“Mean Value Coordinates for Closed Triangular Meshes”

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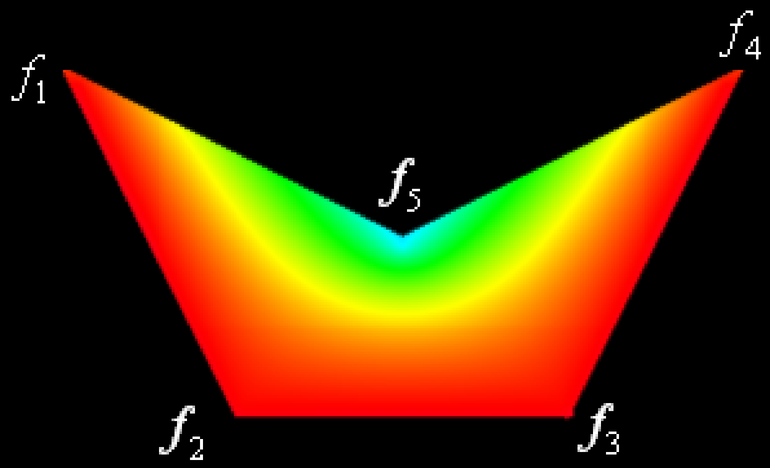
Outline

- Abstract
- Preliminaries
- Previous work
- Mean value Interpolation
- 3D Mean value coordinates for closed triangular meshes
- Applications
- Questions

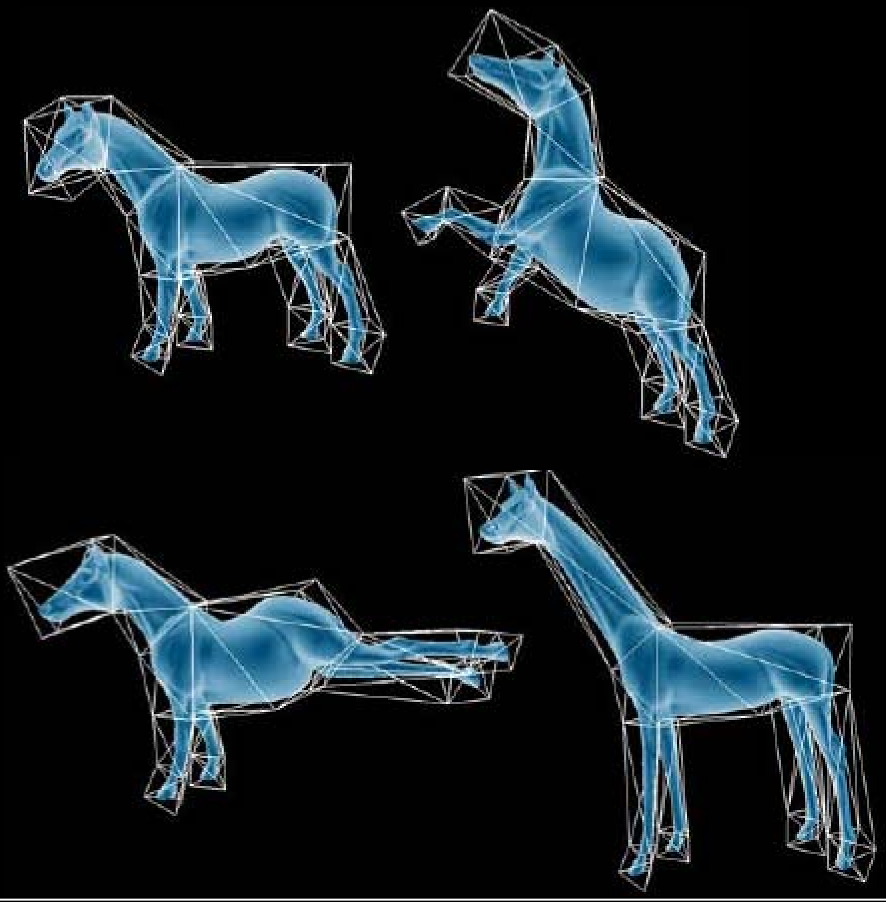
Abstract

- Search for a function that can interpolate a set of values at the vertices of a mesh smoothly into its interior
- Mean value coordinates have been used as an interpolant for closed 2D polygons.



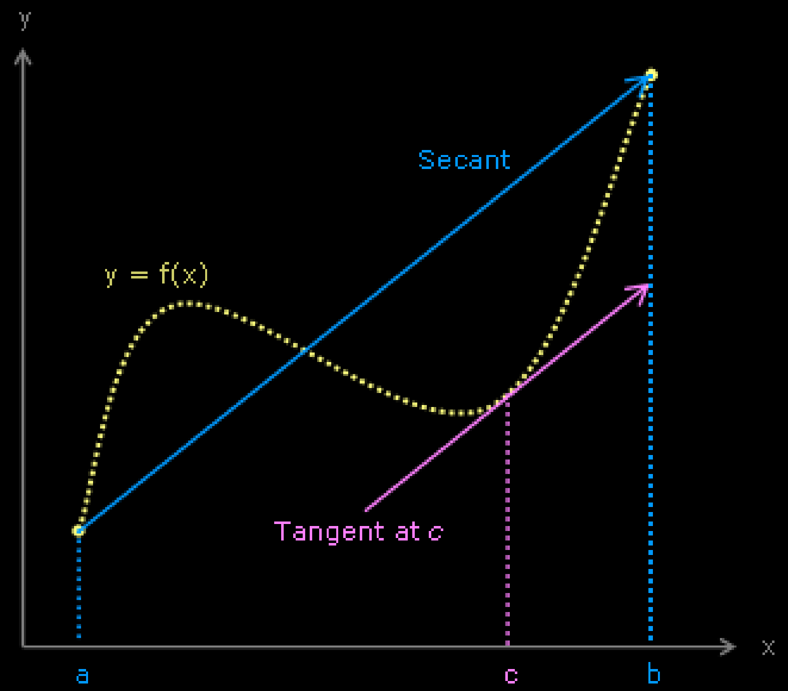
Abstract

- This paper generalizes the mean value coordinates to closed triangular meshes
- Interesting applications to surface deformation and volumetric textures



Mean Value Theorem

- Wikipedia :



For any function that is continuous on [*a*, *b*] and differentiable on (*a*, *b*) there exists some *c* in the interval (*a*, *b*) such that the **secant** joining the endpoints of the interval [*a*, *b*] is parallel to the **tangent** at *c*.

Harmonic functions and Mean Value property

- A harmonic function is twice continuously differentiable function f: U->R which satisfies the laplace's equation

$$\frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2} = 0$$

everywhere on U. This is also often written as

$\nabla^2 f = 0$ or $\Delta f = 0$.

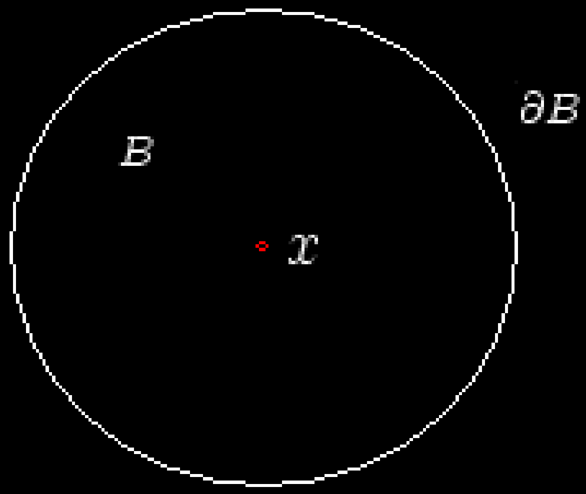
Harmonic functions and Mean Value property

- They attain there maxima/minima only at the boundaries.
- Let B(x,r) be a ball with center x and radius r, contained totally in U,

Harmonic functions and Mean Value property

- Then, $$u(x) = \frac{1}{\omega_n r^{n-1}} \oint_{\partial B(x,r)} u \, dS = \frac{n}{\omega_n r^n} \int_{B(x,r)} u \, dV$$

where ω_n is the surface area of the unit sphere in n dimensions.

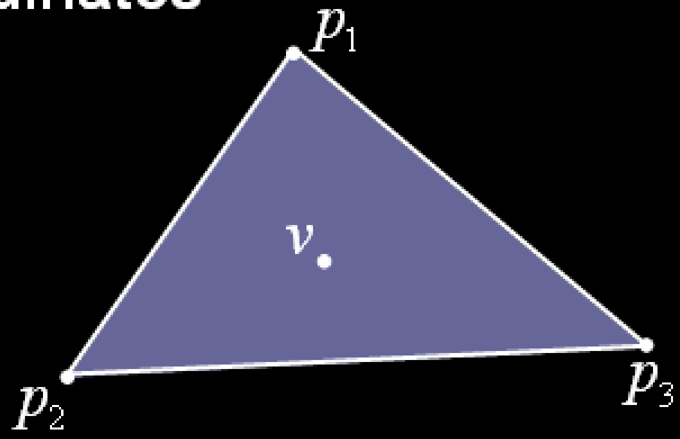


Barycentric Coordinates (Mobius, 1827)

- Given v find weights w_i such that

$$v = \frac{\sum_i w_i p_i}{\sum_i w_i}$$

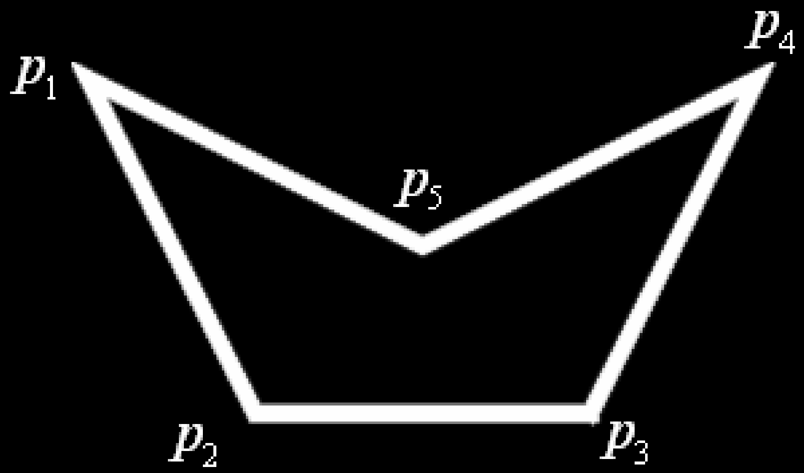
$\frac{w_i}{\sum_j w_j}$ are barycentric coordinates



Boundary Value Interpolation

Given p_i, compute w_i such that

$$v = \frac{\sum_i w_i p_i}{\sum_i w_i}$$



Boundary Value Interpolation

Given p_i, compute w_i such that

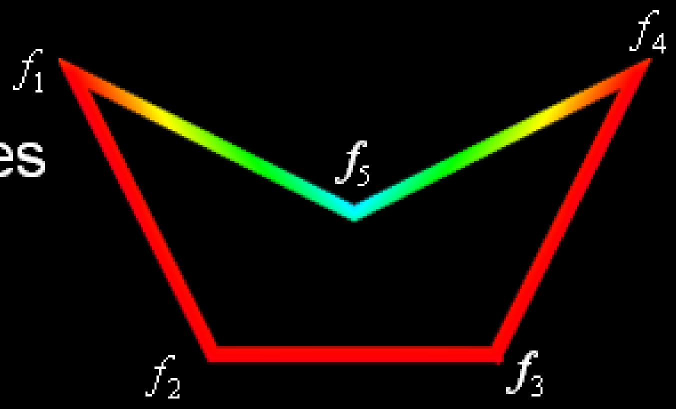
$$v = \frac{\sum_i w_i p_i}{\sum_i w_i}$$

Given values f_i at p_i, construct a function

$$\hat{f}[v] = \frac{\sum_i w_i f_i}{\sum_i w_i}$$

Interpolates values at vertices

Linear on boundary



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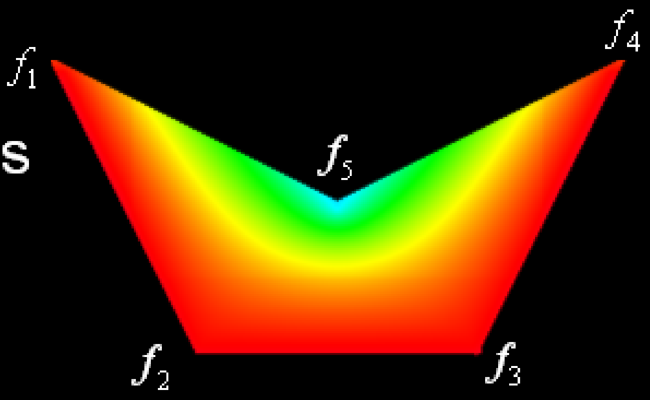
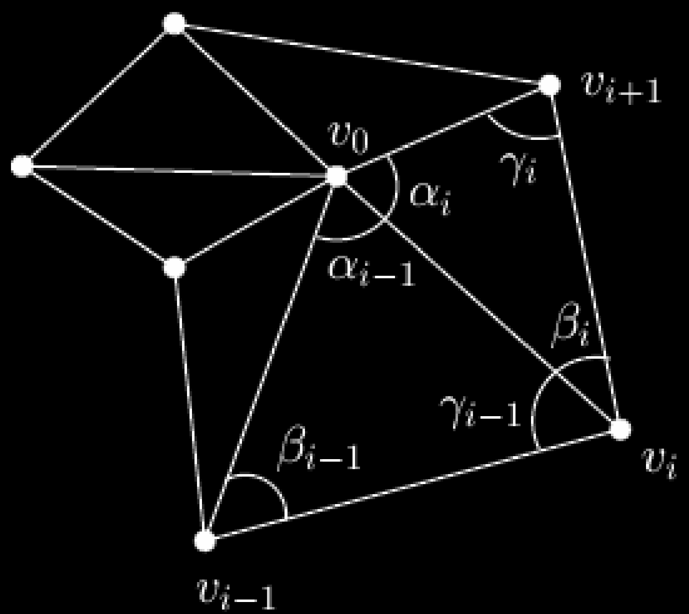
Interpolates values at vertices

Linear on boundary

Smooth on interior



Previous work: Wachpress's solution (1975)



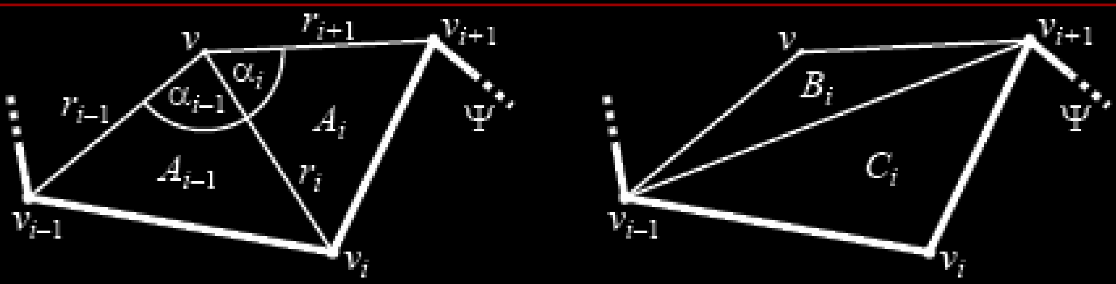
Star-shaped polygon.

weights $\lambda_1, \ldots, \lambda_k \geq 0$

$$\sum_{i=1}^{k} \lambda_i v_i = v_0, \quad \sum_{i=1}^{k} \lambda_i = 1$$

$$\lambda_i = \frac{w_i}{\sum_{j=1}^{k} w_j}, \qquad w_i = \frac{A(v_{i-1}, v_i, v_{i+1})}{A(v_{i-1}, v_i, v_0)A(v_i, v_{i+1}, v_0)} = \frac{\cot \gamma_{i-1} + \cot \beta_i}{||v_i - v_0||^2}$$

Barycentric coordinates for arbitrary polygons in the plane



$$A_i(v), \qquad -B_i(v), \qquad A_{i-1}(v)$$

$$A_i(v)(v_{i-1}-v) - B_i(v)(v_i - v) + A_{i-1}(v)(v_{i+1}-v) = 0 \qquad \text{(Coxeter, 1969)}$$

define

$$w_i(v) = b_{i-1}(v)A_{i-2}(v) - b_i(v)B_i(v) + b_{i+1}(v)A_{i+1}(v)$$

weight functions $b_i : \mathbb{R}^2 \to \mathbb{R}$ can be chosen arbitrarily

$$b_i(v) = \frac{\|v_i - v\|}{A_{i-1}(v)A_i(v)}$$

guarantee $\sum_{i=1}^{n} w_i(v) \neq 0$ for *any* $v \in \mathbb{R}^2$

$$\frac{w_i(v)}{2} = \frac{\tan(\alpha_{i-1}(v)/2) + \tan(\alpha_i(v)/2)}{r_i(v)}$$

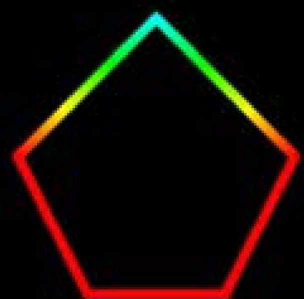
(Hormann 2004)

Floater : Mean Value Coordinates

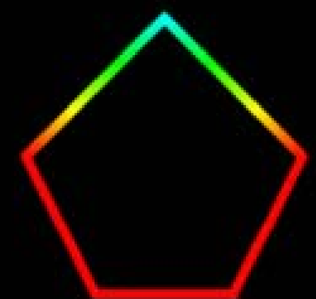
$$\lambda_i = \frac{w_i}{\sum_{j=1}^{k} w_j}, \qquad w_i = \frac{\tan(\alpha_{i-1}/2) + \tan(\alpha_i/2)}{||v_i - v_0||}$$

- These weights were derived by application of mean value theorem for harmonic functions.
- They depend smoothly on the vertices

Previous Work

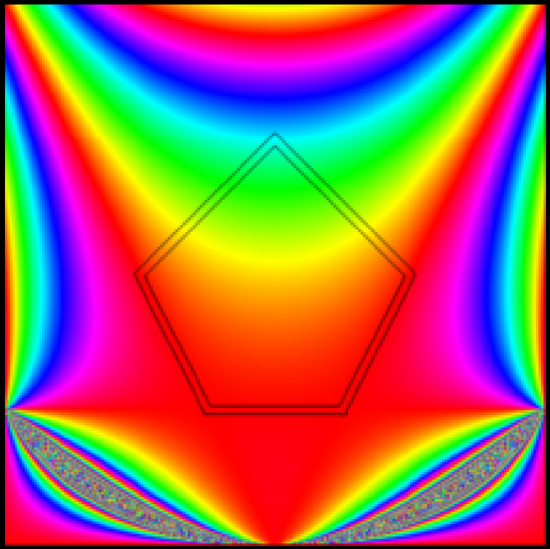


convex polygons
[Wachspress 1975]



closed polygons
[Floater 2003, Hormann 2004]

Previous Work



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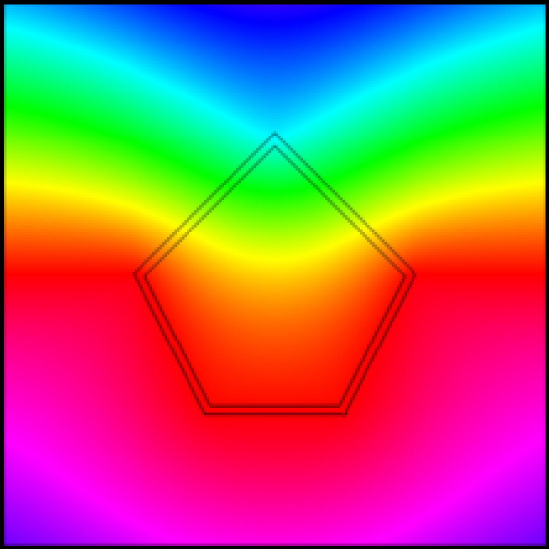

closed polygons
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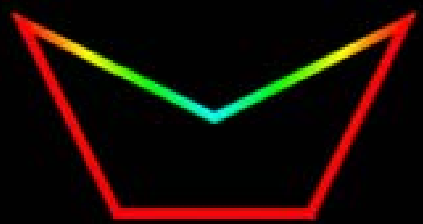
Previous Work



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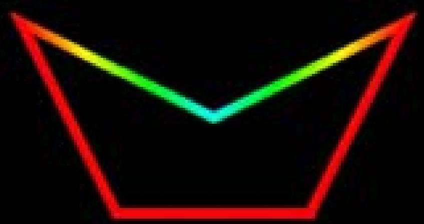
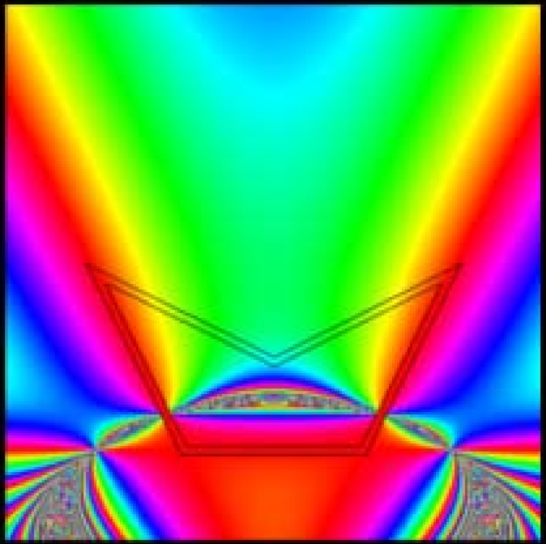


closed polygons
[Floater 2003, Hormann 2004]

Previous Work



convex polygons
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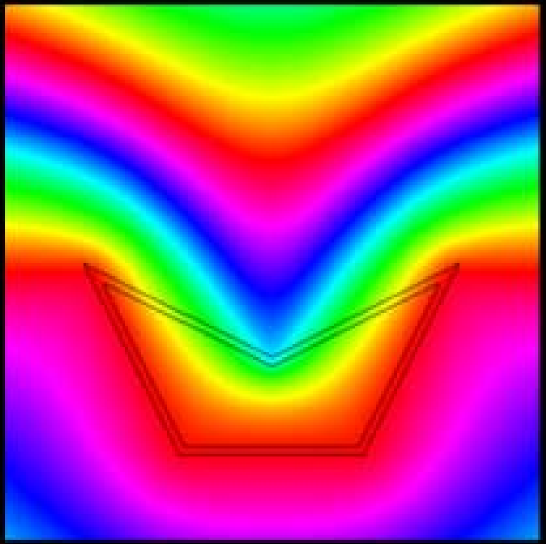
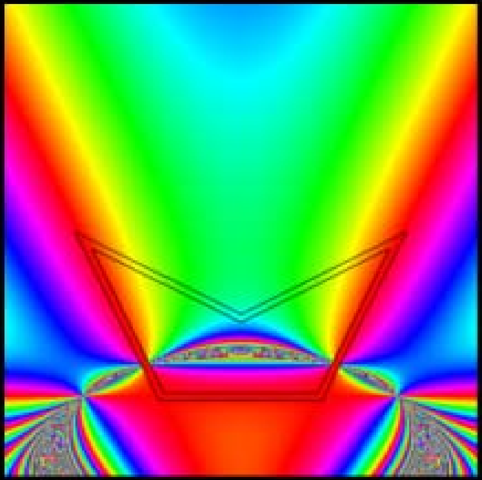
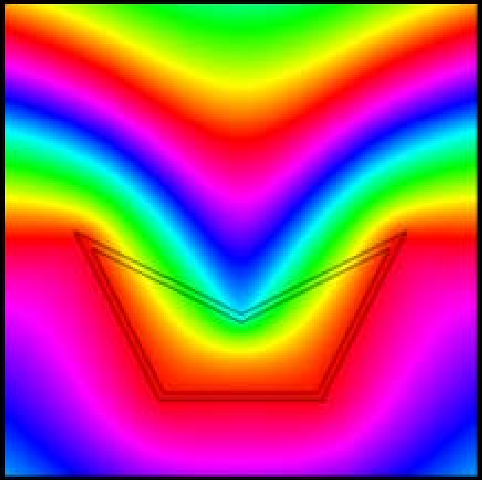


closed polygons
[Floater 2003, Hormann 2004]



convex polygons
[Wachspress 1975]

3D convex polyhedra
[Warren 1996, Warren et al 2004]



closed polygons
[Floater 2003, Hormann 2004]

3D closed triangle meshes
[Floater et al (to appear in CAGD)]

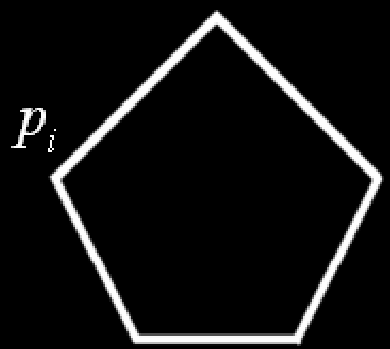
Continuous Barycentric Coordinates

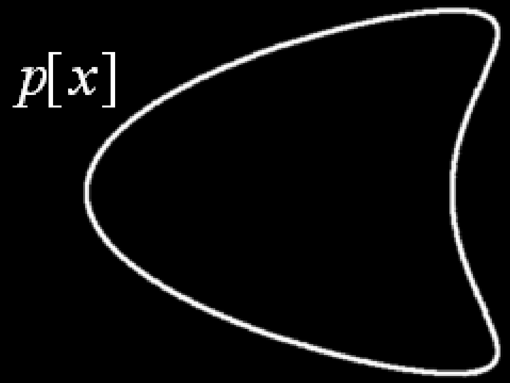
Discrete

$$\hat{f}[v] = \frac{\sum_i w_i f_i}{\sum_i w_i}$$

Continuous

$$\hat{f}[v] = \frac{\int_x w[x,v] f[x] dx}{\int_x w[x,v] dx}$$





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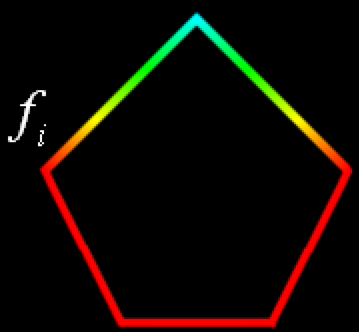

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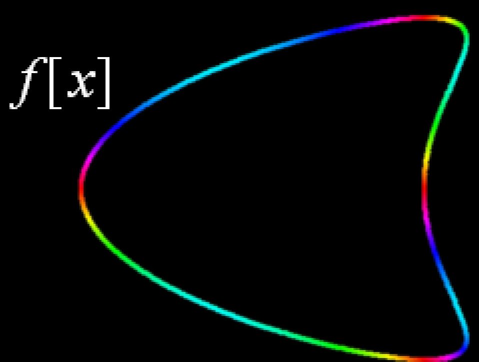
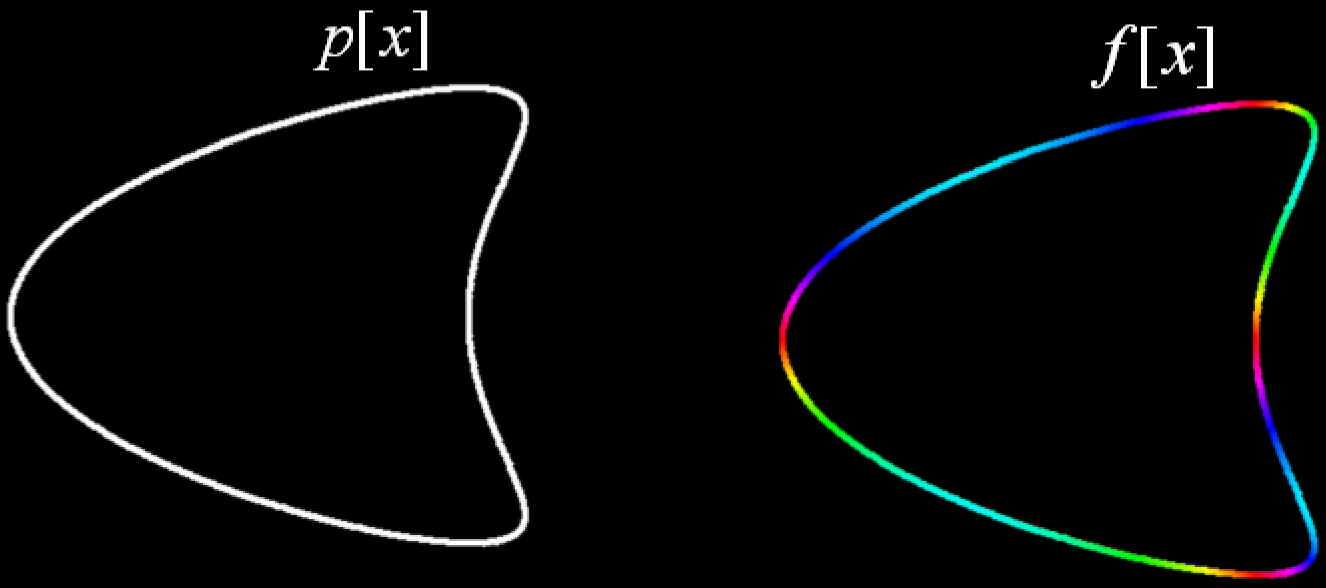


Mean Value Interpolation

$$\hat{f}[v] = \frac{\displaystyle\int_x \frac{f[x]}{|p[x]-v|}\,dS_v}{\displaystyle\int_x \frac{1}{|p[x]-v|}\,dS_v}$$

- Continuous form of mean value coordinates
- Consider evaluation of the numerator

Mean Value Interpolation



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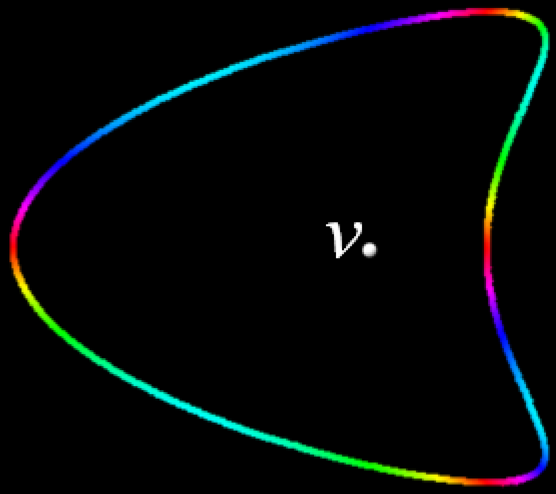

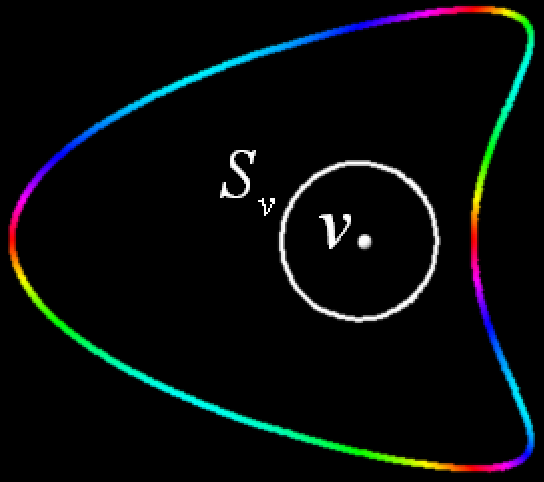

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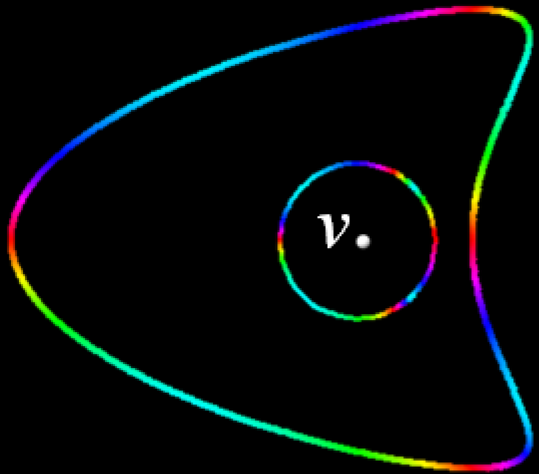
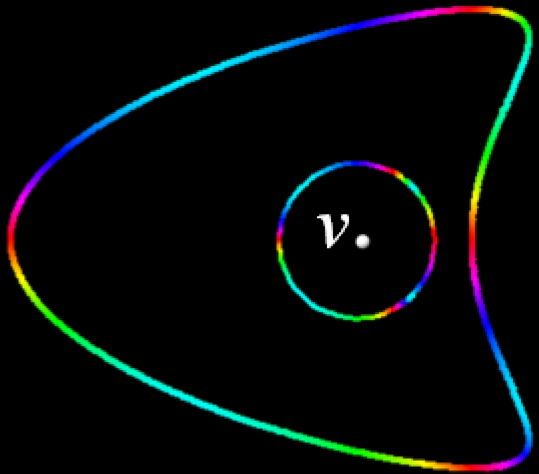


- Project the function f[x] onto the boundary of this circle

Mean Value Interpolation



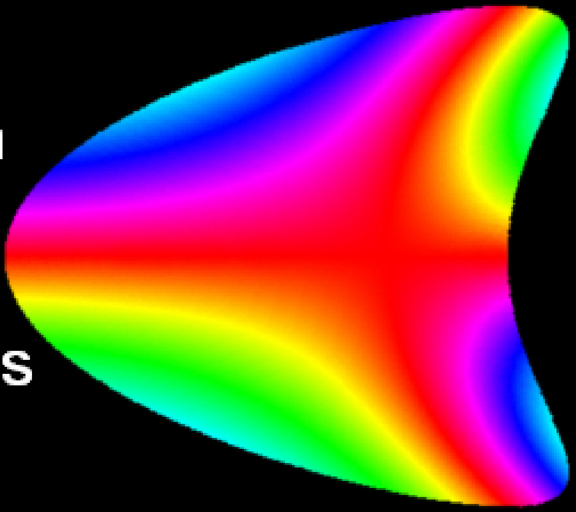
$$\hat{f}[v] = \frac{\int_x \frac{f[x]}{|p[x]-v|} dS_v}{\int_x \frac{1}{|p[x]-v|} dS_v}$$

- Integrate the projected function divided by (p[x]-v) over the circle S_v and then normalize.

Mean Value Interpolation

$$\hat{f}[v] = \frac{\int_x \frac{f[x]}{|p[x]-v|} dS_v}{\int_x \frac{1}{|p[x]-v|} dS_v}$$

- Generates smooth function
- Interpolates boundary
- Reproduces linear functions



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Generates smooth function

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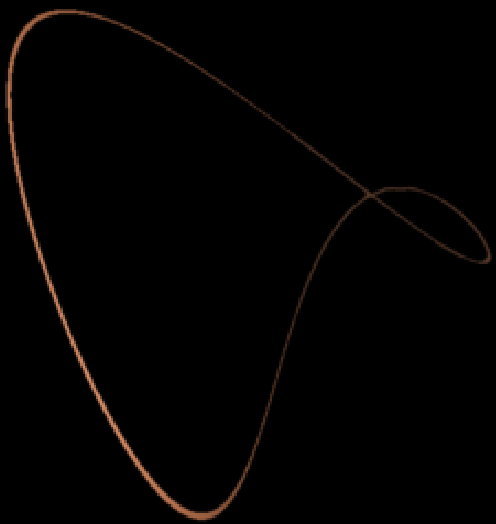

Mean Value Interpolation

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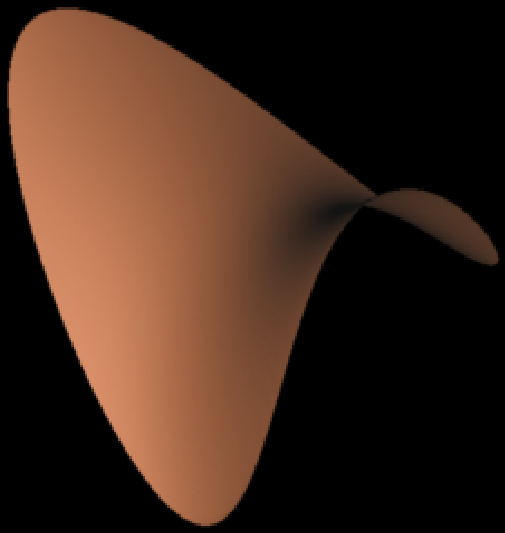
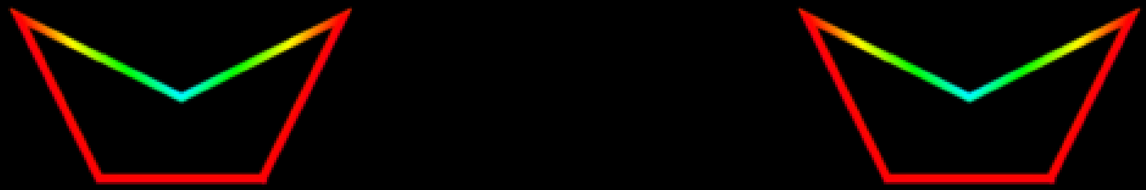


Relation to Discrete Coordinates

MV coordinates $\longrightarrow$ closed-form solution of continuous interpolant for piecewise linear shapes

Discrete Continuous



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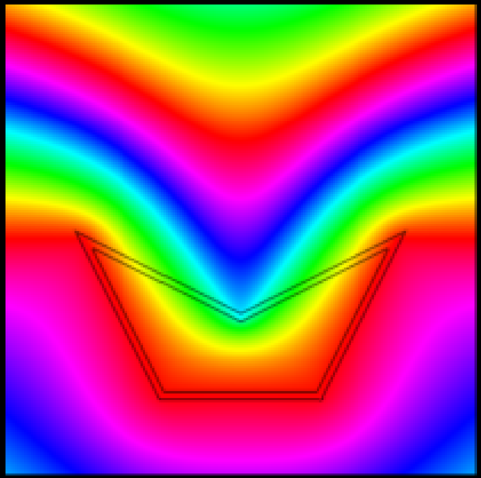
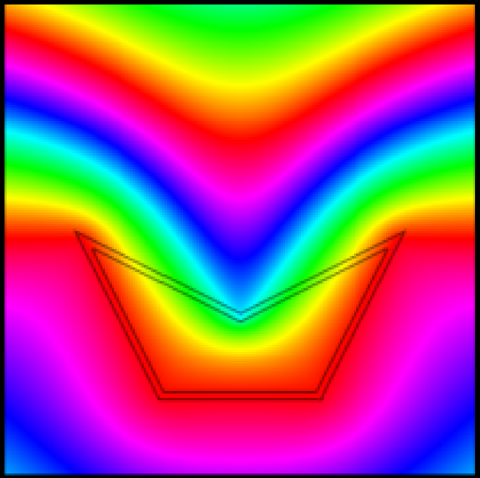


Continuous



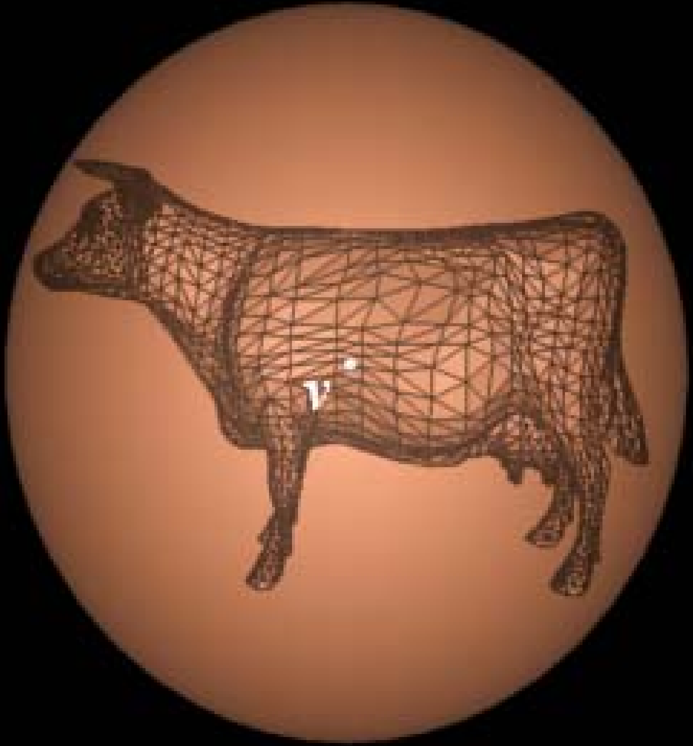
3D Mean Value Coordinates

- Find weigths w_i which allow us to represent any v as a weighted combination of the vertices of a closed triangular mesh and satisfy mean value interpolation

$$v = \frac{\sum_i w_i p_i}{\sum_i w_i} \longrightarrow \sum_i w_i (p_i - v) = 0$$

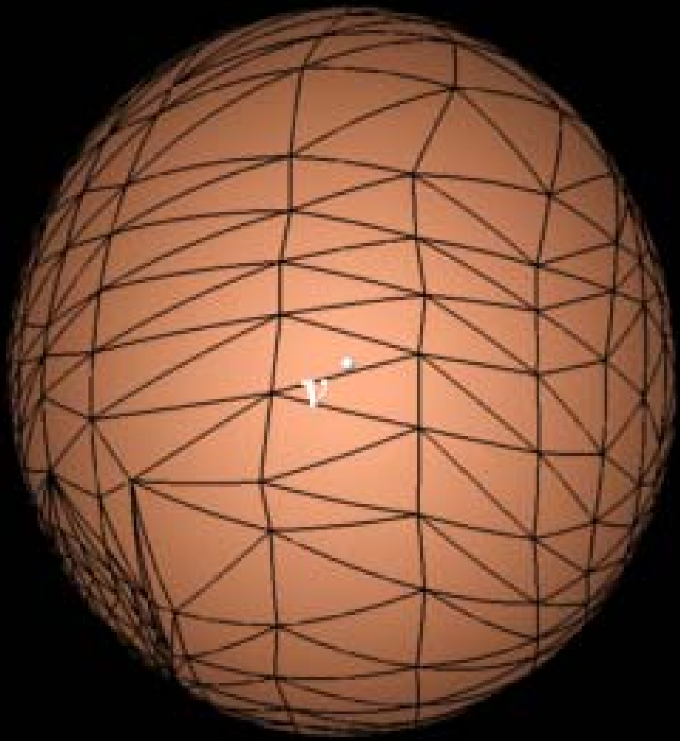
3D Mean Value Coordinates

- Given a triangular mesh and a vertex v in its interior
- Consider a unit sphere centered at vertex v



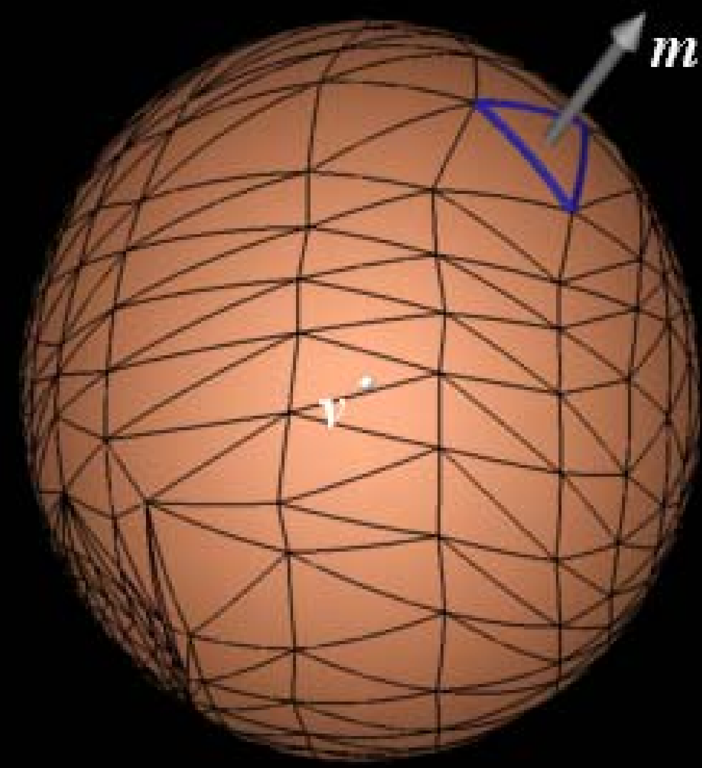
3D Mean Value Coordinates

- Project the mesh onto the surface of the sphere
- Planar triangles -> spherical triangles



3D Mean Value Coordinates

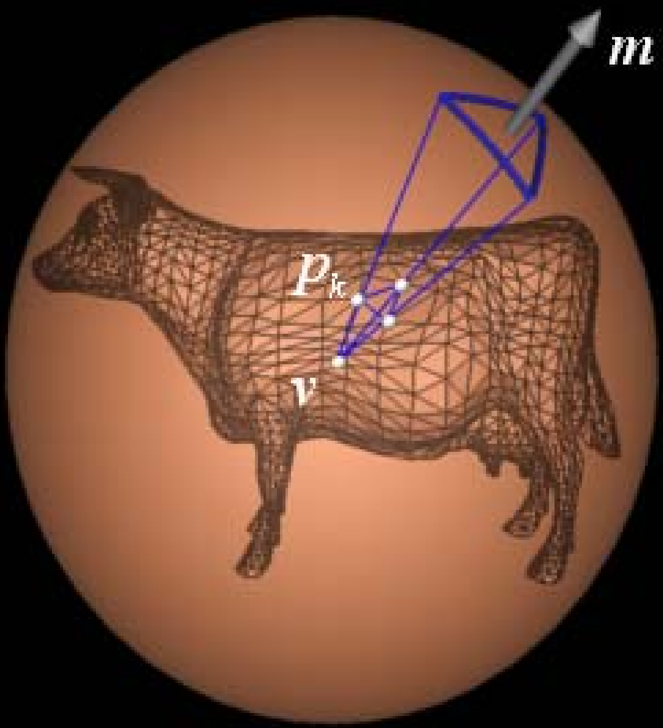
- Define m as the mean vector = integral of unit normal over spherical triangle



3D Mean Value Coordinates

- Given m, represent it as a weighted combination of the vertex v to the vertices p_k of the triangle

$$m = \sum_{k=1}^{3} w_k (p_k - v)$$

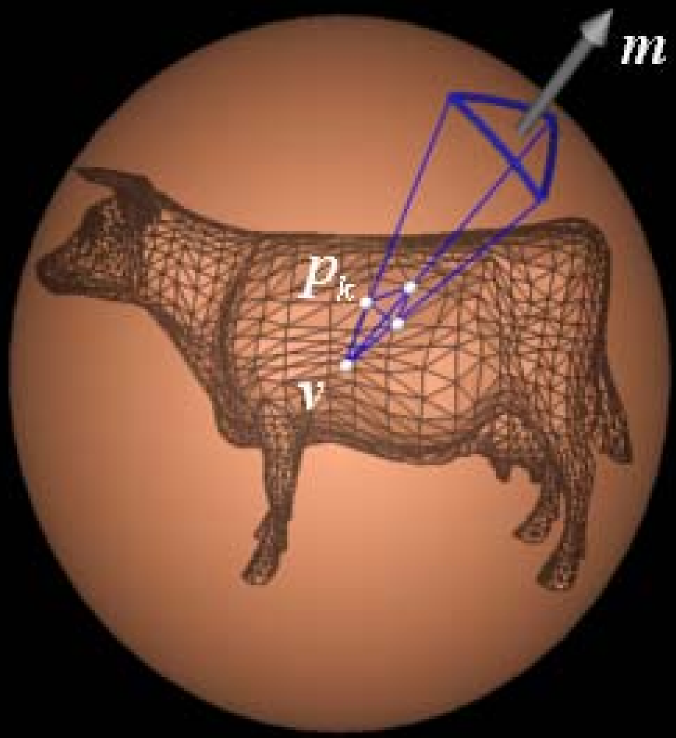


3D Mean Value Coordinates

$$m = \sum_{k=1}^{3} w_k (p_k - v)$$

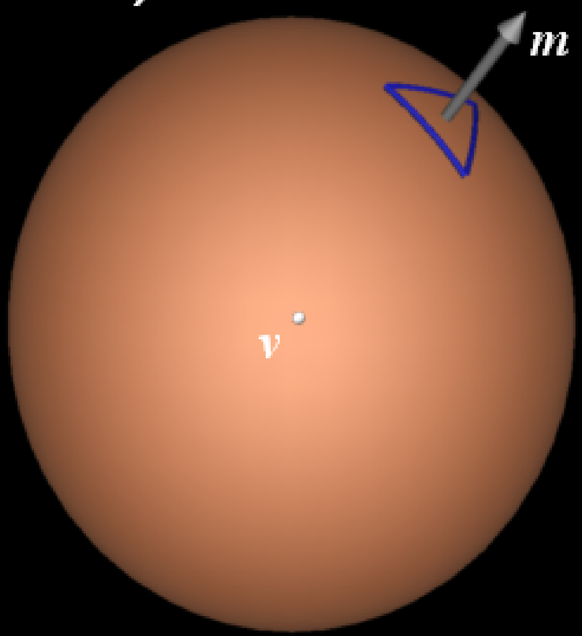
Stokes' Theorem $\sum_j m_j = 0$

$$\sum_i w_i (p_i - v) = 0$$



Computing The Mean Vector

Given spherical triangle, compute mean vector m (integral of unit normal)



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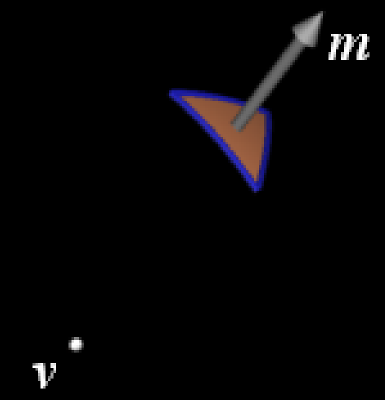
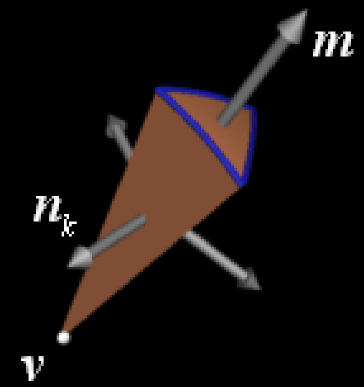


Computing The Mean Vector

Given spherical triangle, compute mean vector m (integral of unit normal)

Build wedge with face normals n_k



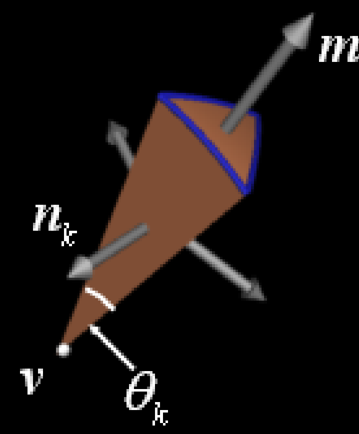
Computing The Mean Vector

Given spherical triangle, compute mean vector m (integral of unit normal)

Build wedge with face normals n_k

Apply *Stokes' Theorem*,

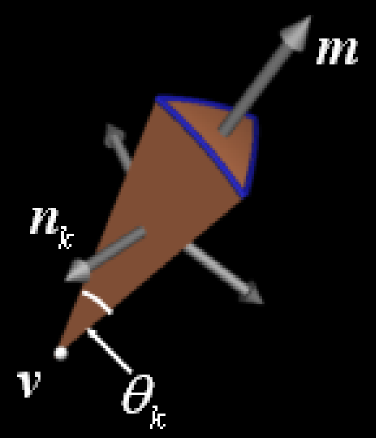
$$\sum_{k=1}^{3} \tfrac{1}{2}\theta_k n_k + m = 0$$



Interpolant Computation

Compute mean vector

$$\sum_{k=1}^{3} \tfrac{1}{2}\, \theta_k \boldsymbol{n}_k + \boldsymbol{m} = 0$$



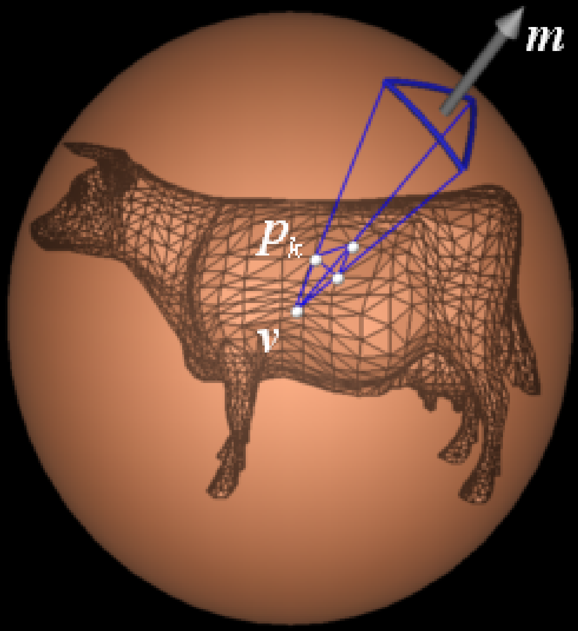
Interpolant Computation

Compute mean vector

$$\sum_{k=1}^{3} \frac{1}{2} \theta_k n_k + m = 0$$

Calculate weights

$$w_k = \frac{n_k \cdot m}{n_k \cdot (p_k - v)}$$



Interpolant Computation

Compute mean vector

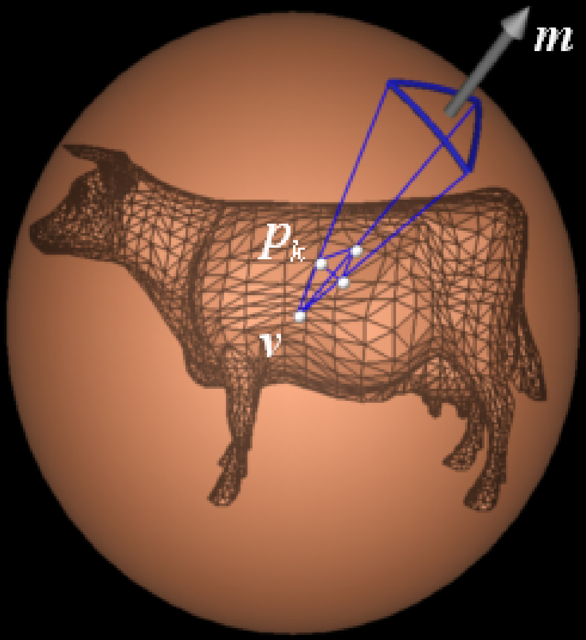
$$\sum_{k=1}^{3} \tfrac{1}{2}\theta_k n_k + m = 0$$

Calculate weights

$$w_k = \frac{n_k \cdot m}{n_k \cdot (p_k - v)}$$

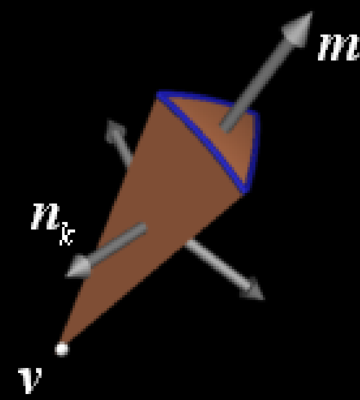
Sum over all triangles

$$\hat{f}[v] = \frac{\sum_j \sum_{k=1}^{3} w_k^j f_k^j}{\sum_j \sum_{k=1}^{3} w_k^j}$$



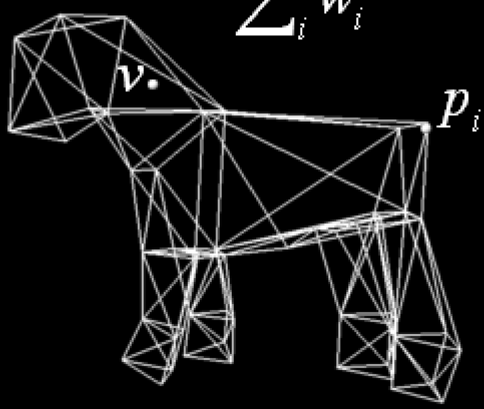
Implementation Considerations

- Special cases
  - v on boundary
- Numerical stability
  - Small spherical triangles
  - Large meshes
- Pseudo-code provided in paper



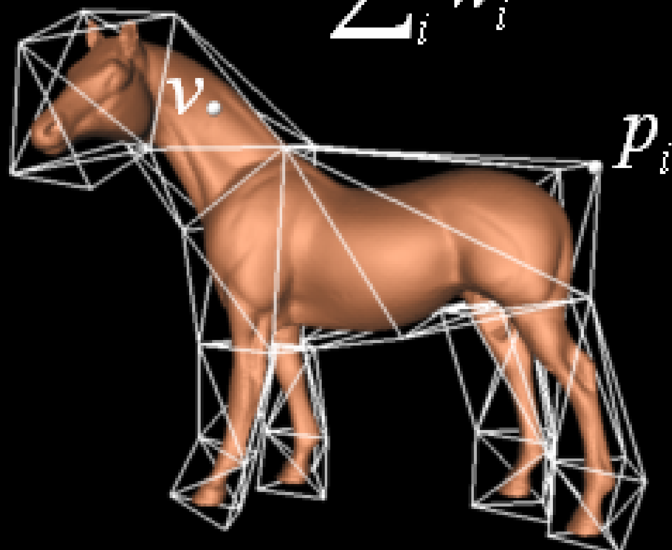
Application: Surface Deformation

$$v = \frac{\sum_i w_i p_i}{\sum_i w_i}$$

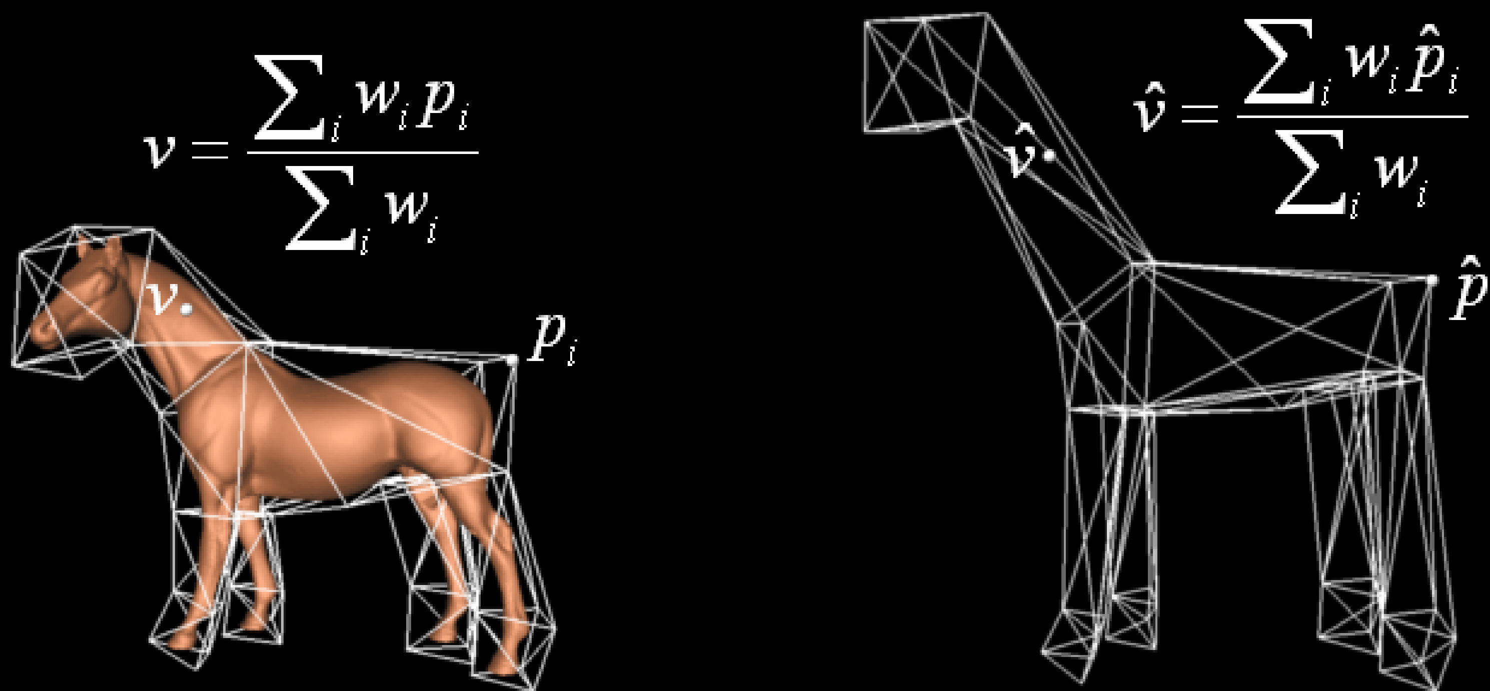


Application: Surface Deformation

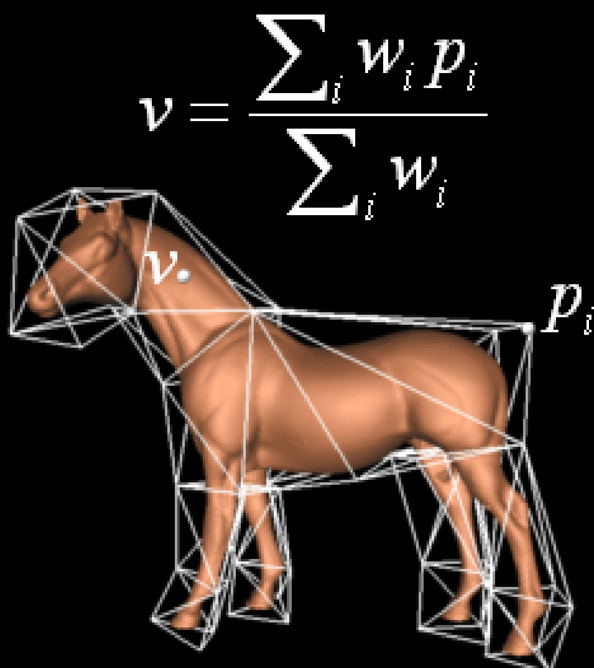
$$v = \frac{\sum_i w_i p_i}{\sum_i w_i}$$

