\varphi}$. Therefore ψ extends φ. ⬠

Corollary 4.5 (Uniqueness of splitting fields): Any two splitting fields of $f(x) \in F[x]$ over F are isomorphic, and so we can say *the* splitting field of $f(x)$ over F.

Proof: Let $\varphi : F \to F$ be the identity map, and apply Theorem 4.4. ⬠

Theorem 4.6: Let F be a field, $f(x) \in F[x]$ with $\deg(f) = n \geq 1$. If E/F is the splitting field of $f(x)$, then $[E : F] \mid n!$.

Proof: By induction on $\deg(f)$. If $\deg(f) = 1$, choose $E = F$ and we have $[E : F] \mid 1!$. Suppose $\deg(f) > 1$ and the statement holds for all $g(x)$ with $\deg(g) < \deg(f)$. Two cases:

Case 1. $f(x)$ is irreducible in $F[x]$. Let $\alpha \in E$ be a root of $f(x)$, and by Theorem 3.3

$$F(\alpha) \cong F[x]/\langle f(x)\rangle$$
$$\text{and } [F(\alpha) : F] = \deg(f) = n$$

since f is the minimal polynomial of α. Write $f(x) = (x - \alpha)g(x)$ with $g(x) \in F(\alpha)[x]$ and $\deg(g) \leq n - 1$. Since E is the splitting field of $g(x)$ over $F(\alpha)$, by induction $[E : F(\alpha)] \mid (n-1)!$ which gives

$$[E : F] = [E : F(\alpha)][F(\alpha) : F] = n \cdot [E : F(\alpha)] \Longrightarrow [E : F] \mid n!.$$

Case 2. $f(x)$ is reducible in $F[x]$. Write $f(x) = g(x)h(x)$ with $g(x), h(x) \in F[x]$ and $\deg(g) = m, \deg(h) = k$, $m + k = n$, and $1 \leq m, k < n$. Let K be the splitting field of $g(x)$ over F. Since $\deg(g) = m$, by induction $[K : F] \mid m!$. Since E is the splitting field of $h(x)$ over K, by induction $[E : K] \mid k!$. Therefore $[E : F] \mid m!k!$ which is a factor of $n!$ since $\binom{n}{m} = \frac{n!}{m!k!}$ is an integer.

Aside: E is the splitting field of $h(x)$ over K because certainly $h(x)$ splits over E, and if L/K were to be a splitting field for $h(x)$ with $L \subsetneq E$, then $f(x)$ would split over L as well. However, E is the splitting field of $f(x)$ over F, a contradiction. ⬠

5 More Field Theory

5.1 Prime fields

Definition (Prime field): The **prime field** of a field F is the intersection of all subfields of F.

Theorem 5.1: If F is a field, its prime field is isomorphic to either $\mathbb{Q}$ or $\mathbb{Z}_p$ for a prime p.

Definition (Character): Given a field F, if its prime field is isomorphic to $\mathbb{Q}$ (resp. $\mathbb{Z}_p$), we say F has characteristic 0 (resp. p), denoted $\mathrm{ch}(F) = 0$ (resp. $\mathrm{ch}(F) = p$).

Remark: When $\mathrm{ch}(F) = p$, for $a, b \in F$,

$$\begin{aligned}(a+b)^p &= a^p + \binom{p}{1} a^{p-1} b + \dots + \binom{p}{p-1} a b^{p-1} + \binom{p}{p} b^p \\ &= a^p + b^p\end{aligned}$$

since $p \mid \binom{p}{i}$ for each $i = 1, \dots, p-1$.

Proof of Theorem 5.1: Let F_1 be a subfield of F. Consider the map

$$\begin{aligned}\chi : \mathbb{Z} &\to F_1 \\ n &\mapsto n \cdot 1\end{aligned}$$

where $1 \in F_1 \subseteq F$. Let $I = \operatorname{Ker} \chi$. Since $\mathbb{Z}/I \cong \operatorname{Im} \chi$, a subring of F_1, $\mathbb{Z}/I$ is an integral domain. Thus I is a prime ideal.

- If $I = \langle 0 \rangle$, then $\mathbb{Z} \subseteq F_1$. Since F_1 is a field, $\mathbb{Q} = \mathrm{Frac}(\mathbb{Z}) \subseteq F_1$.
- If $I = \langle p \rangle$ for a prime p, $\mathbb{Z}_p \cong \mathbb{Z}/\langle p \rangle \cong \operatorname{Im} \chi \subseteq F_1$. ⬠

Proposition 5.2: Let F be a field with $\mathrm{ch}(F) = p$ and $n \in \mathbb{N}$. Then $\varphi : F \to F, u \mapsto u^{p^n}$ is an injective $\mathbb{Z}_p$-homorphism of fields. In particular if F is finite, then φ is a $\mathbb{Z}_p$-isomorphism.

Proof: By (a slight modification of) the previous remark, $\varphi(a+b) = \varphi(a) + \varphi(b)$ and multiplicativity is obvious, so φ is indeed a hom. Also, $1 \notin \mathrm{Ker}(\varphi)$, so $\mathrm{Ker}(\varphi) \neq F \Longrightarrow \mathrm{Ker}(\varphi) = \{0\}$ since F is a field, hence φ is injective. For $a \in \mathbb{Z}_p$, we have $a = a \cdot 1 \Longrightarrow \varphi(a) = a\varphi(1) = a1 = a$ and so φ is a $\mathbb{Z}_p$-hom. ⬠

5.2 Formal derivatives and repeated roots

Definition (Formal derivative): If F is a field, the monomials $\{1, x, x^2, \dots\}$ form an F-basis for $F[x]$. Define the linear operator

$$\begin{aligned}D : F[x] &\to F[x] \\ 1 &\mapsto 0 \\ x^i &\mapsto i x^{i-1}, \forall i \in \mathbb{N}.\end{aligned}$$

Notice that $D(f+g) = D(f) + D(g)$ and $D(fg) = D(f)g + fD(g)$. We call $D(f) =: f'$ the **formal derivative** of f.

Theorem 5.3: Let F be a field, $f(x) \in F[x]$.

(1) If $\mathrm{ch}(F) = 0$, then $f'(x) = 0 \Longleftrightarrow f(x) = c$ for some $c \in F$.

(2) If $\mathrm{ch}(F) = p$, then $f'(x) = 0 \Longleftrightarrow f(x) = g(x^p)$ for some $g(x) \in F[x]$.

Proof:

(1) ($\Longleftarrow$) is clear. For ($\Longrightarrow$), say $f(x) = a_0 + \dots + a_n x^n$. Then

$$f'(x) = a_1 + 2a_2 x + \dots + n a_n x^{n-1} = 0$$

implies that each $ia_i = 0$ for all $i = 1, ..., n$. Since $\mathrm{ch}(F) = 0$ we have $i \neq 0$, and so each $a_i = 0$. Therefore $f(x) = a_0$.

(2) ($\Longleftarrow$) Write $g(x) = b_0 + b_1 x + ... + b_m x^m \in F[x]$. Then

$$\begin{aligned} f(x) &= g(x^p) = b_0 + b_1 x^p + ... + b_m x^{pm} \\ \Longrightarrow f'(x) &= b_1 p x^{p-1} + ... + b_m p m x^{pm-1}. \end{aligned}$$

Since $\mathrm{ch}(F) = p$, we have $p = 0$ so $f'(x) = 0$.

($\Longrightarrow$) For $f(x) = a_0 + ... + a_n x^n$,

$$f'(x) = a_1 + 2a_2 x + ... + n a_n x^{n-1} = 0$$

implies $ia_i = 0$. Since $\mathrm{ch}(F) = p$, $ia_i = 0$ gives $a_i = 0$ unless $p \mid i$. Thus

$$f(x) = a_0 + a_p x^p + a_{2p} x^{2p} + ... + a_{mp} x^{mp} = g(x^p)$$

where $g(x) = a_0 + a_p x + ... + a_{mp} x^m$. ⬠

Definition (Repeated root): Let E/F be a field extension, $f(x) \in F[x]$. We say $\alpha \in E$ is a **repeated root** of $f(x)$ if $f(x) = (x - \alpha)^2 g(x)$ for some $g(x) \in E[x]$.

Theorem 5.4: Let E/F be a field extension, $f(x) \in F[x]$, $\alpha \in E$. Then α is a repeated root of $f(x)$ iff $x - \alpha$ divides both f and f', i.e. $(x - \alpha) \mid \gcd(f, f')$.

Proof: ($\Longrightarrow$) Suppose $f(x) = (x - \alpha)^2 g(x)$. Then

$$\begin{aligned} f'(x) &= 2(x - \alpha) g(x) + (x - \alpha)^2 g'(x) \\ &= (x - \alpha)[2g(x) + (x - \alpha) g'(x)], \end{aligned}$$

so $(x - \alpha) \mid f, f'$.

($\Longleftarrow$) Suppose $(x - \alpha) \mid f, f'$. Write $f(x) = (x - \alpha) h(x)$ with $h(x) \in E[x]$. Then

$$\begin{aligned} f'(x) &= h(x) + (x - \alpha) h'(x) \\ \Longrightarrow h(\alpha) &= f'(\alpha) - (\alpha - \alpha) h'(\alpha) = 0, \end{aligned}$$

since $(x - \alpha) \mid f'$. So α is a root of h, giving $(x - \alpha) \mid h$, hence $f(x) = (x - \alpha)^2 g(x)$ for some $g(x) \in E[x]$. ⬠

Definition (Separable): Let F be a field, $f(x) \in F[x] \setminus \{0\}$. We say $f(x)$ is **separable over F** if it has no repeated roots in any extension of F.

Example: $f(x) = (x - 4)(x - 9)$ is separable in $\mathbb{Q}[x]$.

Corollary 5.5: Let F be a field and $f(x) \in F[x]$. $f(x)$ is separable iff $\gcd(f, f') = 1$.

Remark: The condition of repeated roots depends on the extension of F while gcd involves only F.

Proof: Note $\gcd(f, f') \neq 1 \Longleftrightarrow (x - \alpha) \mid \gcd(f, f')$ for some α in some extension of F. By Theorem 5.4, the result follows. ⬠

Corollary 5.6: If $\mathrm{ch}(F) = 0$, then every irreducible $r(x) \in F[x]$ is separable.

Proof: Let $r(x) \in F[x]$ be irreducible. Then

$$\gcd(r, r') = \begin{cases} 1 \text{ if } r' \neq 0 \\ r \text{ if } r' = 0 \end{cases}$$

If $r'(x) = 0$, then $r(x) = c$ for $c \in F$, but $\deg(r) \geq 1$ as r is irreducible, so we must have $\gcd(r, r') = 1$ and the result follows by Corollary 5.5. ⬠

Example: $\Phi_p(x) = 1 + x + ... + x^{p-1} = \frac{x^p-1}{x-1}$ is irreducible, hence separable. Recall the roots of $\Phi_p(x)$ are $\zeta_p, \zeta_p^2, ..., \zeta_p^{p-1}$ which are all distinct.

5.3 Finite fields

Given a field F, define $F^\times := F \setminus \{0\}$ (the group of units).

Proposition 5.7: If F is a finite field, then $\mathrm{ch}(F) = p$ for some prime p and $|F| = p^n$ for some $n \in \mathbb{N}$.

Proof: Since F is finite, by Theorem 5.1 its prime field is $\mathbb{Z}_p$ for some prime p. Since F is a finite dimensional vector space over $\mathbb{Z}_p$, $F \cong \mathbb{Z}_p^n$ where $n = [F : \mathbb{Z}_p]$. Therefore $|F| = |\mathbb{Z}_p|^n = p^n$. ⬠

Theorem 5.8: Let F be a field and G a finite subgroup of $F^\times$. Then G is cyclic. In particular, the group of units of a finite field is cyclic.

Proof: Wlog we assume $G \neq \{1\}$. Since G is abelian, by the classification of finite abelian groups

$$G \cong \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times ... \times \mathbb{Z}_{n_r}$$

where each $n_i \mid n_{i+1}$ and $n_i > 1$ since $G \neq \{1\}$. Notice every $g \in G$ must then satisfy $g^{n_r} = 1$, so is a root of $x^{n_r} - 1 \in F[x]$. Since $x^{n_r} - 1$ has at most n_r distinct roots in F, we have $|G| \leq n_r$, where the above isomorphism gives $|G| = n_1 \times n_2 \times ... \times n_r$, so it must be that $r = 1$ and $G \cong \mathbb{Z}_{n_1}$ is a cyclic group. ⬠

Corollary 5.9: If F is a finite field, then F is a simple extension of $\mathbb{Z}_p$.

Proof: By taking $u \in F$ to be a generator of $F^\times$, we have $F = \mathbb{Z}_p(u)$. ⬠

Theorem 5.10: Let p be a prime and $n \in \mathbb{N}$.

(1) F is a finite field with $|F| = p^n$ iff F is the splitting field of $x^{p^n} - x$ over $\mathbb{Z}_p$.

(2) Let F be a finite field with $|F| = p^n$, let $m \in \mathbb{N}$ with $m \mid n$. Then F contains a unique subfield K with $|K| = p^m$.

