

# On the Segmented Helix Formed by Stacking Similar Objects

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## Abstract

In nature, helical structures arise when identical structural subunits combine sequentially, the orientational and translational relation between each unit and its predecessor remaining constant. A helical structure is thus generated by the repeated action of a screw transformation acting on a subunit.[1]

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## 1 Introduction

During the Public Invention Mathathon of 2018[2], software was created to view chains of regular tetrahedra joined face-to-face in chains. The participants noticed that whenever the rules for which face to add the next tetrahedron to were periodic, the resulting chain was always a helix.

Although unknown to us at that time, we now call Lord's Observation:

In nature, helical structures arise when identical structural subunits combine sequentially, the orientational and translational relation between each unit and its predecessor remaining constant.[1]

The purpose of this paper is to prove Lord's Observation and provide mathematical tools and software for studying the segmented helices generated in this way.

The fundamental method of finding the properties of a segmented helix from two segments on the helix, which we call Kahn's method[3], is explained and implemented in Mathematica and an interactive, 3D rendering website written in JavaScript which allows both calculation and interactive play and study, with an aim to allow a helix to be designed from an object or an object designed to produce a particular helix. Kahn's method is extended to cover some degenerate situations.

We exploit Lord's Observation to discover symmetry which allows us to apply Kahn's method when subunits are joined face-to-face with the same *twist*. Kahn was investigating proteins, which do not have faces, geometric solids and other macroscopic objects do. This symmetry allows us compute the parameters of the segmented helix purely from properties intrinsic to a single object and the joining rule. This paper thus provides a set of tools to either design a subunit and rule to produce a helix as desired, or to investigate possible subunit configurations that produce a particular discerned helix. Finally, as an example, we use these tools to produce a table of all possible helices generated by vertex-matched face-to-face joints of the Platonic solids, of which only the tetrahelix[6, 7, 8, 4] has been studied to date.

## 2 A Warm-up: Two Dimensions

Considering the problem in two dimensions may be a valuable introduction. Suppose that we consider a polygon that has two edges, called  $A$  and  $B$ , and that we define the length  $L$  of the polygon as the distance between the midpoints of these edges.

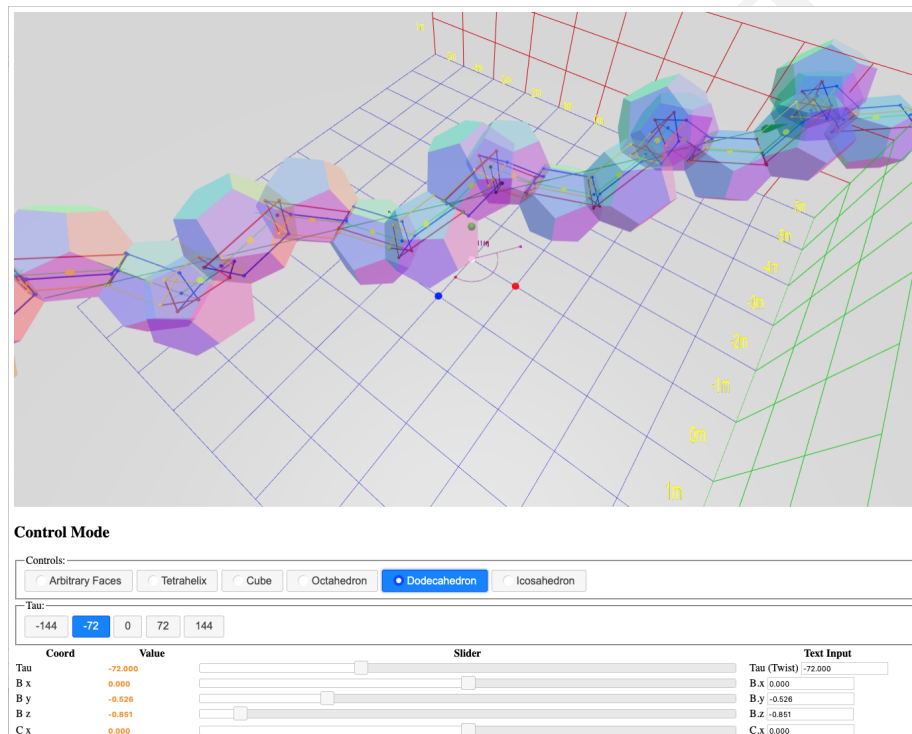


Figure 1: Example Segmented Helix Generated From the Dodecahedron

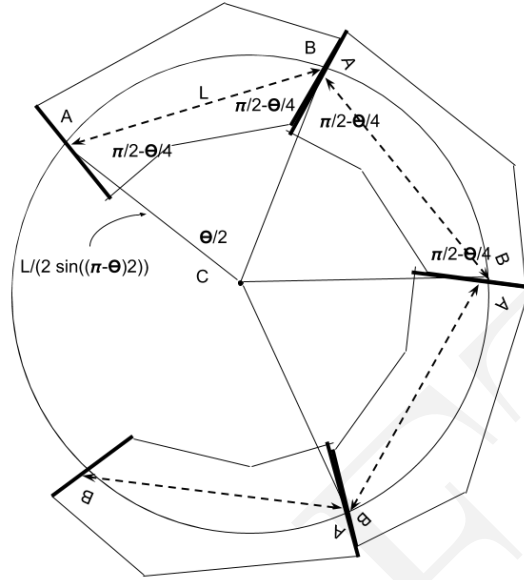


Figure 2: A 2D Analog of a Helix Generated by Repeated Subunits

Suppose that we are only allowed to join these polygons by aligning  $A$  of one polygon to  $B$  of another polygon, with their midpoints coincident. Let us further assume that we disallow inversions of the polygon. Let us imagine that we have a countable number of polygons  $P_i$  indexed from 0. Then what shapes can we make by chaining these polygons together?

Each joint  $J_i$  between polygons  $P_i$  and  $P_{i+1}$  will place the axes of at the same angle,  $\theta$ , since our polygons do not change shape. Let us define  $\theta$  to be positive if we move anti-clockwise from  $P_i$  to  $P_{i+1}$  and negative if we move clockwise. If  $\theta = 0$ , the joints will be collinear.

