The TetraCon: A Hand-Held, Variable-geometry Multipurpose Telepresence Controller

Robert L. Read *email: read.robert@gmail.com
Avinash Baskaran †email: Baskaran.avinash@gmail.com

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Abstract

The TetraCon a hand-held device, with a data collection, processing and wireless communication system and an onboard integrated sensor network, operates as an effective and intuitive physical user interface for the larger Gluss, a biomimetic solid structural robot [citation]. With the incorporation of the Song-Kwon-Kim joint [citation], the Tetracon is capable of adopting a set of fixed configurations which proportionally corresponds to the Gluss, allowing it to control and operate to assume the configurations given in the Gluss-con. is capable of assuming a fixed set of rigid configurations corresponding the simultaneous input of signals

An innovative approach to modular robotics that is capable of both locomotion and exerting and withstanding structural forces is described. Extending the TETROBOT[1, 2, 3, 4] concept of a variable-geometry truss which is composed only of joints and linear actuators with a new 3D-printable embodiment of a spherical joint[5] produces a completely modular, mechanically strong, tentacle-like machine capable of independent locomotion. We call this truss that can ooze like a slug a *gluss*. The *turret joint* is shown to theoretically support a maximum ratio of maximum actuator length to minimum actuator length of $\varphi \equiv \frac{1+\sqrt{5}}{2} \approx 1.618....$ The simplest glussbot capable of crawling and turning, the 3TetGlussBot, is constructed of inexpensive open-source modules comprising an Arduino Mega, a custom PCB, servo controller chips, and a Bluetooth module. A 3-module robot, the 5TetGlussBot, that uses spherical magnetic joints is described which oozes at 27 cm/min (11 in/min).

*read.robert@gmail.com
†Baskaran.avinash@gmail.com

1 Introduction

Between 1996 and 2002 years ago, Arthur C. Sanderson and his colleagues published a series of papers[1, 2, 3] on modular robots. The "TETROBOT" was a variable-geometry truss, in which motion was accomplished by the change in length of linear actuators, connected in a modular geometry based on the tetrahedron and octahedron. Such a system requires a special joint which allows the actuators to remain aimed at the center of joints while supporting a certain amount of rotation about this center.

This paper builds upon that work by introducing 3D-printable embodiments of a recently invented spherical joint[5], and gives some results related to the underlying geometry and math, as well as providing references to all of the open-source materials needed to duplicate and expand on this work. This is an open-source embodiment of the TETROBOT with physically smaller actuators which is more accessible to the hobbyist or researcher with a limited budget. The development of 3D printers, Bluetooth, microprocessors such as the Arduino, and inexpensive commercial actuators has made this possible. A very simple robot having only three tetrahedra is shown to be capable of locomotion.

1.1 Motivation

Imagine a strong, light, metamorphic material that can exert or resist force. You can command it to form any shape, limited only by its flexibility. Using that coformability you can get it to crawl across very rough terrain. You can command it to form into a bridge or to lift and move heavy objects, or to perform all the functions of a crane, a forklift, a backhoe and/or bulldozer.

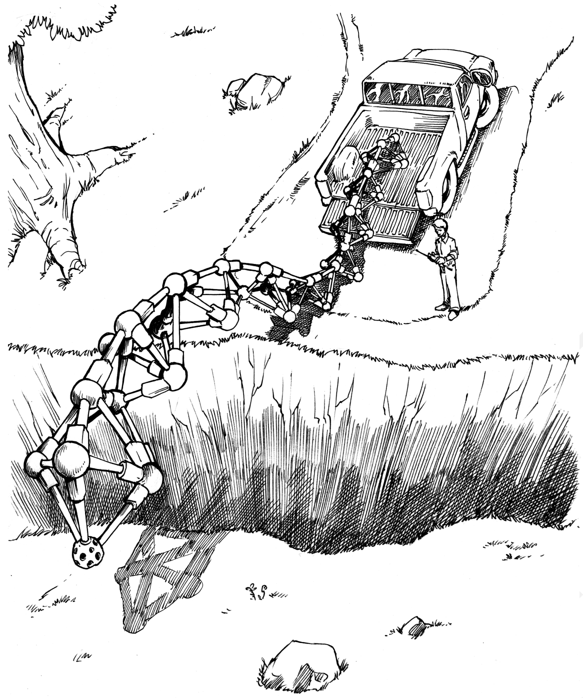

Figure 1: Concept art of GlussBot Spanning a Chasm

It is a truss that crawls like a slug—a *gluss*. If you need more of it, you buy it by the kilo and when it arrives you order it to crawl to your existing gluss. You easily join it there, creating larger, stronger, combined mass.

