

# Calculating the Segmented Helix Formed by Repetitions of Identical Subunits and a Zoo of Platonic Helices

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## Abstract

Eric Lord has observed:

In nature, helical structures arise when identical structural subunits combine sequentially, the orientational and translational relation between each unit and its predecessor remaining constant.[1]

This paper proves Lord's observation as a consequence of screw theory. Constant-time algorithms are given for the segmented helix generated from the intrinsic properties of a stacked object and its conjoining rule are given. Standard results from screw theory[2] and previous work for finding the axis of a helix from points[3] are combined with corollaries of Lord's observation to allow calculations of segmented helices from either transformation matrices or four known consecutive points. The construction of these from the intrinsic properties of the rule for conjoining repeated subunits of arbitrary shape is provided, allowing the complete parameters describing the unique segmented helix generated by arbitrary stackings can be easily calculated. Free-libre open-source interactive software[4] is provided which performs this computation for arbitrary prisms along with 3D visualization[5]. This allows the deduction of intrinsic properties of a repeated subunit from known properties of a segmented helix, as a chemist might want to do. Because the algorithms are efficient, a repeated subunit can be designed to create a segmented helix of desired properties, as a mechanical engineer or robotocist might want. As a verification and demonstration, the software and paper compute, render, and catalog a "zoo" of all platonic helices, such as Boerdijk-Coxeter tetrahelix and various species of helices formed from dodecahedra, for example.

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# 1 Introduction

During the Public Invention Mathathon of 2018[6], software was created to view chains of regular tetrahedra joined face-to-face. The participants noticed that whenever the rules for which face to add the next tetrahedron to were periodic, the resulting chain was always a helix.

Although unknown to us at that time, we now call Lord’s Observation:

In nature, helical structures arise when identical structural subunits combine sequentially, the orientational and translational relation between each unit and its predecessor remaining constant.[1]

The purpose of this paper is to prove Lord’s Observation and provide mathematical tools and software for studying arbitrary segmented helices generated in this way.

Finding the properties of a segmented helix from four contiguous segments on the helix from screw theory[7, 2, 8, 3], is explained and reformulated. An interactive, 3D rendering website written in JavaScript which allows both calculation and interactive play and study[5]. This allows a structure or molecule coincident to a segmented helix to be designed by adjustments to the repeated object, or for the shape of a repeated subunit to be inferred from the intrinsic properties of the segmented helix. Kahn’s method [3] is modified to cover some degenerate situations.

We exploit Lord’s Observation to discover symmetry which allows us to compute the helix when subunits are joined face-to-face with the same *twist*. Kahn was investigating proteins, which do not have faces, but geometric solids and other macroscopic solid objects do. This concept can be generalized to a *joint face angle*, even if the objects conjoined do not technically have flat faces. This symmetry allows us compute the parameters of the segmented helix purely from properties intrinsic to a single object and the joining rule. Finally, as an important demonstration, these tools are used to produce a catalog of all possible helices generated by vertex-matched face-to-face joints of the Platonic solids, of which only the tetrahelix[9, 10, 11, 12, 13] has been completely described to date.

## 2 A Warm-up: Two Dimensions

Considering the problem in two dimensions may be a valuable introduction. Suppose that we consider a polygon that has two edges, called  $A$  and  $B$ , and that we define the length  $L$  of the polygon as the distance between the midpoints of these edges. Suppose that we are only allowed to join these polygons by aligning  $A$  of one polygon to  $B$  of another polygon, with their midpoints coincident. Let us further assume that we disallow inversions of the polygon. Let us imagine that we have a countable number of polygons  $P_i$  indexed from 0. Then what shapes can we make by chaining these polygons together?

Each joint  $J_i$  between polygons  $P_i$  and  $P_{i+1}$  will place the axes of at the same angle,  $\theta$ , since our polygons do not change shape. Let us define  $\theta$  to be positive if we move anti-clockwise from  $P_i$  to  $P_{i+1}$  and negative if we move clockwise. If  $\theta = 0$ , the joints will be collinear.



Figure 1: Example Segmented Helix Generated From the Dodecahedron

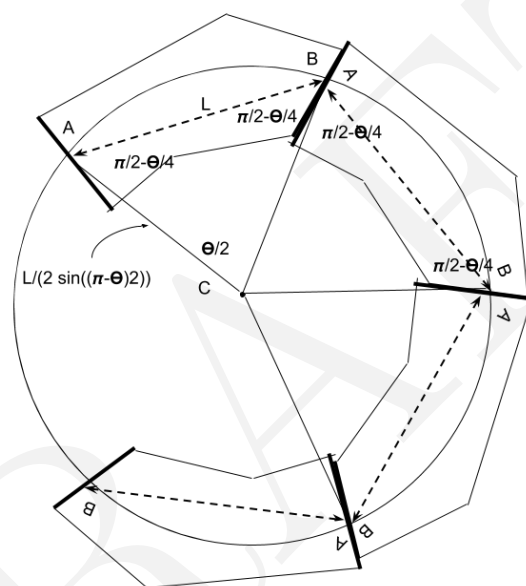


Figure 2: A 2D Analog of a Helix Generated by Repeated Subunits

If  $\theta \neq 0$ , it seems they polygon joints will always lie on a circle. A proof of this is that each polygon has associated with it an isosceles triangle  $A, B, C$ , where  $\angle CBA = \angle CAB = \theta/2$ , and  $\angle ACB = (\pi - \theta)$ .  $AC$  and  $BC$  are not necessarily aligned with an edge of the polygon. The length  $AB$  is  $L$ , and the lengths  $AC$  and  $BC$  are  $(L/2)/\sin(\pi - \theta)/2$ . In any chain of polygons, these triangles all meet at point  $C$ , and there all joints are on the circle centered at  $C$  with radius  $\frac{L}{2\sin \frac{\theta}{2}}$ .

### 3 The Segmented Helix

An analogous result holds in three dimensions.

In this section we consider a helix evaluated only at regular points.

Following the Wikipedia article <https://en.wikipedia.org/wiki/Helix>, we set up a helix parametrically.

$$\begin{aligned} P_x(t) &= r \sin t \\ P_y(t) &= r \cos t \\ P_z(t) &= bt \end{aligned}$$

Such a helix has a radius of  $r$  and slope (if  $r \neq 0$ ) of  $b/r$ . The pitch of helix, the change in  $t$  needed to make one complete revolution, is  $2\pi b$ . Note that a helix may be degenerate in two ways. If  $r = 0$ , these equations become a line. If  $b = 0$ , these equations describe a circle in the  $xy$ -plane. If  $r = 0$  and  $b = 0$ , the figure is a point.

Such helices are continuous, but we are investigating stacks of discrete objects. We in fact wish to derive the parameters for a continuous helix from such discrete objects which constrain discrete points, so we wish to study a helix evaluated at integral points. We call such an object a *segmented helix*. A segmented helix may be thought of as function that given an integer gives back a point in three space.

$$\begin{aligned} P_x(n) &= r \sin n\theta \\ P_y(n) &= r \cos n\theta \\ P_z(n) &= nd \end{aligned}$$

$d$  is the distance or *travel* along the axis between adjacent joints. In this canonical representation this is the  $z$ -axis.  $\theta$  is the rotation around the  $z$ -axis between adjacent points.  $r$  is the radius of the segmented helix. Note that if  $\theta = \pi$ , we have a third form of degeneracy (to the human eye) of a segmented helix which is a zig-zag contained completely within a single plane.

If we think of the segmented helix as describing a polyline in 3-space, we would like to investigate the properties of that polyline. If we consider only the intrinsic shape of the segmented helix, there are three degrees of freedom:  $r, d, \theta$ . We call these the *intrinsic* properties of the segmented helix.

Figure 3 demonstrates our conventional concepts. It is a screen shot taken from our interactive software[5]. The software allows parallax by supporting interactive rotation, which makes the 3D structure easier to understand; we encourage the reader to visit our interactive page as we discuss our naming conventions.

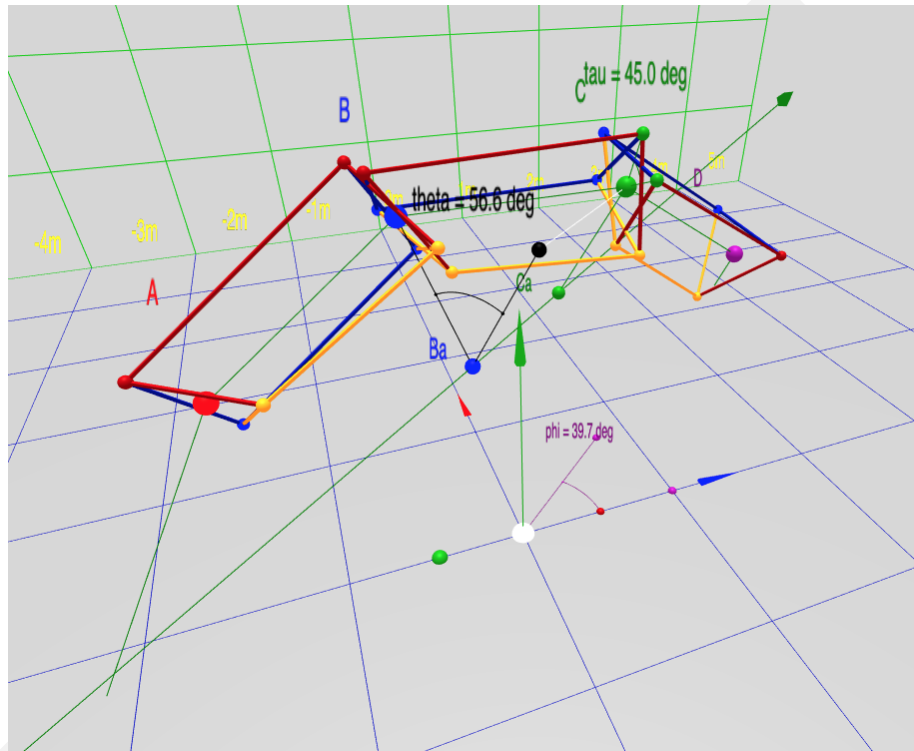


Figure 3: Naming of measures

In Figure 3 and our software, we represent the object as a prism with triangular cross-section, because this is the simplest physically realizable macroscopic object that supports a face-to-face connection. In this diagram, the points  $A, B, C, D$  are represented by the sphere of the same color as the label. The view is roughly in the direction of the axis of the segmented helix, which is drawn as a dark green arrow, pointing in the positive  $z$  and positive  $x$  direction, parallel to the  $xz$  plane. For ease of viewing, the entire segmented helix has been raised by two units on the  $y$  axis. The segment  $\overline{BC}$  has coordinate  $y = 2$ , and is aligned with the  $z$ -axis, and centered in  $z$  direction.