Proof:

(1) ($\Longrightarrow$) Suppose $|F| = p^n$. Then $|F^\times| = p^n - 1$, so every $u \in F^\times$ satisfies $u^{p^n-1} = 1$, thus is a root of $f(x) := x(x^{p^n-1} - 1) = x^{p^n} - x \in \mathbb{Z}_p[x]$. Also, $0 \in F$ is a root of $f(x)$, so every element of F is a root of $f(x)$ which therefore has p^n distinct roots in F. Clearly $f(x)$ cannot split over any smaller field, so F must be the splitting field of $f(x)$ over $\mathbb{Z}_p$.

($\Longleftarrow$) Suppose F is the splitting field of $f(x) := x^{p^n} - x$ over $\mathbb{Z}_p$. Since $\mathrm{ch}(F) = p$, we have

$$f'(x) = p^n x^{p^n-1} - 1 = -1.$$

Thus $\gcd(f, f') = 1$, so by Corollary 5.5 $f(x)$ has p^n distinct roots in F. Let E be the set of all roots of $f(x)$ in F and define

$$\begin{aligned} \varphi : F &\to F \\ u &\mapsto u^{p^n}. \end{aligned}$$

Notice $u \in F$ satisfies $u \in E$ iff $\varphi(u) = u$. This equality condition is closed under $+, -, \times, /$, and so E is a subfield of F of order p^n. Since F is a splitting field, it is generated over $\mathbb{Z}_p$ by the roots of $f(x)$ i.e. the elements of E, so $F = \mathbb{Z}_p(E) = E$, giving $|F| = p^n$.

(2) Let $\alpha \neq 0$ be a root of $x^{p^m} - x$, so α must be a root of $x^{p^m-1} - 1$, giving $\alpha^{p^m-1} = 1$ We recall

$$x^{ab} - 1 = (x^a - 1)\left(x^{ab-a} + x^{ab-2a} + ... + 1\right)$$

so as $m \mid n \Longleftrightarrow n = mk$ for $k \in \mathbb{Z}$, we have

$$p^n - 1 = p^{mk} - 1 = (p^m - 1)M$$

for some $M \in \mathbb{Z}$, and so

$$\alpha^{p^n-1} = \alpha^{(p^m-1)M} = \left(\alpha^{p^m-1}\right)^M = 1^M = 1.$$

Therefore α is a root of $x^{p^n-1} - 1$, and so every root of $x^{p^m} - x$ is a root of $x^{p^n} - x$. Since $x^{p^n} - x$ splits over F, so does $x^{p^m} - x$. Let

$$K := \{u \in F : u^{p^m} - u = 0\}.$$

Then $|K| = p^m$ since the roots of $x^{p^m} - x$ are distinct and by (1), K is a field. Now if $\tilde{K} \subseteq F$ is a subfield with $|\tilde{K}| = p^m$, then $\tilde{K} \subseteq K$, since all elements $v \in \tilde{K}$ satisfy $v^{p^m} - v = 0$. Therefore $\tilde{K} = K$, so K is unique. ⬠

Corollary 5.11 (E.H. Moore): Let p be a prime and $n \in \mathbb{N}$. Then any two finite fields of order p^n are isomorphic. We denote such a field by $\mathbb{F}_{p^n}$.

Proof: Follows by Theorem 5.10 and uniqueness of splitting fields. ⬠

Definition (F^p): Let F be a field with $\mathrm{ch}(F) = p$. Define $F^p := \{b^p : b \in F\}$.

Remark: F^p is a subfield of F as $(a+b)^p = a^p + b^p$ and multiplicativity is obvious.

Theorem 5.12: Let F be a finite field with $\mathrm{ch}(F) = p$.

(1) $F = F^p$.

(2) Every irreducible $r(x) \in F[x]$ is separable.

Proof:

(1) Clearly $F^p \subseteq F$. Every finite field $F = \mathbb{F}_p^n$ is the splitting field of $x^{p^n} - x$ over $\mathbb{Z}_p$ for some prime p and $n \in \mathbb{Z}^+$. Thus for any $a \in F$,

$$a = a^{p^n} = \left(a^{p^{n-1}}\right)^p \in F^p.$$

(2) Let $r(x) \in F[x]$ be irreducible. Now $\gcd(r, r')$ divides $r(x)$, so

$$\gcd(r, r') = \begin{cases} 1 \text{ if } r' \neq 0 \\ r \text{ if } r' = 0 \end{cases}$$

Supposing $r' = 0$, by Theorem 5.3 $r(x) = g(x^p)$ for some $g(x) \in F[x]$, but then

$$r(x) = a_0 + a_1 x^p + \ldots + a_m x^{mp}$$

and since $F = F^p$, each $a_i = b_i^p$ for some $b_i \in F$, giving

$$\begin{aligned} r(x) &= b_0^p + b_1^p x^p + \ldots + b_m^p x^{mp} \\ &= (b_0 + b_1 x + \ldots + b_m x_m)^p, \end{aligned}$$

so $r(x)$ is reducible, a contradiction. Therefore $\gcd(r, r') = 1$, so by Corollary 5.5 we have $r(x)$ is separable. ⬠

Example: We now know that irreducible implies separable in the following cases:

- $\mathrm{ch}(F) = 0$.
- $\mathrm{ch}(F) = p$ *and* F is finite.

However, this is not true if $\mathrm{ch}(F) = p$ and F is infinite. Let F be a field with $\mathrm{ch}(F) = p$ and consider $f(x) = x^p - a \in F[x]$. Since $f'(x) = px^{p-1} = 0$ we have $\gcd(f, f') \neq 1$, so by Corollary 5.5 $f(x)$ is not separable. Furthermore:

(1) If $a \in F^p$, say $a = b^p$. Then $f(x) = x^p - b^p = (x - b)^p$ and so $f(x)$ is reducible.

(2) If $a \notin F^p$, let E/F be a field extension where $x^p - a$ has a root $\beta \in E$. Then $\beta^p = a$, and since $a \notin F^p$ we have $\beta \notin F$, so

$$f(x) = x^p - \beta^p = (x - \beta)^p$$

which is not separable. However, in this situation $x^p - a$ is actually irreducible in $F[x]$.

Proof: Write $x^p - a = g(x)h(x)$ for monic $g(x), h(x) \in F[x]$. We have

$$\begin{aligned} (x-\beta)^p &= g(x)h(x) \\ \implies g(x) &= (x-\beta)^r, \quad h(x) = (x-\beta)^s \end{aligned}$$

for some $0 \leq r, s$ with $r + s = p$. Write

$$g(x) = x^r + r\beta x^{r-1} + ... \in F[x],$$

so $r\beta \in F$. However $\beta \notin F$ so it must be that $r = 0 \in F$. As an integer, then, $r = 0$ or $r = p$. Either way, $g(x) = 1$ or $h(x) = 1$ (one is a unit), so $f(x)$ is irreducible in $F[x]$. ⬠

6 Solvable and Automorphism Groups

6.1 Solvable groups

Definition (Solvable): A group G is **solvable** if there is a tower

$$G = G_0 \supseteq G_1 \subseteq G_2 \supseteq \ldots \supseteq G_m = \{1\}$$

with $G_{i+1} \trianglelefteq G_i$ and G_i/G_{i+1} abelian for all $i = 0, \ldots, m-1$.

Remark: G_{i+1} is not necessarily a normal subgroup of G, but if it is then we get $G_{i+1} \trianglelefteq G_i$ for free.

Example: Consider S_4. Let $V := \mathbb{Z}/\langle 2\rangle \times \mathbb{Z}/\langle 2\rangle$. Note A_4 and V are normal subgroups of S_4, and we have

$$S_4 \supseteq A_4 \supseteq V \supseteq \{e\}.$$

Since $S_4/A_4 \cong \mathbb{Z}_2$, $A_4/V \cong \mathbb{Z}_3$, and $V/\{e\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ are all abelian, S_4 is solvable.

Recall from group theory:

Theorem 6.1 (Second isomorphism theorem): Let $H, K \leq G$ with $K \trianglelefteq G$. Then $HK \leq G$ and $K \trianglelefteq HK$ and $H \cap K \trianglelefteq H$ and $HK/K \cong H/H \cap K$.

Theorem 6.2 (Third isomorphism theorem): Let $K \leq H \leq G$ with $K, H \trianglelefteq G$. Then $H/K \trianglelefteq G/K$ and $(G/K)/(H/K) \cong G/H$.

Theorem 6.3: Let G be a solvable group. If $H \leq G$, then

(1) H is solvable.

(2) Let $N \trianglelefteq G$. Then G/N is solvable.

Proof: As G is solvable there is a tower

$$G = G_0 \supseteq G_1 \supseteq \ldots \supseteq G_m = \{1\}$$

with $G_{i+1} \trianglelefteq G_i$ and G_i/G_{i+1} abelian.

(1) Define $H_i = H \cap G_i$. Since $G_{i+1} \trianglelefteq G_i$, the tower

$$H = H_0 \supseteq G_1 \supseteq \ldots \supseteq H_m = \{1\}$$

satisfies $H_{i+1} \trianglelefteq H_i$. Note that both H_i and G_{i+1} are subgroups of G_i and

$$H_{i+1} = H \cap G_{i+1} = H_i \cap G_{i+1}.$$

By the second isomorphism theorem:

$$H_i/H_{i+1} = H_i/(H_i \cap G_{i+1}) \cong H_iG_{i+1}/G_{i+1} \leq G_i/G_{i+1}.$$

Since G_i/G_{i+1} is abelian, so is H_i/H_{i+1}, hence H is solvable.

(2) Consider the towers

$$G = G_0N \supseteq G_1N \supseteq \ldots \supseteq G_mN = N$$
$$G/N = G_0N/N \supseteq G_1N/N \supseteq \ldots \supseteq G_mN/N = N/N = \{1\}.$$

Since $G_{i+1} \trianglelefteq G_i$ and $N \trianglelefteq G$, we have $G_{i+1}N \trianglelefteq G_iN$ which implies $G_{i+1}N/N \trianglelefteq G_iN/N$. By the third isomorphism theorem,

$$(G_iN/N)/(G_{i+1}N/N) \cong G_iN/G_{i+1}N.$$

Notice that as $G_{i+1} \subseteq G_i$,

$$G_iG_{i+1}N = \{g_1g_2n : g_1 \in G_i, g_2 \in G_{i+1}, n \in N\} = \{g_1n : g_1 \in G_i, n \in N\} = G_iN,$$

so by the second isomorphism theorem,

$$G_iN/G_{i+1}N = G_iG_{i+1}N/G_{i+1}N \cong G_i/(G_i \cap G_{i+1}N).$$

Consider the natural quotient map

$$G_i \to G_i/(G_i \cap G_{i+1}N)$$

which is surjective, and since $G_{i+1} \subseteq G_i \cap G_{i+1}N$, induces a surjective map

$$\varphi : G_i/G_{i+1} \to G_i/(G_i \cap G_{i+1}N)$$

by the universal property of groups. Since G_i/G_{i+1} is abelian, so is $G_i/(G_i \cap G_{i+1}N) = \operatorname{Im}\varphi$. Therefore

$$(G_iN/N)/(G_{i+1}N/N)$$

is abelian. It follows that G/N is solvable. ⬠

Example: Since S_4 is solvable, so are S_3 and S_2.

Theorem 6.4: Let $N \trianglelefteq G$. If N and G/N are both solvable, then G is solvable. In particular, a direct product of finitely many solvable groups is solvable.

Proof: Since N is solvable, we have a tower

$$N = N_0 \supseteq N_1 \supseteq \dots \supseteq N_m = \{1\}$$

with $N_{i+1} \trianglelefteq N_i$ and N_i/N_{i+1} abelian. For a subgroup $H \leq G$ with $N \leq H$, write $\overline{H} = H/N$. Since G/N is solvable, we have a tower

$$G/N = \overline{G} = \overline{G_0} \supseteq \overline{G_1} \supseteq \dots \supseteq \overline{G_r} = N/N = \{1\}$$

with $\overline{G_{i+1}} \trianglelefteq \overline{G_i}$ and $\overline{G_i}/\overline{G_{i+1}}$ abelian. Let $\operatorname{Sub}_N(G) := \{H \leq G : N \leq H\}$ and $\operatorname{Sub}(G)$ be the set of all subgroups of G. Consider the map[1]

$$\begin{aligned} \sigma : \operatorname{Sub}_N(G) &\to \operatorname{Sub}(G/N) \\ H &\mapsto H/N. \end{aligned}$$

For all $i = 0, \dots, r$, define $G_i = \sigma^{-1}\left(\overline{G_i}\right)$. Since $N \trianglelefteq G$ and $\overline{G_{i+1}} \trianglelefteq \overline{G_i}$, we have (see Piazza) $G_{i+1} \trianglelefteq G_i$. Moreover, by the third isomorphism theorem,

$$G_i/G_{i+1} \cong \overline{G_i}/\overline{G_{i+1}}.$$

It follows that

$$G = G_0 \supseteq G_1 \supseteq \dots \supseteq G_r = N = N_0 \supseteq N_1 \supseteq \dots \supseteq N_m = \{1\}.$$

with $G_{i+1} \trianglelefteq G, N_{i+1} \trianglelefteq N_i$, and $G_i/G_{i+1}, N_i/N_{i+1}$ are all abelian. Therefore G is solvable. ⬠

Definition (Simple): A group G is **simple** if $G \neq \{1\}$ and its only normal subgroups are $\{1\}$ and G.

