Formulae for the Helix Formed by Stacking Similar Objects

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1 Introduction

During the Public Invention Mathathon of 2018, software was created to view tetrahedra joined face-to-face in chains. It was noted by the participants that when the rules for which face to add the next tetrahedron to were periodic, the resulting chain was always a helix or a torus. A torus is a degenerate helix.

After some relfeciton, it became clear that any stack of objects of the same length joined at the same angles to each other repetitively form a helix. A specific example is a face-to-face connection of physical objects of the same length where the angles of the two joining faces relative to each other control the angle and rotation of the axis at each joint. We have not yet found this simple idea articulated elsewhere. The purpose of this paper is to prove and to provide formulae for the resulting helix.

2 A Warm-up: 2Dimensions

Considering the problem in two dimensions may be a valuable introduction. Suppose that we consider a polygon that that as two edges, called A and B, and that we define the length L of the polygon as the distance between the midpoints of these edges. Suppose that we are only allowed to join these polygons by aligning A of one polygon to B of another polygon, with their midpoints coincident. Let us further assume that we disallow inversions of the polygon. Let us imagine that we have a countable number of polygons P_i indexed from 0. Then what shapes can we make by chaining these polygons together?

Each joint J_i between polygons P_i and P_{i+1} will place the axes of at the same angle, θ, since our polygons do not change shape. Let us define θ to be positive if we move

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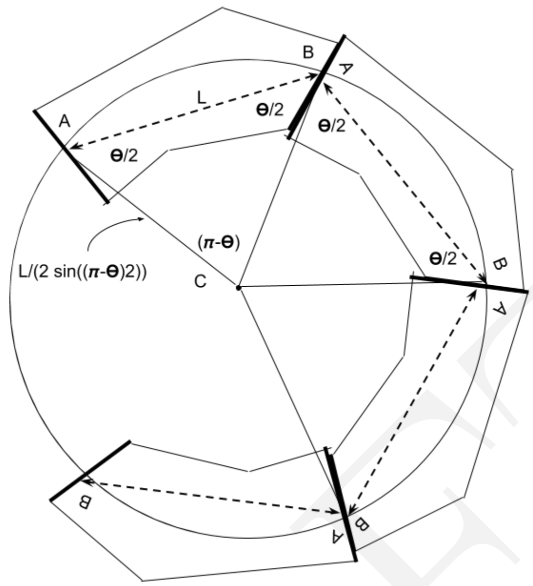

Figure 1: The rotatable prism of three objects

anti-clockwise from P_i to P_{i+1} and negative if we move clockwise. If $\theta = 0$, the joints will be collinear.

If $\theta \neq 0$, it seems they polygon joints will always lie on a circle. A proof of this is that each polygon has associated with it an isoceles triangle A, B, C, where $\angle CBA = \angle CAB = \theta/2$, and $\angle ACB = (\pi - \theta)$. AC and BC are not necessarily aligned with an edge of the polygon. The length AB is L, and the lengths AC and BC are $(L/2)/\sin(\pi - \theta)/2$. In any chain of polygons, these triangles all meet at point C, and there all joints are on the circle centered at C with radius $\frac{L}{2 \sin \frac{\pi - \theta}{2}}$.

An analogous, though far more complicated, result holds in three dimensions.

3 Three dimensions

Consider a slender cylinder or prism with two faces F_0, F_1 cut at arbitrary angles to the axis of the cylinder. Joining two such cylinders at the axes by placing F_0 angainst F_1 produced a joint with a difference in angle between the axes of α and a rotation of the orienation of the faces about the axis of θ.

Theorem 1 (Stacking Helix)**.** *The joints of a sequential stack of objects of length L whose*

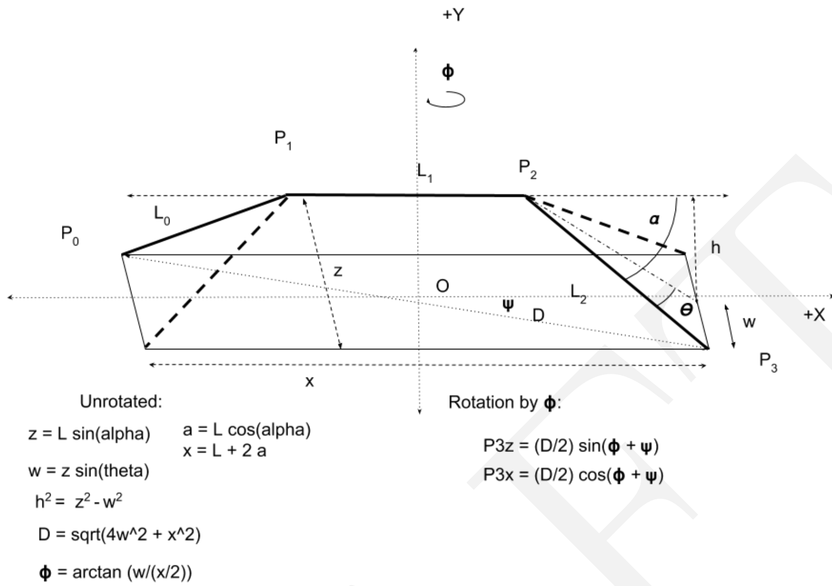

Figure 2: The rotatable prism of three objects

joints axis change by α and orientation rotate by θ are intersected by a helix of radius calculatable by a closed expression of L, α, and θ.

Proof. In Figure , let L_i be the ith instance of the length-L objects. Let α be the change in angle in the axes at a joint (the point were axes meet) measured in the plane containing the axes of both objects. (If these axes are parallel (and therefore coincident) define α to be 0. Let θ be the change in orientation relative to the axis, or, the half-angle formed by L_0 and L_2 when projected onto a plane normal to the axis of L_1. Without loss of generality, choose the measures of these angles in radians so that $0 \leq \theta < \pi/2$ and $0 \leq \alpha < \pi$. Take $\alpha < 0$ to mean that objects L_0 and L_1 bend away from each other, and $\alpha > 0$ to mean that the objects L_0 and L_1 bend toward each other.

If $\theta = 0$ and $\alpha > 0$, the stack will form a circle-like structure or radius $\frac{L}{2\sin\frac{\pi-\alpha}{2}}$, as shown in Section 2, where as if $\theta = 0$ and $\alpha < 0$, then stack will form a sawtooth-like structure.

Note that any angle α is possible, if we do not concern oursleves with the self-collision of physical objects. $\alpha > \pi/2$ means the stack "turns back on itself" to some extent.

If we arrange object L_1 so that its axis lies on the x-axis and its midpoint is at the origin, L_0 and L_1 extend from it symmetrically in the projection onto the yz-plane along the x-axis, which is always possible, and define h to be the height of the faces of L_0 and L_2 along the y-axis and w to be the distances between these faces in the z-dimesnion. Then we have:

$$
\begin{aligned}
z &= L \sin \alpha \\
w &= z \sin \theta \\
h &= \sqrt{z^2 - w^2}
\end{aligned}
$$

Now we seek the formula for the helix which intersects the joints. To find the radius of this helix, we conceptually place our three objects in a cylinder, with the axis of L_1 along the surface of the cylinder aligned with the axis. We can size this cylinder to include the joints at the extreme ends of L_2 and L_0 as well. However, the L_1 axis lies on the surface, but the axes of L_0 and L_2 in general do not. If there exists a helix which intersects all joints, all axes will cut through this cylinder in the same way, creating chords in the projection into the yz-plane of the same length. Name these chords $c_0, c_1, and c_2$.

Because we will need them later, we work out the geometry of this prism-like stack of three objects completely.

- The three objects are named L_0, L_1, and L_2. L_0 has joints P_0 and P_1, L_1 has joints P_1 and P_2, L_2 has joints P_2 and P_3.
- x is the total length of the prism along the x-axis.
- z is the length of the "face" of the prism, and the length of the chord before any rotation ϕ about the y-axis.
- w is half of the width of the prism in the z-dimension.
- h is the height of the prism in the y-dimension.
- D is the length of the diagonal from P_0 to P_3.
- ψ is the angle of $\overline{P_0 P_3}, \arctan(w/(x/2))$.
- ϕ is the amount we will have to rotate the prism about the y-axis.

Given these defintions about our assumptions, the cord lengths are:

$$
\begin{aligned}
\psi &= \arctan\left(w/(x/2)\right) \\
D &= \sqrt{4w^2 + x^2} \\
c_1 &= L\sin\phi \\
c_2 &= \sqrt{h^2 + ((1/2)(D\sin\left(\psi + \phi\right) - L\sin\phi)^2} \\
c_0 &= c_2
\end{aligned}
$$

We can imagine turing our 3-object stack and simulataneously increasing the size of our intersecting cyliner. If we turn the stack about the y-axis by ϕ degrees and keep the cylinder intersection the two faces of L_1, then the length of the L_1 chord will gradually increase. At the same time, the L_0 and L_2 chords will change their length, possibly increasing or decrasing. When $\phi = 0$, $c_0 = z, c_1 = 0, c_2 = z$.

Equating these quantities to find when $c_2 = c_1$, we have a trigonometric equation with a single unknown, ϕ.

$$
L\sin\phi = \sqrt{h^2 + ((1/2)(D\sin\left(\psi + \phi\right) - L\sin\phi)^2}
$$

Thus, by rotating our stack of objects ϕ degrees around the y-axis all four faces of our three objects intersect a cylinder on its surface with equal rotational and axial distance. The axial distance between any two joints on the same object is $L\cos\phi$, and the length of the projected chord is $L\sin\phi$.

The points P_0, P_1, P_2, and P_3 now exist on a cylinder or unknown radius parallel to x-axis, and are evenly spaced along and evenly rotated about the axis of the cylinder. The joints points thus coincide with a general helix.

Let us choose our coordinate system so that the x-axis corresponds to the axis of the helix. The general equation for the helix is:

$$
\begin{aligned}
P_x(n) &= \kappa t \\
P_y(n) &= r\cos t \\
P_z(n) &= r\sin t
\end{aligned}
$$

We seek to discover r and κ based on our knowledge of P_3 and P_2. In particular, we can deduce from the axial spacing there exists some t_0 such that $P_2 = P(t_0)$ and $P_3 = P(3t_0)$. Since we know that after rotation that:

$$
\begin{aligned}
P_{3z} &= (D/2)\sin\phi + \psi \\
P_{3z} &= r\sin 3t_0 \\
P_{2z} &= \sin\phi \\
P_{2z} &= r\sin t
\end{aligned}
$$

We can use symbolic computation to solve this system of 2 equations and 2 unkowns:

$$
\begin{aligned}
r\sin 3t_0 &= (D/2)\sin\phi + \psi \\
r\sin t &= \sin\phi
\end{aligned}
$$

Defining symbols:

$$
\begin{aligned}
E &= (D/2)\sin\phi + \psi \\
F &= \sin\phi
\end{aligned}
$$

Wolfram alpha solves the system:

$$
\begin{aligned}
r\sin 3t_0 &= E \\
r\sin t_0 &= F
\end{aligned}
$$

giving the result ($3F \neq E$ and $F \neq 0$), and ignoring multiples of 2π in t:

$$
\begin{aligned}
r &= \frac{2F^{\frac{3}{2}}}{\sqrt{3F - E}} \\
t_0 &= -2\arctan x \frac{r + \sqrt{-\frac{F^2(F+E)}{E-3F}}}{F}
\end{aligned}
$$

From which, using $P_{2x} = L/2 = \kappa t_0$, we conclude:

$$
\begin{aligned}
r &= \frac{2F^{\frac{3}{2}}}{\sqrt{3F - E}} \\
\kappa &= \frac{L}{4\arctan \frac{r + \sqrt{-\frac{F^2(F+E)}{E-3F}}}{F}}
\end{aligned}
$$

. Putting this all together we have:

$$
\begin{aligned}
a &= L\cos\alpha \\
x &= L + 2a \\
z &= L\sin\alpha \\
w &= z\sin\theta \\
h &= \sqrt{z^2 - w^2} \\
D &= \sqrt{4w^2 + x^2} \\
\phi &= \arctan\frac{z}{L} \\
E &= (D/2)\sin\phi + \psi \\
F &= \sin\phi \\
r &= \frac{2F^{\frac{3}{2}}}{\sqrt{3F - E}} \\
\kappa &= \frac{L}{4\arctan\frac{r+\sqrt{-\frac{F^2(F+E)}{E-3F}}}{F}}
\end{aligned}
$$

Using these values derived exclusively from the inputs $L, \alpha, and\theta$, we can evalute the formula for the general helix only and integral values of n to obtian a formula for precise the joint points of this any such stack.

$$
\begin{aligned}
P_x(n) &= \kappa t0(1 + 2n) \\
P_y(n) &= r\cos t0(1 + 2n) \\
P_z(n) &= r\sin t0(1 + 2n)
\end{aligned}
$$

$\square$

4 Checking against comprehensible values

Unfortunately, the complexity of these formualae exceed the author's comprehension. However, we may check these formuale by graphing them against comprehensible examples. Obvious examples are extreme solutions, where α and θ are 0 or $\pi/2$, for example. We also have the particular non-trivial example of the Boerdick-Coxeter tetrahelix, formed by regular tetrahedra, which has been studied enough to have a known pitch.

5 Implications

One of the implcations of having a formulaic understanding of the math is that it may be possible to design helixes of any radius and pitch by designing periodic (possibly scalene) segments. Combined with slight irregularites, this means that you have a basis of design molecular helices out of "atoms" which correspond to our objects.

This would mean that if you wanted to build a brace of length exactly 3 meters with bars of exactly 1/2 meter you would be able to come as close to this as mathematically possible.

6 Applied to Periodic Regular Simplex Chains

Corollary 1. *Every regular simplex chain formed by a periodic generator has a helical structure.*

Prove or disprove that *every* periodic 3D Generator generates a figure contained within a cylinder of unbounded length but bounded diameter.

Note: Every example that we have tested exhibits this property. We believe it is a property of any repeated structure, not related to simplices. However, we do not yet know the name of this theorem or principle. We conjecture that every stack of repeated truncated prisms forms a helical aperigon, which is hinted at but not stated in the Wikipedia article`https://en.wikipedia.org/wiki/Skew_apeirogon`.

Note: Rob believes a proof that any periodic structure fits within a cylinder is possible, and that it should be possible to give a formulaic bound on the diameter of this cylinder (under some assumptions.) The key to the proof is to use symmetry and focus on the the center of three such objects, observing that the other two must necessarily bend towards or away from each other in a way describable by two angles. A formula for the cylinder as a function of these angles would convincingly complete the proof.

Note: This paper may be critical.

`https://arxiv.org/pdf/1610.00280.pdf`

`https://www.impan.pl/wydawnictwa/preprints_impan/p741.pdf`