On Simple Planar Variable Geometry Trusses

Robert L. Read *

Founder, Public Invention, an educational non-profit.

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1 Introduction

A *variable geometry truss* is a truss in which changes shape by means of the change of the length of members, in contrast to many robot arms which change joint angles. We call a member which can change length an *actuator*. A truss constructed purely by starting with a triangle and and repeatedly adding two members and a new joint to one side of an existing triangle to form a new triangle is a *simple truss*. This paper is concerned only with simple planar trusses. In particular, we focus on trusses isomorphic to the Warren truss (e.g., an unbranching configuration of triangles.) Furthermore, although the world *truss* connotes forces and structural analysis, we are concerned purely with geometry. Altough motivated by robotics, we assume that our trusses and actuators are strong enough that we need not consider the forces acting upon the truss—in other words, we are treating it as mechanism and not a machine.

If we imagine a joint, usually at one end of our truss, to be an end effector, he fundamental goal of this paper is to answer the question:

> How does a change in length of an actuator change the position of an end effector?

2 Formulation

A *truss* structure is a graph and a set of fixed nodes $T = (G, F)$. $G = (V, E)$. V is a set of nodes or joints. E is a set of lines or members which are 2-element subsets of V. At least two nodes are condidered to be fixed in space via a set of fixed nodes $F = (i, x, y) \| i \in V x, y \in \mathbb{R}$.

*read.robert@gmail.com

We number joints from zero and designate them with a subscript. Members are designated by two subscripts, naming the joints they connect.

A configuration function mapping each member of a truss structure to a non-negative length. $C : V \times V \mapsto \mathbb{R}$.

A geometry is a placement of each joint in the Cartesian plane. $G : V \mapsto \mathbb{R} \times \mathbb{R}$.

A *simple truss* is a truss constructed from a triangle by adding a single joint and two members to a side repetitively. A *Warren truss* is an unbranched simple truss, which is isomorphic to a truss that is a single chain of equilateral triangles. Furthermore, we insist that each even node n occurs as a anti-clockwise turn (in the anti-clockwise semiplane) from the vector $\overrightarrow{n-2, n-1}$, and each odd node occurs as a clockwise turn (in the clockwise semiplane) from the previous to nodes.

For a Warren truss, there is a simple algorithm W for computing a geometry from a configuration: $W : C \mapsto G$.

A goal joint is a joint j such that $j \in V$ and a desired position given by a goal function $d : V \mapsto \mathbb{R}^2$. In robotics, a goal joint is often called an *end effector*.

A *scoring function* is a function that that takes a geometry and returns a real, non-negative value based on the goal node positions given by by a goal function and the actual position given by a Geometry. A score of zero is considered a perfect score and the a higher score represents a less sought-after result.

A *linear distance scoring function* is a special scoring function that that takes a geometry and returns a real, non-negative value based solely on a summation of distances between the goal node positions given by by a goal function and the actual position given by a Geometry.

A truss *problem* $P = (T, d, s)$ is a truss with a scoring function and a desired position function. A solution to a problem is a configuration and a geometry. An optimal solution is a solution which minimizes the score.

Our fundamental goal is to develop a formula for the partial derivative of the end effector with respect to the change in length of a member.

A further goal is to be able to render a diagram illustrating the impact on the end-effector of a change of each member, as drawn by rendering a vector from the center point of each member.

A further goal is to have algorithms to determine:

- What is the minimum overall change in length to a group of actuators to solve a Problem?
- Can a Problem be solved with a single change in length?
- Assuming bounds on the lengths of actuators, can we solve Problems?
- What is the workspace of a truss?

In the remainder of this paper we will consider only Warren trusses, linear distance scoring functions, and desired position functions that map only one node, called an end effector, to a desired position. Furthermore, we will assume that the first two nodes are fixed and that the furthest node (by path length) from the first node is the end effector.

3 Moving an External Member

We seek a formula for the change in position of the end effector e with respect to change in the length of a member given a geometry. In the case of a Warren truss, all members can be divided between external members and internal members. A change to the length of an external member is particularly simple.