Example: One can show A_5 is simple, so its only tower is $A_5 \supseteq \{1\}$. As $A_5/\{1\} = A_5$ is not abelian, A_5 is not solvable, so by Theorem 6.3 S_5 cannot be solvable ($A_5 \leq S_5$). Moreover, since all S_n for $n \geq 5$ contains a subgroup isomorphic to S_5, Theorem 6.3 gives that S_n is not solvable.

Corollary 6.5: Let G be a finite solvable group. Then there is a tower

$$G = G_0 \supseteq G_1 \supseteq \dots \supseteq G_m = \{1\}$$

with $G_{i+1} \trianglelefteq G_i$ and G_i/G_{i+1} a *cyclic* group.

[1] This map is a bijection: see https://en.wikipedia.org/wiki/Correspondence_theorem

Proof: We know in such a tower each G_i/G_{i+1} is abelian, so by the classification of finite abelian groups for each $A_i := G_i/G_{i+1}$,

$$A_i \cong C_{k_1} \times \ldots \times C_{k_r}$$

where C_k is a cyclic group of order k, so the result follows.

TODO: why?

⬠

Remark: In the above proof, given a finite cyclic group C by the Chinese remainder theorem

$$C \cong \mathbb{Z}/\langle p_1^{\alpha_1}\rangle \times \ldots \times \mathbb{Z}/\langle p_r^{\alpha_r}\rangle$$

where the p_i are distinct primes. Also, for a cyclic group whose order is a prime power, say $\mathbb{Z}/\langle p^\alpha\rangle$, we have a tower of subgroups

$$\mathbb{Z}/\langle p^\alpha\rangle \supseteq \mathbb{Z}/\langle p^{\alpha-1}\rangle \supseteq \ldots \supseteq \mathbb{Z}/\langle p\rangle \supseteq \{1\}$$

so we can further require the quotient G_i/G_{i+1} in Corollary 6.5 to be a cyclic group of prime order.

6.2 Automorphism groups

Definition ($\mathrm{Aut}_F(E)$): Let E/F be a field extension. If $\psi : E \to E$ is an automorphism and $\psi|_F = \mathrm{id}$ we say ψ is an F-automorphism of E. Under composition, the set of F-automorphisms of E is a group called the **automorphism group of E/F** denoted

$$\mathrm{Aut}_F(E) := \{\psi : E \to E : \psi|_F = \mathrm{id}\}.$$

Lemma 6.6: Let E/F be a field extension, $f(x) \in F[x]$, and $\psi \in \mathrm{Aut}_F(E)$. If $\alpha \in E$ is a root of $f(x)$, then $\psi(\alpha)$ is a root of $f(x)$.

Proof: Write $f(x) = a_0 + \ldots + a_n x^n \in F[x]$ so

$$\begin{aligned} f(\psi(\alpha)) &= a_0 + \ldots + a_n \psi(\alpha)^n \\ &= \psi(a_0) + \ldots + \psi(a_n)\psi(\alpha)^n \quad && (a_i \in F, \psi|_F = \mathrm{id}) \\ &= \psi(a_0 + \ldots + a_n\alpha^n) && (\psi \text{ is a hom}) \\ &= \psi(f(\alpha)) = \psi(0) = 0. \end{aligned}$$

⬠

Lemma 6.7: Let $E = F(\alpha_1, \ldots, \alpha_n)$ be a field extension of F. For $\psi_1, \psi_2 \in \mathrm{Aut}_F(E)$, if $\psi_1(\alpha_i) = \psi_2(\alpha_i)$ for all $i = 1, \ldots, n$ then $\psi_1 = \psi_2$.

Proof: For $\alpha \in E$ there exist $f, 0 \neq g \in F[x_1, \ldots, x_n]$ such that

$$\alpha = \frac{f(\alpha_1, \ldots, \alpha_n)}{g(\alpha_1, \ldots, \alpha_n)}$$

and by the same manipulation as in the proof of Lemma 6.6,

$$\begin{aligned} \psi_1(\alpha) &= \psi_1\left(\frac{f(\alpha_1, \ldots, \alpha_n)}{g(\alpha_1, \ldots, \alpha_n)}\right) \\ &= \frac{f(\psi_1(\alpha_1), \ldots, \psi_1(\alpha_n))}{g(\psi_1(\alpha_1), \ldots, \psi_1(\alpha_n))} \\ &= \frac{f(\psi_2(\alpha_1), \ldots, \psi_2(\alpha_n))}{g(\psi_2(\alpha_1), \ldots, \psi_2(\alpha_n))} \\ &= \psi_2(\alpha) \end{aligned}$$

so $\psi_1 = \psi_2$.

⬠

Corollary 6.8: If E/F is a finite extension then $\mathrm{Aut}_F(E)$ is a finite group.

Proof: Since E/F is finite by Theorem 3.5 there are $\alpha_1, ..., \alpha_n \in E$ so $E = F(\alpha_1, ..., \alpha_n)$ with each α_i algebraic over F. For $\psi \in \mathrm{Aut}_F(E)$ by Lemma 6.6 we have $\psi(\alpha_i)$ is a root of the minimal polynomial of α_i over F. Thus there are only finitely many choices for the value of $\psi(\alpha_i)$. By Lemma 6.7 ψ is completely determined by its values at each α_i, so there are only finitely many ψ, hence $\mathrm{Aut}_F(E)$ is finite. ⬠

Example: The converse of Corollary 6.8 is false, for example $\mathbb{R}/\mathbb{Q}$ is an infinite extension but $\mathrm{Aut}_{\mathbb{Q}}(\mathbb{R}) = \{\mathrm{id}\}$ and in fact $\mathrm{Aut}(\mathbb{R}) = \{\mathrm{id}\}$.

6.3 Automorphism groups of splitting fields

Definition (Aut group of $f(x)$): Let F be a field, $f(x) \in F[x]$. The **automorphism group of $f(x)$ over F** is defined to be $\mathrm{Aut}_F(E)$ where E is the splitting field of $f(x)$ over F.

Recall by the proof of Theorem 4.4 we can show the number of such extensions in its statement is at most $[E : F]$, and one can show equality holds iff every irreducible factor of $f(x)$ is separable over F.

Exercise 6.1: Prove the above statement. ▶

Theorem 6.9: Let E/F be the splitting field of $0 \neq f(x) \in F[x]$. Then $|\mathrm{Aut}_F(E)| \leq [E : F]$ with equality iff every irreducible factor of $f(x)$ is separable.

Proof: In the proof of the previous exercise we count the number of extensions as those extending maps $F(\alpha) \to F_1(\alpha_1)$ mapping a root of an irreducible factor to a corresponding root, where each resulting extension ψ is an element of $\mathrm{Aut}_F(E)$, and the set of all such extensions accounts for all of $\mathrm{Aut}_F(E)$ (the F-automorphisms permuting the roots in valid ways) so the result follows. ⬠

Theorem 6.10: If $f(x) \in F[x]$ has n distinct roots in its splitting field E, then $\mathrm{Aut}_F(E)$ is isomorphic to a subgroup of S_n. In particular, $|\mathrm{Aut}_F(E)| \mid n!$.

Proof: Let $X := \{\alpha_1, ..., \alpha_n\}$ be the distinct roots of $f(x)$ in E. By Lemma 6.6 if $\psi \in \mathrm{Aut}_F(E)$ then $\psi(X) = X$. Let $\psi|_X$ be the restriction of ψ to X, so that

$$\begin{aligned} f : \mathrm{Aut}_F(E) &\to \mathrm{Sym}(X) \cong S_n \\ \psi &\mapsto \psi|_X \end{aligned}$$

is a group homomorphism. Moreover by Lemma 6.7, f is injective so $\mathrm{Aut}_F(E) \cong \mathrm{Im}(f) \leq S_n$. ⬠

Example: Consider $f(x) = x^3 - 2 \in \mathbb{Q}[x]$ and let $E/\mathbb{Q}$ be the splitting field of $f(x)$ over $\mathbb{Q}$. Then

$$E := \mathbb{Q}\left(\sqrt[3]{2}, \zeta_3\right) \text{ and } [E : \mathbb{Q}] = 6.$$

Since $\mathrm{ch}(\mathbb{Q}) = 0$ and $f(x)$ is irreducible, $f(x)$ is separable. By Theorem 6.9 we know

$$\left|\mathrm{Aut}_{\mathbb{Q}}(E)\right| = [E : F] = 6$$

and by Theorem 6.10 since f has 3 distinct roots in E we know $\mathrm{Aut}_{\mathbb{Q}}(E)$ is a subgroup of S_3, which forces $\mathrm{Aut}_{\mathbb{Q}}(E) \cong S_3$.

Example: Let F be a field with $\mathrm{ch}(F) = p$ and $f(x) = x^p - a$ where $a \in F^p \neq F$. Then $f(x)$ is irreducible over F and if β is a root of $f(x)$ we have $f(x) = (x - \beta)^p$ with $\beta \notin F$. Then the splitting field of $f(x)$ over F is $E = F(\beta)$ and $\mathrm{Aut}_F(E) = \{\mathrm{id}\}$ since the only choice is $\beta \mapsto \beta$. In this case we have

$$1 = |\mathrm{Aut}_F(E)| \neq [E : F] = \deg f = p$$

which is fine, since f is not separable (in fact it is completely inseparable).

Definition (Fixed field): Let E/F be a field extension and $\psi \in \mathrm{Aut}_F(E)$. Define

$$E^\psi := \{a \in E : \psi(a) = a\}$$

which is a subfield of E containing F, called the **fixed field of ψ**. If $G \leq \mathrm{Aut}_F(E)$, the **fixed field of G** is defined as

$$E^G := \bigcap_{\psi \in G} E^\psi = \{a \in E : \psi(a) = a, \forall \psi \in G\}.$$

Theorem 6.11: Let $f(x) \in F[x]$ be a polynomial whose irreducible factors are separable. Let E/F be the splitting field of $f(x)$. Then $E^G = F$ where $G := \mathrm{Aut}_F(E)$.

Proof: Write $L := E^G$. Since $F \subseteq L$, if ψ fixes L then it fixes F, so $\mathrm{Aut}_L(E) \leq \mathrm{Aut}_F(E)$. On the other hand if $\psi \in \mathrm{Aut}_F(E)$ then by definition of E^G, for all $a \in L$, $\psi(a) = a$ gives $\psi \in \mathrm{Aut}_L(E)$, so $\mathrm{Aut}_F(E) = \mathrm{Aut}_L(E)$. Note that since all the irreducible factors of $f(x)$ are separable over F and $f(x)$ splits over E, $f(x)$ is also separable over L and has E as its splitting field over L. Thus by Theorem 6.9 we have

$$[E : F] = |\mathrm{Aut}_F(E)| = |\mathrm{Aut}_L(E)| = [E : L]$$

where $[E : F] = [E : L][L : F]$ gives $[L : F] = 1$, and so $L = E^G = F$. ⬠

7 Separable and Normal Extensions

7.1 Separable extensions

Definition (Separable extension): Let E/F be an algebraic extension. For $\alpha \in E$, let $p(x) \in F[x]$ be its minimal polynomial. We say α is **separable** over F if $p(x)$ is separable. If this holds for all $\alpha \in E$, we say E/F is a **separable extension**.

Example: If $\mathrm{ch}(F) = 0$, by Corollary 5.5 every irreducible $p(x) \in F[x]$ is separable, so any algebraic extension E/F is separable.

Theorem 7.1: Let E/F be the splitting field of $f(x) \in F[x]$. If every irreducible factor of $f(x)$ is separable, then E/F is separable.

Proof: Let $\alpha \in E$ and $p(x) \in F[x]$ be its minimal polynomial. Let $\alpha = \alpha_1, ..., \alpha_n$ be the distinct roots of $p(x)$ in E. Define

$$\tilde{p}(x) = (x - \alpha_1) \cdots (x - \alpha_n).$$

We claim $\tilde{p}(x) \in F[x]$. Let $\psi \in \mathrm{Aut}_F(E)$. Since ψ is an automorphism, $\psi(\alpha_i) \neq \psi(\alpha_j)$ if $i \neq j$, so by Lemma 6.6 ψ permutes $\{\alpha_1, ..., \alpha_n\}$. Therefore by extending ψ uniquely to $E[x]$ with $x \mapsto x$ we have

$$\begin{aligned} \psi(\tilde{p}(x)) &= (x - \psi(\alpha_1)) \cdots (x - \psi(\alpha_n)) \\ &= (x - \alpha_1) \cdots (x - \alpha_n) = \tilde{p}(x). \end{aligned}$$

It follows that $\tilde{p}(x) \in E^{\psi}[x]$, but ψ was arbitrary, so $\tilde{p}(x) \in E^{\mathrm{Aut}_F(E)}[x]$. Since E/F is the splitting field of $f(x)$ whose irreducible factors are separable, by Theorem 6.11 we have $\tilde{p}(x) \in F[x]$.

Now $\tilde{p}(x) \in F[x]$ and $\tilde{p}(\alpha) = 0$, so $p \mid \tilde{p}$ since $p(x)$ is the min poly of α. By construction $\tilde{p} \mid p$, so $p(x) = \tilde{p}(x)$ since they are both monic. It follows that $p(x)$ is separable, hence E/F is separable. ⬠

Corollary 7.2: Let E/F be a finite extension so $E = F(\alpha_1, ..., \alpha_n)$. If each α_i is separable over F, then so is E/F.