The positive axes  $x, y$  and  $z$  axes are shown by the red, green and blue axes, respectively. Following computer graphics convention, the  $y$  axis is oriented vertically. The points  $A, B, C, D$  correspond to  $P(0), P(1), P(2)$ , and  $P(3)$  for a segmented helix aligned to this axis (not the  $z$ -axis.) A thin green polyvinyl represents the segmented helix, and thus connects the joints  $A, B, C$ , and  $D$ . The points are wrapping around the axis clockwise, with an angle of  $\theta = 56.6^\circ$  as show by the on-screen protractor as the rotation from one point to the next. As will be explained in Section ??,  $\tau$  is the face-on-face joint rotation of  $45^\circ$ . The helix angle  $\phi = 39.7^\circ$  is rendered by a protractor on the  $y = 0$  plane; this the angle between the axis and of the helix and the  $z$  axis, and therefore the angle of any one segment against the helix angle.

A line is drawn from the blue point  $B$  to an black sphere on the helix axis. The point on the helix axis is unlabeled in our software, which renders it as a black sphere, but is denoted  $B_a$  in this paper. Analogously,  $C_a$  is the point on the axis closest to he green point  $C$ .

A segmented helix located in space is completely determined by these parameters, a vector describing the axis of the helix, and the position of any one joint.

Because the segmented helix is a discrete structure, we reframe the concept of *pitch* as *sidedness*  $s$ : how many segments (sides) make a complete rotation?

The following concepts and conventional variable names for them will be related:

- $L$  is the distance between any two adjacent points (that is, between  $B$  and  $C$ , for example.)
- $r$  is the distance between a point and the helix axis (that is,  $B$  and  $B_a$  for example.)
- $\theta$  rotation about the axis between two joints.
- $c$  is the length of a chord formed by the projection of the segment between two points projected along the axis of the segmented helix (A chemist may recognized this as the distance between residues on a *helical wheel* projection).
- $d$  is the distance along the axis of the helix between any two joints ( $B_a$  and  $C_a$ , for example, rendered as a small black and blue sphere, respectively in Figure 3.
- $\phi$  is the angle between any vector between two adjacent joints and the axis of the helix. In physical screws used in mechanical engineering, this is analogous to the *helix angle* [14].
- $p$  is the pitch of the helix, the distance traveled in one complete rotation.
- $s$  is the number of segments in a complete rotation (in general not rational.)
- Finally, we find it useful to define the *tightness* of a segmented helix as travel divided by radius, a number analogous to the extension of a coil spring or slinky.



A torus-like segmented helix has zero tightness and a zig-zag has maximum tightness. The symbol  $t$  represents tightness.

These quantities are related:

$$c = 2r \sin \frac{\theta}{2} \quad (1)$$

$$L^2 = c^2 + d^2 \quad (2)$$

$$\arctan \frac{c}{d} = \phi \quad (3)$$

$$s = \frac{2\pi}{\theta} \quad (4)$$

$$d = L \cos \phi \quad (5)$$

$$p = ds \quad (6)$$

$$t = d/r \quad (7)$$

$$(8)$$

Measuring  $\phi$  requires us to decide on the sign of the direction of the axis, which is arbitrary and not based on the physical shape.

Any point on the segmented helix has a closest point on the axis of the helix. In particular, we will call the points closest to the joints *joint axis points*. Then  $d$  is the distance along the axis between consecutive joint axis points.

We seek to relate these properties to properties intrinsic to the joint or interface between two segments or objects in the segmented helix. If given an object, the length between the joints  $L$  is intrinsic.

### 3.1 Sign Conventions for Spatially Located Segmented Helices

When thinking about the overall shape of a segmented helix, one is likely to be interested in the absolute magnitude of its intrinsic properties.

However, when doing computer graphics work or kinematic calculations, the sign conventions are critical. Because this paper wishes to emphasize the continuum of shapes produced by changing an object used to generate the segmented helix, and in particular is interested in the degenerate helix which produces toroidal figures, we prefer to be able to discuss the axis of a segmented helix as existing even when the figure has no travel along the axis (that is, when  $d = 0$ ).

Therefore we elect the following conventions:

- A right-handed coordinate system.
- The helix as a normalized vector which never vanishes.
- The travel along the axis ( $d$ ) is negative when the helix is counterclockwise (that is, when motion from joint  $n$  to joint  $n + 1$  appears counterclockwise when  $\theta < \pi$ , zero when toroidal, and positive when the helix is clockwise.
- $\theta$  is never negative.

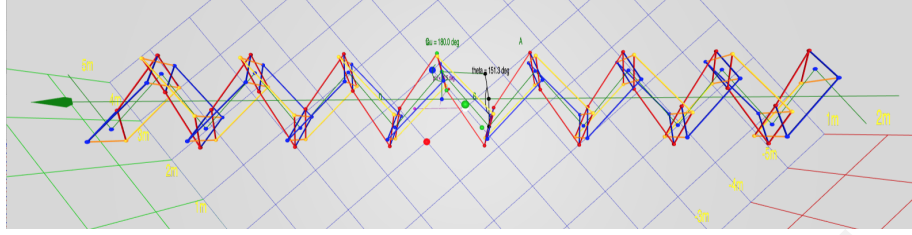


Figure 4: A Planar Zig-Zag

## 4 The Intrinsic Properties of Periodic Chains of Solids

If we have chains of repeated 3D units conjoined identically *periodic chains*, they generate a segmented helix coincident on their joints. Although fairly obvious from Chasles' Theorem, we have not found this stated in writing elsewhere, so we call this Lord's Observation:

**Observation 1** (Lord's Observation). *In nature, helical structures arise when identical structural subunits combine sequentially, the orientational and translational relation between each unit and its predecessor remaining constant.[1]*

Lord's Observation may perhaps be clarified that in fact identical objects conjoined via a rule produce periodic chains of objects that are uniformly intersected by segmented helices and that they may be degenerate in one of three ways that we might not strike us as a helix if we are not seeking them:

1. The segments may form a straight line. (For example, see Figure 8).
2. The segments may be planar about a center, forming a polygon or ring. (For example, see Figure 14).
3. The segments may form a planar saw-tooth or zig-zag pattern of indefinite extent (For example, see Figure 4).

There are two complementary ways of learning about such segmented helices. In one approach, we may have knowledge of the segmented helix, and wish to learn about the subunits and the rule with which the subunits are combined. For example, we may have microscopic objects such as proteins or atoms, and we know from crystallography something about the positioning of these objects, without knowing ahead of time the angles at which these objects would combine in their natural environment. In this case, we use a variant of a linear algebra method[3] for determining the radius, travel, and twist of the segmented helix (these terms will be defined precisely below.)

In the other approach, we may know *a priori* exactly the relevant properties of the objects and the rule which which they combine, and we seek to compactly describe the segmented helix they create. For example, a mathematician may consider a chain of dodecahedra, or a woodworker may cut identical blocks of maple wood, which are to be glued together face-to-face. In these cases everything about the objects and the rules for conjoining is known before the first two objects are glued together. We call this the *joint face normal method*, because it can be simulated by joining two flat faces

together with a specified twist, even if the objects in question do not actually have a physical face (such as molecules).

In both cases, we would like to understand how a change in a face normal or a twist affects the parameters of the segmented helix, and, conversely, we would like to be able to choose the construction of the subunits to achieve a particular segmented helix.

In engineering, sometimes the term *special helix*[15] is use for helical curves on non-cylindrical surfaces. This paper use the term *helix* only in the sense of *cylindrical helix*.

## 5 Periodic Chains Produce Segmented Helices

A periodic chain is in fact a simple object which demonstrates tremendous symmetry. Before using this symmetry in the construction of the segmented helix corresponding to a periodic chain, we prove that such a segmented helix indeed exists.