The change in position is a centered on the goal node representing the place the goal node would move to with a unit change in length. However, the derivative is only valid as an infinitessimal, but as an infinitessimal its magnitude and direction may be usefully added to or compared to other such vectors. We can thus usefully tell which member would change the position of the goal node most rapidly in response to a minute change in different members.

The fundamental observation is that for an external member $(i, i-2)$, a change in length generates a rotation θ about joint $n-1$ whose position is given by (x, y). θ is the angle of the vector from the pivot point to the end-effector with the x-axis. This rotation applies to the triangle defined by three joints: $\triangle i-1, i, e$. Not that this triangle doe not exist as a physical structure in the truss. This rotation about a point can be expressed as a matrix M:

$$M = \begin{bmatrix} \cos\theta & -\sin\theta & -x \cdot \cos\theta + x + y \cdot \sin\theta \\ \sin\theta & \cos\theta & -x \cdot \sin\theta - y \cdot \cos\theta + y \\ 0 & 0 & 1 \end{bmatrix}$$

such that:

$$M \begin{bmatrix} e_x \\ e_y \\ 1 \end{bmatrix} = \begin{bmatrix} e'_x \\ e'_y \\ 1 \end{bmatrix}$$

where (e'_x, e'_y) is the new position of e after the rotation by *theta*. Multiplying this out:

$$\begin{bmatrix} e'_x \\ e'_y \end{bmatrix} = \begin{bmatrix} e_x \cos\theta + -e_y \sin\theta + -x\cos\theta + x + y\sin\theta \\ e_x \sin\theta + e_y \cos\theta + -x\sin\theta + -y\cos\theta + y \end{bmatrix}$$

Collecting terms:

$$\begin{bmatrix} e'_x \\ e'_y \end{bmatrix} = \begin{bmatrix} (e_x - x)\cos\theta + (y - e_y)\sin\theta + x \\ (e_x - x)\sin\theta + (e_y - y)\cos\theta + y \end{bmatrix}$$

This can be easily differentiated with respect to θ:

for Derivatives of Ladder.png

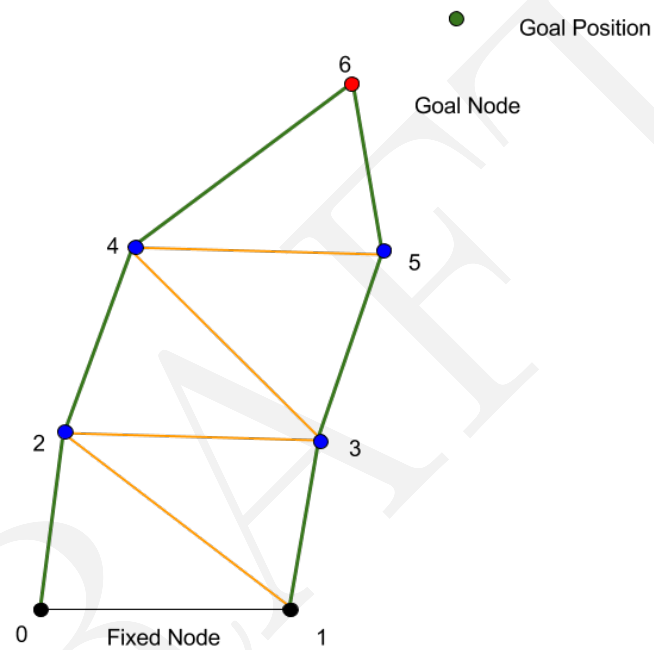

Figure 1: A Trusss Problem

$$\frac{\partial e}{\partial \theta} = \begin{bmatrix} \frac{\partial e_x}{\partial \theta} \\ \frac{\partial e_y}{\partial \theta} \end{bmatrix} = \begin{bmatrix} -(e_x - x)\sin\theta + -(e_y - y)\cos\theta \\ (e_x - x)\cos\theta + -(e_y - y)\sin\theta \end{bmatrix}$$

As we might expect, the magnitude of this vector depends on the distance from the pivot joint (x, y) and the position of the end effector (e_x, e_y) and the direction depends on the direction of the vector from the pivot joint to the end effector, θ.

By using the law of cosines, where $a = \|\overrightarrow{n-2, n-1}\|, b = \|\overrightarrow{n-2, n}\| = l_{i,i-2}, c = \|\overrightarrow{n-1, n}\|$, where b is the member opposite the pivot joint $n-1$ which changes the angle $\angle n-2, n-1, n = \phi_{i-1}$.

$$\cos\phi_{i-1} = \frac{a^2 - b^2 + c^2}{2ac}$$

Using Woflram Alpha to differentiate this, we obtain:

$$\frac{\partial \phi_{i-1}}{\partial l_{i,i-2}} = \frac{b}{ac\sqrt{1 - \frac{(a^2+c^2-b^2)^2}{4a^2c^2}}}$$

So, by the chain rule

$$\frac{\partial e}{\partial l_{i,i-2}} = \begin{bmatrix} \frac{\partial e_x}{\partial l_{i,i-2}} \\ \frac{\partial e_y}{\partial l_{i,i-2}} \end{bmatrix} = \frac{b}{ac\sqrt{1 - \frac{(a^2+c^2-b^2)^2}{4a^2c^2}}} \begin{bmatrix} -(e_x - x)\sin\theta + -(e_y - y)\cos\theta \\ (e_x - x)\cos\theta + -(e_y - y)\sin\theta \end{bmatrix}$$

So this is a closed-form expression for the change in the position for any external member $i, i-2$ we choose.

4 Moving an Internal Member

Moving an internal member is slightly more complicated. In a Warren Truss, moving an internal member is in fact changing the shape of a parallelogram. The best way to think of the overall transformation of an end effector is to imagine it as both a translation and a rotation of the midpoint of one of the edges of the parallelogram.

We need to develop the formulae for the derivatives of the motions of both endpoints and add them together to obtain this. The motion of each point, however, is a pure translation.

Given that we are dealing with infinitessimal motion, the partial derivative of C is easy because it must be at right angles to $\vec{AC}$. Let θ be the anticlockwise angle AC makes with the x-axis.

$$\frac{\partial C}{\partial l_{BC}} = \begin{bmatrix} \frac{\partial C_x}{\partial l_{BC}} \\ \frac{\partial C_y}{\partial l_{BC}} \end{bmatrix} = \begin{bmatrix} \sin\theta \\ \cos\theta \end{bmatrix}$$

angle change.png

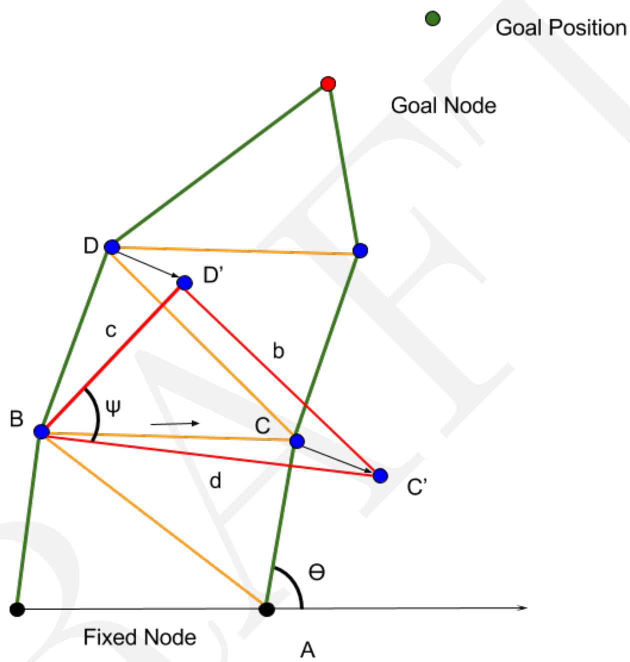

Figure 2: Internal length change

To find the partial derivative of the D's motion, we utlize the fact that the lengths of $\vec{BD}$ and $\vec{CD}$ don't change. We thus have three lengths and two points, from which the third point (D) can be computed. The easily way to do this is to assume that the point B is at the origin, and compute the angle of $\vec{BD'}$. This is the sum of the angle $\angle D'BC'$ which can be obtained from the law of cosines, and the angle of $\vec{BC'}$ with the x axis.

$$\angle D'BC' = \arccos \frac{c^2 + d^2 - b^2}{2cd}$$

$$\angle \vec{BC'} = \operatorname{atan2}(C'_y - B_y, C'_x - B_x)$$

$$\psi = \angle \vec{BD'} = \angle D'BC' + \angle \vec{BC'}$$

$$\psi = (\arccos \frac{c^2 + d^2 - b^2}{2cd} + \operatorname{atan2}(C'_y - B_y, C'_x - B_x))$$

$$D' = \delta \cdot \begin{bmatrix} \sin \angle \vec{BD'} \\ \cos \angle \vec{BD'} \end{bmatrix} + \begin{bmatrix} B_x \\ B_y \end{bmatrix} = \delta \cdot \begin{bmatrix} \sin \psi \\ \cos \psi \end{bmatrix} + \begin{bmatrix} B_x \\ B_y \end{bmatrix}$$

$$D' = \delta \cdot \begin{bmatrix} \sin \psi \\ \cos \psi \end{bmatrix} + \begin{bmatrix} B_x \\ B_y \end{bmatrix}$$

Given that δ is the differential change in length, this can be rewritten as derivative:

$$\frac{\partial D}{\partial l_{BC}} = \begin{bmatrix} \frac{\partial D_x}{\partial l_{BC}} \\ \frac{\partial D_y}{\partial l_{BC}} \end{bmatrix} = \begin{bmatrix} \sin \psi \\ \cos \psi \end{bmatrix}$$

Now to understand how a change to the $\vec{BC}$ length effects the end effector, we can add the partial derivatives of C and D in order to get the translational change to the midpoint P of $\vec{CD}$. Then separately we must compute the change in the angle of the vector $\vec{CD}$. These things together will allow us to compute the partial derivative of the end effector e.

$$\frac{\partial P}{\partial l_{BC}} = \frac{\partial C}{\partial l_{BC}} = \begin{bmatrix} \frac{\partial C_x}{\partial l_{BC}} \\ \frac{\partial C_y}{\partial l_{BC}} \end{bmatrix} \begin{bmatrix} \frac{\partial D_x}{\partial l_{BC}} \\ \frac{\partial D_y}{\partial l_{BC}} \end{bmatrix} = \begin{bmatrix} \sin \theta \\ \cos \theta \end{bmatrix} + \begin{bmatrix} \sin \psi \\ \cos \psi \end{bmatrix}$$

Moving the member $\vec{BC}$ in general induces a rotation about P as well.

$$\rho = \angle \vec{CD} - \angle \vec{C'D'}$$

$$\rho = \angle \operatorname{atan2} D'_y - C'_y D'_x - C'_x$$

Since:

$$\frac{d}{dx}\operatorname{atan2} g(x)f(x) = \frac{g(x)f'(x) - f(x)g'(x)}{f(x)^2 + g(x)^2}$$

, we have:

$$\frac{\partial \rho}{\partial l_{BC}} = \frac{(D'_x - C'_x)\frac{\partial (D'_y - C'_y)}{\partial l_{BC}} - (D'_y - C'_y)\frac{\partial (D'_x - C'_x)}{\partial l_{BC}}}{(D'_y - C'_y)^2 + (D'_x - C'_x)^2}$$

I always get confused between differentials and derivatives. However, if we evaulate this instantaneously, we can replace D' with D and C' with C. We already have closed form expressions for the other partials.

$$\frac{\partial \rho}{\partial l_{BC}} = \frac{(D'_x - C'_x)\frac{\partial (D'_y - C'_y)}{\partial l_{BC}} - (D'_y - C'_y)\frac{\partial (D'_x - C'_x)}{\partial l_{BC}}}{(D'_y - C'_y)^2 + (D'_x - C'_x)^2}$$

So we have:

$$\frac{\partial \rho}{\partial l_{BC}} = \frac{(D_x - C_x)(\sin\psi - \sin\theta) - (D_y - C_y)(\cos\psi - \cos\theta)}{(D_y - C_y)^2 + (D_x - C_x)^2}$$

The change to the end effector is the composition of the translation of the midpoint and the rotaion about the midpoint by this angle.

I now should check this by making graphs of this against reasonalbe values. This is probably best done with a computer program. Is it better to do this interactively or to produce a plot?

5 References