Proof: Let $p_i(x)$ be the minimal polynomial of each α_i $(1 \leq i \leq n)$. Define $f(x) = p_1(x) \cdots p_n(x)$, where each $p_i(x)$ is separable. Let L be the splitting field of $f(x)$ over F. By Theorem 7.1 L/F is separable as every irreducible factor of $f(x)$ is separable. Since $E = F(\alpha_1, ..., \alpha_n)$ is a subfield of L, we have that E is separable. ⬠

Corollary 7.3: Let E/F be an algebraic extension and $L \subseteq E$ be all the $\alpha \in E$ with α separable over F. Then L is a field.

Proof: Let $\alpha, \beta \in L$. Then $\alpha \pm \beta, \alpha\beta, \frac{\alpha}{\beta}$ $(\beta \neq 0) \in F(\alpha, \beta)$. By Corollary 7.2, $F(\alpha, \beta)$ is separable, so $F(\alpha, \beta) \subseteq L$. Thus $\alpha \pm \beta, \alpha\beta, \frac{\alpha}{\beta}$ $(\beta \neq 0) \in L$. ⬠

Definition (Primitive element): If $E = F(\gamma)$ is a simple extension, we say γ is a **primitive element** of E/F.

Theorem 7.4 (Primitive element theorem): If E/F is a finite, separable extension, then $E = F(\gamma)$ for some $\gamma \in E$. In particular, if $\mathrm{ch}(F) = 0$, then any finite extension E/F is a simple extension.

Example: We have that $\mathbb{Q}\left(\sqrt{2}, \sqrt{3}, \sqrt[3]{5}, i, \sqrt{3+i}\right) = \mathbb{Q}(\beta)$ for some β.

Proof: We have seen in Corollary 5.9 that a finite extension of a finite field is always simple, so wlog let F be an infinite field. Since $E = F(\alpha_1, ..., \alpha_n)$ with each $\alpha_i \in E$ it suffices to consider the case when $E = F(\alpha, \beta)$ and the general case follows by induction.

Let $E = F(\alpha, \beta)$ for $\alpha, \beta \notin F$. We claim there exists $\lambda \in F$ such that $\gamma = \alpha + \lambda\beta$ with $\beta \in F(\gamma)$. With this claim, we have

$$\alpha = \gamma - \lambda\beta \in F(\gamma) \Longrightarrow F(\alpha, \beta) \subseteq F(\gamma)$$
$$\gamma = \alpha + \lambda\beta \in F(\alpha, \beta) \Longrightarrow F(\gamma) \subseteq F(\alpha, \beta).$$

Thus $E = F(\alpha, \beta) = F(\gamma)$.

Proof of claim: Let $a(x), b(x)$ be the minimal polynomials of α, β over F. Since $\beta \notin F$, $\deg(b) > 1$ so there is a root $\tilde{\beta}$ of $b(x)$ such that $\tilde{\beta} \neq \beta$. Choose $\lambda \in F$ such that

$$\lambda \neq \frac{\tilde{\alpha} - \alpha}{\tilde{\beta} - \beta}$$

for all roots $\tilde{\alpha}, \tilde{\beta}$ of $a(x), b(x)$ with $\tilde{\beta} \neq \beta$ in some splitting field of $a(x), b(x)$ over F. This is possible as there are finitely many choices of $\tilde{\alpha}, \tilde{\beta}$, but F is infinite. Define $\gamma := \alpha + \lambda\beta$. Consider $h(x) := a(\gamma - \lambda x) \in F(\gamma)[x]$. Then

$$h(\beta) = a(\gamma - \lambda x) = a(\alpha) = 0.$$

However, for $\tilde{\beta} \neq \beta$, since $\gamma - \lambda\tilde{\beta} = \alpha + \lambda\left(\beta - \tilde{\beta}\right) \neq \tilde{\alpha}$ by the choice of λ, we have

$$h\left(\tilde{\beta}\right) = a\left(\gamma - \lambda\tilde{\beta}\right) \neq 0.$$

Thus $h(x)$ and $b(x)$ have β as their only common root in any extension of $F(\gamma)$. Let $b_1(x)$ be the minimal polynomial of β over $F(\gamma)$. Then $b_1(x) \mid h(x), b(x)$. Since E/F is separable and $b(x) \in F[x]$ is irreducible, $b(x)$ has distinct roots and so does $b_1(x)$. The roots of $b_1(x)$ are common to $h(x)$ and $b(x)$, but β being the only common root forces $b_1(x) = x - \beta$. Since $b_1(x) \in F(\gamma)[x]$ we get $\beta \in F(\gamma)$ and this completes the proof. ⬠

7.2 Normal extensions

Definition (Normal extension): Let E/F be an algebraic extension. We say E/F is a **normal extension** if for any irreducible $p(x) \in F[x]$, either $p(x)$ has no roots or all roots in E. Equivalently, if $p(x)$ has a root in E then $p(x)$ splits over E.

Theorem 7.5: A finite extension E/F is normal iff it is the splitting field of some $f(x) \in F[x]$.

Proof: ($\Longrightarrow$) Suppose E/F is normal. Write $E = F(\alpha_1, ..., \alpha_n)$ and let $p_i(x) \in F[x]$ be the minimal polynomial of a_i $(1 \leq i \leq n)$ and $f(x) = p_1(x) \cdots p_n(x)$. Since E/F is normal, each $p_i(x)$ splits over E. Let

$$\alpha_i = \alpha_{i,1}, \alpha_{i,2}, ..., \alpha_{i,r_i} (1 \leq i \leq n)$$

be the roots of $p_i(x)$ in E. Then

$$\begin{aligned} E &= F(\alpha_1, ..., \alpha_n) \\ &= F\left(\alpha_{1,1}, ..., \alpha_{1,r_1}, ..., \alpha_{n,1}, ..., \alpha_{r_n}\right) \end{aligned}$$

which is the splitting field of $f(x)$ over F.

($\Longleftarrow$) Suppose E/F is the splitting field of $f(x) \in F[x]$. Let $p(x) \in F[x]$ be irreducible with a root $a \in E$. Let K/E be the splitting field of $p(x)$ over E, say

$$p(x) = (x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_n)$$

where $\alpha = \alpha_1 \in E$ and $\alpha_2, ..., \alpha_n \in K = E(\alpha_1, \alpha_2, ..., \alpha_n)$. Since

$$F(\alpha) \cong F[x]/\langle p(x) \rangle \cong F(\alpha_2)$$

we have an F-isomorphism $\theta : F(\alpha) \to F(\alpha_2), \alpha \mapsto \alpha_2$. Note $f(x)p(x) \in F[x] \subseteq F(\alpha)[x], F(\alpha_2)[x]$, so we can view K as the splitting field of $f(x)p(x)$ over both $F(\alpha)$ and $F(\alpha_2)$. Then by Theorem 4.4 there is an isomorphism $\psi : K \to K$ extending θ. In particular, $\psi \in \mathrm{Aut}_F(K)$, so ψ permutes the roots of $f(x)$. Since E is generated over F by these roots, by Lemma 6.6 we have $\psi(E) = E$. It follows that $\alpha_2 = \psi(\alpha) \in E$ since $\alpha \in E$. Similarly, we can prove $\alpha_i \in E$ $(3 \le i \le n)$. Thus $K = E$ and $p(x)$ splits over E, so E/F is normal. ⬠

Example: We claim every quadratic extension is normal. Let E/F be a field extension with $[E : F] = 2$. For $\alpha \in E \setminus F$, we have $E = F(\alpha)$. Let $p(x) = x^2 + ax + b$ be the minimal polynomial of α over F. If β is another root of $p(x)$, then $p(x) = (x - \alpha)(x - \beta) = x^2 - (\alpha + \beta)x + \alpha\beta$ so $\beta = -a - \alpha = \frac{b}{\alpha} \in E$. Hence E/F is normal.

Example: Consider $\mathbb{Q}\left(\sqrt[4]{2}\right)/\mathbb{Q}$. This is not a normal extension since $x^4 - 2$ is irreducible but has non-real roots. Notice something strange: the extension $\mathbb{Q}\left(\sqrt[4]{2}\right)/\mathbb{Q}$ is made up of two quadratic extensions:

$$\mathbb{Q}\left(\sqrt[4]{2}\right)/\mathbb{Q}\left(\sqrt{2}\right) \text{ and } \mathbb{Q}\left(\sqrt{2}\right)/\mathbb{Q}$$

which are both normal, but their "composition" is not.

The previous example shows if K/F and E/K are normal, then E/F is not necessarily normal. However, a sort of converse statement does hold.

Proposition 7.6: If E/F is a normal extension and K is an intermediate field, then E/K is normal.

Proof: Let $p(x) \in K[x]$ be irreducible with a root $\alpha \in E$. Let $f(x) \in F[x] \subseteq K[x]$ be the minimal polynomial of α over F. Then $p(x) \mid f(x)$, and since E/F is normal $f(x)$ splits over E, hence so does $p(x)$, thus E/K is normal. ⬠

Remark: Note that K/F in Proposition 7.6 may not be normal. Consider $F := \mathbb{Q}, K := \mathbb{Q}\left(\sqrt[4]{2}\right), E := \mathbb{Q}\left(\sqrt[4]{2}, i\right)$. Then E/F is the splitting field of $x^4 - 2$ hence normal, and E/K is normal, but K/F is *not* normal.

Proposition 7.7: Let E/F be a finite normal extension and $\alpha, \beta \in E$. TFAE:
(1) There exists $\psi \in \mathrm{Aut}_F(E)$ with $\psi(\alpha) = \beta$.

(2) α and β have the same minimal polynomial over F.

In this case we say α and β are **conjugate over F**.

Proof:
($1 \Longrightarrow 2$) Let $p(x)$ be the minimal polynomial of α over F and $\psi \in \mathrm{Aut}_F(E)$ with $\psi(\alpha) = \beta$. By Lemma 6.6 β is also a root of $p(x)$, but $p(x)$ is monic and irreducible over F, so it must be the minimal polynomial of β over F.

($2 \Longrightarrow 1$) Suppose α, β have the same minimal polynomial $p(x)$ over F. Since

$$F(\alpha) \cong F[x]/\langle p(x) \rangle \cong F(\beta)$$

we have an F-isomorphism $\theta : F(\alpha) \to F(\beta), \alpha \mapsto \beta$. Since E/F is a finite normal extension, by Theorem 7.5 E is the splitting field of some $f(x)$ over F. We can also view E as the splitting field of $f(x)$ over $F(\alpha)$ and $F(\beta)$, so by Theorem 4.4 there is an isomorphism $\psi : E \to E$ extending θ, giving $\psi \in \mathrm{Aut}_F(E)$ with $\psi(\alpha) = \beta$, so we are done. ⬠

We have seen not every field extension is normal, but we would like to work with these, which motivates the following definition.

Definition (Normal closure): A **normal closure** of a finite extension E/F is a finite normal extension N/F satisfying
- E is a subfield of N
- For any intermediate field L of N/E, if L is normal over F, then $L = N$

That is, N is the smallest field containing E such that N/F is normal.

Example: The normal closure of $\mathbb{Q}\left(\sqrt[3]{2}\right)/\mathbb{Q}$ is $\mathbb{Q}\left(\sqrt[3]{2}, \zeta_3\right)/\mathbb{Q}$.

Theorem 7.8: Every finite extension E/F has a normal closure N/F that is unique up to E-isomorphism

Proof: Write $E = F(\alpha_1, ..., \alpha_n)$ and let $p_i(x)$ be the minimal polynomial of each a_i over F. Write $f(x) := p_1(x) \cdots p_n(x)$ and let N/E be the splitting field of $f(x)$ over E. Since $\alpha_1, ..., \alpha_n$ are roots of $f(x)$, N is also the splitting field of $f(x)$ over F, so by Theorem 7.5 N/F is normal.

Let $L \subseteq N$ be a subfield containing E. Then L contains all the α_i and if L is normal over F, each $p_i(x)$ then splits over L, so $f(x)$ splits over L, giving $N \subseteq L \Longrightarrow L = N$.

For uniqueness, let N/E be the splitting field of $f(x)$ over E as above. Let N_1/F be another normal closure of E/F. Since N_1 is normal over F and contains each α_i, $f(x)$ splits over N_1, so N_1 contains a splitting field $\tilde{N}$ of $f(x)$ over F, thus over E. By Corollary 4.5 N and $\tilde{N}$ are E-isomorphic. Since $\tilde{N}$ is a splitting field of $f(x)$ over F, by Theorem 7.5, $\tilde{N}$ is normal over F, so $N_1 \subseteq \tilde{N} \Longrightarrow \tilde{N} = N_1 \cong N$. ⬠

8 Galois Correspondence

8.1 Galois extensions

Recall for a finite extension E/F we have shown
- Theorem 7.5: E is the splitting field of some $f(x) \in F[x]$ iff E is normal
- Theorem 7.1: If E is the splitting field of some $f(x) \in F[x]$ whose irreducible factors are separable then E/F is separable.

Definition (Galois group): An algebraic extension E/F is **Galois** if it is normal and separable. If E/F is a Galois extension, we define the **Galois group of E/F** as $\mathrm{Gal}_F(E) := \mathrm{Aut}_F(E)$.