If  $\theta \neq 0$ , it seems they polygon joints will always lie on a circle. A proof of this is that each polygon has associated with it an isosceles triangle  $A, B, C$ , where  $\angle CBA = \angle CAB = \theta/2$ , and  $\angle ACB = (\pi - \theta)$ .  $AC$  and  $BC$  are not necessarily aligned with an edge of the polygon. The length  $AB$  is  $L$ , and the lengths  $AC$  and  $BC$  are  $(L/2)/\sin(\pi - \theta)/2$ . In any chain of polygons, these triangles all meet at point  $C$ , and there all joints are on the circle centered at  $C$  with radius  $\frac{L}{2 \sin \frac{\theta}{2}}$ .

If we al

## 2.1 The Inscribed Figure

TODO: Bring the reasoning from [4] here to define the inscribed figure.

### 3 The Segmented Helix

An analogous, though far more complicated, result holds in three dimensions.

In this section we consider segmented helix, or a helix evaluated only at regular points.

Following the Wikipedia article <https://en.wikipedia.org/wiki/Helix>, we set up a helix parametrically.

$$\begin{aligned}P_x(t) &= r \sin t \\P_y(t) &= r \cos t \\P_z(t) &= bt\end{aligned}$$

Such a helix has a radius of  $r$  and slope (if  $r \neq 0$ ) of  $b/r$ . The pitch of helix, the change in  $t$  needed to make one complete revolution, is  $2\pi b$ . Note that a helix may be degenerate in two ways. If  $r = 0$ , these equations become a line. If  $b = 0$ , these equations describe a circle in the  $xy$ -plane. If  $r = 0$  and  $b = 0$ , the figure is a point.

Such helices are continuous, but we are investigating stacks of discrete objects. We in fact wish to derive the parameters for a continuous helix from such discrete objects which constrain discrete points, so we wish to study a helix evaluated at integral points. We call such an object a *segmented helix* or *shelix* (*SHELL-lix*), *shelices* (*SHELL-ih-sees*). A segmented helix may be thought of as function that given an integer gives back a point in three space.

$$\begin{aligned}P_x(n) &= r \sin n\theta \\P_y(n) &= r \cos n\theta \\P_z(n) &= nd\end{aligned}$$

$d$  is the distance or *travel* along the  $z$ -axis between adjacent joints, and  $\theta$  is the rotation around the  $z$ -axis between adjacent points.  $r$  is the radius of the shelix. Note that if  $\theta = \pi$ , we have a third form of degeneracy (to the human eye) of a shelix which is a zig-zag contained completely within a single plane.

If we think of the shelix as describing a polyline in 3-space, we would like to investigate the properties of that polyline.

Because the shelix is a discrete structure, we reframe the concept of *pitch* as *sidedness*  $s$ : how many segments (sides) make a complete rotation?

- $L$  is the distance between any two adjacent points.
- $\theta$  rotation about the axis between two joints.
- $c$  is the length of a chord formed by the projection of the segment between two points into the  $xy$ -plane.
- $\phi$  is the angle in the plane containing two joints which is perpendicular to  $xy$ -plane (in other words the plane parallel to the  $z$  axis containing two points.)
- $s$  is the number of segments in a complete rotation (in general not rational.)

These quantities are related:

$$c = 2r \sin \frac{\theta}{2} \quad (1)$$

$$L^2 = c^2 + d^2 \quad (2)$$

$$\arctan \frac{c}{d} = \phi \quad (3)$$

$$s = \frac{2\pi}{\theta} \quad (4)$$

Now we are attempting to relate these properties to properties intrinsic to the joint or interface between two segments or objects in the shelix. If given an object, the length between the joints  $L$  is intrinsic.

### 3.1 The Interior of a Shelix

What is the radius a cylinder that would just fit inside a shelix without any of the segments of the shelix touching the interior of the cylinder? An approach to solving this problem from [4] may be useful. As shown in [4], if the rotation angle  $\theta/\pi$  is rational, the interior may in fact be n-gon prism, otherwise it will be a circle.

## 4 The Intrinsic Properties of Periodic Chains

If we have chains of repeated 3D units conjoined identically, they generate a helix. Although it may have been known earlier, we call this Lord's Observation:

**Observation 1** (Lord's Observation). *In nature, helical structures arise when identical structural subunits combine sequentially, the orientational and translational relation between each unit and its predecessor remaining constant.[1]*

Lord's Observation may perhaps be clarified that in fact identical objects conjoined via a rule produce periodic chains of objects that are uniformly intersected by segmented helices (shelices) and that they may be degenerate in one of three ways that we might not strike us as a helix if we are not seeking them:

1. The segments may form a straight line.
2. The segments may be planar about a center, forming a polygon or ring.
3. The segments may form a planar saw-tooth or zig-zag pattern of indefinite extent.

TODO: Add figures for each of these cases, maybe in one figure.

There are two complementary ways of learning about such shelices. In one approach, we may have knowledge of the shelix, and wish to learn about the subunits and the rule with which the subunits are combined. For example, we may have microscopic objects such as proteins or atoms, and we know, from crystallography something about the positioning of these objects, without knowing ahead of time the angles at which these objects would combine in their natural environment. In this case, we use a method we call Kahn's method[3] for determining the radius, travel, and twist of the shelix (these terms will be defined precisely below.)

In the other approach, we may know *a priori* exactly the relevant properties of the objects and the rule which they combine, and we seek to compactly describe the shelix they create. For example, a mathematician may consider a chain of dodecahedra, or a woodworker may cut identical blocks of maple wood, which are to be glued together face-to-face. In these cases everything about the objects and the rules for conjoining is known before the first two objects are glued together. We call this the *face normal method*, because it can be simulated by joining two flat faces together with a specified twist, even if the objects in question do not actually have a physical face.

In both cases, we would like to understand how a change in a face normal or a twist affects the parameters of the shelix, and, conversely, we would like to be able to choose the construction of the subunits to achieve a particular shelix.

In engineering, sometimes the term “special helix” [5] is used for helical curves on non-cylindrical surfaces. This paper uses the term “helix” only in the sense of “cylindrical helix”.

