The Generalized Spherical Rotation Hole-and-Post Cover Problem

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Abstract

It is possible to generalize a ball-and-socket mechanical joint to allow multiple members to be attached concentrically so that each radiates directly fromt he center of the ball and supports a limited amount of angular displacement. Because such joints a completely concentric, they have the mechanical advantage that force is purely compressive or tensile in the members, making them tensegrities. It is not obvious, however, how much angular displacement can be supported by a physical joint. In other words, can a given joint support a given configuration of members coming in from different directions? This paper formalizes this problem mathematically and addresses its solution using numerical optimization methods.

1 Introduction

2 Problem Statement

Define the Generalized Spherical Rotation Hole-and-Post Cover Problem:

Problem 1 *Given a unit radius sphere A with a sequence of n circular holes cut out whose centers are given by the unit vectors C_i and half-angles (less than $\pi/2$) are given by A_i and a unit radius sphere B with a corresponding sequence of n dots or posts given by unit vectors D_i, is there a rotation about the origin of the A that exposes (or covers) each dot or post with the corresponding hole? If not, what rotation mimimizes the sum of the squares of the distances from the dot to the corresponding circles if they are not exposed?*

Although this problem could have an algorithmic solution, the problem only has three dimensions (the Euler angles of the rotation of A to a cover, for example), and therefore is likely very efficiently solved by numerical optimazation methods, even without computation of a derivative, given an appropriate objective function to minimize. Although a proper algorithm would be more elegant, it is probably beyond the author without a great deal of thought, and our primary purpose is to make our actual robot function properly without moving into configurations which cause it to crack itself apart. A numerical solution is expected to be adequately efficient for that purpose. The second part of the problem statement clearly anticipates a numeric solution.

3 An Objective Function

We need an objective function which takes Euler angles α, β, γ, the sphere A and the sphere B and produces a penalty which is the sum of the squares of the distance to the circle of the dot on B to the circle on A for each pair i. This will be zero if the dot is actually in the circle (or on the section of the sphere defined by the circles.

From the Euler angles we construct a rotation matrix which can be applied to the center of each circle to rotate it. Assume we are in a right-handed frame of reference. We choose to think in terms of extrinsic rotations about the fixed axes, and arbitrarily choose the $z_2 - x - z_1$ sequence.

Wikipedia explains how to compute a rotation matrix M for from α, β and γ directly. Let $\angle(a, b)$ denote the angle between (unit) vectors a and b. The distance between two unit vectors is twice the sin of the half angle.

$$p(a, b, \theta) = \begin{cases} 0 & \text{if } \angle(a, b) <= \theta \\ (2 \sin \frac{\angle(a,b) - \theta}{2})^2 & \text{if } \angle(a, b) > \theta \end{cases}$$

$$f(x) = \sum_{i=1}^{n} p(D_i, MC_i, A_i)$$

4 Conclusion

Write your conclusion here.