Remark:
(1) By Theorem 7.1 and Theorem 7.5, a finite Galois extension E/F is the splitting field of some $f(x) \in F[x]$ whose irreducible factors are separable.
(2) If E/F is a finite Galois extension, by Theorem 6.9 $|\mathrm{Gal}_F(E)| = [E : F]$.
(3) If E/F is the splitting field of a separable polynomial $f(x) \in F[x]$ with degree n, by Theorem 6.10 $\mathrm{Gal}_F(E)$ is a subgroup of S_n.

Example: Let E be the splitting field of $(x^2-2)(x^2-3)(x^2-5)$ over $\mathbb{Q}$. Then $E = \mathbb{Q}\left(\sqrt{2}, \sqrt{3}, \sqrt{5}\right)$ and $[E : F] = 8$.

Exercise 8.1: Prove the above.

For $\psi \in \mathrm{Gal}_{\mathbb{Q}}(E)$, we have

$$\psi\left(\sqrt{2}\right) = \pm\sqrt{2}$$
$$\psi\left(\sqrt{3}\right) = \pm\sqrt{3}$$
$$\psi\left(\sqrt{5}\right) = \pm\sqrt{5}.$$

Since $\left|\mathrm{Gal}_{\mathbb{Q}}(E)\right| = [E : F] = 8$, we have $\mathrm{Gal}_{\mathbb{Q}}(E) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

Theorem 8.1 (E. Artin): Let E be a field, G a finite subgroup of $\mathrm{Aut}(E)$. Then E/E^G is a finite Galois extension and $\mathrm{Gal}_{E^G}(E) = G$. In particular, $\left[E : E^G\right] = |G|$.

Proof: Let $n = |G|$ and $F := E^G$. For $\alpha \in E$, consider the G-orbit of α, i.e.

$$\{\psi(\alpha) : \psi \in G\} = \{\alpha = \alpha_1, \alpha_2, ..., \alpha_m\}$$

where the a_i are distinct and $m \leq n$. Consider $f(x) = (x - \alpha_1)\cdots(x - \alpha_m)$. For any $\psi \in G$, ψ permutes the roots of $f(x)$. Since the coefficients of $f(x)$ are symmetric w.r.t each α_i, $f(x) \in E^G[x] = F[x]$. To show $f(x)$ is the minimal polynomial of α over F, consider a factor $g(x) \in F[x]$ of $f(x)$. Wlog, we can write

$$g(x) = (x - \alpha_1)\cdots(x - \alpha_\ell).$$

If $\ell < m$, since each α_i are in the G-orbit of α, there is $\psi \in G$ such that

$$\{\alpha_1, ..., \alpha_\ell\} \neq \{\psi(\alpha_1), ..., \psi(\alpha_\ell)\}.$$

It follows that

$$\psi(g(x)) = (x - \psi(\alpha_1))\cdots(x - \psi(\alpha_\ell)) \neq g(x)$$

so $g(x) \notin F[x]$. Therefore $\ell = m$, so $g(x) = f(x)$ and so $f(x)$ is the minimal polynomial of α over F. Since $f(x)$ is separable and splits over E, E/F is separable, hence a Galois extension.

Claim: $[E : F] \leq n$.

Assume the claim is true. We have shown E/F is a finite Galois extension, so E is the splitting field of some polynomial whose irreducible factors are separable over F. Since

$$F = E^G = \{\alpha \in E : \psi(\alpha) = \alpha \ \forall \psi \in G\},$$

G is a subgroup of $\mathrm{Gal}_F(E)$. By Theorem 6.9 we have

$$n = |G| \leq |\mathrm{Gal}_F(E)| = [E : F] \leq n.$$

Therefore $\mathrm{Gal}_F(E) = G$ and $[E : F] = n$.

Proof of claim: Suppose bwoc $[E : F] > n = |G|$. We can choose $\beta_1, ..., \beta_{n+1} \in E$ that are linearly independent over F. Consider the system

$$\psi(\beta_1)v_1 + ... + \psi(\beta_{n+1})v_{n+1} = 0 \quad \forall \psi \in G$$

of n linear equations in $n+1$ variables v_i. This must have a non-zero solution in E, so let

$$\gamma := (\gamma_1, ..., \gamma_{n+1})$$

be such a solution with the *minimum* number of non-zero entries, say r. Notice $r \geq 2$ as if there is only one such entry, the sum cannot be 0. Wlog, we can assume $\gamma_1 = ... = \gamma_r \neq 0$ and $\gamma_{r+1} = ... = \gamma_{n+1} = 0$. Therefore

$$\psi(\beta_1)\gamma_1 + ... + \psi(\beta_r)\gamma_r = 0 \quad \forall \psi \in G \quad (1)$$

and by dividing the solution by γ_r, we can assume $\gamma_r = 1$. Also, since $\beta_1, ..., \beta_r$ are linearly independent over F and $\beta_1\gamma_1 + ... + \beta_r\gamma_r = 0$ (take $\psi = \mathrm{id}$ in (1)) there is at least one $\gamma_i \notin F$ (if each $\gamma_i \in F$, they must all be 0). Since $r \geq 2$, wlog we can assume $\gamma_1 \notin F$. Choose $\varphi \in G$ with $\varphi(\gamma_1) \neq \gamma_1$. Applying φ to (1) we get

$$(\varphi \circ \psi)(\beta_1)\varphi(\gamma_1) + ... + (\varphi \circ \psi)(\beta_r)\varphi(\gamma_r) = 0 \quad \forall \psi \in G.$$

Notice since ψ runs through all of G, so does $\varphi \circ \psi$ (the left action is a permutation), so we can rewrite the above as

$$\psi(\beta_1)\varphi(\gamma_1) + ... + \psi(\beta_r)\varphi(\gamma_r) \quad \forall \psi \in G. \quad (2)$$

Subtracting (2) from (1),

$$\psi(\beta_1)(\gamma_1 - \varphi(\gamma_1)) + ... + \psi(\beta_r)(\gamma_r = \varphi(\gamma_r)) = 0 \quad \forall \psi \in G.$$

Since $\gamma_r = 1$ we have $\gamma_r - \varphi(\gamma_r) = 0$, and since $\gamma_1 \notin F$ we have $\gamma_1 - \varphi(\gamma_1) \neq 0$. Therefore

$$(\gamma_1 - \varphi(\gamma_1), ..., \gamma_{r-1} - \varphi(\gamma_{r-1}), 0, ..., 0)$$

is a solution of the system with fewer non-zero entries than γ, a contradiction of our choice. ⬠

Remark: Let E be a field and G finite subgroup of $\mathrm{Aut}(E)$. For $\alpha \in E$, let $\{\alpha = \alpha_1, ..., \alpha_n\}$ be the G-orbit of α in the set of all conjugates of α. Then we see in the proof of Theorem 8.1 that the minimal polynomial of α over E^G is

$$(x - \alpha_1) \cdots (x - \alpha_n) \in E^G[x].$$

Definition (Symmetric functions): Let $t_1, ..., t_n$ be variables. We define the **elementary symmetric functions** in $t_1, ..., t_n$ as

$$\begin{aligned} s_1 &= t_1 + ... + t_n \\ s_2 &= \sum_{1 \leq i < j \leq n} t_i t_j \\ &\vdots \\ s_n &= t_1 \cdots t_n. \end{aligned}$$

This gives $f(x) = (x - t_1) \cdots (x - t_n) = x^n - s_1 x^{n-1} + s_2 x^{n-1} + \ldots + (-1)^n s_n$.

Example: Let $E = F(t_1, \ldots, t_n)$ be the function field in n variables $t_1, \ldots, t_n$ over a field F. Consider the symmetric group $G := S_n$ as the subgroup of $\mathrm{Aut}(E)$ which permutes the variables $t_1, \ldots, t_n$ and fixes F. We are interested in finding E^G.

From the proof of Theorem 8.1, the coefficients of the minimal polynomial of t_1 lie in E^G. The G-orbit of t_1 is $\{t_1, \ldots, t_n\}$. By the above remark, we see

$$\begin{aligned} f(x) &= (x - t_1) \cdots (x - t_n) \\ &= x^n - s_1 x^{n-1} + s_2 x^{n-1} + \ldots + (-1)^n s_n \in L[x] \end{aligned}$$

is the minimal polynomial of t_1 over E^G, where $L := F(s_1, \ldots, s_n)$. Notice that $L \subseteq E^G$.

Claim: $L = E^G$.

Proof: Notice E is the splitting field of $f(x)$ over L. Since $\deg(f) = n$, by Theorem 4.6

$$[E : L] \leq n!.$$

On the other hand, by Theorem 8.1

$$\left[E : E^G\right] = |G| = |S_n| = n!.$$

Since $L \subseteq E^G$, $n! = \left[E : E^G\right] \leq [E : L] \leq n!$ and so $L = E^G$. ⬠

8.2 The fundamental theorem

Notation: Let $\mathrm{Int}(E/F)$ denote the set of intermediate fields of E/F and $\mathrm{Sub}(G)$ the set of all subgroups of G.

Theorem 8.2 (The fundamental theorem of Galois theory): Let E/F be a finite Galois extension and $G = \mathrm{Gal}_F(E)$. Then the maps

$$\begin{aligned} \mathrm{Int}(E/F) \to \mathrm{Sub}(G) \qquad & L \mapsto L^* := \mathrm{Gal}_L(E) \\ \mathrm{Sub}(G) \to \mathrm{Int}(E/F) \qquad & H \mapsto H^* := E^H \end{aligned}$$

are inverses of each other and reverse the inclusion relation. In particular, for $L_1, L_2 \in \mathrm{Int}(E/F)$ with $L_2 \subseteq L_1$ and $H_1, H_2 \in \mathrm{Sub}(G)$ with $H_2 \subseteq H_1$, we have

$$[L_1 : L_2] = [L_2^* : L_1^*] \quad \text{and} \quad [H_1 : H_2] = [H_2^* : H_1^*].$$

We have the following diagram:

$$\begin{array}{ccc} E & \text{—} & \{1\} = \mathrm{Gal}_E(E) \\ \uparrow & & \downarrow \\ L_1 & & L_1^* = \mathrm{Gal}_{L_1}(E) \\ \uparrow & & \downarrow \\ L_2 & & L_2^* = \mathrm{Gal}_{L_2}(E) \\ \uparrow & & \downarrow \\ F & & G = \mathrm{Gal}_F(E) \end{array}$$

Proof: Let $L \in \mathrm{Int}(E/F)$ and $H \in \mathrm{Sub}(G)$. Recall Theorem 6.11: if $G_1 = \mathrm{Gal}_{F_1}(E_1)$ then $E_1^{G_1} = F_1$, therefore

$$(L^*)^* = (\mathrm{Gal}_L(E))^* = E^{\mathrm{Gal}_L(E)} = L.$$

Similarly by Theorem 8.1 if $G_1 \subseteq \mathrm{Aut}(E_1)$ then $\mathrm{Gal}_{E_1^{G_1}}(E_1) = G_1$, so

$$(H^*)^* = \left(E^H\right)^* = \mathrm{Aut}_{E^H}(E) = H.$$

Therefore the maps are bijections and inverses of each other.

Let $L_1, L_2 \in \mathrm{Int}(E/F)$. Since E/F is the splitting field of some polynomial $f(x) \in F[x]$ whose irreducible factors are separable, E/L_1 and E/L_2 are also Galois extensions since E is also the splitting field of $f(x)$ over L_1 and L_2.

We have $L_2 \subseteq L_1 \Longrightarrow \mathrm{Gal}_{L_1}(E) \subseteq \mathrm{Gal}_{L_2}(E)$ as if an automorphism fixes L_1 then it will fix L_2, so $L_2 \subseteq L_1 \Longrightarrow L_1^* \subseteq L_2^*$. Also,

$$[L_1 : L_2] = \frac{[E : L_2]}{[E : L_1]} = \frac{\left|\mathrm{Gal}_{L_2}(E)\right|}{\left|\mathrm{Gal}_{L_1}(E)\right|} = \frac{|L_2^*|}{|L_1^*|} = [L_2^* : L_1^*].$$

For $H_1 \in H_2 \in \mathrm{Sub}(G)$, note $H_2 \subseteq H_1 \Longrightarrow E^{H_1} \subseteq E^{H_2} \Longrightarrow H_1^* \subseteq H_2^*$. Also,

$$[H_1 : H_2] = \frac{|H_1|}{|H_2|} = \frac{|\mathrm{Gal}_{E^{H_1}}(E)|}{|\mathrm{Gal}_{E^{H_2}}(E)|} = \frac{[E : E^{H_1}]}{[E : E^{H_2}]} = [E^{H_2} : E^{H_1}] = [H_2^* : H_1^*].$$ ⬠

Remark: Consider $E/\mathbb{Q}$ where $E := \mathbb{Q}\left(\sqrt{2}, \sqrt{3}\right)$. Recall $\left|\mathrm{Gal}_{\mathbb{Q}}(E)\right| = 4$ so the Galois group is either $\mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$, but the automorphisms are of order 2, so $\mathrm{Gal}_{\mathbb{Q}}(E) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Consider the subgroups:

$$\begin{array}{ccccc} & & \mathbb{Z}_2 \times \mathbb{Z}_2 & & \\ & \diagup & | & \diagdown & \\ \mathbb{Z}_2 \times 0 & & \langle(1,1)\rangle & & 0 \times \mathbb{Z}_2 \\ & \diagdown & | & \diagup & \\ & & \langle(0,0)\rangle & & \end{array}$$

Since there are finitely many subgroups of $\mathrm{Gal}_{\mathbb{Q}}(E)$, there are finitely many intermediate fields between $\mathbb{Q}$ and E.