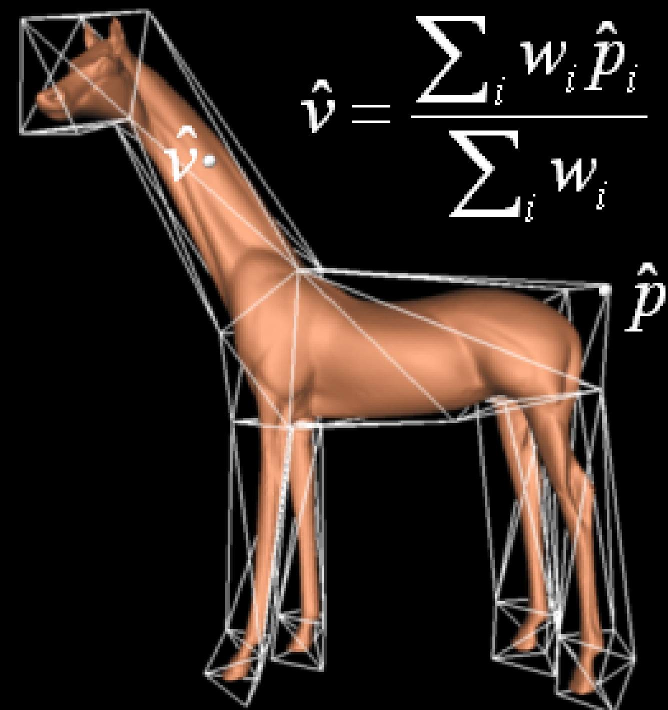


Application: Surface Deformation

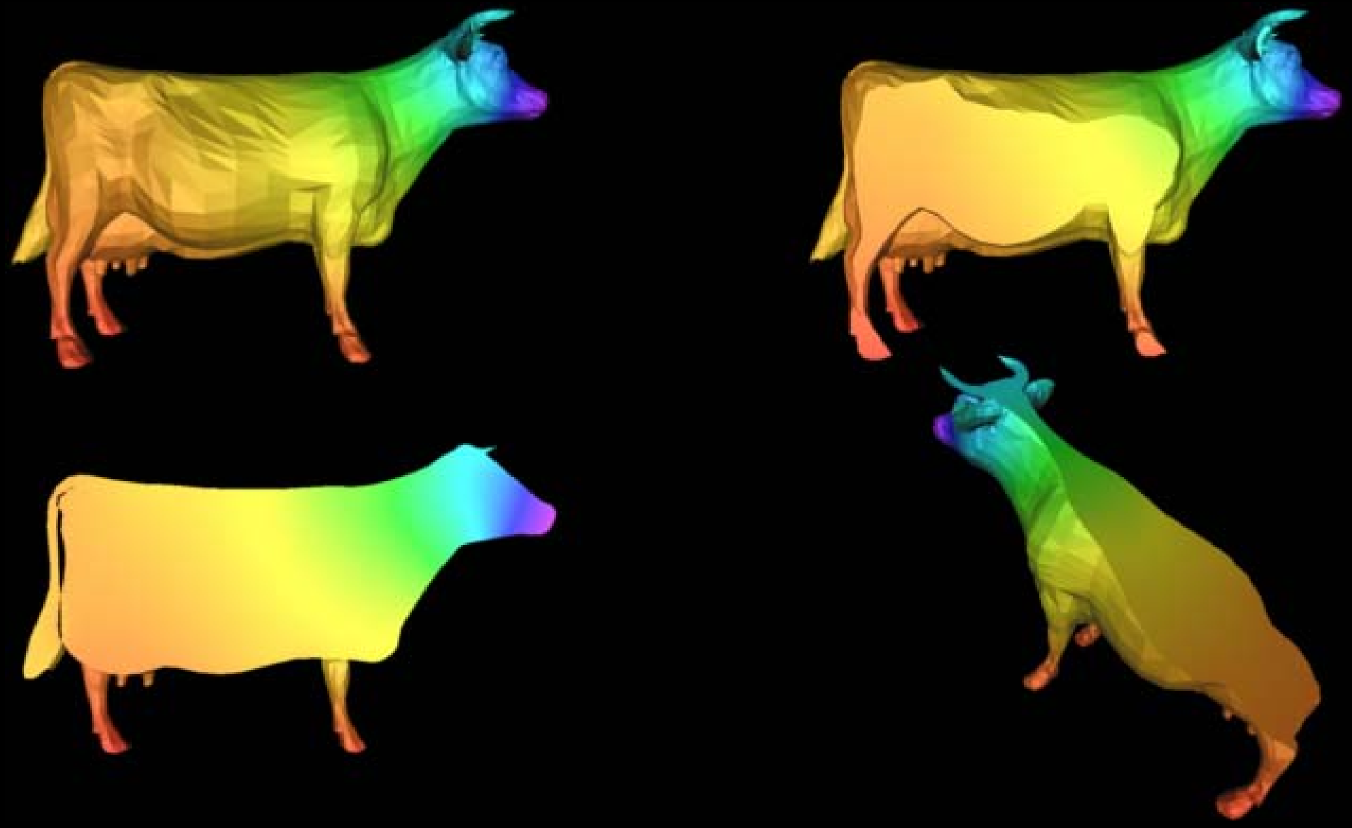


Application: Surface Deformation





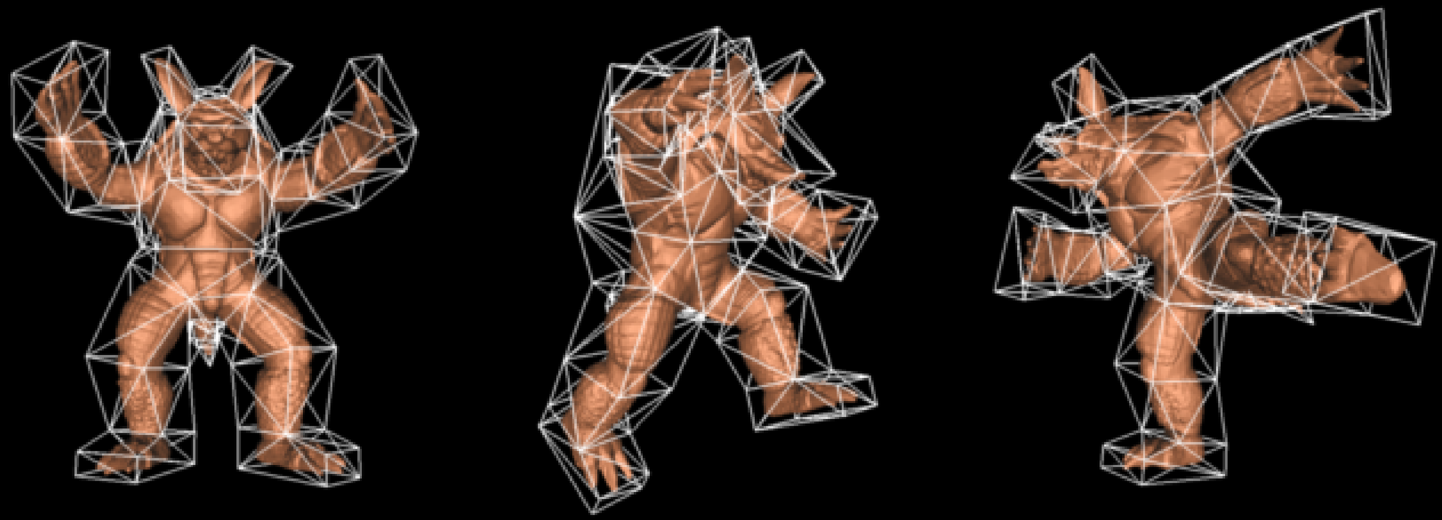
Applications Boundary Value Problems



Applications Solid Textures

Applications Surface Deformation

Control Mesh	Surface	Computing Weights	Deformation
216 triangles	30,000 triangles	1.9 seconds	0.03 seconds



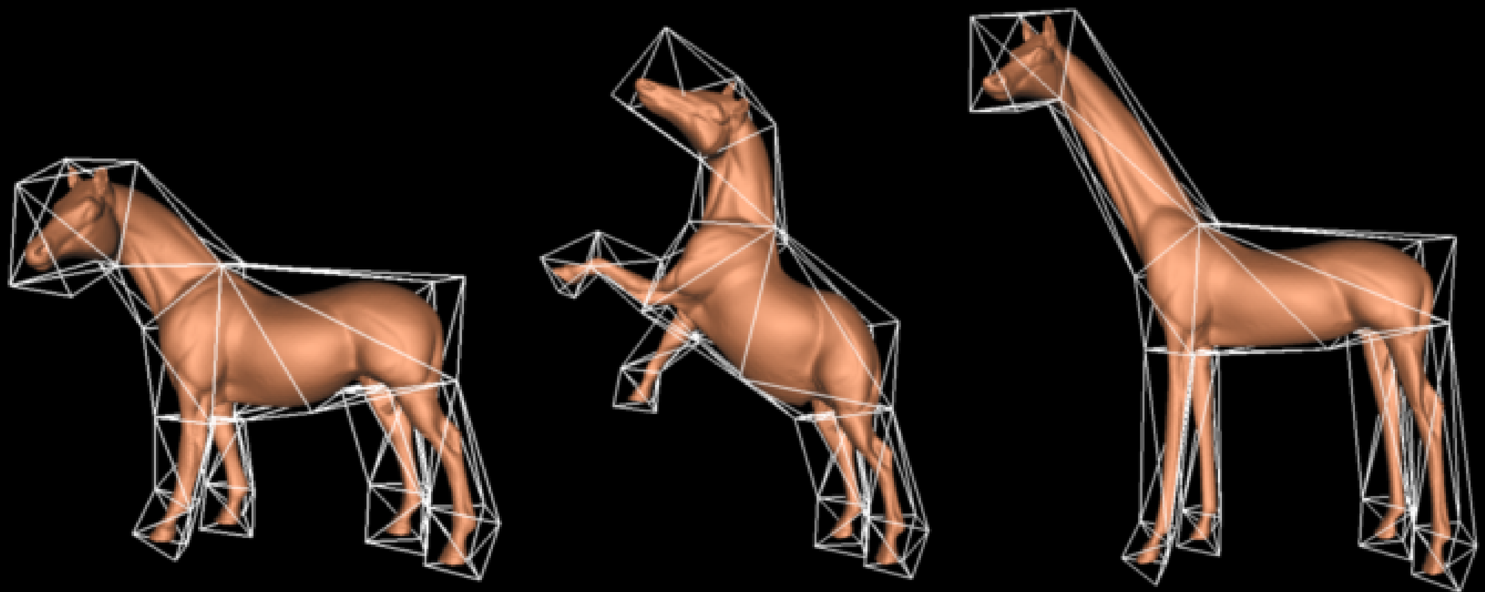
Applications Surface Deformation

Control Mesh	Surface	Computing Weights	Deformation
216 triangles	30,000 triangles	1.9 seconds	$\boxed{\text{0.03 seconds}}$

Real-time!

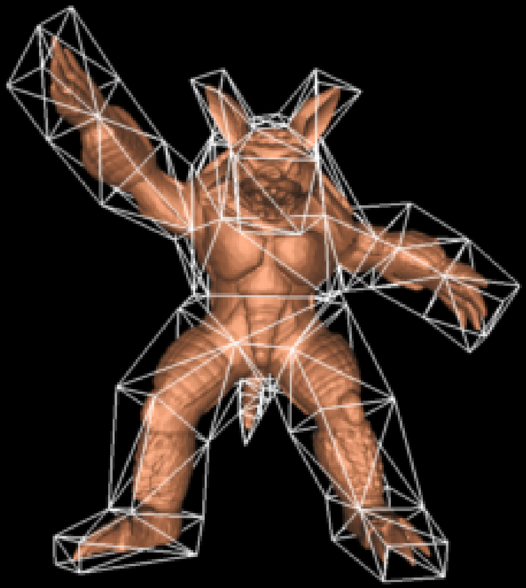
Applications Surface Deformation

Control Mesh	Surface	Computing Weights	Deformation
98 triangles	96,966 triangles	3.3 seconds	0.09 seconds



Summary

- Integral formulation for closed surfaces
- Closed-form solution for triangle meshes
  - Numerically stable evaluation
- Applications
  - Boundary Value Interpolation
  - Surface Deformation



Thank You

- Questions?