The advantages of snakebots have been widely recognized [6]. In general, these have been constructed with angular joints. In this paper we propose a different, truss-like approach to providing similar mobility that uses only linear actuators and spherical joints that eliminates non-axial forces so that only compressive and tensile forces act on the actuators. This potentially combines the advantages of forceful machines with snake-like mobility.

Other geometries, such as moving planes, are possible with the same material.

Additional videos at the YouTube channel, Public Invention, reachable from the above link, further motivate the *gluss* concept.

1.2 Concept: Gluss = Slug + Truss

Imagine a metamorphic or polymorphic machine that forcefully assumes a variety of shapes. It moves like a mollusc or amoeba, oozing into position as commanded. It is technically a “machine” because it can exert force reliably, but it may be thought of as a material, because unlike most robots its components are not differentiated.

Although someday an actual chemical substance may do this, today it can be constructed from commercial components and 3D-printable parts. This paper introduces the *gluss* approach to building metamorphic dynamic robots and static machines.[1]

Massively scalable robots have often been proposed. Our particular approach is to use linear actuators, which are rod-like machines that can make themselves shorter or longer. These are tied together using a relatively new joint [5] which allows, for example, as many as 12, but more realistically 4 or 6, members to be joined together sturdily at a single point. A 3-D printed embodiment presented here, called the *turret joint*, allows the change of angle required for the gluss to ooze about. Some gluss consists of some actuators joined together with some turret joints and whatever batteries and control microelectronics are needed. In the first crawling robot discussed here, the *3TetGlussBot*, there are two controllers and two batteries in addition to the 12 actuators and the 6 multi-member turret joints, but 3TetGlussBot is only a small amount of gluss compared to what we hope to build. Additionally, we present *5TetGlussBot*, an obvious extension of the 3Tet geometry implemented with magnetic joints.

[1] “Gluss” is a portmanteau of “Slug” and “Truss” because we are attempting to build a truss, or space frame, that is capable of moving like a slug or octopus. The word *gluss* should be used as a substantive noun in English, much like the word *clay* is used. The use of *glusses*, the plural of *gluss*, should be rare and refer to different kinds of metamorphic material, such as the expression “four clays” suggests four distinct types of clay without specifying how many kilograms of each one means.

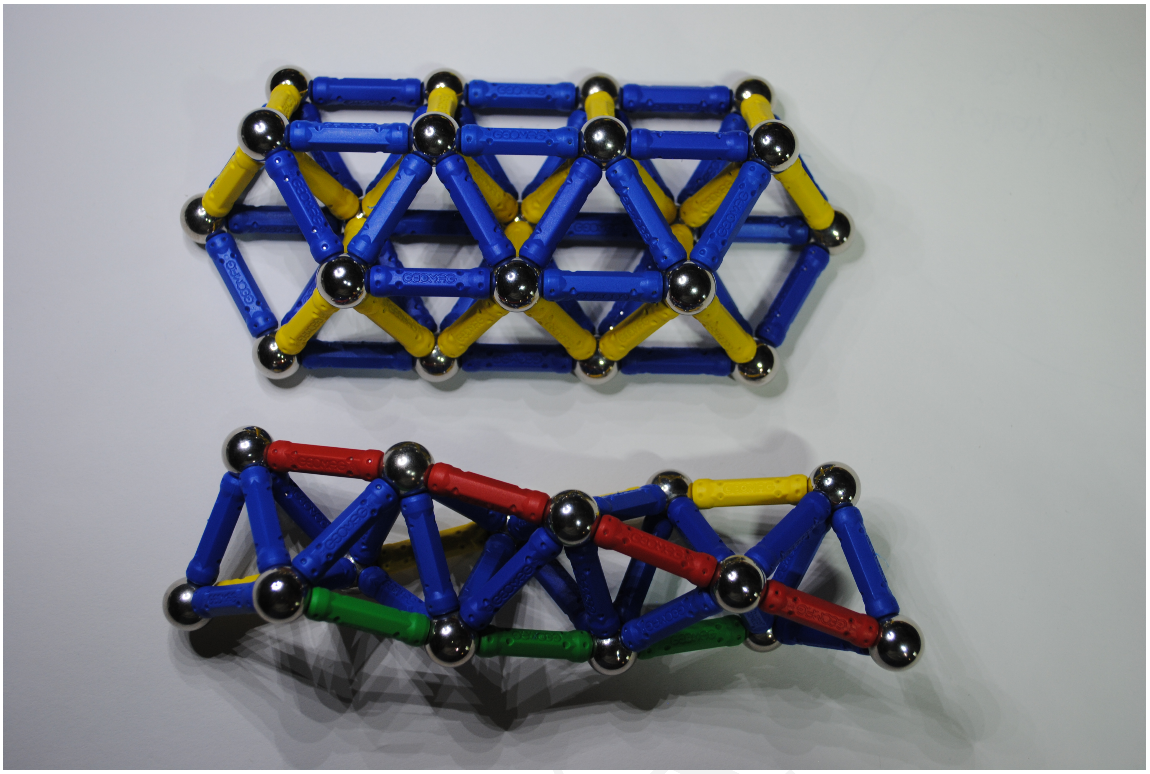

Figure 2: The Octet Truss (above) and Tetrahelix (below)

Although unlimited geometries are possible, two geometries (See Figure 2) have the advantage of being regular, thus allowing us to seamlessly connect any number of actuators into any amount of gluss with some hope of effective software control.

1. Design concept
2. Control:
    (a) Sensors
        i. Mechanism
        ii. Housing
        iii. Wiring
        iv. Circuit analysis (needed?)
    (b) Signals
        i. Multiplexing

        ii. Wireless communication

3. Turret joint
    (a) Gluss-con scaling/proportions to the Gluss
    (b) Universal joint / modularity
4. Sleeves
    (a) Modularity
5. User-friendliness
    (a) (experimentation?)
6. Applications
    (a) Assistive
        i. Body-attached assistive / tertiary limb device
        ii. Pole mounted arm
        iii. Physically independent device
    (b) Medical
        i. Rehabilitation
            A. Motor skill development for learning imparied children
            B. Skeleto-muscular rehab
            C. Physiotherapy
        ii. Prosthetics / limb replacement
        iii. Medical casts / sleeves
        iv. Surgical assistant device
    (c) Future work
        i. Optical sensing
            A. Computer - vision based detection

2 The ***Turret Joint***

2.1 The Need

The way to make something large, light, and strong is to make it inherently rigid by building it out of triangles. In a single plane, this is called a *truss* [7], and more generally is called a *space frame*. Space frames made completely from triangles tend to be rigid even if the joints that connect members allow motion, such as a pin joint or a ball-and-socket joint. This is an advantage because non-axial strain (that is, a slight change in the angular geometry of the frame) cannot cause the joint to fail, as it can with a welded joint.

But we seek a space frame that can change its shape dramatically. Imagine a radio tower in which each girder has been replaced with an actuator that can get longer or shorter. Such a tower could bend its top down to the ground, or even tie itself into a knot. To accomplish this, the joints must support significant but limited range of angular motion.

The spherical joint invented by Song, Kwon and Kim [5] is such a joint, the essence of which is rendered in their patent drawing, Figure 3. We name this joint the *Turret Joint*.

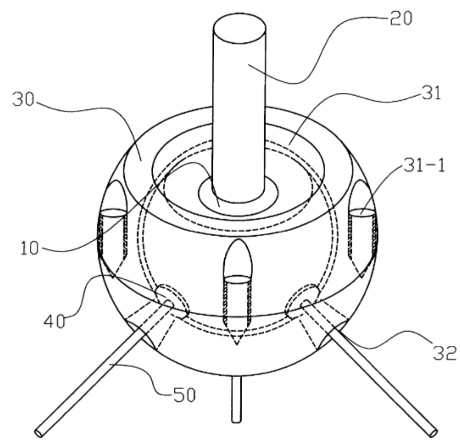

Figure 3: Song, Kwon, Kim, patent image.

When properly configured to support regular nets of actuators, it allows the gluss to be a moving space frame. It happens that the specific actuators we use are geared such that when no power is applied, they strongly resist outside forces that would change their length, essentially becoming rigid members. The resulting gluss can move into position and then be powered off to be a temporarily static space frame.

One could also use this joint with members which are not actuators. For example, we first constructed the joint with carbon fiber rods. In essence it is then a construction kit with continuously variable member lengths liberated from using a finite set of angles.

2.2 Geometry

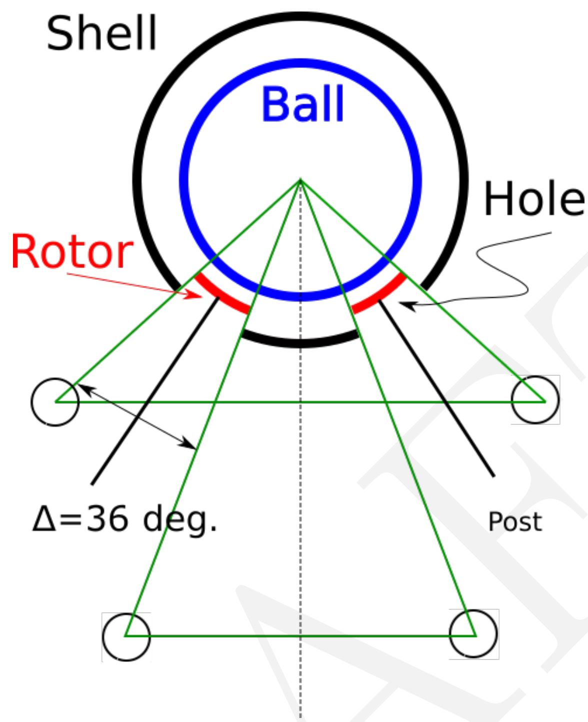

Figure 4: Turret Joint Planar Geometry

How versatile can we make the turret joint?

In particular, since we are attempting to build gluss which is regular in its use of actuators, we may ask: What is the maximum range of motion in our actuators which we can usefully employ in our gluss? To work independent of scale, we use the symbol Q to denote the ratio of the actuator length at its longest to its length at its shortest. The particular Actuonix$^{\mathrm{TM}}$ actuators we use have a Q of 1.5. But is that the maximum Q that we could utilize? Or is it already too high?

One way to approach this problem is to consider a single triangle formed by joints and actuators. The joint must support the most acute triangle that can be formed with the three actuators and the most obtuse triangle that can be formed with the actuators.

In fact it is a surprising result that we prove in Appendix A that the maximum Q which can be utilized by an ideal turret joint happens to be the famous golden ratio, $\varphi \equiv \frac{1+\sqrt{5}}{2} \approx 1.618...$, and the maximum deviation for any one member coming into the joint is 36°. Thus in Figure 4 the triangles drawn are in fact a Golden Triangle and a Golden Gnomon. A real-world joint, which will support less variation because the "post" must have a certain thickness and the "rotor" must have a lip slightly larger than the hole in order to remain locked in place. Furthermore, the joint adds a certain necessary thickness, the minimum length from joint-center to joint-center will be somewhat greater than from actuator tip to actuator tip.

However, the theoretical ideal result is a valuable approach to physical computation and makes a Q for a physical actuator of 1.5 seem quite appropriate.

2.3 Embodiment

Although the joint could be machined or formed in some other way, 3D Printers have made the construction of the Turret Joint far easier. We have designed a complete set of components needed to 3D print the joint and the rotors to attach to the linear actuators. These models are created with OpenSCAD, a functional parametric modeling program.

Our experience has been that the common plastics PLA and ABS are adequate for the Turret Joint, but have found that nylon, which is far tougher and less prone to cracking, is superior for the rotors which bolt directly to the actuators.

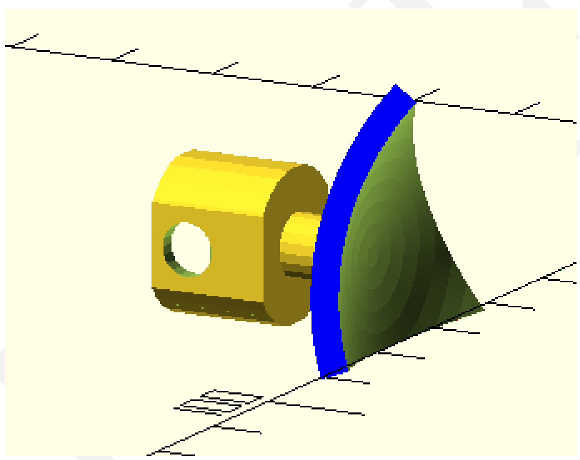

Figure 5: Triangular Rotor Model.

We have innovated the design of the rotor by using a triangular section of a sphere as the rotor rather than a circular section, as shown in Figure 5. Assuming that each actuator is free to rotate about its axis as well as revolve about the center of the ball joint, this shape does not limit motion even in the most pinched configurations. The triangular rotor provides greater extent of contact, which presumably makes the joint motion smoother and less likely to bind.

Figure 6 shows most of the parts. The nylon triangular rotor is white and rests upon the red ball. The green part is a Tetrahelix lock, and the yellow parts are the locks for the Octet Truss geometry.

Figure 6: 3D Printed Parts

In Section **??**, one of our actuators and also a carbon rod with 3D-printed tubular mounts are shown in Figure **??**. The carbon rod and mounts can be used as a construction system to build a static structure. By cutting the rods to any length (with $Q \leq \varphi$), you can build a static version of any geometry that the actuators are capable of dynamically achieving.

2.4 Specific Geometries

Although the possible ways to configure actuators and joints is limitless, the simplest thing is to use regular, repeatable geometries. The two most obvious are the Boerdijk-Coxeter helix (more easily called the *tetrahelix*) `https://en.wikipedia.org/wiki/Boerdijk%E2%80%93Coxeter_helix` and the *Octet Truss* [8]. It is instructive to compare Figure 7 with the photo of the same object made with the Geomag™ toy shown in Figure 2.

Roughly speaking, the Tetrahelix is a good way to make a long shaft or tentacle, and the octet truss is a good way to make a planar shape. The purpose of a tentacle is to curl, and the we have no word for a stingray-like plane that rolls itself up into a cylinder or cone or forms a barrel vault. Buckminster Fuller discusses both the tetrahelix and octet truss [9] in terms of static structures and geometries.

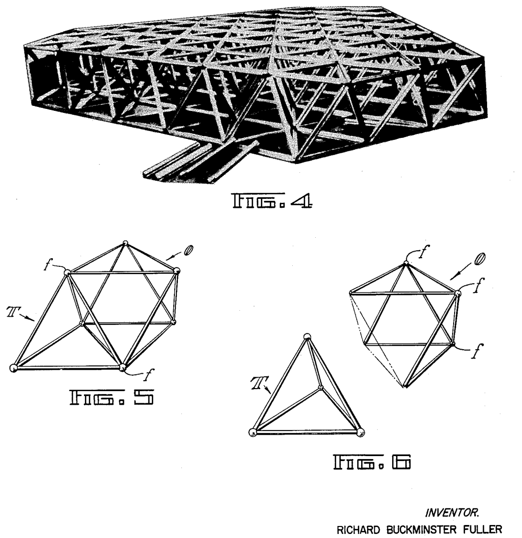

Figure 7: The Octet Truss, selected from Buckminster Fuller's patent.

The Octet Truss reflects the geometry called the *cuboctahedron*.

Because the Turret Joint does not support infinite revolution of all members, the joint, or the "lock" or "stator" in particular, must be adjusted to the regular geometry that one is implementing. Figure 8 indicates this difference. The image is from TurrentJoint.scad: `https://github.com/PubInv/turret-joint/blob/master/Models/TurretJoint.scad`, an open-source file that can be used to 3D print all of the turret joint components. It shows a lock part for the Tetrahelix geometry on the left, and the more expansive Octet Truss geometry on the right. Observe that some of the holes are semicircular, half in this part, and half in the "cap" parts not shown here.

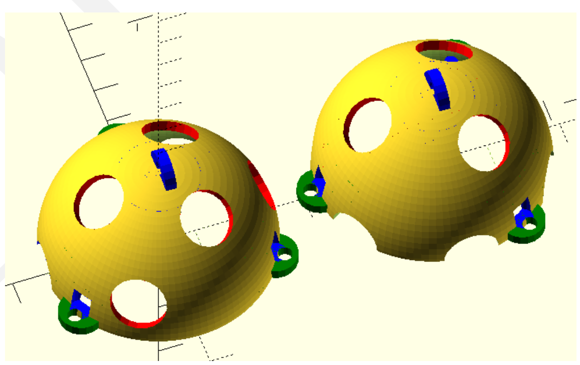

Figure 8: Lock Comparison

We have additionally experimented with magnetic joints similar to those use in the Geomag$^{\text{TM}}$ toy. These joints use 1/2” diameter by 1” long neodymium magnets restrained within small cages such that the circular face may contact a 2” diameter hollow steel ball. This provides a pull force greater than the 50 Newton force applied by our linear actuators.

Although this joint functions well and easily achieves the intention of making the robot quick to assemble and disassemble for travel, it is easily broken by non-axial forces exerted from outside the robot. Since climbing over obstacles and stairs will necessarily involve side forces on the actuators, we intend to move away from the magnetic joints.

Furthermore, it would be very expensive to scale upward (and similarly becomes less expensive when scaling downward.) We believe the magnetic joint could be a good engineering solution for glussbots smaller than the current general scale of 50 Newton, 400mm-average-length actuators.

2.5 Open Source Realizations

3 Related Research

Between 1996 and 2002 years ago, Arthur C. Sanderson and his colleagues published a series of papers[1, 2, 3] on modular robots. The “TETROBOT” was a variable-geometry truss, in which motion was accomplished but the change in length of linear actuators, connected in a modular geometry based on the tetrahedron and octahedron. A quadrupedal robot was constructed completely out of the tetrahedral/octahedral geometry. The TETROBOT robots successfully walked and even rolled. The TETROBOT hardware was significantly heavier and more powerful that then hardware used here. The glussbots have so far demonstrated no greater functionality, although we have demonstrated that very simple robots consisting of only 3 tetrahedra can locomote.

The technology presented in this article has drastically lowered the cost, thus making the glussbot/TETROBOT concept accessible to hobbyists and researchers on a limited budget.

The TETROBOT used a joint called the CMS joint. Although possibly superior in not allowing an extra degree or rotational freedom, it would be a challenge to use the CMS joint with the Actuonix actuators because the pushrod must fully retract, or the length of the pushrod would be have to be extended with an attachment. Sanderson’s students used actuators that extended from the middle, avoiding this problem. If the Gluss Project ever develops its own actuators, it should explore using this joint.

If you read the introduction of the brilliant book by Shigoe Hirose[10] substituting “even simpler soft squiggly thing that might not be as cylindrical as a snake” for the word “snake”, you will have an excellent motivation for the gluss concept. More generally, much of the work developed for snakebot locomotion[6] is directly reusable, in the sense that a long enough tetrahelix can model a snake, and further inspires the idea of using simpler models mapped into a gluss model to perform complex movements.

Buckminster Fuller also promoted *tensegrity*, and some research on Tensegrity Robots has been done, the work of Paul, Valero-Cuevas, and Lipson[11] being a good starting point. This work has developed into a serious effort[12] by NASA to explore tensegrity robots for extraterrestrial exploration.

Tensegrities are closely related to the gluss concept, more researched, and potentially more performant. In fact a gluss could be considered a special case of a tensegrity, using vanishingly short cables and, in the terminology of [11], *strut-collocated actuation*. It has been reasonable to produce a static gait for the 3TetGlussBot and 5TetGlussBot because its behavior is not very dynamic: it is so slow and strong that velocity is irrelevant at the current scale. Reported tensegrity robots have focused on dynamic, "hopping" and rolling gaits.

It is possible that gluss is easier to work with for an actual human being on the ground. Although of course both systems will use computer control systems, one can imagine a large robot crawling into place imperfectly, and some workperson making a manual adjustment: "Actuator #37, get shorter!" This is intellectually more difficult for a tensegrity, wherein changing a cable length has less predictable impact on the tensegrity geometry. However, many of the future steps outlined in Section 4 apply to both gluss and tensegrity robots.

This paper presents the gluss as a "machine", rather than a "mechanism". That is, it motivates gluss by asserting it can exert and resist force, yet currently treats gluss positioning as a purely kinematic, rather than dynamic problem. The actuators currently in use are geared such that they are so slow and powerful that the behavior is not really dynamic. If static analysis of a resulting geometry is needed, for example to ask if structure used as a bridge will bear a load, a finite element approach[13] will be adequate.

If one chooses to attempt to exert a high enough force or to move more quickly, classic robot control theory which models forces and velocities based on Lagrangian mechanics will be required.

4 Future Steps

5 Contact and Getting Involved

Public Invention is a free-libre, open-source research, hardware, and software project that welcomes volunteers. It is our goal to organize projects for the benefit of all humanity without seeking profit or intellectual property. To assist, contact <read.robert@gmail.com>.

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A Turret Joint Geometric Limitations

A.1 Geometric Preparation

Defining all angles against a center line between two rotor holes in a plane, let:

$A \equiv$ Min Actuator Length. Without loss of generality, assume this is 1.0.
$Z \equiv$ Max Actuator Length
$Q \equiv \frac{Z}{A}$, The ratio of the actuator lengths, noting that $Q \geq 1$. Since $A = 1$, $Q \equiv Z$.
$G \equiv$ Greatest Angle Reachable by Center of Rotor
$L \equiv$ Least Angle Reachable by Center of Rotor
$P \equiv$ Angle from the Post to the edge of the Rotor
$\theta \equiv$ Angle of Inmost Edge of the Rotor Hole
$\psi \equiv$ Angle of Outmost Edge of the Rotor Hole

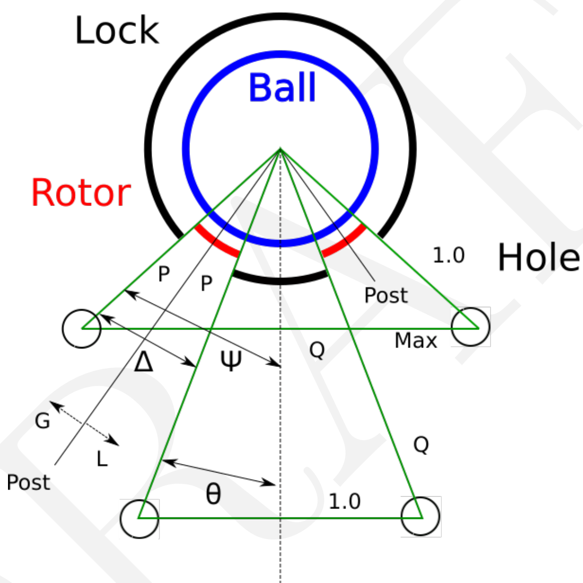

Figure 9: Turret Joint Geometry Constraints

What θ, ψ maximizes Q and what is Q?

We will define the angle of the hole to be:

$$\Delta \equiv \psi - \theta \qquad \text{(Delta Definition)}$$

From the diagram:

We then place engineering constraints upon these variables to represent physical conditions which define the limits of the joint. We name these constraints *meet*, *bump*, *capture*.

The *meet* condition means that post is actually within the hole.

The *bump* condition means that the rotors don't bump into each other in their extreme position.

The *capture* condition means that the the rotor can't fall out of the hole.

In both cases, we have have the capture constraint:

$$\Delta \geq G - L \tag{capture}$$

To realize the most acute position, we have constraints:

$$\theta \leq L \tag{acute meet}$$

$$\frac{\Delta}{2} \leq \theta \tag{acute bump}$$

To realize the least acute position, we have constraints:

$$\psi \leq G \tag{obtuse meet}$$

It is perhaps obvious that the rotors cannot bump in the most obtuse position, but we observe that:

$$360^\circ - 2 \cdot \psi \geq \frac{G - L}{2} \tag{obtuse bump}$$

is invariably true because $2 \cdot \psi < 180^\circ$ because psi is a half angle of a triangle, and likewise $\frac{G-L}{2} < 180^\circ$.

A.2 Maximum usable Q $= \varphi$

In the previous subsection we set up geometric conditions generally so that we could consider them from both a Classical Trigonometry and from Rational Trigonometry. Although we used the world "angle", we did not rely additive properties of angles except in the definition of Delta and the "obtuse bump" condition. This will allow us to perform an analysis in Rational Trigonometry in the next section.

When we attempt to set G and L to be as wide apart as possible by asserting $G \equiv \psi$ and $L \equiv \theta$, we must ensure that all of thse constaints are true. The two meet conditions become true by equality. Likewise the capture condition is trivially true by substitution. The obtuse bump condition can never be false. However, the acute bump conditions remains:

$$\frac{\Delta}{2} \leq \theta \quad \equiv \quad \frac{G - L}{2} \leq L \quad \equiv \quad G \leq 3 \cdot L \tag{acute bump}$$

$$G \equiv \arcsin \frac{Q}{2}$$

$$L \equiv \arcsin \frac{1}{Q \cdot 2}$$

Returning to our definitions of G and L,

$$\arcsin \frac{Q}{2} \leq 3 \cdot \arcsin \frac{1}{Q \cdot 2}$$

So our question becomes what is the maximum Q that satisfies this inequality. Noting from the graph of the arcsin function in the relevant ranges where $Q \geq 1$ that G is a monotonically increasing function and L is a monotonically decreasing function of Q (since Q is in the denominator), the maximum Q will be the solution to:

$$\arcsin \frac{Q}{2} = 3 \cdot \arcsin \frac{1}{Q \cdot 2} \tag{original}$$

Upon investigating this with a spreadsheet we noted that Q suspiciously approached the famous number 1.618..., the golden ratio, φ. We investigated further with the help of Wolfram Alpha Pro (https://www.wolframalpha.com). Wolfram Alpha gave us a closed-form solution to *original* of $\frac{1+\sqrt{5}}{2}$ that we recognized as φ, but oddly would not show us the set of transformations. After verifying that $Q = \varphi$ was a solution in that way, we used the special property of φ that $\varphi = \frac{1}{\varphi} + 1$, to rewrite our constaint as:

$$\arcsin \frac{Q}{2} = 3 \cdot \arcsin \frac{Q-1}{2} \tag{assuming $Q = \varphi$}$$

which led Wolfram Alpha Pro to indeed provide a long and complex chain of trigonometric transformations to show that $Q \equiv \frac{1+\sqrt{5}}{2} \equiv \varphi$ is actually an algebraic solution to this equation.

Thus the surprising result is obtained that, for a turret joint where the problem of the rotors physically bumping is not solved by some other means, the highest Q that we can take advantage of is φ, and that it would be perfectly correct in Figure 9 to label the slenderest triangle as a Golden Triangle and the obtuse triangle as a Golden Gnomon. Furthermore, $\theta = 18°$, and $\psi = 54°$, and $\Delta = 36°$.

This is a valuable ideal to strive for, but a physically realizable joint may not support such a high Q, because any naive physically realizable turret joint is likely to have a rotor diameter with a lip significantly larger than the hole, and the post will have an actual thickness that must be considered in computing G and L. Nonetheless this analysis is a starting point for analyzing more realistic joints, even if such joints will have to be dealt with numerically rather than having an elegant analytic solution.

A.3 Rational Trigonometry

A magician may pull a rabbit out of a hat, but a mathematician should avoid it. The Classical Trigonometry result in the last section feels like pulling a rabbit out of a hat because it relies on results beyond our ability of human calculation. Norman J. Wildberger has a presented a simpler re-formulation of trigonometry called Rational Trigonometry[14] that proposes to avoid such rabbits.

This section assumes the reader is familiar with Rational Trigonometry.

In our diagram and our previous formulation, we can almost treat the angles of that diagrams as spreads. However, since spreads are non-linear, we cannot use *Delta Definition*, or *acute bump* in their formulation from that section.

Now if we consider *theta*, G and L as spreads, we have the definition of the spread as the quadrance of the opposite leg over the quadrance of the hypotenuse:

$$\theta \quad \equiv \quad \frac{(1/2)^2}{Q^2} \quad \equiv \quad \frac{1}{4 \cdot Q^2} \tag{rational-theta}$$

$$\psi \quad \equiv \quad (\frac{Q}{2})^2 \quad \equiv \quad \frac{Q^2}{4} \tag{rational-psi}$$

Rather we assert to avoid the problem of the rotors bumping in the case of the most acute triangle we assert:

$$P \leq \theta \tag{rational acute bump}$$

Our goal is to find the maximum Q satisfying all constraints. It is clear that θ goes down as Q goes up, and that ψ goes up as Q goes up. P goes up as Q goes up (this remains to be rigorously shown).

Therefore Q is maximized when $P = \theta$.

We can thus use Theorem 56 (Three equal spreads) [14] on θ:

$$\psi \equiv \theta(3 - 4 \cdot \theta)^2$$

together with *rational-psi* via substitution to obtain a series of purely algebraic manipulations:

$$\frac{Q^2}{4} = \theta(3 - 4 \cdot \theta)^2$$

then by substituting *rational-theta*:

$$\frac{Q^2}{4} = \frac{1}{4 \cdot Q^2}(3 - 4 \cdot \frac{1}{4 \cdot Q^2})^2$$

performing elementary algebraic simplification we obtain:

$$Q^4 = (3 - \frac{1}{Q^2})^2$$

Perform the substitution $x = Q^2$:

$$x^2 = (3 - \frac{1}{x})^2$$

Take the square root of both sides and converting to common fraction:

$$x = \pm\frac{(3 \cdot x - 1)}{x}$$

Cross multiplying and simplifying:

$$x^2 - 3 \cdot x = -1$$

Completing the square by adding $\frac{9}{4}$ to both sides:

$$x^2 - 3 \cdot x + \frac{9}{4} = \frac{5}{4}$$

Writing left hand side as a square:

$$(x - \frac{3}{2})^2 = \frac{5}{4}$$

Take the square root of both sides:

$$\pm(x - \frac{3}{2}) = \pm\frac{\sqrt{5}}{2}$$

Adding $\frac{3}{2}$ to both sides:

$$x = 1 + \frac{1}{2} + \frac{\sqrt{5}}{2}$$

Being previously alerted to the existing of φ in our solution, we recognize this as $1 + \varphi = \varphi^2$.

$$Q = \sqrt{x} = \sqrt{1 + \varphi}$$

Substituting back into $x = Q^2$,

$$Q = \varphi$$

Note that we have taken square roots twice in this operation, and therefore must check that the negative solutions are not also valid solutions. Wolfram Alpha Pro combined with the fact that only $Q > 1.0$ allows us to conclude this is the only valid solution.

We have thus obtained the same algebraic result as the trigonometrical more understandably and with less recourse to computer aids.