We recall that if E/F is a finite Galois extension and $L \in \mathrm{Int}(E/G)$ then L/F is *not* necessarily Galois. When is this true?

$$\begin{array}{ccl} E & \leftrightarrow & \{1\} = \mathrm{Gal}_E(E) \\ \uparrow & & \quad\ \downarrow \\ L & \leftrightarrow & L^* = \mathrm{Gal}_L(E) \\ \uparrow & & \quad\ \downarrow \\ F & \leftrightarrow & G = \mathrm{Gal}_F(E) \end{array}$$

From the above picture, if L/F is Galois, the corresponding group is G/L^* which is well-defined iff $L^* \trianglelefteq G$. We will work towards showing L/F is Galois iff $L^* \trianglelefteq G$.

Proposition 8.3: Let E/F be a finite Galois extension and $G = \mathrm{Gal}_F(E)$. Let $L \in \mathrm{Int}(E/F)$. For $\psi \in G$, we have $\mathrm{Gal}_{\psi(L)}(E) = \psi \, \mathrm{Gal}_L(E) \psi^{-1}$.

Proof: For any $\alpha \in \psi(L)$, $\psi^{-1}(\alpha) \in L$. If $\varphi \in \mathrm{Gal}_L(E)$ we then have

$$\varphi(\psi^{-1}(\alpha)) = \psi^{-1}(\alpha) \Longrightarrow \psi\varphi\psi^{-1}(\alpha) = \alpha.$$

Therefore $\psi\varphi\psi^{-1} \in \mathrm{Gal}_{\psi(L)}(E)$ for all $\varphi \in \mathrm{Gal}_L(E)$. Therefore

$$\psi \, \mathrm{Gal}_L(E) \psi^{-1} \leq \mathrm{Gal}_{\psi(L)}(E).$$

Since $\psi \in \mathrm{Aut}_F(E)$ is a vector space isomorphism, it acts as a change of basis over E/L, so $[E : L] = [E : \psi(L)]$. Thus

$$\begin{aligned}\left|\psi \operatorname{Gal}_L(E)\psi^{-1}\right| &= |\operatorname{Gal}_L(E)| \\ &= [E : L] \\ &= [E : \psi(L)] \\ &= \left|\operatorname{Gal}_{\psi(L)}(E)\right|,\end{aligned}$$

so we have $\psi \operatorname{Gal}_L(E)\psi^{-1} = \operatorname{Gal}_{\psi(L)}(E)$. ⬠

Theorem 8.4: Let E/L, L, L^* be defined as in Theorem 8.2. Then L/F is a Galois extension iff $L^* \trianglelefteq \operatorname{Gal}_F(E)$. In this case, $\operatorname{Gal}_F(L) \cong \operatorname{Gal}_F(E)/L^*$

Proof: We have

$$\begin{aligned} L/F \text{ is normal} &\iff \psi(L) = L, \quad \forall \psi \in \operatorname{Gal}_F(E) \\ &\iff \operatorname{Gal}_{\psi(L)}(E) = \operatorname{Gal}_L(E), \quad \forall \psi \in \operatorname{Gal}_F(E) \\ &\iff \psi \operatorname{Gal}_L(E)\psi^{-1} = \operatorname{Gal}_L(E), \quad \forall \psi \in \operatorname{Gal}_F(E) \\ &\iff \operatorname{Gal}_L(E) = L^* \trianglelefteq \operatorname{Gal}_F(E). \end{aligned}$$

Now $L \subseteq E$ where everything in E is separable over F, so L/F is separable. In the case L/F is normal it is therefore Galois, and the restriction map

$$\begin{aligned}\operatorname{Gal}_F(E) &\to \operatorname{Gal}_F(L) \\ \psi &\mapsto \psi|_L\end{aligned}$$

is well-defined, as $\psi(L) = L$. Moreover, it is surjective and its kernel is $\operatorname{Gal}_L(E)$ (the maps that fix L). Therefore $\operatorname{Gal}_F(E)/L^* \cong \operatorname{Gal}_F(L)$. ⬠

Example: For a prime p, let $q := p^n$. Consider the finite field $\mathbb{F}_q$ of q elements which is an extension of $\mathbb{F}_p$ of degree n. Recall the Frobenius automorphism

$$\begin{aligned}\sigma_p : \mathbb{F}_q &\to \mathbb{F}_q \\ \alpha &\mapsto \alpha^{p^n}.\end{aligned}$$

For $\alpha \in \mathbb{F}_q$, we have $\sigma_p^n(\alpha) = \alpha^{p^n} = \alpha$ so $\sigma_p^n = \mathrm{id}$. For $1 \leq m < n$, we have $\sigma_p^n(\alpha) = \alpha^{p^m}$. Since $x^{p^m} - x$ has at most p^m roots in $\mathbb{F}_q$, there exists $\alpha \in \mathbb{F}_q$ such that $\alpha^{p^m} \neq \alpha$, and so $\sigma_p^m \neq \mathrm{id}$. Therefore σ_p has order n in $\operatorname{Aut}_{\mathbb{F}_p}(\mathbb{F}_q) =: G$. It follows that

$$n = \left|\langle \sigma_p \rangle\right| \leq |G| = \left[\mathbb{F}_q : \mathbb{F}_p\right] = n,$$

where the last equality is by Proposition 5.7. Thus $G = \langle \sigma_p \rangle$ is a cyclic group of order n. Consider a subgroup $H \leq G$ of order d. Then $d \mid n$ and $[G : H] = \frac{n}{d}$. By Theorem 8.2,

$$\frac{n}{d} = [G : H] = [H^* : G^*] = \left[\mathbb{F}_q^H : \mathbb{F}_q^G\right] = \left[\mathbb{F}_q^H : \mathbb{F}_p\right].$$

Therefore $H^* = \mathbb{F}_q^H = \mathbb{F}_{p^{n/d}}$. We have:

$$\begin{array}{ccc} \mathbb{F}_q & \leftrightarrow & \{1\} \\ \uparrow & & \downarrow \\ H^* & \leftrightarrow & H \\ \uparrow & & \downarrow \\ \mathbb{F}_p & \leftrightarrow & G \end{array}$$

Example: Let E be the splitting field of $x^5 - 7$ in $\mathbb{Q}$ over $\mathbb{C}$. Then $E = \mathbb{Q}(\alpha, \zeta_5)$ where $\alpha = \sqrt[5]{7}$ and $\zeta_5 = \exp\left(\frac{2\pi i}{5}\right)$. The minimal polynomials of α and ζ_5 over $\mathbb{Q}$ are $x^5 - 7$ and $x^4 + x^3 + x^2 + x + 1$ respectively. We have

$$\begin{tikzcd}[column sep=small]
 & E = \mathbb{Q}(\alpha, \zeta_5) \arrow[dl, dash] \arrow[dr, dash] & \\
\mathbb{Q}(\alpha) \arrow[dr, dash, "5"'] & & \mathbb{Q}(\zeta_5) \arrow[dl, dash, "4"] \\
 & \mathbb{Q} &
\end{tikzcd}$$

Since $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 5$ and $[\mathbb{Q}(\zeta_5) : \mathbb{Q}] = 4$ are divisors of $[E : \mathbb{Q}]$ we have $20 \mid [E : \mathbb{Q}]$, so $[E : \mathbb{Q}(\zeta_5)] \geq 5$. Also, $E = \mathbb{Q}(\zeta_5)(\alpha)$ and the minimal polynomial of α over $\mathbb{Q}(\zeta_5)$ divides $x^5 - 7$, so has degree at most 5. Therefore $[E : \mathbb{Q}(\zeta_5)] = 5$, giving $[E : \mathbb{Q}] = 20$. It follows that $\mathrm{Gal}_{\mathbb{Q}}(E)$ is a group of order 20.

Each $\psi \in G$ is determined by where it sends α and ζ_5. Write $\psi_{k,s}$ for the map ψ that sends

$$\psi(\alpha) = \alpha\zeta_5^k, \quad k \in \mathbb{Z}_5$$
$$\psi(\zeta_5) = \zeta_5^s, \quad s \in \mathbb{Z}_5^*.$$

Define

$$\sigma := \psi_{1,1} = \begin{cases} \alpha \mapsto \alpha\zeta_5 \\ \zeta_5 \mapsto \zeta_5 \end{cases}$$

$$\tau := \psi_{0,2} = \begin{cases} \alpha \mapsto \alpha \\ \zeta_5 \mapsto \zeta_5^2 \end{cases}$$

Exercise 8.2: Verify that $\tau\sigma = \sigma^2\tau$ and $G = \langle \sigma, \tau : \sigma^5 = \tau^4 = 1, \tau\sigma = \sigma^2\tau \rangle$.

It follows that $G = \{\sigma^a\tau^b : a \in \mathbb{Z}_5, b \in \mathbb{Z}_4\}$. Since $|G| = 20$, by Lagrange's theorem a subgroup can only have one of the follwing orders:

$$1, 2, 4, 5, 10, 20.$$

Let n_p denote the number of Sylow p-subgroups. Recall by the third Sylow theorem since $20 = 2^2 \cdot 5$ that $n_5 \mid 4$ and $n_5 \equiv 1 \bmod 5$, so the only choice is $n_5 = 1$, so there is a unique Sylow 5-subgroup, say P_5, of order 5. By the Sylow theorems $P_5 \trianglelefteq G$ and since $\langle \sigma \rangle$ is a subgroup of order 5, we have that $P_5 = \langle \sigma \rangle$.

Also, $n_2 \mid 5$ and $n_2 \equiv 1 \bmod 2$, so $n_2 \in \{1, 5\}$. Now if $n_2 = 1$, there is a single Sylow 2-subgroup say $P_4 = \langle \tau \rangle \cong \mathbb{Z}_4$, where $P_4 \trianglelefteq G$. Since $|P_4 \cap P_5| = 1$ we have

$$G \cong P_4 \times P_5 \cong \mathbb{Z}_4 \times \mathbb{Z}_5 \cong \mathbb{Z}_{20},$$

which contradicts that G is not abelian. Thus there are 5 Sylow 2-subgroups each of order 4. We have seen that τ is of order 4, so $\langle \tau \rangle$ is a Sylow 2-subgroup and the others must be conjugate to $\langle \tau \rangle$. Note since all elements of G are of the form $\sigma^a\tau^b$ we have

$$\sigma^a\tau^b\tau\tau^{-b}\sigma^{-a} = \sigma^a\tau\sigma^{-a}, \quad a \in \mathbb{Z}_5.$$

Now using $\tau\sigma = \sigma^2\tau$,

$$\langle \sigma^4\tau\sigma^{-4} \rangle = \langle \sigma^{-1}\tau\sigma \rangle = \langle \sigma\tau \rangle = \langle \psi_{1,2} \rangle.$$

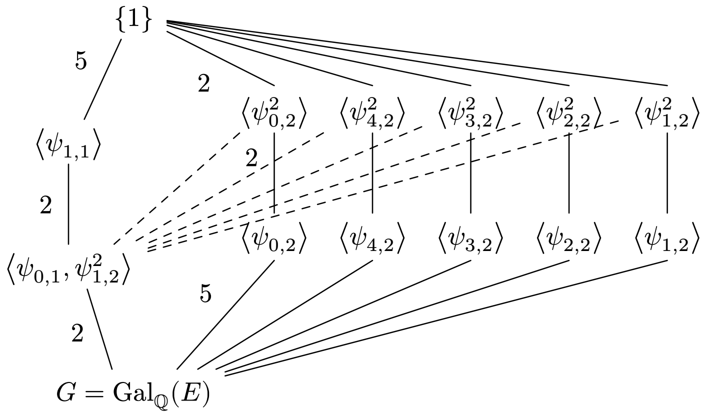
Similarly, we see the Sylow 2-subgroups are

$$\langle \psi_{0,2} \rangle, \langle \psi_{1,2} \rangle, \langle \psi_{2,2} \rangle, \langle \psi_{3,2} \rangle, \langle \psi_{4,2} \rangle.$$

Moreover, since a subgroup of G of order 2 is contained in a Sylow 2-subgroup,

$$\langle \psi_{0,2}^2 \rangle, \langle \psi_{1,2}^2 \rangle, \langle \psi_{2,2}^2 \rangle, \langle \psi_{3,2}^2 \rangle, \langle \psi_{4,2}^2 \rangle$$

are all the subgroups of G of order 2. For a subgroup H of G with order 10, since P_5 is the only subgroup of order 5 we have $H \supseteq P_5 = \langle \sigma \rangle$, so $\sigma^a\tau^b \in H \iff \tau^b \in H$. The only element of the form τ^b of order 2 is τ^2, so $H = \langle \sigma, \tau^2 \rangle$. We have now found all subgroups of G, so combining everything we get the following diagram:



The dotted lines indicate that $\langle \psi_{0,1}, \psi_{1,2}^2 \rangle$ contains each subgroup of order 2. The edge labels are the index of each subgroup inclusion.