**Theorem 1** (Segmented Helix). *Consider  $N$  identical objects which each have two points,  $A$  and  $B$ , called joints. Call  $\overrightarrow{AB}$  the axis of this object. Consider the frame of reference for this object to have its axis on the  $z$ -axis with  $B$  in the positive direction, the midpoint of the at the origin, and the up-vector pointing in the positive  $y$  direction.*

*Consider any rule that conjoins  $A$  of object  $i + 1$  to  $B$  such that from the frame of reference of  $i$ , the object  $i + 1$  and anything rigidly attached to it is always in the same position in the frame of reference for  $i$ . Informally,  $i + 1$  “looks the same” to  $i$ , no matter what  $i$  we choose,  $i < N$ . Call a chain of  $N$  identical rigid objects conjoined via a rule that conjoins  $A$  to  $B$  in such a way that every vector of  $B$  is always in the same position relative to a frame of reference constructed from  $A$  a periodic chain.*

*Any periodic chain of three or more objects has a unique segmented helix who whose segments correspond to the axes of these objects.*

*Proof.* We will proceed by induction.

Base Case ( $k == 3$ ):

Take an object  $AB$ . By Chasles’ theorem[16] there is a screw axis  $S$ , a set of rotations, and a transposition  $d$  which moves the first object to position where the second object  $BC$  is. Take one of the rotation angles of smallest value. Construct the points  $A'$ ,  $B'$  and  $C'$  as the closest points to  $A, B$  and  $C$  on this axis. These points are collinear by construction.

Now add the object  $CD$  to object  $BC$  by our the rule of periodic chains. Consider the points  $B'$  and  $C'$  from  $A$ ’s frame of reference. Let  $d = \|\overrightarrow{C'} - \overrightarrow{B'}\|$ . Construct the point  $D'$  on our screw axis as the point closest to  $D$  on that line.

Now because  $C'D'$  in  $BC$ ’s frame of reference must look like  $B'C'$  in  $A$ ’s frame of reference, the distance  $\|\overrightarrow{D'} - \overrightarrow{C'}\| = d$ . From  $A$ ’s frame of reference,  $A'B'C'$  are collinear, the points  $B'C'D'$  must be collinear in  $B$ ’s frame of reference.

In any frame of reference, if  $A'B'C'$  are collinear and  $B'C'D'$  are collinear, then  $A'B'C'D'$  are collinear.

Now, looking backward from  $CD$  towards  $A$ , the distance  $A'B'$  must be the same as the distance  $B'C'$  so as to not violate our rule. So  $d = \|A'B'\| = \|B'C'\| = \|C'D'\|$ . Similarly because by construction  $r = \|BB'\| = \|CC'\|$ , and  $AA'$  is a rigid

transformation of  $BB'$ , so  $r = \|AA'\|$ . By symmetry,  $r = \|DD'\|$ . Compute  $\theta$  as the rotation about  $S$  that takes  $BB'$  into  $CC'$ . By our rule of attachment,  $\theta$  also takes  $CC'$  into  $DD'$  and  $AA'$  into  $BB'$ .

Now construct a segmented helix, the radius  $r$ , distance, and angle  $\theta$ . This segmented helix can be positioned coincident with  $S$  so that  $H_0 = A$ . Then  $H_1 = B$ ,  $H_2 = C$ , and  $H_3 = D$ .

Therefore, for the base case of three objects, there is a segmented helix whose segments coincide with the axes of the objects.

Inductive Case  $(k + 1)$ : Assume there is a segmented helix coinciding with the first  $k$  objects, and consider the frame of reference of the  $k$ th object. The axis and any other rigid property of the  $k + 1$ th object stands in relation to object  $k$  as  $k$  stood to  $k - 1$ . By induction, the  $k$ th object had a segment of a segmented helix corresponding to its axis. Attach vectors  $V_{Ak}$  and  $V_{Bk}$  from the joints of  $k$  to the axis of the helix perpendicularly. Define these vectors in the frame of reference for  $k$ .

To the  $k - 1$ th, the tips of  $V_{Ak}$  and  $V_{Bk}$  define a line segment which lies on the axis of the segmented helix  $H$ , with the tip of  $V_{Ak}$  coincident with the tip of  $V_{B(k-1)}$ .

By our rule and by induction, since this is true of the  $k - 1$ th object, it is true of the  $k$ th object. Therefore the  $k + 1$ th objects  $V$  vectors point to a line segment which lies on the axis of  $H$ , extending it in the same direction. The axis of the  $k + 1$ th object therefore coincides with the  $k + 1$ th segment of  $H$ .

Therefore, by induction, identical objects conjoined by the same rule always coincide with some segmented helix, whose parameters are discoverable.  $\square$

This theorem leads to the following corollary. In engineering the term *helix angle* refers to the angle between a line tangent to a continuous helix and the axis of the helix. In segmented helices, this is the same as the angle between the axis of each object in a periodic chain and the axis of the segmented helix coincident to it.

**Corollary 1** (Segment Similarity). *The helix angle of a any object axis in a periodic chain is the same.*

*Proof.* The axes of each object coincide with a segment of a segmented helix. A segmented helix is completely symmetric no matter in which direction of the axis you look down or which point on the axis you begin at. The angle between each pair of objects is exactly the same.  $\square$

Corollary 1 is perhaps not obvious when one considers period chains of objects which are, taken as individuals, highly asymmetric. For example, The  $B$  face does not have to be the same size as the  $A$  face. In fact, the object itself might be shaped like the letter “C”, and not completely enclose the axis. Taking the idea further, the object might be spiky like a stellated polyhedron or a sea urchin, and still be joined by joints relatively close to the center of the object. In this paper we do not concern ourself with the issue of self-collision of the objects, which would have to be considered if one attempted to make a period chain of sea urchins.

Corollary 1 will be used in our development of *PointAxis* algorithm and in our computation of segmented helix properties and to justify balancing face normals to produce an intrinsic out-vector and to apply the *PointAxis* algorithm without actually assigning objects Cartesian coordinates.

## 6 Computing Screws and Segmented Helices from Transformation Matrices

The rule for how objects in a periodic chain are joined may be conveniently captured as a transformation matrix. In general, a human engineer will have to compute this transformation matrix from some other information, such as the face-to-face conjoining rule. We discuss how to do this from joint-face normal vectors in Section 8. However, a transformation matrix clearly capture the idea of “repetition”. Since by definition the objects in the chain are the same shape, moving one object into a new position and place an identical copy of that object in that position are practically the same.

Using standard screw theory[2, 8], a screw can be computed from such a transformation matrix. This consists of the axis of the screw  $S$ , a point on the screw axis  $P$ , the rotation  $\theta$  around the axis, and the transposition, or travel, along the axis of one transformation.

Neither a transformation matrix or its corresponding screw completely define all of the intrinsic properties of a segmented helix. In particular, a matrix  $M$  maps any point  $p$  to a point  $p'$ . Since this applies to all points no matter how far from the screw axis or the axis of rotation and such transformation preserve distance to this axis (the radius), the radius of a segmented helix is not determined by a transformation matrix or a screw. Since the helix angle  $\phi$  changes with radius for a helix of a given pitch,  $\phi$  is not determined.

However, if we have a screw and one point on the axis of the screw fixing it in space and the location of one joint, all of the properties of the segmented helix are completely determined.

In our software, we have coded the calculation of the screw directly from a transformation matrix, and the additional routines which determine all intrinsic properties from a joint position (making certain arbitrary alignment choices without loss of generality.)

Although not original to this paper, the author found it difficult to find clear documentation on how to calculate the screw from the transformation matrix. We therefore include an exposition here, in hope it will be useful, and a valuable adjunct to the code to a programmer seeking to duplicate this functionality.

### 6.1 Computing the Screw Axis from a Transformation Matrix

Our goal is to compute a normalized vector  $\hat{u}$  aligned with the screw axis of the transformation effected by a transformation matrix  $R$ .

In order to be robust, it is valuable to check that the transformation matrix is a rigid transformation[17], as Chasles’ theorem applies only to rigid transformations.

(TODO: Note that the wikipedia article on rotations tells us we must treat this as a special case, but does not give this “zig zag” treatment, instead saying we must “it is necessary to diagonalize  $R$  and find the eigenvector corresponding to an eigenvalue of 1.” That seems more complicated than what I am doing there, though perhaps it is better. Probably it avoids the problem of accidentally guessing  $P$  to be on the axis.)

The angle of rotation is computable from the trace of  $R$ ,  $Tr(R)$ .

$$\theta = \arccos \frac{Tr(R) - 1}{2} \quad (9)$$

Technically,  $\arccos$  is multi-valued, but we will take  $\theta$  to be in its principle range,  $0 \leq \theta \leq \pi$ . If  $\theta$  is 0 or a multiple of  $\pi$ , then we the *zig-zag* degenerate case, the method of compute  $H$  from the rotation basis of  $R$  is numerically unstable. However, in this case we can compute the direction vector  $H$  as the difference vector between an arbitrary point  $P$  and its transformation performed twice. Informally, this is a “zig, then a zag”.

$$H = (R(R(P))) - P \quad (10)$$

(Note that in general  $H$  is not normalized,  $H \neq \hat{H}$ . Also recall that multiplication by a transformation matrix produces a point that in a new position representing the transformation.)

In other cases,  $u$  can be computed from the direction cosines of  $R$ [18]:

$$R = \begin{bmatrix} a & b & c' & x \\ d' & e & f & y \\ g & h & i & z \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (11)$$

$$H = \begin{bmatrix} h - f \\ c' - g \\ d' - b \end{bmatrix} \quad (12)$$

(The variables  $c'$  and  $d'$  are marked to distinguish them from the symbols for the chord  $c$  and the travel along the axis  $d$ .) The magnitude of  $\|H\| = 2 \sin \theta$ , which vanishes when  $\theta$  is 0 or multiple of  $\pi$ , hence our need to treat those cases differently.

