Edgeworth Box with Scaled Cobb–Douglas Utilities

1 Preferences and Endowments

There are two agents, A and B, and two goods. Total endowments are normalized to

$$x_A + x_B = 1, \qquad y_A + y_B = 1.$$

Preferences are given by scaled Cobb–Douglas utility functions.

Agent A

$$u_A(x_A, y_A) = \left(x_A^{\alpha} y_A^{1-\alpha}\right)^s, \qquad \alpha = 0.25.$$

Agent B

$$u_B(x_B, y_B) = \left(x_B^{\beta} y_B^{1-\beta}\right)^s, \qquad \beta = 0.75.$$

Equivalently,

$$u_A = x_A^{\alpha s} y_A^{(1-\alpha)s}, \qquad u_B = x_B^{\beta s} y_B^{(1-\beta)s}.$$

2 Marginal Rates of Substitution

For agent A,

$$\mathrm{MRS}_A = \frac{\partial u_A / \partial x_A}{\partial u_A / \partial y_A} = \frac{\alpha}{1-\alpha} \frac{y_A}{x_A}.$$

For agent B,

$$\mathrm{MRS}_B = \frac{\partial u_B / \partial x_B}{\partial u_B / \partial y_B} = \frac{\beta}{1-\beta} \frac{y_B}{x_B}.$$

Note that the scaling parameter s cancels out and does not affect marginal rates of substitution.

3 Contract Curve

Pareto efficiency requires $\mathrm{MRS}_A = \mathrm{MRS}_B$. With $a = 0.25$ and $\beta = 0.75$,

$$\frac{1}{3}\frac{y_A}{x_A} = 3\frac{1-y_A}{1-x_A}.$$

Rearranging,

$$y_A(1-x_A) = 9x_A(1-y_A).$$

Solving for y_A yields the contract curve:

$$\boxed{y_A = \frac{9x_A}{1+8x_A}, \qquad x_A \in (0,1).}$$

4 Utilities Along the Contract Curve

Substitute $y_A = \frac{9x}{1+8x}$.

Agent A

$$\begin{aligned} u_A(x) &= \left(x^{1/4}\left(\frac{9x}{1+8x}\right)^{3/4}\right)^s \\ &= \boxed{9^{3s/4}\frac{x^s}{(1+8x)^{3s/4}}}. \end{aligned}$$

Agent B

$$\begin{aligned} u_B(x) &= \left((1-x)^{3/4}\left(\frac{1-x}{1+8x}\right)^{1/4}\right)^s \\ &= \boxed{\frac{(1-x)^s}{(1+8x)^{s/4}}}. \end{aligned}$$

5 Core (Pareto-Improving Segment)

Let the initial endowment be $E = (x_e, y_e)$.

Endowment utilities are

$$u_A^e = (x_e^{1/4}y_e^{3/4})^s, \qquad u_B^e = ((1-x_e)^{3/4}(1-y_e)^{1/4})^s.$$

The core is the subset of the contract curve such that both agents weakly gain.

Endpoints

The lower endpoint x_L satisfies

$$\boxed{9^{3s/4}\frac{x_L^s}{(1+8x_L)^{3s/4}} = (x_e^{1/4}y_e^{3/4})^s}.$$

The upper endpoint x_U satisfies

$$\boxed{\frac{(1-x_U)^s}{(1+8x_U)^{s/4}} = ((1-x_e)^{3/4}(1-y_e)^{1/4})^s}.$$

The core is therefore

$$\boxed{x_L \le x_A \le x_U, \qquad y_A = \frac{9x_A}{1+8x_A}.}$$

6 Utility Possibility Frontier

The utility possibility frontier (UPF) is the image of the contract curve in utility space.

$$\boxed{\begin{aligned} u_A(x) &= 9^{3s/4}\frac{x^s}{(1+8x)^{3s/4}}, \\ u_B(x) &= \frac{(1-x)^s}{(1+8x)^{s/4}}, \qquad x \in (0,1). \end{aligned}}$$

7 Key Observation

The scaling parameter s affects utility levels but not:

- the contract curve,
- the core allocations,
- or Pareto efficiency.

Only ordinal preferences (MRS) matter for efficiency.

same, now with two scaling parameters

8 Preferences and Endowments

There are two agents, A and B, and two goods. Total endowments are normalized to

$$x_A + x_B = 1, \qquad y_A + y_B = 1.$$

Preferences are given by Cobb–Douglas utilities with agent-specific scaling parameters.

Agent A

$$u_A(x_A, y_A) = \left(x_A^{\alpha} y_A^{1-\alpha}\right)^{s_A}, \qquad \alpha = 0.25.$$

Agent B

$$u_B(x_B, y_B) = \left(x_B^{\beta} y_B^{1-\beta}\right)^{s_B}, \qquad \beta = 0.75.$$

Equivalently,

$$u_A = x_A^{\alpha s_A} y_A^{(1-\alpha)s_A}, \qquad u_B = x_B^{\beta s_B} y_B^{(1-\beta)s_B}.$$

9 Marginal Rates of Substitution

For agent A,

$$\mathrm{MRS}_A = \frac{\partial u_A/\partial x_A}{\partial u_A/\partial y_A} = \frac{\alpha}{1-\alpha}\frac{y_A}{x_A}.$$

For agent B,

$$\mathrm{MRS}_B = \frac{\partial u_B/\partial x_B}{\partial u_B/\partial y_B} = \frac{\beta}{1-\beta}\frac{y_B}{x_B}.$$

Key observation The scaling parameters s_A and s_B cancel out. Thus, Pareto efficiency depends only on α and β.

10 Contract Curve

Pareto efficiency requires $\text{MRS}_A = \text{MRS}_B$. Substituting $\alpha = 0.25$ and $\beta = 0.75$:

$$\frac{1}{3}\frac{y_A}{x_A} = 3\frac{1-y_A}{1-x_A}.$$

Rearranging,

$$y_A(1-x_A) = 9x_A(1-y_A).$$

Solving for y_A yields the contract curve:

$$\boxed{y_A = \frac{9x_A}{1+8x_A}, \qquad x_A \in (0,1).}$$

11 Utilities Along the Contract Curve

Substitute $y_A = \frac{9x}{1+8x}$.

Agent A

$$\begin{aligned} u_A(x) &= \left(x^{\alpha}\left(\frac{9x}{1+8x}\right)^{1-\alpha}\right)^{s_A} \\ &= \boxed{9^{(1-\alpha)s_A}\frac{x^{s_A}}{(1+8x)^{(1-\alpha)s_A}}} \qquad (\alpha = 0.25). \end{aligned}$$

Agent B

$$\begin{aligned} u_B(x) &= \left((1-x)^{\beta}\left(\frac{1-x}{1+8x}\right)^{1-\beta}\right)^{s_B} \\ &= \boxed{\frac{(1-x)^{s_B}}{(1+8x)^{(1-\beta)s_B}}} \qquad (\beta = 0.75). \end{aligned}$$

12 Core (Pareto-Improving Segment)

Let the initial endowment be $E = (x_e, y_e)$.

Endowment utilities:

$$u_A^e = (x_e^{\alpha} y_e^{1-\alpha})^{s_A}, \qquad u_B^e = ((1-x_e)^{\beta}(1-y_e)^{1-\beta})^{s_B}.$$

Endpoints of the Core

The lower endpoint x_L is defined by agent A being indifferent:

$$\boxed{9^{(1-\alpha)s_A}\frac{x_L^{s_A}}{(1+8x_L)^{(1-\alpha)s_A}} = (x_e^{\alpha}y_e^{1-\alpha})^{s_A}}.$$

The upper endpoint x_U is defined by agent B being indifferent:

$$\boxed{\frac{(1-x_U)^{s_B}}{(1+8x_U)^{(1-\beta)s_B}} = ((1-x_e)^{\beta}(1-y_e)^{1-\beta})^{s_B}}.$$

The core is therefore:

$$\boxed{x_L \le x_A \le x_U, \qquad y_A = \frac{9x_A}{1+8x_A}.}$$

13 Utility Possibility Frontier

The utility possibility frontier (UPF) is the image of the contract curve in utility space.

$$\boxed{\begin{aligned} u_A(x) &= 9^{(1-\alpha)s_A}\frac{x^{s_A}}{(1+8x)^{(1-\alpha)s_A}}, \\ u_B(x) &= \frac{(1-x)^{s_B}}{(1+8x)^{(1-\beta)s_B}}, \qquad x \in (0,1). \end{aligned}}$$

14 Summary

- The contract curve is independent of s_A and s_B.
- The core allocations are unchanged by scaling.
- Scaling parameters affect only the *cardinal representation* of utilities.
- Pareto efficiency depends solely on marginal rates of substitution.

same, now with 1 scaling parameter for A, and a=b

15 Preferences and Endowments

There are two agents, A and B, and two goods. Total endowments are normalized to

$$x_A + x_B = 1, \qquad y_A + y_B = 1.$$

Preferences are given by Cobb–Douglas utilities with one agent-specific scaling parameter.

Agent A

$$u_A(x_A, y_A) = \left(x_A^{\alpha} y_A^{1-\alpha}\right)^{s_A}, \qquad \alpha = 0.5.$$

Agent B

$$u_B(x_B, y_B) = \left(x_B^{\beta} y_B^{1-\beta}\right), \qquad \beta = 0.5.$$

Equivalently,

$$u_A = (x_A y_A)^{\frac{1}{2}s_A} \qquad u_B = (x_B y_B)^{\frac{1}{2}}$$

16 Marginal Rates of Substitution

For agent A,

$$\text{MRS}_A = \frac{\partial u_A/\partial x_A}{\partial u_A/\partial y_A} = \frac{y_A}{x_A}.$$

For agent B,

$$\text{MRS}_B = \frac{\partial u_B/\partial x_B}{\partial u_B/\partial y_B} = \frac{1 - y_A}{1 - x_A}.$$

Key observation The scaling parameters s_A cancels out.

17 Contract Curve

Pareto efficiency requires $\mathrm{MRS}_A = \mathrm{MRS}_B$. Substituting $\alpha = 0.5$ and $\beta = 0.5$:

$$\frac{y_A}{x_A} = \frac{1 - y_A}{1 - x_A}.$$

Rearranging,

$$y_A(1 - x_A) = x_A(1 - y_A).$$

Solving for y_A yields the contract curve:

$$\boxed{y_A = x_A, \qquad x_A \in (0, 1).}$$

18 Utilities Along the Contract Curve

Substitute $y_A = x$.

Agent A

$$u_A(x) = x^{s_A}$$

Agent B

$$u_B(x) = 1 - x$$

19 Core (Pareto-Efficient Segment)

Let the initial endowment be $E = (x_e, y_e)$.

Endowment utilities:

$$u_A^e = (x_e y_e)^{\frac{1}{2} s_A}, \qquad u_B^e = ((1 - x_e)(1 - y_e))^{\frac{1}{2}}.$$

Endpoints of the Core

The lower endpoint (x_L, y_L) is defined by agent A being indifferent:

$$x_L = y_L = (u_A^e)^{\frac{1}{s_A}}$$

The upper endpoint (x_U, y_U) is defined by agent B being indifferent:

$$x_U = y_U = 1 - u_B^e$$

The core is therefore:

$$x_L \leq x_A \leq x_U, \qquad y_A = x_A.$$

20 Utility Possibility Frontier

The utility possibility frontier (UPF) is the image of the contract curve in utility space.

$$u_A(x) = (1 - u_B(x))^{s_A}$$

21 Summary

- The contract curve is independent of s_A and s_B.
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