Additional Homework Assignment

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原式

$$h'(t) = Ah(t) + Bx(t)$$

兩邊同乘e^{-At}

$$e^{-At}h'(t) = e^{-At}Ah(t) + e^{-At}Bx(t)$$

等式移動

$$e^{-At}h'(t) - e^{-At}Ah(t) = e^{-At}Bx(t)$$

等式左邊用 Inverse product rule

$$\frac{d}{dt}[e^{-At}Ah(t)] = e^{-At}Bx(t)$$

兩邊同時積分

$$e^{-At}h(t) - e^{0}h(0) = \int_0^t e^{-A\gamma}Bx(\gamma)\,d\gamma$$

同乘e^{At}

$$h(t) - e^{At}e^{0}h(0) = e^{At}\int_0^t e^{-A\gamma}Bx(\gamma)\,d\gamma$$

等式移動、整理

$$h(t) = e^{At}h(0) + \int_0^t e^{A(t-\gamma)}Bx(\gamma)\,d\gamma$$

定義$h[k] := k(k\Delta)$，Δ為 step size，列出$h[k]$與$h[k+1]$

$$h[k] = e^{Ak\Delta}h(0) + \int_0^{k\Delta} e^{A(k\Delta-\gamma)}Bx(\gamma)\,d\gamma$$

$$h[k+1] = e^{A(k+1)\Delta}h(0) + \int_0^{(k+1)\Delta} e^{A((k+1)\Delta-\gamma)}Bx(\gamma)\,d\gamma$$

改寫$h[k+1]$

$$h[k+1] = e^{A(k+1)\Delta}h(0) + \int_0^{k\Delta} e^{A((k+1)\Delta-\gamma)}Bx(\gamma)\, d\gamma$$
$$+ \int_{k\Delta}^{(k+1)\Delta} e^{A((k+1)\Delta-\gamma)}Bx(\gamma)\, d\gamma$$

因為$e^{A((k+1)\Delta-\gamma)} = e^{A\Delta}e^{A(k\Delta-\gamma)}$*故可改寫*$h[k+1]$*為*

$$h[k+1] = e^{A(k+1)\Delta}h(0) + e^{A\Delta}\int_0^{k\Delta} e^{A(k\Delta-\gamma)}Bx(\gamma)\, d\gamma$$
$$+ \int_{k\Delta}^{(k+1)\Delta} e^{A((k+1)\Delta-\gamma)}Bx(\gamma)\, d\gamma$$

再次改寫$h[k+1]$

$$h[k+1] = e^{A\Delta}e^{Ak\Delta}h(0) + e^{A\Delta}\int_0^{k\Delta} e^{A(k\Delta-\gamma)}Bx(\gamma)\, d\gamma$$
$$+ \int_{k\Delta}^{(k+1)\Delta} e^{A((k+1)\Delta-\gamma)}Bx(\gamma)\, d\gamma$$

灰色部分*為*$e^{A\Delta}h[k]$*，得新的*$h[k+1]$*為*

$$h[k+1] = e^{A\Delta}h[k] + \int_{k\Delta}^{(k+1)\Delta} e^{A((k+1)\Delta-\gamma)}Bx(\gamma)\, d\gamma$$

設$v(\gamma) = k\Delta + \Delta - \gamma$*、*$du = -d\gamma$*，改寫*$h[k+1]$*積分部分*

$$\int_{k\Delta}^{(k+1)\Delta} e^{A\left((k+1)\Delta-\gamma\right)}Bx(\gamma)\, d\gamma$$
$$= \int_{v(k\Delta)}^{v\left((k+1)\Delta\right)} e^{Av}\, dvBx[k]$$
$$= -(\int_\Delta^0 e^{Av}\, dv)Bx[k]$$
$$= (\int_0^\Delta e^{Av}\, dv)Bx[k]$$

計算積分，計算$v = \Delta$*減去*$v = 0$

$$\frac{e^{A\Delta}}{A} - \frac{e^{A(0)}}{A} = \frac{e^{A\Delta} - I}{A} = A^{-1}(e^{A\Delta} - I)$$

合併回原式

$$h[k+1] = e^{A\Delta}h[k] + A^{-1}(e^{A\Delta} - I)Bx[k]$$

得灰色為最終解

$$\bar{A} = e^{A\Delta} = \exp(A\Delta)$$

$$\bar{B} = A^{-1}(e^{A\Delta} - I)B = (\Delta A)^{-1}(e^{A\Delta} - I)\Delta B$$