Now for an intermediate field L of $E/\mathbb{Q}$, we consider $L^* = \mathrm{Gal}_L(E)$. For example, for $\mathbb{Q}(\zeta_5)$ note that $\psi_{1,1}(\zeta_5) = \zeta_5$, so $\mathbb{Q}(\zeta_5)^* \supseteq \langle \psi_{1,1} \rangle$. Since

$$|\langle \psi_{1,1} \rangle| = [\langle \psi_{1,1} \rangle : \{1\}] = 5 = [E : \mathbb{Q}(\zeta_5)] = [\mathbb{Q}(\zeta_5)^* : \{1\}]$$

we have $\mathbb{Q}(\zeta_5)^* = \langle \psi_{1,1} \rangle$. Also,

$$\psi_{1,2}(\alpha \zeta_5^r) = \alpha \zeta_5 \zeta_5^{2r} = \alpha \zeta_5^{2r+1}.$$

If $\psi_{1,2}$ fixes $\alpha\zeta_5^r$, then $r \equiv 2r + 1 \bmod 5$ i.e. $r \equiv 4 \bmod 5$. Thus $\mathbb{Q}(\alpha\zeta_5^4) \supseteq \langle \psi_{1,2} \rangle$. Since

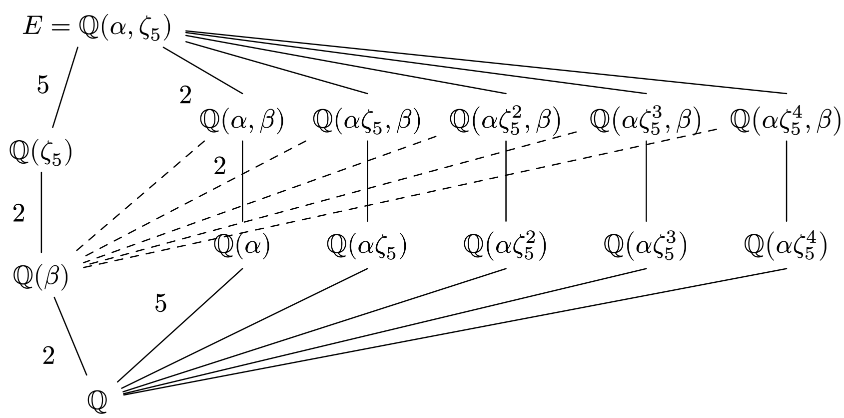
$$|\langle \psi_{1,2} \rangle| = [\langle \psi_{1,2} \rangle : \{1\}] = 4 = [E : \mathbb{Q}(\alpha\zeta_5^4)] = \left[\mathbb{Q}(\alpha\zeta_5^4)^* : \{1\}\right]$$

we have $\mathbb{Q}(\alpha\zeta_5^4)^* = \langle \psi_{1,2} \rangle$. Using the same argument we can get $\langle \psi_{r,2} \rangle^*$ for $r \in \mathbb{Z}_5$.

Consider $\beta = \zeta_5 + \zeta_5^{-1} \in \mathbb{R}$. We have

$$\begin{aligned} \beta^2 + \beta - 1 &= (\zeta_5 + \zeta_5^{-1})^2 + \zeta_5 + \zeta_5^{-1} - 1 \\ &= \zeta_5^2 + \zeta_5^{-2} + 2 + \zeta_5 + \zeta_5^{-1} - 1 \\ &= 1 + \zeta_5 + \zeta_5^2 + \zeta_5^3 + \zeta_5^4 = 0 \end{aligned}$$

since the minimal polynomial of ζ_5 is $1 + x + x^2 + x^3 + x^4$. Since $x^2 + x - 1$ has no rational roots, we have $[\mathbb{Q}(\beta) : \mathbb{Q}] = 2$, and similarly $[\mathbb{Q}(\alpha, \beta) : \mathbb{Q}(\alpha)] = 2$. We therefore obtain the following diagram of intermediate fields of $E/\mathbb{Q}$:



where the inclusions are reversed compared to the subgroup diagram by Theorem 8.2.

9 Cyclic Extensions

Definition: A Galois extension E/F is called abelian, cyclic, or solvable if $\mathrm{Gal}_F(E)$ has the corresponding property.

Lemma 9.1 (Dedekind): Let K, L be fields and $\psi_i : L \to K$ be distinct non-zero homomorphisms for $1 \le i \le n$. If $c_i \in K$ and

$$c_1\psi_1(\alpha) + \dots + c_n\psi_n(\alpha) = 0 \quad \forall \alpha \in L,$$

then $c_1 = \dots = c_n = 0$.

Proof: Suppose bwoc there are $c_1, \dots, c_n \in K$ not all 0 so

$$c_1\psi_1(\alpha) + \dots + c_n\psi_n(\alpha) = 0 \quad \forall \alpha \in L.$$

Let $m \ge 2$ be minimal so

$$c_1\psi_1(\alpha) + \dots + c_m\psi_m(\alpha) = 0 \quad \forall \alpha \in L. \quad (1)$$

Since m is minimal, each $c_i \ne 0$ $(1 \le i \le m)$ as we may re-order the ψ_i. Given $\psi_1 \ne \psi_2$, we can choose $\beta \in L$ such that $\psi_1(\beta) \ne \psi_2(\beta)$, and we may assume $\psi_1(\beta) \ne 0$ since they cannot both be zero. By (1) we have

$$c_1\psi_1(\alpha\beta) + \dots + c_m\psi_m(\alpha\beta) = 0 \quad \forall \alpha \in L,$$

and dividing by $\psi_1(\beta)$,

$$c_1\psi_1(\alpha) + c_2\frac{\psi_2(\beta)}{\psi_1(\beta)}\psi_2(\alpha) + \dots + c_m\frac{\psi_m(\beta)}{\psi_1(\beta)}\psi_m(\alpha) = 0 \quad \forall \alpha \in L. \quad (2)$$

Taking $(1) - (2)$,

$$c_2\left(1 - \frac{\psi_2(\beta)}{\psi_1(\beta)}\right)\psi_2(\alpha) + \dots + c_m\left(1 - \frac{\psi_m(\beta)}{\psi_1(\beta)}\right)\psi_m(\alpha) = 0 \quad \forall \alpha \in L.$$

As $c_2\left(1 - \frac{\psi_2(\beta)}{\psi_1(\beta)}\right) \ne 0$, we have a contradiction of the minimality of m, so such non-zero $c_1, \dots, c_n$ do not exist and the lemma follows. ⬠

Theorem 9.2: Let $n \in \mathbb{N}$ and F be a field with $\mathrm{ch}(F) \nmid n$. Assume $x^n - 1$ splits over F.

(1) If a Galois extension E/F is cyclic of degree n, then $E = F(\alpha)$ for some $\alpha \in E$ with $\alpha^n \in F$. In particular, $x^n - \alpha^n$ is the minimal polynomial of α over F.

(2) If $E = F(\alpha)$ with $\alpha^n \in F$, then E/F is a cyclic extension of degree d with $d \mid n$ and $\alpha^d \in F$. In particular, $x^d - \alpha^d$ is the minimal polynomial of α over F.

Proof: Let $\zeta_n \in F$ be a primitive n^{th} root of unity. Note that since $\mathrm{ch}(F) \nmid n$, the polynomial $x^n - 1$ has separable irreducible factors, so $1, \zeta_n, \zeta_n^2, \dots, \zeta_n^{n-1}$ are all distinct.

(1) Let E/F be cyclic with $G \coloneqq \mathrm{Gal}_F(E) = \langle \psi \rangle \cong C_n$. Apply Lemma 9.1 to $K = L = E$ and the ψ_i all the elements of G and $c_i = \zeta_n^{-i}$ for $i = 0, \dots, n-1$. By the contrapositive, since $c_i \ne 0$, there exists $u \in E$ so

$$\alpha \coloneqq u + \zeta_n^{-1}\psi(u) + \dots + \zeta_n^{-(n-1)}\psi^{n-1}(u) \ne 0.$$

We have

$$\begin{aligned} \psi^0(\alpha) &= \mathrm{id}(\alpha) = \alpha. \\ \psi(\alpha) &= \psi(u) + \zeta_n^{-1}\psi^2(\alpha) + \dots + \zeta_n^{-(n-1)}\psi^n(u) = \alpha\zeta_n. \\ \psi^2(\alpha) &= \alpha\zeta_n^2, \dots, \psi^{n-1}(\alpha) = \alpha\zeta_n^{n-1}. \end{aligned}$$

Thus $\alpha, \alpha\zeta_n, ..., \alpha\zeta_n^{n-1}$ are conjugate to each other, so have the same minimal polynomial $p(x)$ over F. Since the $\alpha\zeta_n^i$ are distinct, it follows that $\deg(p) = n$. Also, since $p(x) \in F[x]$,

$$p(0) = \pm\alpha^n \zeta_n^{\frac{n(n-1)}{2}} \in F,$$

but $\zeta_n \in F$ gives $\alpha^n \in F$. Since α is a root of $x^n - \alpha^n \in F[x]$ and $\deg(p) = n$, we have $p(x) = x^n - \alpha^n$. Moreover, since $F(\alpha) \subseteq E$ and

$$[F(\alpha) : F] = \deg(p) = n = |G| = [E : F],$$

we have $E = F(\alpha)$.

(2) Suppose $\alpha^n \in F$. Let $p(x) \in F[x]$ be the minimal polynomial of α over F. Since α is a root of $x^n - \alpha^n \in F[x]$, $p(x) \mid (x^n - \alpha^n)$, so the roots of $p(x)$ are of the form $\alpha\zeta_n^i$ for some i and we have

$$p(0) = \pm\alpha^d \zeta_n^k$$

for some $k \in \mathbb{Z}$ where d is the number of roots of $p(x)$ i.e. $d = \deg(p)$. Since $p(0) \in F$ and $\zeta_n \in F$, it follows that $\alpha^d \in F$. Since $x^d - \alpha^d \in F[x]$ has α as a root, $p(x) \mid \left(x^d - \alpha^d\right)$. As $\deg(p) = d$ and $p(x)$ is monic, we have $p(x) = x^d - \alpha^d$.

Claim: $d \mid n$. Suppose not, so $n = qd + r$ for $q \in \mathbb{Z}$ and $0 < r < d$. Then

$$\alpha^r = \alpha^{n-qd} = \alpha^n \left(\alpha^d\right)^q \in F,$$

but then $p(x) \mid (x^r - \alpha^r)$, contradicting that $\deg(p) = d > r$. Write $n = md$. Since $p(x) = x^d - \alpha^d$, the roots of $p(x)$ are

$$\alpha, \alpha\zeta_n^m, \alpha\zeta_n^{2m}, ..., \alpha\zeta_n^{(d-1)m}.$$

As $\zeta_n \in F$, $E = F(\alpha)$ is the splitting field of the separable polynomial $p(x) \in F[x]$, so E/F is Galois. If $\psi \in G := \mathrm{Gal}_F(E)$ satisfies $\psi(\alpha) = \alpha\zeta_n^m$, then $G = \langle\psi\rangle \cong C_d$. Thus E/F is a cyclic extension of order d. ⬠

Theorem 9.3: Let F be a field with $\mathrm{ch}(F) = p$.

(1) If $x^p - x - a \in F[x]$ is irreducible, then its splitting field E/F is a cyclic extension of degree p.

(2) If E/F is a cyclic extension of degree p, then E/F is the splitting field of some irreducible $x^p - x - a \in F[x]$ for some $a \in F$.

Proof:

(1) Let $f(x) := x^p - x - a$ and α be a root of $f(x)$. Then since $\mathrm{ch}(F) = p$, by Fermat's little theorem,

$$\begin{aligned} f(\alpha + t) &= (\alpha + t)^p - (\alpha + t) - a \\ &= \alpha^p + t^p - \alpha - t - a = f(\alpha) + t^p - t = 0 \end{aligned}$$

for all $1 \leq t \leq p - 1$. Since $f(x)$ has at most p distinct roots, these are all of them and it follows that

$$E = F(\alpha, \alpha + 1, ..., \alpha + p - 1) = F(\alpha)$$

is the splitting field of $f(x)$ over F and $[E : F] = \deg(f) = p$. Since C_p is the only group of order p, we have $\mathrm{Gal}_F(E) \cong C_p$; indeed $\mathrm{Gal}_F(E) = \langle\psi\rangle$ when $\psi(\alpha) = \alpha + 1$.