## 5 Periodic Chains Produce Segmented Helices

A periodic chain is in fact a simple object which demonstrates tremendous symmetry. Before using this symmetry in the construction of the segmented helix, we prove that such a segmented helix indeed exists.

**Theorem 1.** *Consider  $N$  identical objects which each have two points,  $A$  and  $B$ , called joints, and a vector  $U$  not collinear with  $A$  and  $B$ . Call  $\overline{AB}$  the axis of this object. Consider the frame of reference for this object to have its axis on the  $z$ -axis with  $B$  in the positive direction, the midpoint of the axis at the origin, and the up-vector pointing in the positive  $y$  direction.*

*Consider any rule that conjoins  $A$  of object  $i + 1$  to  $B$  such that from the frame of reference of  $i$ , the object  $i + 1$  anything rigidly attached to it is always in the same position in the frame of reference for  $i$ . Informally,  $i + 1$  “looks the same” to  $i$ , no matter what  $i$  we choose,  $i < N$ . Call a chain of  $N$  identical rigid objects conjoined via a rule that conjoins  $A$  to  $B$  in such a way that every vector of  $B$  is always in the same position relative to a frame of reference constructed from  $A$  a periodic chain.*

*Any periodic chain of three or more objects has a unique segmented helix whose segments correspond to the axes of these objects.*

*Proof.* We will proceed by induction.

Base Case ( $k == 3$ ):

Take an object  $AB$ . By Chasles’ theorem, there is a screw axis  $S$  and slide  $d$  which moves the first object to position where the second object  $BC$  is. Construct the points  $A'$ ,  $B'$  and  $C'$  as the closest points to  $A$ ,  $B$  and  $C$  on this axis. These points are collinear by construction.

Now add the object  $CD$  to object  $BC$  by our rule of periodic chains. Consider the points  $B'$  and  $C'$  from  $A$ ’s frame of reference. Let  $d = \|\overline{C'} - \overline{B'}\|$ . Construct the point  $D'$  on our screw axis as the point closest to  $D$  on that line.

Now because  $C'D'$  in  $BC$ ’s frame of reference must look like  $B'C'$  in  $A$ ’s frame of reference, must look like  $B'C'$ , the distance  $\|\overline{D'} - \overline{C'}\| = d$ . From  $A$ ’s frame of

reference,  $A'B'C'$  are collinear, the points  $B'C'D'$  must be collinear in  $B$ 's frame of reference.

In ANY frame of reference, if  $A'B'C'$  are collinear and  $B'C'D'$  are collinear, then  $A'B'C'D'$  are collinear.

Now, looking backward from  $CD$  towards  $A$ , the distance  $A'B'$  must be the same as the distance  $B'C'$  so as to not violate our rule. So  $d = \|A'B'\| = \|B'C'\| = \|C'D'\|$ . Similarly because by construction  $r = \|BB'\| = \|CC'\|$ , and  $AA'$  is a rigid transformation of  $BB'$ , so  $r = \|AA'\|$ . By symmetry,  $r = \|DD'\|$ . Compute  $\theta$  as the rotation about  $S$  that takes  $BB'$  into  $CC'$ . By our rule of attachment,  $\theta$  also takes  $CC'$  into  $DD'$  and  $AA'$  into  $BB'$ .

Now construct a segmented helix, the radius  $r$ , distance, and angle  $\theta$ . This segmented helix can be positioned coincident with  $S$  so that  $H_0 = A$ . Then  $H_1 = B$ ,  $H_2 = C$ , and  $H_3 = D$ .

Therefore, for the base case of three objects, there is a segmented helix whose segments coincide with the axes of the objects.

Inductive Case  $(k+1)$ : Assume there is a segmented helix coinciding with the first  $k$  objects, and consider the frame of reference of the  $k$ th object. The axis and any other rigid property of the  $k+1$ th object stands in relation to object  $k$  as  $k$  stood to  $k-1$ . By induction, the  $k$ th object had a segment of a segmented helix corresponding to its axis. Attach vectors  $V_{Ak}$  and  $V_{Bk}$  from the joints of  $k$  to the axis of the helix perpendicularly. Define these vectors in the frame of reference for  $k$ .

To the  $k-1$ th, the tips of  $V_{Ak}$  and  $V_{Bk}$  define a line segment which lies on the axis of the segmented helix  $H$ , with the tip of  $V_{Ak}$  coincident with the tip of  $V_{B(k-1)}$ .

By our rule and by induction, since this is true of the  $k-1$ th object, it is true of the  $k$ th object. There the  $k+1$ th objects  $V$  vectors point to a line segment which lies on the axis of  $H$ , extending it in the same direction. The axis of the  $k+1$ th object therefore coincides with the  $k+1$ th segment of  $H$ .

Therefore, by induction, identical objects conjoined by the same rule always coincide with some segmented helix, whose parameters are discoverable by Kahn's method.  $\square$

This theorem leads to the following corollary.

**Corollary 1.** *The axes of the object in a periodic chain are symmetric to the axis of the segmented helix coincident to it. The signed angle between the axes of two adjacent objects is the same in both directions.*

*Proof.* The axes of each object coincide with a segment of a segmented helix. A segmented helix is completely symmetric no matter if which direction of the axis you look down. The angle between each pair of objects is exactly the same.  $\square$

Note that the objects themselves may not be symmetric; for example, The  $B$  face does not have to be the same size as the  $A$  face. In fact, the object itself might be shaped like the letter "C", and not complete enclose the axis. Taking the idea further, the object might be spiky like a stellated polyhedron or a sea urchin, and still be joined by joints relatively close to the center of the object. In this paper we do not concern ourself with the issue of self-collision of the objects, which would have to be considered if one attempted to make a period chain of sea urchins.



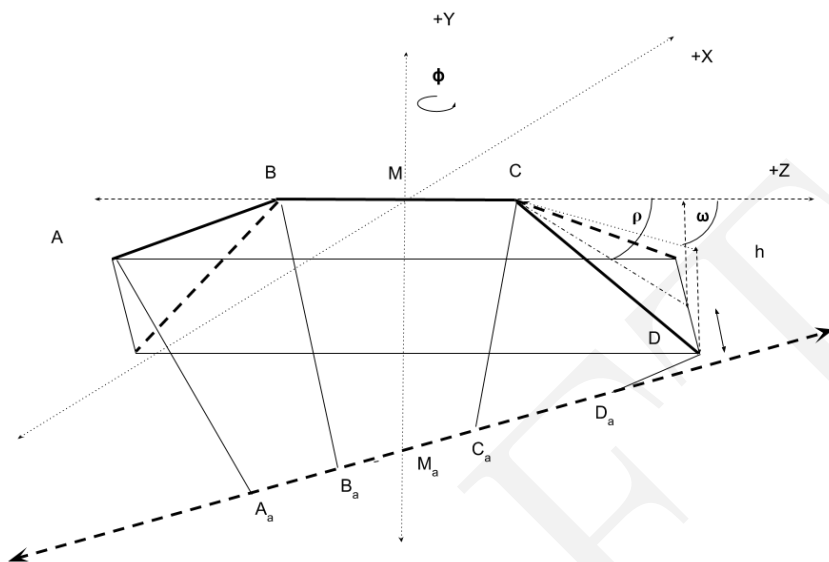


Figure 3: Three Symmetric Members

Corollary 1 will be used in our development of KahnAxis algorithm and in our computation of shelix properties and to justify balancing face normals to produce an intrinsic out-vector and to apply the KahnAxis algorithm without actually assigning objects Cartesian coordinates.

## 6 KahnAxis: Computing Segmented Helices from Periodic Chains

Kahn[3] has given a method for computing the axis of a helix in the context of chemistry. This method uses the observation that the angle bisectors of the segments on a segmented helix are perpendicular and intersect the axis of the helix. Kahn’s method was a prelude to some error-tolerant methods applicable to the realm of organic chemistry. In this paper we are concerned with pure geometry. Also, Kahn was writing in 1989, and we now have more convenient compute tools. We give here a modification of Kahn’s algorithm, called *KahnAxis*, which relies on our ability, working in the realm of pure geometry, to position the segments on the axes to simplify the derivation and computation.

In the derivations below, we rely on certain facts about the segmented helix formed by the stack of objects, the first of which is key:

- Without loss of generality, we may think of any member whose faces and twist generate a non-degenerate helix as being “above” the axis of the helix. We furthermore choose to place the object in this figure so that  $B_y = C_y$ , that is, that the members are symmetric about the  $z$ -axis.  $A$  and  $D$  are “balanced” across the  $YZ$ -plane, and  $A_x = -D_x$  and  $A_y = D_y$ .
- Every joint  $(A, B, C, D)$  is the same distance  $r$  from the axis  $H$  of the helix.
- Every member is in the same angular relation  $\phi$  to the axis of the helix.
- Since every member cuts across a cylinder around the axis, the midpoint of every member is the same distance from the axis which is general a little less than  $r$ . In particular the midpoint  $M$  whose closest point on the helix axis  $m$  is on the  $y$ -axis and  $\overline{Mm} < \overline{Bb}$ .
- The points  $(A_a, B_a, C_a, D_a)$  on the axis closest to the joints  $(A, B, C, D)$  are equidistant about the axis and centered about the  $y$ -axis. In particular,  $\|B - B_a\| = \|C - C_a\|$ .

From the observations that  $\|B - B_a\| = \|C - C_a\|$  we concluded that the helix axis is in a plane parallel to the  $XZ$ -plane, it intersects the  $y$ -axis, but in general is not parallel to the  $z$ -axis.

Because the angle bisectors of each joint are in general skew, and intersect the axis perpendicularly, it is clear we can use linear algebra and the algorithm for the closest points on two skew lines to find  $B_a$  and  $C_a$ .

However, we can take advantage of the fact that a segmented helix has tremendous symmetry, and the angle bisectors are very far from being two generally skew lines. In fact, by taking advantage of the fact that the generating rule for an object chain requires similarity in every joint, we can arrange the objects as in 2.

KahnAxis takes a length and a point  $D$  known to be in a specific relation to  $B$  and  $C$ . From there, he goes on to compute various parameters.

We have modified this method by carefully arranging our axes so that we can compute  $\phi$ , the angle between the helical axis and the  $z$  axis. This, in combination with symmetry and the knowledge that the helical axis is in the  $XZ$  plane, lets us compute the points on the axis corresponding to the joints directly from  $\phi$ .

This algorithm coded below, is simple enough that Mathematica can actually produce symbolic closed-form formula for all computed valued in terms of  $L, x, y, z$ , but they are less comprehensible to the human eye than this algorithm, although their existence opens the possibility that, for example, the derivative representing the change in  $r$  with a change in  $D$  could be calculated.

The fundamental insight that the axis of the helix  $H$  can be computed by a cross product of the angle bisector vectors ( $Bb$  and  $Cb$ ) applies only when the angle-bisectors have a non-zero length and when they are not anti-parallel. When the are of zero length, this is the degenerate case of a straight line coinciding with all segments. This occurs only when  $z = L$ . When  $Bb$  and  $Cb$  are parallel (pointing in opposite directions), the zig-zag degeneracy occurs. This occurs only when  $y = 0$ .

Once  $H$  has been calculated, the signed travel along the axis  $da$  is the scalar projection of a segment  $(C - B)$  onto  $H$ . From this  $\phi$  is directly calculatable.  $\phi$  allows a direct calculation of the  $x, y$  and  $z$  components of the point  $Ba$  on the axis pointed to

by  $Bb$ .  $r$  is the distance between  $Ba$  and  $B$ .  $c$  and  $\theta$  are easily computed from these values.

TODO: Move the tests that I have coded in JavaScript into Mathematica, handle special cases as separate math routines to report here.

```
ChordFromLDaxis[L_, Da_] := Sqrt[L^2 - Da^2]
RotationFromRadiusChord[R_, C_] := 2 ArcSin[C/(2R)]
KahnAxis[L_, {x_, y_, z_}] := Block[{A, B, C, D, Cb, Bb, H,
                                     Bax, Bay, Baz, Ba,
                                     r, theta, da, c, phi},

  D = {x, y, z};
  A = {-x, y, -z};
  B = {0, 0, -L/2};
  C = {0, 0, L/2};
  Cb = (B+D)/2 - C;
  Bb = (A+C)/2 - B;
  H = Normalize[Cross[Bb,Cb]]; // Why am I calling Normalize twice?
  da = (C - B) . Normalize[H];
  phi = ArcCos[da/L];
  Bax = Sin[phi] da / 2;
  Baz = -Cos[phi] da / 2;
  Bay = Bb[[2]] (Bax / Bb[[1]]);
  Ba = {Bax, Bay, Baz};
  r = Norm[Ba - B];
  c = ChordFromLDaxis[L, da];
  theta = RotationFromRadiusChord[r, c];
  {r, theta, da, c, phi, H}
]
```

Note that in Figure 2 there is great room for confusion in terms of plane  $\omega$  is actually measured against. The three triangles  $\triangle BB_aC$ ,  $\triangle BM_aC$ , and  $\triangle BM_aC$  are all in general not co-planar, that is, they are all at slight angles to each other.

## 6.1 The Face Normal Method

KahnAxis takes a point  $D$  known to be in a specific, balance relation to  $B$  and  $C$ . A chemist might know 4 such points from crystallography, and be able to move them into this symmetric position along the  $z$ -axis.

However, we might instead know something of the subunits and how they are conjoined, without actually knowing where points  $A$  and  $D$  are.

If we are given a physical or mathematical object that stacks, the faces may be specified by a normal vector or by the angles of the face relative to the axis. The normal vector is a more linear-algebraic approach; the face angles may be more natural to chemists and mechanical engineers. Since the two methods are interchangeable, it is largely a matter of convenience. However, we seek specifically to develop a formula which allows helices to be designed, and the face-angle approach seems more suited to the carpenter grasping a hand saw or the molecular biologist designing a molecule.

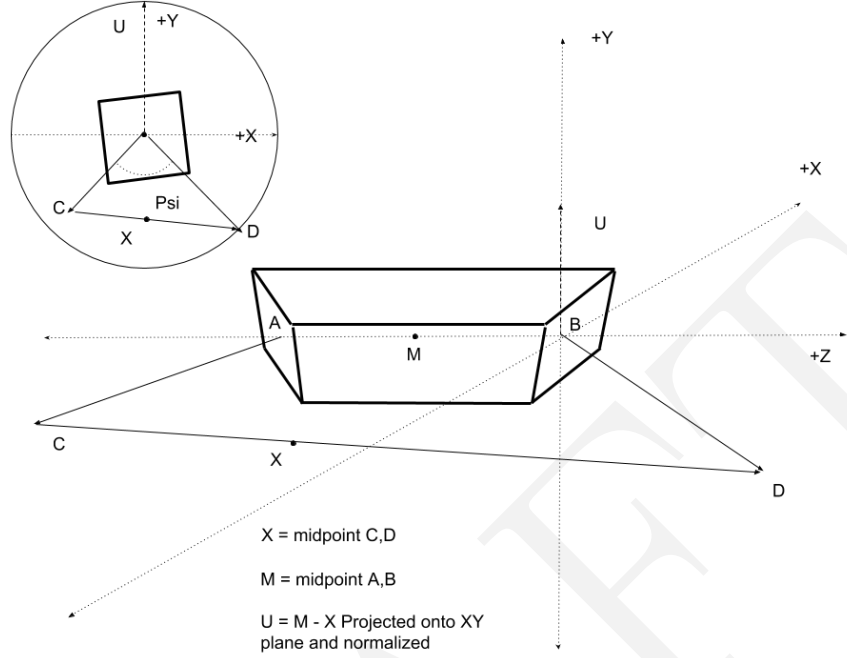


Figure 4: The rotatable prism of three objects

As shown in figures and , We place a joint point on each face, and call these points  $A$  and  $B$ . We define the axis of the object as  $\overline{AB}$ . Imagine the object placed on the  $z$  axis in a right-handed coordinate system, so that  $A$  is in the negative  $z$  direction and  $B$  is at the origin. Then the cut of the  $B$  face can be described by two angles.  $\alpha$  is the angle in the  $XZ$  plane, and  $\beta$  is the angle in the  $YZ$  plane. In other words, if a box or cuboid were drawn with three edges aligned with the axis and at the origin, and the vector defining the face-normal defined the diagonal of the box or cuboid,  $\alpha$  is the angle with the  $z$ -axis of the face diagonal in the  $XZ$  plane (or the projection of the body diagonal into that plane), and  $\beta$  is the angle of the  $z$ -axis with the  $YZ$ -plane.

The face angles for  $A$  are denoted  $\alpha_A, \beta_A$ , and likewise the independent face angles for  $B$  are  $\alpha_B, \beta_B$ .

In the angle method, we start with these intrinsic properties of an object, and additionally the rule for how objects are laid face-to-face. That is, knowing the length between two joint points and a vector normal to the faces of the two joints, we almost have enough to determine the unique stacking of objects. The final piece is that we must know the *twist*. That is, when face  $A$  of a second objects is placed on face  $B$  of a first object so that they are flush (that is, their normals are in opposite directions), it remains the case that the second object can be rotated about the normals. To define the joining rule, we must attach an *up vector* to each object. Then a joining rule is “place the second object against the first, joint point coincident to joint point, and twist it so that its up vector differs by  $\tau$  degrees from the up vector of the first

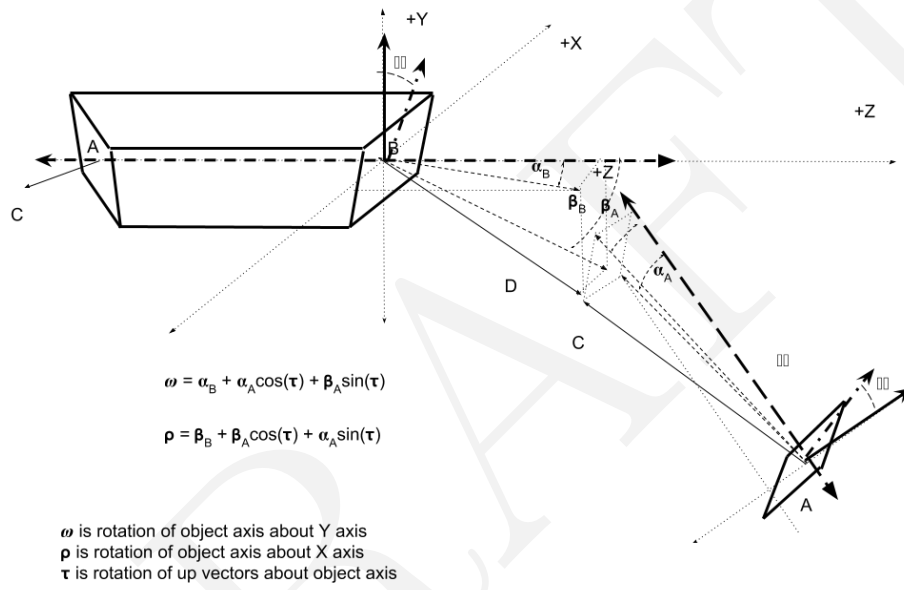


Figure 5: Joint Geometry

object.” In this definition, the up vectors are considered to be measured against the plane containing the two axes meeting in a joint.

Define the *joint plane* to be the plane which contains the two axes meeting in a joint. Define the *joint line* to be the line through the joint perpendicular to the joint plane. Define the *joint angle* to be the angle of the first axis to the second measured about the joint line. The twist  $\tau$  is the change in the angle of a vector attached to the object rotated about the joint line by the joint angle. That is, take any vector attached to the first object, place it at the joint, rotate it about the joint line via the joint angle.  $\tau$  is the difference between the angle of this vector measured against the joint plane and the angle of the up vector of the second object measured against the joint plane.

If the objects are macroscopic objects which have faces, this is the same as the rotation of the axis of the second object relative to the first in the plane of the coincident faces.

- An object with two identified faces, labeled  $F_A$  and  $F_b$ . These faces are in fact planes, with  $\alpha_A, \beta_A, \alpha_B, \beta_B$ , specifying the angular separation of the normal from the  $XZ$  and  $YZ$  planes. We assume that normals point out of the object rather than in. The angles in the method may be considered the angles of the face diagonals of the axis-aligned cuboid whose body diagonal is the vector normal.
- The length  $L$  of an object, measured from joint point  $A$  to joint point  $B$ .
- A joint twist  $\tau$  defining the change in computed out-vector between objects, measured at the joint face.

## 6.2 Rotating into balance from face normal vectors

In order to use the KahnAxis algorithm, we need a way to compute points  $A$  and  $D$  in balance around the axis  $BC$ . From Lord’s Observation, we know that after placing  $BC$  along the  $z$ -axis, there is always an angle  $\psi$  which will rotate the points  $A$  and  $D$  into balance.

It is relatively easy to work out the math to adjoin objects face-to-face based on the face normals; we will assume this as a subroutine called “adjoinPrism” which takes  $\tau$  (the rotation inside the plane of the joint).

The key insight to finding  $\psi$  is to note that we can consider the projections of the  $B$  and  $C$  face normal vectors projected into the  $XY$  plane, and rotate these so that they are balanced around the negative  $y$  unit vector. Even if the lengths of the projections of the face normals in  $XY$  are different, this mechanism works.

TODO: Properly code in Mathematica and place code here. Figure out how to express this mathematically.

## 7 Checks and Explorations

In this section add graphs. Also, a check against the BC helix. Possibly software should be used to produce a 3D simulation of the issues.

## 8 Applying to The Boerdijk-Coxeter Tetrahelix

TODO: This is highly redundant.

The Boerdijk-Coxeter tetrahelix is a chain of conjoined regular tetrahedra which has been much studied[6, 7, 8, 4] and happens to have irrational measures, making it an ideal test case for our algorithms. Because the face-normals can be calculated and the positions of the elements of the BC helix directly calculated, we can use it to test our algorithms, and in fact these algorithms give the correct result.

However, it should be cautioned that the helix which Coxeter identified[6] goes through every node of every tetrahedron. Constructing the helix that goes through only “rail” nodes allows the tetrahelix to be modified[4]. However, the shelices defined in this paper do neither; rather, it is most natural to imagine them moving through the centroid of face of a tetrahedron. The rotation of a segment thus matches the BC Helix ( $\arctan -3/2$ ), but the radius of the generating shelix in the paper would be smaller than those shelices that intersect the nodes.

In light of Lord’s observation and the Shelix algorithm, we can now reconsider the BC Helix, and in fact a variety of segmented helices which should perhaps be called *Platonic delices* or if you prefer *Platonic shelices*.

This complementary view is to think of the BC Helix not as the helix that intersects the vertices of the tetrahedron as Coxeter did[6], nor a single rail as may be valuable to engineers[4], but rather as a helix through the center point of the faces of the tetrahedron. This is a shelix of very small radius compared to the other two approaches, but it has the advantage that it is far more general. For example, it is clearly defined if one used truncated tetrahedra.

More generally, the same approach gives the formula for the shelix created by placing dodecahedra or icosahedra or octahedra face-to-face in a regular pattern. The table of parameters of such shelices is provided below.

Note this also makes clear that in these cases we must also specify the *twist*, even if we insist on perfect face-to-face matching. Thinking of it this way, there are actually 3 tetrahedral shelices, depending on which twist is chosen (keeping the faces matching). In the case of the tetrahedron, this creates the clockwise BC Helix, the anti-clockwise BC Helix, and the not-quite-close tetrahedral torus.

In the case of the icosahedron, there are in fact many possibilities, as one need not choose the precisely opposite face as the joining face, and one may choose up to three twists.

All of this is a consequence of Lord’s observation that *any* repeated transformation produces a shelix.

Unfortunately, the complexity of these formulae exceed the author’s comprehension. However, we may check these formulae by graphing them against comprehensible examples. Obvious examples are extreme solutions, where  $\alpha$  and  $\theta$  are 0 or  $\pi/2$ , for example. We also have the particular non-trivial example of the Boerdijk-Coxeter tetrahelix, formed by regular tetrahedra, which has been studied enough to have a known pitch.

## 9 Implications

One of the implications of having a formulaic understanding of the math is that it may be possible to design helices of any radius and pitch by designing periodic (possibly scalene) segments. Combined with slight irregularities, this means that you have a basis of design molecular helices out of “atoms” which correspond to our objects.

This would mean that if you wanted to build a brace of length exactly 3 meters with bars of exactly 1/2 meter you would be able to come as close to this as mathematically possible.

## 10 Applied to Periodic Regular Simplex Chains

It is now clear that in fact ANY repeated rule applied to chains of tetrahedra, or any other object, is simply producing a larger repeated subunit, and that Kahn’s method and Face Normal method may be applied to the larger subunit.

In some circumstances we may wish to design a shelix based on “molecules” comprised of “atoms”, rather than using fully versatile objects. For example, it might be convenient for us to recombine pre-fabricated regular tetrahedra or some other unit, rather than constructing units having arbitrary face angles. If our construction method allow, we might ask what can be accomplished with a set of “atoms” and a variation in the twist  $\tau$ [9].

**Corollary 2.** *Every regular simplex chain formed by a periodic generator has a helical structure.*

## 11 Relating to the Screw Transform

TODO: Lord has described these operations in terms of screw transforms. In a sense this is the opposite or what we want to do, but it should be possible to interrelate the two mechanisms. Ideally would have an algorithm that operates on Screw Transforms and gives the shelix, and be able to produce a Screw Transform from a shelix.

## 12 Future Work

We propose that the math developed in this paper can be used to build an exhaustive table of the properties Platonic shelices, that is, segmented helices constructed solely out of regular Platonic solids. Such tetrahelices, icosahelices, octahelices and dodechelices have been mentioned in a number of papers[9, 10, 11], but not exhaustively studied in the purely helical form. Because in some cases Platonic delices may be found in nature or related to structures found in nature[12], it would be convenient to have a table, and images, of all such Platonic shelices for reference.

Note: Must read this: [https://www.researchgate.net/profile/Peter\\_Kahn/publication/220667044\\_Defining\\_the\\_axis\\_of\\_a\\_helix/links/5b86ab1e299bf1d5a730ff2e/Defining-the-axis-of-a-helix.pdf](https://www.researchgate.net/profile/Peter_Kahn/publication/220667044_Defining_the_axis_of_a_helix/links/5b86ab1e299bf1d5a730ff2e/Defining-the-axis-of-a-helix.pdf)[3].



Note this extremely important observation: “Since  $V_1$  and  $V_2$  are both perpendicular to the axis, their cross product will have the direction of the axis”. (Here  $V_1$  and  $V_2$  are angle bisectors.) This should make my work much simpler! To some extent this suggest that this work is not as original as I had thought.

Note further that Equations 7 and 8 of this paper give BETTER equations for radius  $r$  and the distance  $d$  than what I have so far given.

Note: Here is an example of a question asked on Math Stack Exchange which is essentially answered by this paper:

<https://math.stackexchange.com/questions/878051/why-does-a-3d-line-of-segments-with-constant-radius-878079#878079>

The answerer in fact predicts the linear bisection method which I have outlined.

Note: This must be studied immediately:

<https://math.stackexchange.com/questions/1041780/how-to-prove-the-bisector-vector-of-the-axis-1042231#1042231>

This assumes helical to begin with so not of much use.

Note: Must read this: [https://www.researchgate.net/profile/Peter\\_Kahn/publication/220667044\\_Defining\\_the\\_axis\\_of\\_a\\_helix/links/5b86ab1e299bf1d5a730ff2e/Defining-the-axis-of-a-helix.pdf\[3\]](https://www.researchgate.net/profile/Peter_Kahn/publication/220667044_Defining_the_axis_of_a_helix/links/5b86ab1e299bf1d5a730ff2e/Defining-the-axis-of-a-helix.pdf[3]).

<https://www.win.tue.nl/~wstomv/publications/mathmitering-final.pdf> [https://www.clinbiomech.com/article/S0268-0033\(98\)00080-1/abstract](https://www.clinbiomech.com/article/S0268-0033(98)00080-1/abstract) <https://gist.github.com/peteristhegreat/3b76d5169d7b9fc1e333> <https://www.sciencedirect.com/science/article/pii/S0022309303008573> <https://www.sciencedirect.com/science/article/pii/S0022309307005583>

This reference is EXTREMELY IMPORTANT <https://link.springer.com/article/10.1023/A:1015863923728>

This may be worth reading: <https://link.springer.com/article/10.1007/PL00011063>

Some discussion of “screw transformations” <http://dergipark.gov.tr/download/article-file/56483>

CRITICAL: <https://ieeexplore.ieee.org/stamp/stamp.jsp?tp=&arnumber=56653>

THIS IS SUPER IMPORTANT. I STILL CAN'T TELL IF IT DOES WHAT WE ARE DOING. [http://www.12000.org/my\\_notes/screw\\_axis/index.htm](http://www.12000.org/my_notes/screw_axis/index.htm)

This paper considers only simple unbranched chains of objects. In the future it might be valuable apply similar software and mathematics to branching systems such as Lindemayer systems[13].

Chains of tetrahedra are interesting because they are structurally strong; however, branched and rejoined structures are equally interesting. The current work might assisted in the design of trusses which begin at a point, trifurcate into chains, and eventually rejoin, create spindle-enveloped space frames which would resist buckling.

## 13 References that need to be studied or reviewed

NOTE: This is a discussion of representing joint angles, it is not obvious how valuable it is: [https://www.clinbiomech.com/article/S0268-0033\(98\)00080-1/abstract](https://www.clinbiomech.com/article/S0268-0033(98)00080-1/abstract)

NOTE: This needs to be read and understood, it is not clear how related it is: <https://ieeexplore.ieee.org/document/56653>

This is a long, expensive book, but it may be quite relevant[14]: [https://books.google.com/books?hl=en&lr=&id=1LZ1SZ70RrQC&oi=fnd&pg=PP1&ots=0hSEwJv1UB&sig=xNG9UWv\\_H10XHwa0i0BJN7TW6xA#v=onepage&q&f=false](https://books.google.com/books?hl=en&lr=&id=1LZ1SZ70RrQC&oi=fnd&pg=PP1&ots=0hSEwJv1UB&sig=xNG9UWv_H10XHwa0i0BJN7TW6xA#v=onepage&q&f=false)

Note: There is another long, deep book that needs to be obtained and studied[15]. <https://books.google.com/books?hl=en&lr=&id=FHP1DWvz1bEC&oi=fnd&pg=PP1&ots=TsOnodavEZ&sig=H086UUV1qRVWGqY-Tv02nb7x7NA#v=onepage&q&f=false>

Note: This work uses the term “segmented” helix, perhaps that is a known term and I need to switch to it:

[https://www.researchgate.net/publication/236066626\\_Segmented\\_helical\\_structures\\_formed\\_by\\_ABC\\_star\\_copolymers\\_in\\_nanopores](https://www.researchgate.net/publication/236066626_Segmented_helical_structures_formed_by_ABC_star_copolymers_in_nanopores)

This is a discussion of segmented coils in a protein structure:

<https://www.sciencedirect.com/science/article/pii/S0022283688903701>

This talks about tuning the period of a helix inside a nanopore:

<https://aip.scitation.org/doi/abs/10.1063/1.4794785>

A modern helix structure protein paper:

<https://www.sciencedirect.com/science/article/pii/S1476927108000583>

“Local Frustration Determines Molecular and Macroscopic Helix Structures”

<https://pubs.acs.org/doi/abs/10.1021/jp4040503>

“Analyzing Protein Structure Using Almost Delaunay Tetrahedra”

[https://www.researchgate.net/profile/Alexander\\_Tropsha/publication/250901525\\_Analyzing\\_Protein\\_Structure\\_Using\\_Almost-Delaunay\\_Tetrahedra/links/5578584408ae75215870347c/Analyzing-Protein-Structure-Using-Almost-Delaunay-Tetrahedra.pdf](https://www.researchgate.net/profile/Alexander_Tropsha/publication/250901525_Analyzing_Protein_Structure_Using_Almost-Delaunay_Tetrahedra/links/5578584408ae75215870347c/Analyzing-Protein-Structure-Using-Almost-Delaunay-Tetrahedra.pdf)

“Simulation of Suspensions of Helical Rigid Fibers” Y Al-Hassan : British Journal of Mathematics and Computer Science (PDF downloaded)

“HELFIT: Helix fitting by a total least squares method” : This needs to be studied closely! <https://www.sciencedirect.com/science/article/pii/S1476927108000418>

QHELIX: A Computational Tool for the Improved Measurement of Inter-Helical Angles in Proteins <https://link.springer.com/article/10.1007/s10930-007-9097-9>

Note: “On the Screw Axes and Other Special Lines Associated With Spatial Displacements of a Rigid Body” <http://manufacturingscience.asmedigitalcollection.asme.org/article.aspx?articleid=1439697>

Note: An historical review of the theoretical development of rigid body displacements from Rodrigues parameters to the finite twist <https://www.sciencedirect.com/science/article/pii/S0094114X0500087X>

Note: Might need to get this book. [https://link.springer.com/chapter/10.1007/978-3-319-31126-5\\_1](https://link.springer.com/chapter/10.1007/978-3-319-31126-5_1)

## 13.1 To Be Done

Need to understand what an “alpha coil” protein structure is. (Done: An alpha coil is one of the most common and studied protein forms, roughly providing distance in the protein, consisting of a helix formed by amino acids, which tend to be cross bonded.

Need to answer the Math stack exchange question.

Need to understand possibility of further simplifying specification of object.

## 14 Acknowledgements

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## References

- [1] Eric A Lord. Helical structures: the geometry of protein helices and nanotubes. *Structural Chemistry*, 13(3-4):305–314, 2002.
- [2] Robert L. Read. The story fo a public, cooperative mathathon, December 2019. [Online; posted 12-December-2019].
- [3] Peter C Kahn. Defining the axis of a helix. *Computers & chemistry*, 13(3):185–189, 1989.
- [4] Robert Read. Transforming optimal tetrahelices between the boerdijk-coxeter helix and a planar-faced tetrahelix. *Journal of Mechanisms and Robotics*, 2018.
- [5] Lizhi Gu, Peng Lei, and Qi Hong. Research on discrete mathematical model of special helical surface. In *Green Communications and Networks*, pages 793–800. Springer, 2012.
- [6] HSM Coxeter et al. The simplicial helix and the equation  $\tan(n \theta) = n \tan(\theta)$ . *Canad. Math. Bull*, 28(4):385–393, 1985.
- [7] Garrett Sadler, Fang Fang, Julio Kovacs, and Klee Irwin. Periodic modification of the Boerdijk-Coxeter helix (tetrahelix). *arXiv preprint arXiv:1302.1174*, 2013.
- [8] R.B. Fuller and E.J. Applewhite. *Synergetics: explorations in the geometry of thinking*. Macmillan, 1982.
- [9] Michael Elgersma and Stan Wagon. The quadrahelix: A nearly perfect loop of tetrahedra. *arXiv preprint arXiv:1610.00280*, 2016.
- [10] Hassan Babiker and Stanisław Janeczko. *Combinatorial cycles of tetrahedral chains*. IM PAN, 2012.
- [11] EA Lord and S Ranganathan. Sphere packing, helices and the polytope  $\{3, 3, 5\}$ . *The European Physical Journal D-Atomic, Molecular, Optical and Plasma Physics*, 15(3):335–343, 2001.
- [12] EA Lord and S Ranganathan. The  $\gamma$ -brass structure and the boerdijk-coxeter helix. *Journal of non-crystalline solids*, 334:121–125, 2004.
- [13] Przemysław Prusinkiewicz and James Hanan. *Lindenmayer systems, fractals, and plants*, volume 79. Springer Science & Business Media, 2013.
- [14] Stephen Hyde, Z Blum, T Landh, S Lidin, BW Ninham, S Andersson, and K Larsson. *The language of shape: the role of curvature in condensed matter: physics, chemistry and biology*. Elsevier, 1996.
- [15] Jean-François Sadoc and Rémy Mosseri. *Geometrical frustration*. Cambridge University Press, 2006.