Although the matrix  $R$  in general produces both a rotation and a translation in space, distance between  $P$  and  $R(P)$  in general depends on how far  $P$  is from the axis of rotation. However, the travel  $d$  along the axis of the rotation does not depend on  $P$ . It can be computed as a dot product:

$$d = \overrightarrow{BC} \cdot \hat{H} \quad (13)$$

These are the only properties that can be computed from the matrix  $R$  alone; but once we have the length along the object axis (*not* the helix axis)  $L$  of an object we can compute our computations, based on relations already given in Section 3.

The chord of the segmented helix (that is, length of an object of axis length  $L$  projected along  $H$ ) is:

$$c = \sqrt{L^2 - d^2} \quad (14)$$

Knowing the chord  $c$  and the amount of rotation  $\theta$  allows us to compute the radius:

$$r = \frac{c}{2 \sin \theta} \quad (15)$$

, unless the chord is 0, in which case the radius is 0. The helix angle  $\phi$  is a function of  $c$  and  $d$ :

$$\phi = \arctan \frac{c}{d} \quad (16)$$

Sidedness and pitch ( $s$  and  $p$ ) follow directly.

Finally, the vector  $\hat{H}$  gives the direction of the axis of the helix, but does not give us a point which fixes it in space. It is most convenient to accept a point  $B$  which is a joint and produce the point  $B_a$  which is the point on the helix axis closest to  $B$ .

To do this, we conceptually construct the midpoint  $M$  of  $\overrightarrow{BC}$  and utilize the fact that the vector  $Q$  from it to its closest point on the axis  $M_a$  is perpendicular to the containing both  $\overrightarrow{BC}$  and  $H$ . Therefore its direction is constructible via cross product, and its length  $l$  is computable from the radius and chord.  $M_a$  is the midpoint of  $B_a$  and  $C_a$ , so we can just move back half the travel  $d$  along the vector  $u$  to get  $B_a$ .

$$C = R(B) \quad (17)$$

$$\overrightarrow{BC} = C - B \quad (18)$$

$$M = \frac{B + C}{2} \quad (19)$$

$$Q = \overrightarrow{BC} \times u \quad (20)$$

$$l = \sqrt{r^2 - \frac{c^2}{2}} \quad (21)$$

$$Q' = -\frac{Q}{l} \quad (22)$$

$$B_a = M + Q' - \frac{d}{2}\hat{H} \quad (23)$$

Thus given only the transformation  $R$ , the length  $L$ , and one joint  $B$ , one may compute all of the intrinsic properties ( $r, \theta, d, c, \phi$ ) of the segmented helix and position it in space via  $H$  and  $B_a$ .

As is common in kinematics[19], there are many ways to represent the same physical or mathematical situation. Four consecutive joint positions also completely determine a segmented helix, as presented below. Because joints can be computed from transformation matrices and transformation matrices from joints, which method of calculation is preferable would be a matter of choice and clarity. We have in fact coded both and used the comparison as automated tests in our software to ensure the correctness of our coding.

A mechanical engineer, robotocist, or computer graphics expert is likely to find the computation from the transformation matrix more natural and convenient. A chemist or crystallographer is more likely to have learned the position of four points and wish to compute from that.

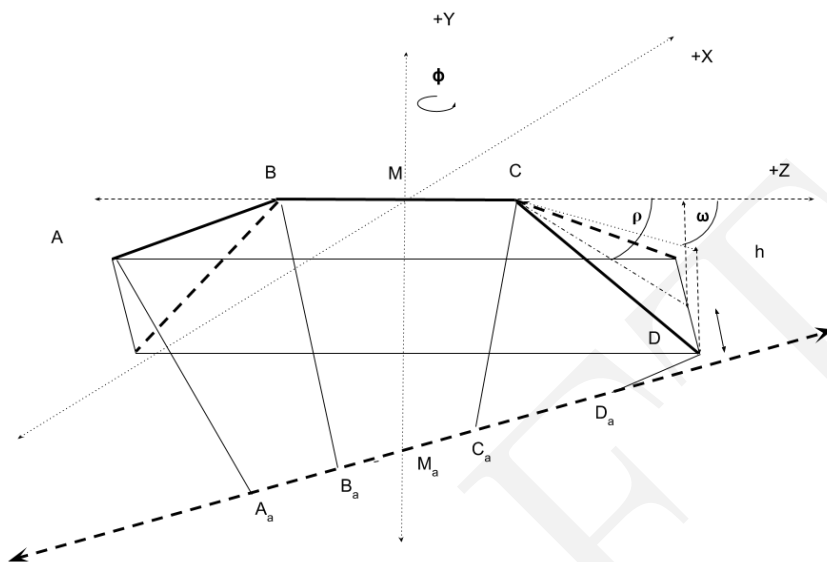


Figure 5: Three Symmetric Members

## 7 *PointAxis*: Computing Segmented Helices from Joints

Kahn[3] has given a method for computing the axis of a helix in the context of chemistry. This method uses the observation that the angle bisectors of the segments on a segmented helix are perpendicular and intersect the axis of the helix. Because the helices may not be perfect and because the measurement of positions may not be perfectly accurate, it is common for chemists to use regression and fitting methods to fit helix parameters to observed positions on the helix. Kahn’s method was a prelude to some error-tolerant methods applicable to the realm of organic chemistry. In this paper we are concerned with pure geometry. Also, Kahn was writing in 1989, and we now have more convenient computing tools. We give here a modification of Kahn’s algorithm, called *PointAxis*, which relies on our ability, working in the realm of pure geometry, to position the segments on the axes to simplify the derivation and computation.

### 7.1 A Sketch of the 4-Point Method

Using tools from linear algebra and well-documented algorithms, a sketch of the finding the segmented helix from four consecutive known points  $A, B, C, D$  is:

- Compute the bisectors of the angle  $B_b$  of  $\angle ABC$  and the angle  $C_b$  of  $\angle BCD$ . If



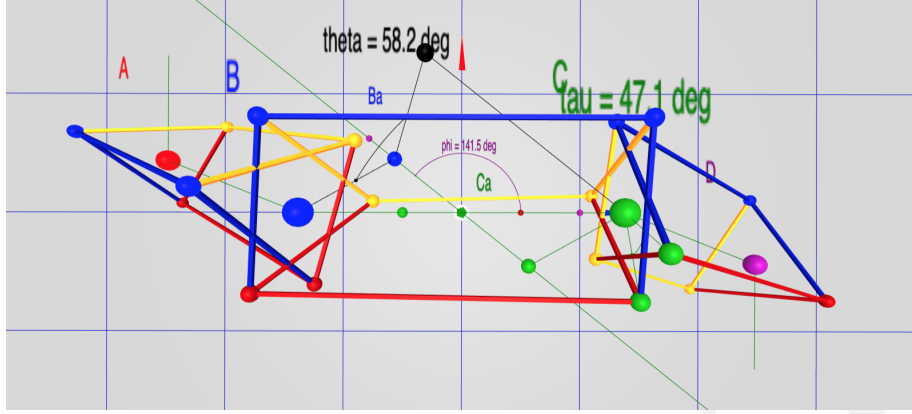


Figure 6: A Balanced Configuration

the points are collinear, we have a special case.

- Because these angle bisectors point at the axis of the segmented helix, their cross product is a vector in the direction of the axis. If  $B_b$  and  $C_b$  are parallel or anti-parallel the cross product is not defined and we have special cases.
- Otherwise the vectors  $B_b$  and  $C_b$  are skew, the algorithm for the closest points on two skew lines provides two points  $B_a$  and  $C_a$  on these vectors which are the closest points on those lines and are also points on the axis.
- The distance between  $B_a$  and  $B$  is the radius, and the distance between  $B_a$  and  $C_a$  is the travel  $d$  along the axis.
- The angle between  $\overrightarrow{B - B_a}$  and  $\overrightarrow{C - C_a}$  is  $\theta$ .

## 7.2 The 4-Point Method

Therefore four consecutive points completely determine at least one segmented helix. We will concern ourselves only with the helix that makes the least rotation between points. We provide a modification of the *PointAxis* algorithm which takes four such points (without loss of generality  $B$  and  $c$  are assumed to be centered on the  $z$ -axis, and that a rotation has been performed to balance  $A$  and  $D$  to that  $A.x = -D.x$ ,  $A.z = -D.z$ , and  $A.y = D.y$ . Thus the input to *PointAxis* in fact has three only three degrees of freedom, which determine the three intrinsic properties  $r, d, \theta$  which completely define the shape of a segments helix (but not its location in space.) A practitioner may be able to set up the coordinate system such that the four points are in balance; if not, Section 8.1 discuss how to transform them into balance.

Figure 6 shows a downward view of a balanced configuration (though raised above the origin instead of at the origin.)  $A$  (the red sphere) is a reflection of  $D$  (the purple sphere) across the midpoint. Both  $A$  and  $D$  are hanging downward. If the structure where hung on a point at the origin, it would be physically balanced. Note it is always possible to achieve this balance, even though the object itself is not symmetric; in this figure the normal of the  $B$  face is not symmetric with  $C$  face.