(2) Let $G := \mathrm{Gal}_F(E) = \langle\psi\rangle \cong C_p$. Apply Lemma 9.1 to $K = L = E$, ψ_i all elements of G, and $c_1 = ... = c_p = 1$. Since $c_i \neq 0$, there is $v \in E$ so that

$$\beta = v + \psi(v) + ... + \psi^{p-1}(v) \neq 0.$$

This equation gives $\psi^i(\beta) = \beta$ for all $\psi^i \in G$ $(0 \leq i \leq p - 1)$, so we must have $\beta \in F$. Set $u := \frac{v}{\beta}$. Since $\beta \in F$,

$$\begin{aligned} u+\psi(u)+\ldots+\psi^{p-1}(u) &= \frac{v}{\beta}+\psi\left(\frac{v}{\beta}\right)+\ldots+\psi^{p-1}\left(\frac{v}{\beta}\right) \\ &= \frac{1}{\beta}(v+\psi(v)+\ldots+\psi^{p-1}(v)) = \frac{\beta}{\beta} = 1. \end{aligned}$$

Set $\alpha := 0 \cdot u - 1 \cdot \psi(u) - 2\psi^2(u) - \ldots - (p-1)\psi^{p-1}(u)$. Then

$$\begin{aligned} \psi(\alpha) &= -\psi^2(u) - 2\psi^3(u) - \ldots - (p-1)\psi^p(u) \\ \implies \psi(\alpha) - \alpha &= \psi(u) + \psi^2(u) + \ldots + \psi^{p-1}(u) + \psi^p(u) = 1, \end{aligned}$$

i.e. $\psi(\alpha) = \alpha + 1$. Since $\mathrm{ch}(F) = p$, we have

$$\psi(\alpha^p) = \psi(\alpha)^p = (\alpha+1)^p = \alpha^p + 1,$$

and therefore

$$\psi(\alpha^p - \alpha) = \psi(\alpha^p) - \psi(\alpha) = \alpha^p + 1 - (\alpha+1) = \alpha^p - \alpha.$$

Thus $\alpha^p - \alpha$ is fixed by ψ. Since $G = \langle\psi\rangle$, $a := \alpha^p - \alpha \in F$ and α is a root of $x^p - x - \alpha \in F[x]$. Since $[E : F] = p$ and $F(\alpha) \subseteq E$, we have $[F(\alpha) : F] \mid p$, but $\alpha \notin F$ and p is prime, so $[F(\alpha) : F] = p$ and $F(\alpha) = E$, where $x^p - x - a$ is the minimal polynomial of α over F. ⬠

10 Solvability by Radical

10.1 Radical extensions

Definition (Radical extension): A finite extension E/F is **radical** if there is a tower of fields

$$F =: F_0 \subseteq F_1 \subseteq \ldots \subseteq F_m := E$$

such that $F_i = F_{i-1}(\alpha_i)$ $(a_i \in F_i)$ and $\alpha_i^{d_i} \in F_{i-1}$ for some $d_i \in \mathbb{N}$ for each $1 \leq i \leq m$.

Lemma 10.1: If E/F is a finite separable radical extension, then its normal closure N/F is also radical.

Proof: Since E/F is a finite separable extension, by Theorem 7.4 $E = F(\beta)$ for some $\beta \in E$. Since E/F is a radical extension, there is a tower

$$F =: F_0 \subseteq F_1 \subseteq \ldots \subseteq F_m := E$$

with each $F_i = F_{i-1}(\alpha_i)$ and $\alpha_i^{d_i} \in F_{i-1}$ for some $d_i \in \mathbb{N}$. Let $p(x) \in F\{x\}$ be the minimal polynomial of β and let

$$\beta =: \beta_1, \beta_2, \ldots, \beta_n$$

be the roots of $p(x)$. The normal closure of F is

$$N = E(\beta_2, \ldots, \beta_n) = F(\beta_1, \beta_2, \ldots, \beta_n)$$

by Theorem 7.5, as this is the splitting field of $p(x)$ over F. Also, there is an F-isomorphism

$$\begin{aligned} \sigma_j : F(\beta) &\to F(\beta_j) \\ \beta &\mapsto \beta_j \end{aligned}$$

for all $2 \leq j \leq n$. Since N can be viewed as the splitting field of $p(x)$ over $F(\beta)$ and $F(\beta_j)$, by Theorem 4.4 there is $\psi_j : N \to N$ extending σ_j for $2 \leq j \leq n$. Thus $\psi_j \in \mathrm{Gal}_F(N)$ and $\psi_j(\beta) = \beta_j$. We have the following tower of fields:

$$\begin{aligned} F = F_0 \subseteq F_1 \subseteq \ldots \subseteq F_m = E = F(\beta_1) &= F(\beta_1)\psi_2(F_0) \\ \subseteq F(\beta_1)\psi_2(F_1) \subseteq F(\beta_1)\psi_2(F_2) \subseteq \ldots &\subseteq F(\beta_1)\psi_2(F_m) \\ &= F(\beta_1)F(\beta_2) \\ &= F(\beta_1, \beta_2) \\ &= F(\beta_1, \beta_2)\psi_3(F_0) \\ \subseteq F(\beta_1, \beta_2)\psi_3(F_1) \subseteq \ldots &\subseteq F(\beta_1, \beta_2)\psi_3(F_m) \\ \subseteq \ldots &\subseteq F(\beta_1, \beta_2, \ldots, \beta_n) = N. \end{aligned}$$

Note that since $F_i = F_{i-1}(\alpha_i)$ and $\alpha_i^{d_i} \in F_{i-1}$, we have

$$\begin{aligned} F(\beta_1, \ldots, \beta_{j-1})\psi_j(F_i) &= F(\beta_1, \ldots, \beta_{j-1})\psi_j(F_{i-1}(\alpha_i)) \\ &= \left(F(\beta_1, \ldots, \beta_{j-1})\psi_j(F_{i-1})\right)\left(\psi_j(\alpha_i)\right) \\ \text{and } \left(\psi_j(\alpha)\right)^{d_i} = \psi_j\left(\alpha_i^{d_i}\right) &\in \psi_j(F_{i-1}). \end{aligned}$$

Therefore N/F is a radical extension. ⬠

Remark: By Lemma 10.1 to consider a finite separable radical extension, we could instead consider its normal closure which is Galois.

Definition (Solvable by radicals): Let F be a field and $f(x) \in F[x]$. We say $f(x)$ is **solvable by radicals** if there is a radical extension E/F such that $f(x)$ splits over E.

Note: It is possible that $f(x) \in F[x]$ is solvable by radicals, but its splitting field is *not* a radical extension.

Remark: We recall that an expression involving only the field operations and roots is a radical. Let F be a field and $f(x) \in F[x]$ have separable irreducible factors. It $f(x)$ is solvable by radicals, by the definition of radical extension, $f(x)$ has a radical root. Conversely, if $f(x)$ has a radical root, this root lies in some radical extension E/F. By Lemma 10.1, the normal closure N/F of E/F is radical. Since $f(x)$ splits over N, $f(x)$ is solvable by radicals.

10.2 Radical solutions

We have seen in A8 that:

Lemma 10.2: Let E/F be a field extension and K, L be intermediate fields of E/F. Suppose K/F is a finite Galois extension. Then KL is a finite Galois extension of L and $\mathrm{Gal}_L(KL)$ is isomorphic to a subgroup of $\mathrm{Gal}_F(E)$.

Definition (Galois group of a polynomial): Let E/F be the splitting field of $f(x) \in F[x]$ whose irreducible factors are separable. The **Galois group of $f(x)$** is defined to be $\mathrm{Gal}_F(E)$, denoted $\mathrm{Gal}(f)$.

Theorem 10.3: Let F be a field with $\mathrm{ch}(F) = 0$ and $f(x) \in F[x] \setminus \{0\}$. Then $f(x)$ is solvable by radical iff $\mathrm{Gal}(f)$ is solvable.

Proposition 10.4: Let $f(x) \in Q[x]$ be an irreducible polynomial of prime degree p. If $f(x)$ has precisely two non-real roots in $\mathbb{C}$, then $\mathrm{Gal}(f) \cong S_p$.

Proof: Recall $S_p = \langle (12), (12...p) \rangle$, so to show $\mathrm{Gal}(f) \cong S_p$ it suffices to find a 2-cycle and a p-cycle. Let α be a root of $f(x)$. Since $f(x)$ is irreducible of degree p,

$$[\mathbb{Q}(\alpha) : \mathbb{Q}] = \deg(f) = p,$$

so $p \mid |\mathrm{Gal}(f)|$ and by Cauchy's theorem there is an element in $\mathrm{Gal}(f)$ of order p, i.e. a p-cycle. Also, the complex conjugate map σ will interchange the two non-real roots of $f(x)$ and fix the rest, so it is of order 2 i.e. a 2-cycle. By changing notation if necessary, $(12), (12...p) \in \mathrm{Gal}(f)$. It follows that $\mathrm{Gal}(f) \cong S_p$. ⬠

Example: Consider $f(x) := x^5 - 2x^3 - 24x - 2 \in \mathbb{Q}[x]$ which is irreducible by Eisenstein's. Notice

$$f(-1) = 19, \qquad f(1) = -23,$$
$$\lim_{x \to -\infty} f(x) = -\infty, \lim_{x \to \infty} f(x) = \infty,$$

so there are at least 3 real roots of $f(x)$ (at least one in each of the intervals $(-\infty, -1), (-1, 1), (1, \infty)$). If $\alpha_1, ..., \alpha_5$ are the roots, by considering the coefficients of x^4 and x^3,

$$\sum_{i=1}^{5} \alpha_i = 0 \quad \text{and} \quad \sum_{i<j} \alpha_i \alpha_j = 2.$$

Now from the first sum,

$$\left(\sum_{i=1}^{5} \alpha_i \right)^2 = \sum_{i=1}^{5} \alpha_i^2 + 2 \sum_{i<j} \alpha_i \alpha_j = 0,$$

and so $\sum_{i=1}^{5} \alpha_i^2 = -4$, thus not all roots of $f(x)$ are real. It follows that $f(x)$ has 3 real roots and 2 non-real roots, so by Proposition 10.4 $\mathrm{Gal}(f) \cong S_5$. As S_5 is not solvable, by Theorem 10.3 $f(x)$ is *not* solvable by radicals.

From the above example, we see a polynomial of degree 5 is not always solvable by radicals. Since $S_5 \subseteq S_n$ for all $n \geq 5$, we have:

Theorem 10.5 (Abel-Ruffini): A general polynomial $f(x)$ with $\deg(f) \geq 5$ is not solvable by radicals.

Example: The polynomial $f(x) := x^7 - 2x^4 - 7x^3 + 14$ is solvable by radicals since $f(x) = (x^3 - 2)(x^4 - 7)$ is a product of polynomials that are solvable by radicals.

Appendix A: Solutions to Exercises

2.1:

(a) Say $a = ua'$ and $b = vb'$, u, v units. Then $ab = uva'b'$ by commutativity of the ring, so $ab \sim a'b'$.

(b) Suppose $a \mid b$. Then $b = ca$ for some $c \in R$, so $vb' = b = ca = cua'$, giving $v^{-1}cua' = b'$, so $a' \mid b'$. The converse is identical.

◀

2.2: Let $x = a + b\sqrt{d}$ and $y = u + v\sqrt{d}$.

$$\begin{aligned} N(x)N(y) &= (a^2 - db^2)(u^2 - dv^2) \\ &= a^2u^2 - da^2v^2 - db^2u^2 + d^2b^2v^2 \\ N(xy) &= N\Big(au + dbv + (av + bu)\sqrt{d}\Big) \\ &= (au + dbv)^2 - d(av + bu)^2 \\ &= a^2u^2 + \cancel{2abuvd} + d^2b^2v^2 - d\big(a^2v^2 + \cancel{2abuv} + b^2u^2\big) \\ &= a^2u^2 - da^2v^2 - db^2u^2 + d^2b^2v^2 = N(x)N(y). \end{aligned}$$

◀

2.3: Take $R := \mathbb{Q} + x\mathbb{R}[x]$. Then x is irreducible in R and $x \mid 2x^2$ where $2x^2 = \left(\sqrt{2}x\right)^2$ but $x \nmid \sqrt{2}x$. If this were true we'd have $qx = \sqrt{2}x$ for some $q \in \mathbb{Q}$, but $\sqrt{2} \notin \mathbb{Q}$. ◀

2.4: Let $d := p_1^{\min\{\alpha_1,\beta_1\}} \cdots p_k^{\min\{\alpha_k,\beta_k\}}$, where it is clear $d \mid a, b$. If $c \mid a, b$ write

$$c \sim p_1^{\gamma_1} \cdots p_k^{\gamma_k}.$$

Notice this product cannot contain any primes that are not $p_1, \dots, p_k$ as otherwise we'd have some $q \mid c$ but $q \nmid a, b$. Suppose wlog p_1 with $\gamma_1 > \min\{\alpha_1, \beta_1\}$. Then wlog say α_1 is the minimum, so as $cm = a$ for some $m \in R$,

$$\begin{aligned} p_1^{\gamma_1} \cdots p_k^{\gamma_k} m &= p_1^{\alpha_1} \cdots p_k^{\alpha_k} \\ \implies p_1^{\gamma_1 - \alpha_1} \cdots p_k^{\gamma_k} m &= p_2^{\alpha_2} \cdots p_k^{\alpha_k} \quad (\star) \end{aligned}$$

since $p_1^{\alpha_1} \neq 0$ and R is an integral domain. Then p_1 divides the LHS of $(\star)$ but not the RHS as the p_i are pairwise non-associated, a contradiction. Thus each $\gamma_i \leq \min\{\alpha_i, \beta_i\}$, so $c \mid d$. ◀

3.1: Suppose $\sqrt{3} = x + y\sqrt{2}$ where $x, y \in \mathbb{Q}$. Notice $y \neq 0$ as otherwise $x = \sqrt{3} \in \mathbb{Q}$ and $x \neq 0$ as otherwise $y = \frac{\sqrt{3}}{\sqrt{2}} \notin \mathbb{Q}$, so

$$\begin{aligned} 3 &= x^2 + 2xy\sqrt{2} + 2y^2 \\ \implies \frac{3 - x^2 - 2y^2}{2xy} &= \sqrt{2} \end{aligned}$$

where the above LHS is rational but the RHS is not, a contradiction. ◀

3.2: ◀

6.1:

◀

8.1:

◀

8.2: