Three Inflatable Spheres as a Theoretical Basis for a Soft Stewart Platform

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Abstract

A Stewart Platform[3] is a fundamental mechanism for varying the angle between two objects. A soft Stewart Platform can be made of two discs and three inflatable spheres. Stacking such devices might make a soft tentacle. Soft robots are meant to deform under force, but it is useful to have an analytic description of a plane in contact with three spheres before deforming force is applied. In 1881, the problem of computing the plane tangent to three spheres was set as an exercise in a textbook, em Practical Solid Geometry[1], but not solved. We have solved this problem in JavaScript, producing an interactive, browser-based web page that dynamically solves the problem[2]. All of the code is released under the GNU Public License.

1 Introduction

2 Prelude: Three Touching Circles

Our goal is to be able to determine the orientation of plane resting on top of three spheres of different radii which are touching each other. Because there is a plane through any three points and we have three spheres, we can construct the plane through the center of these points. The projection of the edges of the spheres onto this plane form three touching circles. Knowing the position of these circles is a valuable prelude to solving the three dimenstional problem.

To solve this problem most conveniently, we place the first circle on the negative x-axis, and the second circle on the positive x-axis, with the circles intersecting at the origin. The third circle is place in the positive y direction. It's center will not be on the y-axis itself unless the radii of the first two circles are equal.

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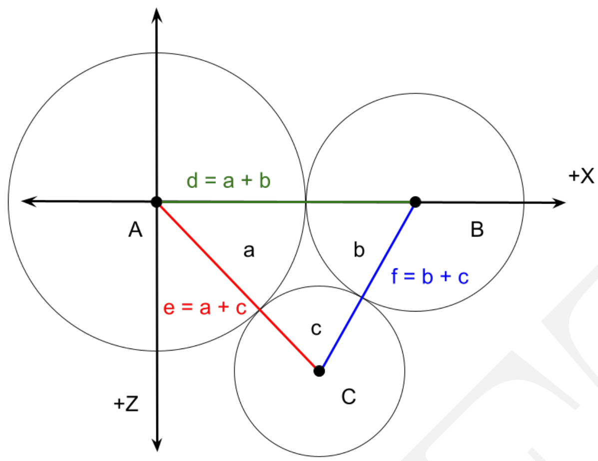

Figure 1: Three Touching Circles

We seek a formula for the coordinates of the third circle in terms of three input radii a, b, c.

Because the distance between adjacent circles is the sum of their radii, define:[1]

$$d = a + b \tag{1}$$
$$e = a + c \tag{2}$$
$$f = b + c \tag{3}$$
$$\tag{4}$$

Then we can use the cosine law to compute the angle $\angle ABC = \theta$:

$$\theta = \arccos \frac{a^2 + b^2 - c^2}{2bc} \tag{5}$$
$$\theta = \arccos \frac{(b+c)^2 + (a+c)^2 - (a+b)^2}{2(a+c)(a+b)} \tag{6}$$
$$\tag{7}$$

It is clear that once θ has been calculated:

$$C_z = a \sin\theta \tag{8}$$

[1]This is a confusing use of the variables a, b, and c.

Allowing us to form a right triangle $\triangle ACD$ and use the Pythagorean theorem:

$$b^2 = C_z^2 + C_x^2 \tag{9}$$

$$C_x = \sqrt{b^2 - C_z^2} \tag{10}$$

$$\tag{11}$$

3 Three Touching Spheres

Our fundamental goal now is to describe three touching spheres. As robotocists, our interest is in the slope of the plane of the tops of these spheres as if they were resting on a table. Then by inflating or deflating spheres, we would be able to control the direction of a plane or platform. Such a device is sometimes called a parallel manipulator, of which a Stewart Platform[3] (`https://en.wikipedia.org/w/index.php?title=Stewart_platform&oldid=898429010`) is the best-known example.

The fundamental action of a parallel manipulator is to tilt a plane or platform in a desired direction based on changes in the radii of the spheres.

The problem finding the planes tangent to three touching spheres is given as an exercise in an advanaced textbook on solid geometry from 1881[1]. It does not give a solution, but it gives a hint: to consider the cones enveloping three spheres taken two at a time.

Choosing the coordinate system of the XZ plane through the center of the spheres greatly simplifies the derivations, because the center of the spheres always form tangent circles in this plane. Call this plane the *center plane.* The position of A is fixed, B is constrained to the x-axis, and C is contrained to the xz-plane. The center of these circles in the xz plane can be calculated from the radii independent of the tilt they induce. In this coordinate system, the y-coordinate of the center of all spheres is 0. Furthermore, a cone tangent to two spheres has its axis and apex in the xz-plane. The projection of all three spheres into this plane produces three touching circles. We seek an expression for the normal of the plane of the tops of these spheres as a function purely of the three radii. Call this plane the *top plane.* We can imagine the spheres resting on a fixed surface called the *bottom plane.* The tilt of the top and bottom plane relative to the center of the spheres is alway a mirror image of each other across the coordinate XZ-plane.

If the tilt of the top plane relative to the coordinate plane is given by a rotation about the z axis of θ and then a rotation about x axis of γ, the tilt of the top plane relative to the bottom plane is given by the zx extrinsic Euler angles $(2\theta, 2\gamma)$.[2]

3.1 Physical Embodiment

To embody this system precisely, the sphere A could be pinned to the origin of the bottom plane. The contact point of the sphere B with the bottom plane

[2] Is this really true?

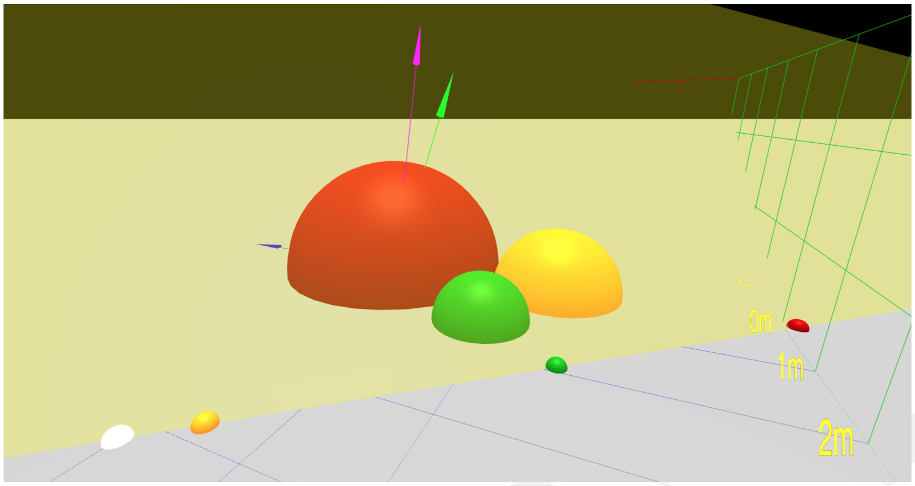

Figure 2: Three Touching Spheres

could be placed on a linear slider on the x-axis. The conact point C could be constrained to rest on the botom plane by gravity or some elastic cables. Choose the convention that the C object is to the right hand (clockwise) side of the AB line. As the spheres change change, the origin of the center plane will experience minor translation relative to the bottom plane, but will not experience rotation about the y-axis. If needed the coordinates of the spheres in the bottom plane can be easily calculates from the radii and the computed extrinsic Euler angles θ and γ.

3.2 Variable definitions

We assume that the spheres centered at points A, B, and C of radii a, b, and c. We setup our coordinate system as a right- handed coordinate system with XZ plane contaning the center of all spheres. The A sphere is placed at the origin, so that $O = A$, and the B sphere is place on the positive X axis. Without loss of generality assume $a \geq b \geq c$. Following computer graphics convention, we think of the Y dimension as vertical.

Taking any two spheres defines a cone whose apex is in the XZ plane. Call the apex of the AB cone U, the AC cone V, and the BC cone W.

It is a beautiful fact that the th apices U, V and W are colinear on a line we call the *apex line*, depicted in both Figure **??** or Figure **??**. Further, by your choice of coordinates, this line is in the XZ plane. Finally, the top and bottom planes intersect at this line, because those planes are tangent to the three cones.

We assert that these three points form a line, which we call the *apex line*, in the XZ plane and that this line is the intersection of the *tangent plane* touching all three spheres at a single point with the XZ plane. We will use points U and V in our calculation.

3.3 Axis Angle of Cone Enveloping Two Spheres

In order to compute the apex line, it is helpful to note that the apex angle ψ of a cone which envelopes (by being tangent to) two tangent spheres of radii r, s, is:

$$\psi = \arcsin \frac{s-r}{s+r} \tag{12}$$
$$\tag{13}$$

where $r < s$ without loss of generality. Note that when $r = s$ there is a special case, the cone tangent to both is degenerate (that is, a cylinder, or a cone of $\psi = 0$ apex angle.)

3.4 Strategy

We seek to compute the normal of the tangent plane. Observe that this plane is tangent to all three cones. Observe that the AB cone intersects the A sphere in a circle on the surface of the A perpendicular to and centered on the X axis.

The half-angle of a cone tangent to two tangent spheres is computed from the radii directly:

$$\theta_{ab} = \arcsin \frac{a-b}{a+b} \tag{14}$$
$$\theta_{bc} = \arcsin \frac{b-c}{b+c} \tag{15}$$

A vector of length a that is always rotated about the origin is always a point on the sphere. The first operation is move this vector perpendicular to the AB cone. A vector N in the Y direction and rotated counterclockwise about the Z-axis by θ_{ab} is perpendicular to the AB cone.

However, we must rotate this vector N about the X axis by an unknown amount, γ, in order to orient the vector to line up properly with the desired tilt of the tangent plane despite not being a pure rotation about the Z axis. Since this angle is computed in the YZ plane, we compute a projection of the point V into that plane, forming a triangle in the YZ plane. Call this point I_z, the intersection of the apex line with the Z-axis. Let h be the height of the intersection of the y-axis with the AB cone. Then γ is the angle we need to rotate the plane by so that it intersects with the Z-axis at a point, referred to as I_z. By basic geometry of simialr triangles, this is computed from the right triangle of the origin with h and I_z. Having computed θ_{ab} and γ, these two degrees of rotation give us the plane tangent to all three spheres, and its corresponding tilt for our desired soft Stewart Platform.

Vectors $\overrightarrow{U}$, $\overrightarrow{V}$, and $\overrightarrow{W}$ represent the vectors from the center of the spheres A, B, and C to the apices of the cones tangent to A and B, B and C, and C and A respectively.

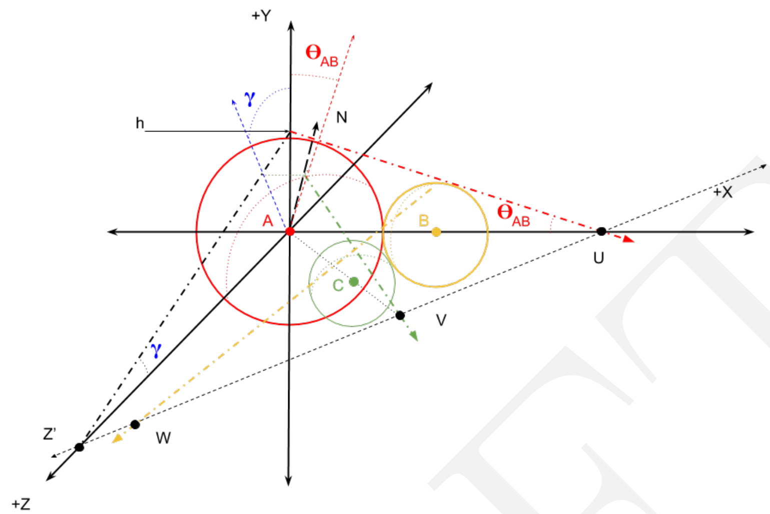

Figure 3: Rotation Math

$$
\begin{aligned}
\overrightarrow{U} &= \overrightarrow{A} + \hat{\overrightarrow{AB}} \frac{a}{\sin\theta_{ab}} && (16) \\
\overrightarrow{V} &= \overrightarrow{B} + \hat{\overrightarrow{BC}} \frac{b}{\sin\theta_{bc}} && (17) \\
& && (18)
\end{aligned}
$$

The point H is the intesection of the AB cone with the y axis. This is right triangle of the origin with the points U and H. (The hypotenuse of this triangle is $\overrightarrow{HU}$, and side adjacent to θ_{AB} is $\overrightarrow{OU}$.) It is computed by projecting the contact point of the top plane in the xy-plane onto the xz-plane. This allows γ to be computed as a pure rotation about the x-axis.

Equations **??** and **??** give the Euler angles θ_{ab} and γ for the tilt of the tangent plane as a function of the three radii a, b, c.

4 The Inverse Problem

Note: The math presented here fails when $c >= b$. We need to find an elegant way to deal with this.

To build a useful robotic platform in by controlling the radii of the spheres by inflating and deflating them, we wish to solve the inverse problem. That is, given θ_{ab} and γ as desired Euler angles, we want to compute a, b, c which acheive these angles.

Since these are smooth functions, the problem could be solved easily by a Newton-Raphson style solver based on **??** and **??**. However, an analytic solution is always superior.

The radius b can be computed directly from θ_{ab} because of our choice of coordinates:

$$U_x = \frac{a}{\sin\theta_{ab}} \tag{19}$$

$$b = \frac{(U_x - a)\sin\theta_{ab}}{\sin\theta_{ab} + 1} \tag{20}$$

Given the plane defined by a known radius a, θ_{ab} and γ, the point C is constrained by asserting that the perpendicular distance to this plane of C is c. Additionally, in the central plane we have two other constraints, that C is placed so that it touches the circle centered on A and the circle centered on B.

Then we know that the point C but have distance $a + c$ from the point A (the origin) and distance $b + c$ from the known point B. These two constraints can be exprssed:

$$a + c = \sqrt{x^2 + z^2} \tag{21}$$

$$b + c = \sqrt{(a + b - x)^2 + z^2} \tag{22}$$

$$\tag{23}$$

If we were to plot the point C as c increases, we would find a gentle curve moving away from the intersection of the circle A and B. An additional constraint will allow us to select a point on this curve.

This gives us three equations, which should be more than enough to solve for the point C.

$$H_y = U_x \tan\theta_{ab} \tag{24}$$

$$Z' = \frac{H_y}{\sin\gamma} \tag{25}$$

$$\gamma = \arcsin\frac{Z'}{H_y} \tag{26}$$

$$\tag{27}$$

Let $H = [0, H_y, 0]$.

Let N be the normal to the plane (pointing up in our diagram when $a >= b >= c$.)

$$\overrightarrow{N} = \overrightarrow{Z' - U} \times \overrightarrow{Z' - H} \tag{28}$$

$$\tag{29}$$

Since we have the normal of the plane three points (U, Z, H_y) in the plane, computing the distance from the point C to the plane has a simple formula. We'll use the point $H = [0, H_y, 0]$ as our main point. The equation of our plane is $\overrightarrow{N} \cdot \overrightarrow{X} = k$.

$$k = \frac{N \cdot U}{N_y} \tag{30}$$

$$c = \frac{|N \cdot P - k|}{|N|} \tag{31}$$

$$c = \frac{|C_x N_x + C_z N_z - k|}{1} \tag{32}$$

$$\tag{33}$$

By substituting **??** into equations **??** and **??**, we obtain:

$$a + -(xN_x + zN_z - k) = \sqrt{x^2 + z^2} \tag{34}$$

$$b + -(xN_x + zN_z - k) = \sqrt{(a + b - x)^2 + z^2} \tag{35}$$

$$\tag{36}$$

This is a system of two nonlinear equations with unkowns x and z. Solving this system is beyond the capacities of the authors, but not Mathematica, although the answer that Mathematica produces in its raw form has dozens of terms. Furthermore, there are actually two solutions, corresponding to the two sides of the AB line which the circle C could be on and satsify these constratints. We have selected the solution that place C in the positive Z direction, as constrained.

However, by introducing new variables to represent repeated portions of this long expression, the expressions be made tractable and easily programmed.

$$\begin{aligned}
J &= (a - bN_x + b + k) \\
L &= abN_z^2(a - b)^2 \\
M &= \sqrt{L\left((k - aN_x)J + abN_z^2\right)} \\
D &= (aN_x + a - bN_x + b)^2 - 4abN_z^2 \\
x &= \frac{\left(a^2 + a(b + k) - bk\right)(aN_x + a - bN_x + b) - 2M - 2abN_z^2(a + b)}{D} \\
R &= -2a^2bkN_z^2 + a^3b(-1 + N_x)N_z^2 - b(-1 + N_x)M \\
S &= a(2b^2kN_z^2 - b^3(-1 + N_x)N_z^2 + (1 + N_x)M) \\
z &= \frac{2(R + S)}{N_z(a - b)D}
\end{aligned}$$

The variable c can be compute from x and z as:

$$c = x^2 + z^2 - a$$

This math has been checked in Javascript and it appears to be correct in computing c, though it is sometimes off in the third or even second digit, which I assume is a numerical problem rather than a problem in the formal math.

4.1 Special when $\gamma = 0$

In the above math when γ is 0, then N_z is zero, creating a division by zero. In this case we can use the proportionality of similar triangles to assert:

$$a = c + \frac{ax}{U_x}$$

Combined with our previous equations in the plane relating x and z to c:

$$\begin{aligned} a + c &= \sqrt{x^2 + z^2} \\ b + c &= \sqrt{(a + b - x)^2 + z^2} \end{aligned}$$

we obtain the following equation:

$$c = \frac{a(a + b)(a - U_x)}{a^2 - ab + aU_x + bU_x}$$

4.2 Special when $\theta = 0$

When $\theta = 0$, the point U is at infinity or cannot be found. In thise case $b = a$. We use the simple fact that:

$$z = \frac{a}{\sin \gamma}$$

to obtain:

$$c = \sqrt{d^2 + x^2} - a$$

When both $\theta = 0$ and $\gamma = 0$, then $c = b = a$.

5 Usage

These equations can be used to compute the radii b and c from a given radius a and either a desired normal or euler angles. When implemented as a soft robot, it must be understand that the center plane is conceptual. The roboticists probably want to think of moving the top plane relative to the bottom plane, which can be implemented as a simple rigid disc. The angles and normals between the top and bottom and will be twice the angles[3] measured against the center plane.

Furthermore, an implementation based on soft inflatable spheres will not be really acheive perfect spheres at any pressure. Nonetheless, because these equations are trivial to implement in a micro-controller, they make a good basis for a control algorithm. A Stewart platform implemented in this would be expected to be have “softly” in response to pressure, and yet have a predictable shape in the absence of external forces.

References

[1] Joseph Payne. *Practical Solid Geometry. Orthographic and Isometric Projection.* 4th ed. Murby’s ”Science and Art Departemnt” Series of Text-Books. Available free as an electronic book, see Problem CXLVIII (148), Page 195. 32 Bouverie Street, Fleet Street, E.C.; Thomsas Murby, Jan. 1881. URL: `https://play.google.com/store/books/details?id=8TQDAAAAQAAJ&rdid=book-8TQDAAAAQAAJ&rdot=1`.

[2] Robert L. Read and Megan Cadena. *Plane Tanget to 3 Spheres.* [Online; accessed 13-November-2019]. 2019. URL: `https://pubinv.github.io/softrobotmath/` (visited on 10/13/2019).

[3] Wikipedia contributors. *Stewart platform — Wikipedia, The Free Encyclopedia.* [Online; accessed 9-October-2019]. 2019. URL: `https://en.wikipedia.org/w/index.php?title=Stewart_platform&oldid=898429010`.

[3] Megan, is this really true? Can we simply double the angles? I think so, but that might not be true.