In the derivations below, we rely on certain facts about the segmented helix formed by the stack of objects, the first of which is key:

- Without loss of generality, we may think of any member whose faces and twist generate a non-degenerate helix as being “above” the axis of the helix. We furthermore choose to place the object in this figure so that  $B_y = C_y$ , that is, that the members are symmetric about the  $z$ -axis.  $A$  and  $D$  are “balanced across the  $YZ$ -plane, and  $A_x = -D_x$  and  $A_y = D_y$ .
- Every joint  $(A, B, C, D)$  is the same distance  $r$  from the axis  $H$  of the helix.
- Every member is in the same angular relation  $\phi$  to the axis of the helix.
- Since every member cuts across a cylinder around the axis, the midpoint of every member is the same distance from the axis which is general a little less than  $r$ . In particular the midpoint  $M$  whose closest point on the helix axis  $m$  is on the  $y$ -axis and  $Mm < \overline{Bb}$ .
- The points  $(A_a, B_a, C_a, D_a)$  on the axis closest to the joints  $(A, B, C, D)$  are equidistant about the axis and centered about the  $y$ -axis. In particular,  $\|\overrightarrow{B - B_a}\| = \|\overrightarrow{C - C_a}\|$ .

From the observations that  $\|\overrightarrow{B - B_a}\| = \|\overrightarrow{C - C_a}\|$  we concluded that the helix axis is in a plane parallel to the  $XZ$ -plane, it intersects the  $y$ -axis, but in general is not parallel to the  $z$ -axis.

Because the angle bisectors of each joint are in general skew, and intersect the axis perpendicularly, it is clear we can use linear algebra and the algorithm for the closest points on two skew lines to find  $B_a$  and  $C_a$ .

However, we can take advantage of the fact that a segmented helix has tremendous symmetry, and the angle bisectors are very far from being two generally skew lines. In fact, by taking advantage of the fact that the generating rule for an object chain requires similarity in every joint, we can arrange the objects as in Figure 5.

*PointAxis* takes a length and a point  $D$  known to be in a specific relation to  $B$  and  $C$ .

We have carefully arranging our axes so that we can compute  $\phi$ , the angle between the helical axis and the  $z$  axis. This, in combination with symmetry and the knowledge that the helical axis is in the  $XZ$  plane, lets us compute the points on the axis corresponding to the joints directly from  $\phi$ .

This algorithm coded below is simple enough that Mathematica[20] can actually produce symbolic closed-form formula for all computed valued in terms of  $L, x, y, z$ , but they are less comprehensible to the human eye than this algorithm, although their existence opens the possibility that, for example, the derivative representing the change in  $r$  with a change in  $D$  could be calculated.

### 7.3 Degenerate Cases

Define the angle bisector vectors:

$$B_b = B - (A + C)/2 \quad (24)$$

$$C_b = C - (B + D)/2 \quad (25)$$

$$(26)$$

The fundamental insight that the axis of the helix  $H$  can be computed by a cross product of the angle bisector vectors ( $B_b$  and  $C_b$ ) applies only when the angle-bisectors have a non-zero length and when they are not anti-parallel. When the are of zero length, this is the degenerate case of a straight line coinciding with all segments. In this case the segmented helix has This occurs only when  $A_x = 0 \wedge A_y = 0 \wedge A_z = -3L/2$ . In this case:

$$r = 0 \quad (27)$$

$$\theta = \text{undefined} \quad (28)$$

$$d = L \quad (29)$$

$$c = 0 \quad (30)$$

$$\phi = 0 \quad (31)$$

$$H = \begin{bmatrix} 0 \\ 0 \\ L \end{bmatrix} \quad (32)$$

$$B_a = B = \begin{bmatrix} 0 \\ 0 \\ -L/2 \end{bmatrix} \quad (33)$$

$H$  is the direction vector of the helix axis. In this case we do not have enough information to define  $\theta$ , unless it is through other information. For example, when using the joint-face normal method which specifies the twist  $\tau$  at the faces, then  $\theta = \tau$ .

When  $B_b$  and  $C_b$  are parallel (pointing in opposite directions), the zig-zag degeneracy occurs. Since we are assuming the balance of  $A$  and  $D$ , this occurs only when

$A_y = 0$ . In this case (denoting the  $x$  component of the  $B_a$  vector as  $B_{a[x]}$ ):

$$B_a = \begin{bmatrix} B_{a[x]} \\ B_{a[y]} \\ B_{a[z]} \end{bmatrix} \quad (34)$$

$$H = C - A \quad (35)$$

$$d = (C - B) \times H \quad (36)$$

$$r = \|B_b\|/2 \quad (37)$$

$$c = 2r \quad (38)$$

$$\phi = \text{atan2}(H_z, H_x) - \pi/2 \quad (39)$$

$$c = 0 \quad (40)$$

$$\theta = \pi \quad (41)$$

$$B_{a[x]} = \frac{d\sqrt{1 - (d/L)^2}}{2} \quad (42)$$

$$B_{a[y]} = 0 \quad (43)$$

$$B_{a[z]} = -\frac{d^2}{2L} \quad (44)$$

$$(45)$$

However, the most general case is simpler, and can be worked out with standard linear algebra operations. In the math below which is a direct analog of our coded solution, we have utilized the tremendous symmetry of the “balance” condition to use mostly scalar operations. There is some hope that this would allow closed-form expressions to be produced, perhaps with the aid of a symbolic computation system such as Mathematica[20]. If completed, this would allow us to give closed-form solution to the intrinsic properties of all the 28 Platonic helices enumerated in Sec12.

Once  $H$  has been calculated, the signed travel along the axis  $da$  is the scalar projection of a segment  $(C - B)$  onto  $H$ . From this  $\phi$  is directly calculatable.  $\phi$  allows a direct calculation of the  $x, y$  and  $z$  components of the point  $B_a$  on the axis pointed to by  $B_b$ .  $r$  is the distance between  $Ba$  and  $B$ .  $c$  and  $\theta$  are easily computed from these values.

$$H = \begin{bmatrix} -2B_{b[y]}B_{b[z]} \\ 0 \\ 2B_{b[y]}B_{b[x]} \end{bmatrix} \quad (46)$$

$$d = \frac{LB_{b[x]}}{\sqrt{B_{b[x]}^2 + B_{b[z]}^2}} \quad (47)$$

$$\phi = \text{atan2}(H_z, H_x) - \pi/2 \quad (48)$$

$$c = \sqrt{L^2 - d^2} \quad (49)$$

$$(50)$$

In this approach to calculation, it is easiest for us to compute the axis point  $B_a$  corresponding to  $B$  and use it to complete our computations.

From trigonometry and utilizing the facts that  $\phi = \arccos d/L$  and that  $\sin \arccos x = \sqrt{1-x^2}$  it can be shown that the  $x$  and  $z$  component of  $B_a$  are:

$$B_{a[x]} = \frac{d\sqrt{1-(d/L)^2}}{2} \quad (51)$$

$$B_{a[z]} = -\frac{d^2}{2L} \quad (52)$$

However, this computation exposes another special case: when the helix angle  $\phi$  is  $\pi/2$ , the segmented helix is a torus-like. In this case the axis point  $B_a$  is in fact on the  $y$ -axis, and we need only compute  $B_{a[y]}$ :

$$B_{a[y]} = \frac{LB_{b[y]}}{2B_{b[z]}} \quad (53)$$

When we are not toroidal, we must take  $B_{b[x]}$  into account, but it is non-zero, so we can divide by it. By imagining a plane pressed downward from the object axis to the helix axis, we see that  $B_{a[y]}$  is proportional to a ratio of the angle bisector  $B_{b[y]}/B_{b[x]}$  times the  $B_{a[x]}$  value:

$$B_{a[y]} = \frac{B_{b[y]}B_{a[x]}}{B_{b[x]}} \quad (54)$$

Having computed all of  $B_a$ , the remaining intrinsic properties are easily calculated:

$$r = \|B - B_a\| \quad (55)$$

$$\theta = 2 \arcsin \frac{c}{2r} \quad (56)$$

$$(57)$$

#### 7.4 The 4-Point Test

It is useful to have a test if four proposed points really do lie on a segmented helix, to see if they allow valid inputs to *PointAxis* algorithm to be computed via rigid transformation to the  $z$ -axis.

**Theorem 2** (Segmented Helix Test). *Four arbitrary sequential points  $A, B, C, D$  are consecutive joints of a segmented helix if and only if they segments  $\overline{AB}$ ,  $\overline{BC}$ , and  $\overline{CD}$  are equal length and the scalar projection of  $\overrightarrow{CD}$  onto  $\overrightarrow{BC}$  is equal to the negation of the scalar projection of  $\overrightarrow{AB}$  onto  $\overrightarrow{BC}$ .*

*Proof.* “If” Case (points on helix imply scalar projections are negations of each other): If  $A, B, C, D$  are consecutive joints on a segmented helix, then the angle  $\eta$  between any

two consecutive segments is the same. Then:

$$\begin{aligned}\|\vec{AB}\| &= \|\vec{CD}\| && \text{Given} \\ \|\vec{AB}\| \cos \eta &= \|\vec{CD}\| \cos \eta && \eta \text{ the same for each joint} \\ \vec{AB} \cdot \vec{BC} &= \vec{CD} \cdot \vec{BC} && \text{def. of scalar product}\end{aligned}$$

“only if” Case (equal scalar projections and length imply coincident segmented helix):

If the scalar products and the lengths are equal, then the cosines of the angles between segments are equal. In the range 0 to  $\pi$ ,  $\cos \theta = \cos \phi$  implies  $\theta = \phi$ . Therefore the angles between the segments are equal.

By the previous argument of correctness for the *PointAxis* algorithm, a rigid transformation always exists which balances three such segments, there always exists a helix axis that is in the  $xz$  plane the intersects the  $y$  axis and is the same distance from  $A$ ,  $B$ ,  $C$  and  $D$ . This axis is the axis of a segmented helix which rotates each point similarly, provides the same translation along the axis, and maintains the same radius. Hence a segmented helix exists whose joints are  $A$ ,  $B$ ,  $C$ , and  $D$ .  $\square$

## 7.5 Comparison

There is one reason one might prefer the transformation matrix method or the point method over the other: with modern computer algebra systems such as Mathematica[20] it might be possible to use these “algorithms” to produce closed-form expressions of closed-form (algebraic) inputs. For example, the Platonic solids all have lengths and face normals which can be specified exactly in closed (though irrational) form. Thus it might be possible to produce an expression for the radius of one of the Platonic Dodecahelicies of unit edge length. We have not undertaken this work.

## 8 The Joint Face Normal Method

*PointAxis* takes a point  $A$  known to be in a specific, balanced relation to  $B$  and  $C$ . A chemist might know 4 such points from crystallography, and be able to move them into this symmetric position along the  $z$ -axis.

However, we might instead know something of the subunits and how they are conjoined, without actually knowing where points  $A$  and  $D$  are.

We start with these intrinsic properties of an object, and additionally the rule for how objects are laid face-to-face. That is, knowing the length between two joint points and a vector normal to the faces of the two joints, we almost have enough to determine the unique stacking of objects. The final piece is that we must know the *twist*. That is, when face  $A$  of a second objects is placed on face  $B$  of a first object so that they are flush (that is, their normals are in opposite directions), it remains the case that the second object can be rotated about the normals. To define the joining rule, we must attach an *up vector* to each object. Then a joining rule is “place the second object against the first, joint point coincident to joint point, and twist it so that its up

vector differs by  $\tau$  degrees from the up vector of the first object.” In this definition, the up vectors are considered to be measured against the plane containing the two axes meeting in a joint.

Define the *joint plane* to be the plane which contains the two axes meeting in a joint. Define the *joint line* to be the line through the joint perpendicular to the joint plane. Define the *joint angle* to be the angle of the first axis to the second measured about the joint line. The twist  $\tau$  is the change in the a vector attached to the object rotated about the joint line by the joint angle. That is, take any vector attached to the first object, place it at the joint, rotate it about the joint line via the joint angle.  $\tau$  is the difference between the angle of this vector measured against the joint plane and the angle of the up vector of the second object measured against the joint plane.

If the objects are macroscopic objects which have faces, this is the same as the rotation of the axis of the second object relative to the first in the plane of the coincident faces. We can identify intrinsic properties:

- An object with two identified faces, labeled  $B$  and  $C$ . Assume there normalized vectors  $N_B$  and  $N_C$  from each of these points that is aligned with the axis of the conjoined object attached to that face. This normals might be enforced by the fact that flat faces are joined in the joint plane. However, molecules don’t have faces; this conjoining relationship may be enforced some other way.
- The length  $L$  of an object, measured from joint point  $A$  to joint point  $B$ .
- A joint twist  $\tau$  defining the change in computed out-vector between objects, measured at the joint face.

## 8.1 Rotating into Balance from Face Normal Vectors

In order to use the *PointAxis* algorithm, we need a way to compute points  $A$  and  $D$  in balance around the axis  $BC$ . A key insight is that Lord’s Observation tells us that no matter how lopsided and different the normal vectors  $N_B$  and  $N_C$  for the joint faces are and no matter what  $\tau$  we choose, when we conjoin objects their relationship is always the same. After placing  $BC$  along the  $z$ -axis, there is always an angle  $\psi$  which will rotate the points  $A$  and  $D$  into balance (that is,  $(A_x = -D_x) \wedge (A_z = -D_z) \wedge (A_y = D_y)$ ).

For computer programmers with a graphics library supporting transformation matrices such as THREE.js[21], it is relatively easy to code the math to adjoin objects face-to-face based on the face normals, simulating the physical act of matching flat faces between macroscopic objects.

1. Create the transformation the aligns and centers  $\overrightarrow{BC}$  on the  $z$ -axis.
2. Create a translation of  $B$  to  $C$ .
3. Create a rotation of the  $z$ -axis to  $N_B$ .
4. Create a rotation of of  $-N_c$  about  $N_B$ .
5. Create a rotation of  $\tau$  around that axis.

Composing these transformation matrices via multiplication creates a transformation matrix which takes  $B$  to  $C$  and  $C$  to  $D$ .

We will assume this as a subroutine called *adjoinPrism* which takes  $\tau$  (the rotation inside the plane of the joint). A byproduct of balancing the points is the a transformation matrix that takes  $C$  into  $D$ . Having done this, one can compute the screw axis, and hence all of the segmented helix properties, from either the transformation matrix or the four points. Our code does both and compares the result as a test.

## 8.2 The Key to Balance

The key insight to finding  $\psi$  to note that we can consider the projections of the  $B$  and  $C$  face normal vectors projected into the  $XY$  plane, and rotate these so that they are balanced around the negative  $y$  unit vector. Such a projection into the cross-section of the helix is closely related to the *helical wheel*[22] plot in the study of alpha helices in proteins. Even if the lengths of the projections of the face normals in  $XY$  are different, this mechanism works, because by Lord's Observation the points  $A$  and  $D$  must be symmetric about the segment  $BC$ .

By composing this balancing operation with the face-adjoining transformation matrix,  $A$  and  $D$  are placed in balance. The screw axis may now be computed from either the four points  $A, B, C, D$  or from the transformation matrix created to balance them.

## 8.3 On the Choice of the Screw Axis Direction

Given only a physical segmented helix without position in space, we may arbitrarily choose the direction for the axis. Changing our decision will make the screw axis point in the negative direction, change the sign of the travel  $d$  along the axis, and choose  $\phi$  to be  $\pi - \phi$ .

As can be seen from interactive play with our software[5], it is entirely possible for the travel along the axis of the helix to be 0—in fact, choosing  $\tau \approx 0$  produces toruses, which have no travel along the axis.

We could represent this by making the length of the vector representing the axis be the length of the travel  $d$ . However, this would have the drawback than when  $d$  is zero we would be unable to determine the axis of revolution of the torus. Although somewhat arbitrary, we have chosen instead to represent the axis as a normalized vector of unit length, and to allow the travel to be negative. This has the benefit that changing  $\tau$  through (something close to) zero smoothly changes  $d$ . However, it creates the problem that as  $\tau$  approaches (something close to)  $\pi$  from different directions the signs of the axes are different. That is,  $\tau \approx \pi$  and  $\tau \approx -\pi$  describe exactly the same shape, but in our calculation they will have different signs for the axes. The radius, pitch and absolute value of the travel, which are intrinsic to the shape, will be the same, but the axis vector,  $\phi$ , and the sign of  $d$  will be different.

# 9 Checks and Explorations

## 9.1 Changing $\tau$ Smoothly Changes Tightness

**Theorem 3** (Twist Spectrum). *For any choice of face normals having non-zero  $x$  or  $y$  components, changing  $\tau$  smoothly varies the segmented helix between a torus and*



*zig-zag cases.*

*Proof.* Define  $\alpha$  to be the angle of the project of the point  $A$  into the  $XY$ -plane in our standard positioning. Define  $\beta$  to be the angle of the  $D$  projection. Define  $\psi$  to be the angle between these:  $\psi = \alpha - \beta$ . By the continuity of mechanical processes, there can be not discontinuity in the shape as  $\tau$  is minutely varied. Rotating the twist  $\tau$  through an entire revolution returns us to the starting configuration. Therefore such a revolution must achieve all values of  $\psi$  between 0 and  $\pi$ . One of these values is similar to a torus, and one is a zig-zag.  $\square$

**Theorem 4** (Torus Corollary). *For any choice of face normals, there is choice of the twist  $\tau$  for which the segmented helix is a planar polygon, thereby resembling a torus.*

*Proof.* Define  $\alpha$  to be the angle of the project of the point  $A$  into the  $XY$ -plane in our standard positioning. Define  $\beta$  to be the angle of the  $D$  projection. Define  $\psi$  to be the angle between these:  $\psi = \alpha - \beta$ . By Lord's Observation, any change to  $\tau$  still produces a segmented helix, therefore a balancable configuration. by Lord's Observation,  $\frac{\partial \alpha}{\partial \tau} = -\frac{\partial \beta}{\partial \tau}$ . Therefore  $\frac{\partial \psi}{\partial \tau} = 2\frac{\partial \alpha}{\partial \tau}$ . By observation and the continuity of mechanical processes,  $\alpha(\tau)$  is a monotone function. Therefore it is always possibly to continuously change  $\tau$  to either increase or decrease  $\psi$ . The  $\tau$  that makes  $\psi = 0$  produces a torus.  $\square$

It would be more useful and elegant to have a formula for the  $\tau$  that produces a torus as a function of the joint face normals; the authors have not been able to produce such a formula. However, we in the calculator page, we numerically provide the  $\tau$  that produces the minimum tightness (torus-like) and maximum tightness (zig-zag) to the nearest degree, with the labels *Minimum Tightness  $\tau$*  and *Maximum Tightness  $\tau$* .

## 9.2 Qualitative Observations

When the normals are coplanar vectors, then the minimum tightness  $\tau$  is always 0, and the maximum tightness occurs when  $\tau = \pm\pi$ . These values deviate from 0 and  $\pi$  roughly in proportion to the non-coplanarity of the normals.

Varying  $\tau$  smoothly varies the tightness of the coiling of the helix, moving through vary linear cases towards a torus, to a torus, to a very linear case on the other side.

In fact it is possible to that there is always a "tightest coil" which does not self-intersect. If we had many objects, we could pack them into a convenient space by computing the  $\tau$  of the tightest coil and stacking them this way. If we had a means to change  $\tau$ , perhaps via motors in a robot arm, we would have a smoothly telescoping and contracting robot arm or linear actuator. If we had a repated molecular subunit that changed shape in response to an exteranl magnetic or electric field or chemicals in the surrounding environment, we would have a telescoping nanomachine.

## 9.3 A Brute-force approach to finding Helix Angle from Twist

The calculation methods described in this paper hardly warrant the name "algorithm" when considered from the computational complexity; the are all constant time (for a

given fixed precision.) Although involving trigonometric functions, they demand no iteration.

This makes it practical to solve some problems by brute-force iteration. An example already calculated is the twist  $\tau$  which makes an object product a toroid-like segmented helix, or on the other hand the  $\tau$  value that maximizes linear extent and tightness. Future work may allow analytic formulae for  $\tau$  as a function of tightness to be developed; but in the meantime it is easy enough to simply evaluate objects at many different  $\tau$  values to find a desired helix angle  $\phi$ , such as 0. One can of course bring standard numerical optimization to bear, because an objective function that depends on the parameters of the segmented helix can be computed in constant time.

## 10 Applying to The Boerdijk-Coxeter Tetrahelix

The Boerdijk-Coxeter tetrahelix is a periodic chain of conjoined regular tetrahedra which has been much studied[9, 10, 11, 12] and happens to have irrational measures, making it an ideal test case for our algorithms. Because the face-normals can be calculated and the positions of the elements of the BC helix directly calculated, we can use it to test our algorithms, and in fact these algorithms give the same rotation.

However, it should be cautioned that the helix which Coxeter identified[9] goes through every node of every tetrahedron. Constructing the helix that goes through only “rail” nodes allows irregular tetrahelices to be designed[12]. However, the segmented helices defined in this paper do neither; rather, it is most natural to imagine them moving through the centroid of face of a tetrahedron. This complementary view is to think of the BC Helix not as the helix that intersects the vertices of the tetrahedron as Coxeter did[9], nor a single rail as may be valuable to engineers[12], but rather as a helix through the center point of the faces of the tetrahedron. This is a segmented helix of very small radius (0.0943) compared to the other two approaches ( $\frac{3\sqrt{3}}{10} \approx 0.5196$ ) which measure the radius to the point of the tetrahedron, but it has the advantage that it is far more general. For example, it is clearly defined if one used truncated tetrahedra. The rotation of a segment matches the BC Helix analytical solution ( $\theta = \arctan -3/2 \approx 131.8103$ ), because a screw transformation does not depend on selecting a point for the radius.

In light of Lord’s Observation and the Segmented Helix algorithms, we can now consider the BC Helix and a variety of other segmented helices generated by face-to-face stacking of Platonic solids, examples of called *Platonic segmented helices*,

Note this also makes clear that in these cases we must also specify the *twist*, even if we insist on perfect face-to-face matching. Thinking of it this way, there are actually three tetrahedral segmented helices, depending on which twist modulo  $120^\circ$  is chosen (keeping the faces matching). In the case of the tetrahedron, this creates the clockwise BC Helix, the anti-clockwise BC Helix, and the not-quite-closed tetrahedral torus, similar in appearance to but quite the same as *toriodal polyhedra*[23]. (Five tetrahedra famously lack  $\approx 7.356$  degrees of being a perfect toriodial polyhedron as can be seen from our software which computes  $\theta = 70.5288$  for this case, so the gap is  $360 - 5 \cdot \theta \approx 7.356$ )

In the case of the icosahedron, there are in fact many possibilities, as one need not

choose the precisely opposite face as the joining face, and one may choose up to three twists. The “zoo” or Platonic helices can be studied via the calculations described here, and our software makes these interactively selectable (see Section12).

## 11 Implications

One of the implications of having an easily-calculable understanding of the math is that it may be possible to design helices of any radius and pitch by designing periodic (possibly scalene) segments. Combined with slight irregularities, this means that you have a basis of design molecular helices out of “atoms” which correspond to our objects.

This would mean that if you wanted to build a brace of length exactly 3 meters with bars of exactly 1/2 meter you would be able to come as close to this as mathematically possible.

A modular robot constructed out of repetitions of the same shape-changing module will always product a helix whose precise shape can be controlled by uniformly changing the shape of all of the modules.

## 12 The Platonic Helices

In order to demonstrate the utility of the calculations explained in this paper, we have explored periodic chains of the five regular Platonic solids joined face-to-face so that their vertices coincide, which form *Platonic helices*. Such tetrahelices, icosahelices, octahelices and dodechelices have been mentioned in a number of papers[24, 25, 26], but not exhaustively studied in the purely helical form. We propose the name *cubahelix* for the helix made from cubes, as opposed to the equivalent but cumbersome *hexahedronahelix*. Because in some cases Platonic segmented helices may be found in nature or related to structures found in nature[27, 13], it would be convenient to have a table, and images, of all such Platonic segmented helices for reference.

To construct a periodic chain from a Platonic solid, one must decide which faces are joined by the rule. Additionally, one must determine a twist  $\tau$  as part of the rule, and this twist must be chosen from a small set if the vertices are to coincide. The set of vertex-matching twists differs slightly depending on the face chosen for octahedron, dodacahedron, and icosahedron (but not for the tetrahedron and the cube.) The number of twists in a set will always be equal to the number of sides on a face.

Therefore the number of Platonic helices is in principle a summation of a number of faces times a number of sides, or  $4 \cdot 3 + 6 \cdot 4 + 8 \cdot 3 + 12 \cdot 5 + 20 \cdot 3 = 12 + 24 + 24 + 60 + 60 = 180$ . However, many of the possibly helices will be indistinguishable of we consider only the shapes produced, as opposed to considering labeled faces. Furthermore, every non-toroidal helix will come in clockwise and counter-clockwise version. However, we do not consider rules such as “attach face zero to face zero” which would constitute “doubling back”[24]. The transformation matrix for such a rule would be the identity matrix. It produces an object axis of zero length, a radius of zero, and a travel of zero. It produces perfect self-intersection; that is the entire degenerate helix would appear to be a simply a single Platonic solid.

Using the math in this paper, it was easy to evaluate all 180 helices, place them in a table, and group them by radius and travel (collapsing chirality). The result is 28 unique shapes. In this number, no provision was made to exclude self-intersection, which does occur, but might not matter to an aerospace engineer building a collapsible space frame of rods and joints. In the language of Elgersma and Wagon[24], not all of the 28 Platonic helices *embedded*.

With those caveats, the helices in Table 1, exemplified by accompanying figures and renderable interactively on our calculator page, thus represents an exhaustive catalog, colloquially called a “zoo”, of all Platonic helices.

Many of these helices have been previously individually mentioned and even rendered in the literature, though not necessarily fully calculated. In this table, column *C-Face* refers to the face as numbered by the THREE.js software[21], which is somewhat arbitrary. The *# Analogs* column gives the number of Platonic Helices with the same shape, or the enantiomer of it, that is, the same shape in either the clockwise or anticlockwise direction. This list can be expanded completely in our interactive software.

Clearly, for each solid there is a change in twist which keeps the faces aligned if they start aligned. This is the  $2\pi/n$ , where  $n$  is the number of sides in a face. The base twist that creates a perfect face-to-face match depends not only on the solid but the face we choose to conjoin to. For each of the Platonic solids, we have just computed this base angle by visual inspection and trial and error. The twist  $\tau$  if a species of helix is given in the column below.

## 12.1 Qualitative Descriptions and Interesting Shapes

For fun and to facilitate conversation, we have given all 28 of these Platonic helices nicknames that we find descriptive<sup>1</sup>. A few of these are interesting enough to be worthy of particular mention, and comparing them shows the possibility of designing structures using nothing but Platonic solids. The math in this paper works equally well with irregular shapes as well, allowing continuous spectra of designed structures from repeated shapes or molecules.

- “The Blockhelix” (Figure 7) is cubic rectilinear structure in which all angles are right angles; nonetheless a segmented helix hides inside it perhaps not apparent to the human eye at first glance.
- “Pearlshaft” (Figure 8). Conjoining parallel faces always produces a *shaft*. This icosahedron, being relatively round, resembles a string of pearls.
- However, shaft-like helices exist which do not join opposite faces. “The Dodecashaft” (Figure 9) is a remarkably tight non-self-intersecting dodecahelix with very narrow gaps between objects. Such a configuration might be formed by nanofibers under pressure.

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<sup>1</sup>These impressions are likely to be incomprehensible to a non-native speaker of English, or even to someone not born in America in the 1960’s. We apologize for this, and hope the value of connoting the shape outweighs this cultural bias.

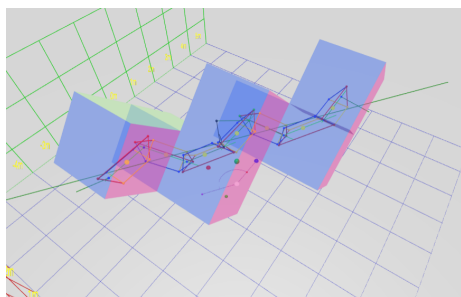


Figure 7: The Blockhelix

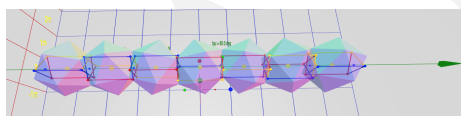


Figure 8: The Pearls shaft

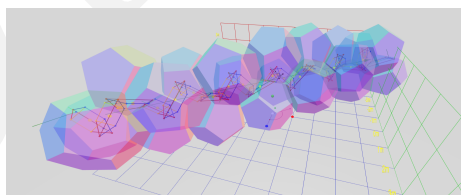


Figure 9: The Dodecashaft  
1

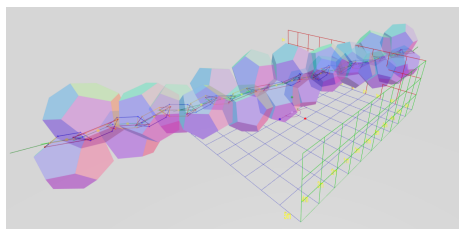


Figure 10: The Dodecadoubler

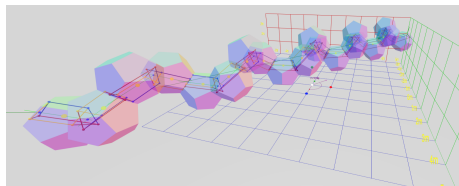


Figure 11: The Dodecacorkscrew

- “The Dodecadoubler” (Figure 10) presents the appearance of being a double helix, even though in fact it is a single helix with a simple twist of  $72^\circ$  from the Dodecashaft
- “The Dodecacorkscrew” (Figure 11) is a contrasting example of a loose helix, reminiscent of a corkscrew for opening wine bottles.
- The “Quasi-planar” (Figure 12) icosahelix presents a slowly twisting metahelix, so perhaps 10 icosahedron could be said to “lay flat”. If this were a molecule or a physical structure made of less-than-perfect rigid members it might be possible to force it into a pure planar configuration, thus wrapping a cylinder or a plane, studding it with icosahedra.
- “Two Strands” (Figure 13) is similar to the “Dodecadoubler” but even more visually striking. It is reminiscent of a depiction of a DNA double helix.

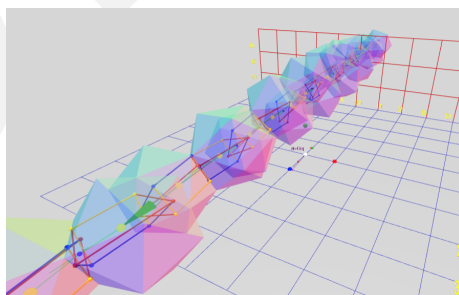


Figure 12: Quasi-planar icosahelix

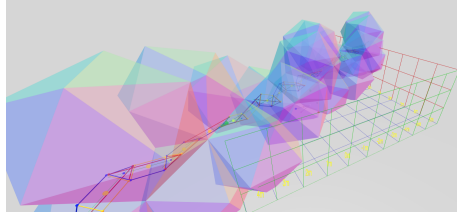


Figure 13: Two Strands

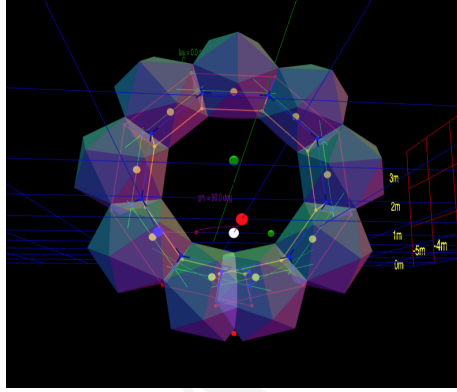


Figure 14: The Wheel

- “The Wheel” (Figure 14) resembles a modern car tire in proportions. All Platonic solids and indeed all shapes have torus-like configurations. In general they do not “close” perfectly; that is, there is a gap that prevents the final faces from fitting together perfectly. However, one could make tiny adjustment to repeated shape to close this gap.

### 13 Future Work

The algorithms and software described herein allow numerical calculation of the intrinsic properties of these Platonic helices, but it would be even better to describe them in with closed-form expressions, as Coxeter did for the Boerdijk-Coxeter tetrahelix. The math and the algorithms are simple enough that if coded in a symbolic algebra system like Mathematica, or with careful work, closed-form expressions could be produced for all the regular Platonic helices. These would be interesting if they happen to be short; we have no reason to believe they will be. The same work could be done for the Archimedean solids. The current work would serve as a useful validation check and intuition-builder for such work.

There may exist a closed-form expression for twist  $\tau$  which produces a desired helix angle  $\psi$ ; we have provided only an iterative algorithm for it, which is practical but less elegant.

## 14 To Do

- Modify that so that it truly respects  $\psi$  as an input.
- Provide explanation, graphically, if need be, for the computation of  $B_{ax}, B_{ay}, B_{az}$  in the general cases.
- Clean up the code. 3 days
- Go through each reference 1 day
- Try to get list of references to the computation of a screw from an arbitrary matrix, and compare and contrast to our code. 1 day
- Need to understand possibility of further simplifying specification of object.
- Note further that Equations 7 and 8 of this paper[3] give BETTER equations for radius  $r$  and the distance  $d$  than what I have so far given. Note: I've studied this; I'm not sure there is anything worth doing there.
- Need to get this paper, by hook or by crook, and probably cite: Note: An historical review of the theoretical development of rigid body displacements from Rodrigues parameters to the finite twist <https://www.sciencedirect.com/science/article/pii/S0094114X0500087X>
- Improve qualitative section, talk about toruses - 8 hours
- Consider better naming mechanism for faces - 4 hours

## 15 References that need to be studied or reviewed

This is a long, expensive book, but it may be quite relevant[28]: [https://books.google.com/books?hl=en&lr=&id=1LZ1SZ70RrQC&oi=fnd&pg=PP1&ots=0hSEwJv1UB&sig=xNG9UWv\\_H10XHwa0i0BJN7TW6xA#v=onepage&q&f=false](https://books.google.com/books?hl=en&lr=&id=1LZ1SZ70RrQC&oi=fnd&pg=PP1&ots=0hSEwJv1UB&sig=xNG9UWv_H10XHwa0i0BJN7TW6xA#v=onepage&q&f=false)

This is a discussion of segmented coils in a protein structure:

<https://www.sciencedirect.com/science/article/pii/0022283688903701>

A modern helix structure protein paper:

<https://www.sciencedirect.com/science/article/pii/S1476927108000583>

"Simulation of Suspensions of Helical Rigid Fibers" Y Al-Hassan : British Journal of Mathematics and Computer Science (PDF downloaded)

"HELFIT: Helix fitting by a total least squares method" : This needs to be studied closely! <https://www.sciencedirect.com/science/article/pii/S1476927108000418>

QHELIX: A Computational Tool for the Improved Measurement of Inter-Helical Angles in Proteins <https://link.springer.com/article/10.1007/s10930-007-9097-9>

Note: "On the Screw Axes and Other Special Lines Associated With Spatial Displacements of a Rigid Body" <http://manufacturingscience.asmedigitalcollection.asme.org/article.aspx?articleid=1439697>

Note: An historical review of the theoretical development of rigid body displacements from Rodrigues parameters to the finite twist <https://www.sciencedirect.com/science/article/pii/S0094114X0500087X>



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Table 1: The Platonic Helices

Name	Solid	#	Analogs	C-Face	#	$\tau$	radius	$\theta$	d	$\phi$
Tetrahelix	Tet		6	1	-120	0.094	131.810	-0.516	161.565	
Tetratorus	Tet		3	1	0	0.471	70.529	0.000	90.000	
Boxbeam	Cub		4	1	-180	0.000	0.000	1.000	0.000	
Staircase	Cub		4	2	-180	0.000	0.000	0.707	0.000	
Blockhelix	Cub		8	2	-90	0.236	120.000	-0.577	144.736	
Cubatorus	Cub		4	2	0	0.500	90.000	0.000	90.000	
Octabeam	Oct		3	5	-60	0.000	0.000	1.155	0.000	
Octaspikey	Oct		6	1	-120	0.148	146.443	-0.603	154.761	
Octamedium	Oct		6	2	-120	0.163	131.810	-0.894	161.565	
Octagear	Oct		3	1	0	0.408	109.471	0.000	90.000	
Treestar	Oct		3	2	0	0.816	70.529	0.000	90.000	
Dodecabeam	Dod		5	8	-108	0.000	0.000	1.589	0.000	
Dodecadoubler	Dod		10	1	-144	0.113	161.301	-0.805	164.550	
The Alternater	Dod		10	2	-144	0.118	149.520	-1.333	170.306	
Dodecashaft	Dod		10	1	-72	0.351	129.657	-0.543	130.501	
Dodecagear	Dod		5	1	0	0.491	116.565	0.000	90.000	
Dodecacorkscrew	Dod		10	2	-72	0.546	93.026	-1.095	144.110	
Dodecadonut	Dod		5	2	0	1.286	63.435	0.000	90.000	
Pearlshaft	Ico		3	13	-60	0.000	0.000	1.589	0.000	
Quasi-planar	Ico		6	8	165	0.049	167.764	1.294	4.347	
Two Strands	Ico		6	1	120	0.137	159.446	0.499	28.340	
Slow Twist	Ico		6	12	120	0.169	124.309	1.454	11.641	
Rock Candy	Ico		12	2	120	0.204	146.443	0.830	25.239	
Icosa Tree Star	Ico		3	1	0	0.304	138.190	0.000	90.000	
Icosacorkscrew	Ico		6	8	-75	0.512	99.253	-1.037	143.042	
Planar point cluster	Ico		6	2	0	0.562	109.471	0.000	90.000	
Big Icosacorkscrew	Ico		6	8	45	0.803	82.064	0.756	54.343	
The Wheel	Ico		3	12	0	2.080	41.810	0.000	90.000	