Three Inflatable Spheres as a Theoretical Basis for a Soft Stewart Platform

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Abstract

A Stewart Platform[4] is a fundamental mechanism for varying the angle between two objects. A soft Stewart Platform can be made of two discs and three inflatable spheres. Stacking such devices might make a soft tentacle. Soft robots are meant to deform under force, but it is useful to have an analytic description of a plane in contact with three spheres before deforming force is applied. In 1881, the problem of computing the plane tangent to three spheres was set as an exercise in a textbook, em Practical Solid Geometry[1], but not solved. We have solved this problem in JavaScript, producing an interactive, browser-based web page that dynamically solves the problem[2]. All of the code is released under the GNU Public License.

1 Introduction

2 Prelude: Three Touching Circles

Our goal is to be able to determine the orientation of plane resting on top of three spheres of different radii which are touching each other. Because there is a plane through any three points and we have three spheres, we can construct the plane through the center of these points. The projection of the edges of the spheres onto this plane form three touching circles. Knowing the position of these circles is a valuable prelude to solving the three dimensional problem.

To solve this problem most conveniently, we place the first circle on the negative x-axis, and the second circle on the positive x-axis, with the circles intersecting at the origin. The third circle is place in the positive y direction. It's center will not be on the y-axis itself unless the radii of the first two circles are equal.

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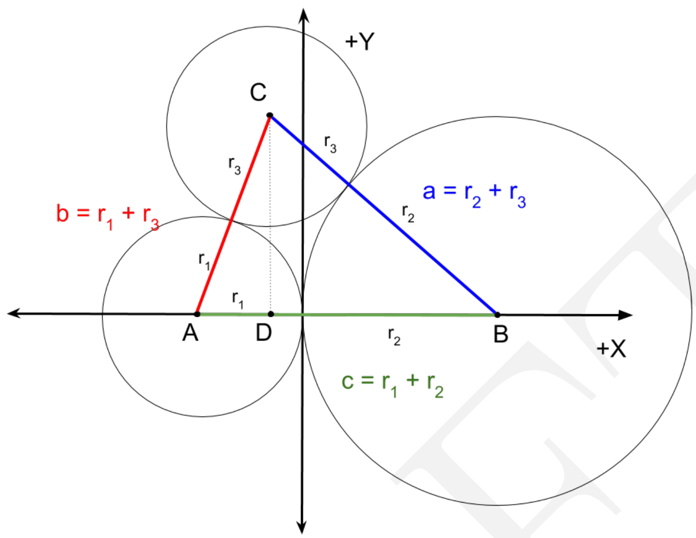

Figure 1: Three Touching Circles

We seek a formula for the coordinates of the third circle in terms of three input radii r_1, r_2, r_3.

Because the distance between adjacent circles is the sum of their radii, define:

$$a = r_2 + r_3 \tag{1}$$

$$b = r_1 + r_3 \tag{2}$$

$$c = r_1 + r_2 \tag{3}$$

Then we can use the cosine law to compute the angle $\angle ABC = \theta$:

$$\theta = \arccos \frac{a^2 + b^2 - c^2}{2bc} \tag{4}$$

$$\theta = \arccos \frac{(r_2 + r_3)^2 + (r_1 + r_3)^2 - (r_1 + r_2)^2}{2(r_2 + r_3)(r_1 + r_3)} \tag{5}$$

$$\tag{6}$$

It is clear that once θ has been calculated:

$$C_y = a \sin \theta \tag{7}$$

Allowing us to form a right triangle $\triangle ACD$ and use the Pythagorean theorem:

$$b^2 = C_y^2 + (r_1 - C_x)^2 \tag{8}$$

$$b^2 - C_y^2 = (r_1 - C_x)^2 \tag{9}$$

$$\sqrt{b^2 - C_y^2} = r_1 - C_x \tag{10}$$

$$C_x = r_1 - \sqrt{b^2 - C_y^2} \tag{11}$$

3 A Progression of Problems

Because we now believe we can model the main problem as the rotation of a plane tangent to a cone about the cone, there are a number of easier problems which may lead us to a solution:

1. In 2D, rotate a tangently line to a circle until it is coincient on an outside point. That is, given the center of a circle C, a radius r, another point p, what angular rotation (measured against x-axis) gives the rotation that makes a line tangent to C touch p?
2. Given two spheres, what is the equation for the cone tangent to both?
3. Is the problem better thought of as the intersection of several spheres?

4 Three Touching Spheres

Note: This problem is mentioned in an advanaced textbook on solid geometry from 1881[1].

Our fundamental goal now is to describe three touching spheres. As robotocists, our interest is in the slope of the plane of the tops of these spheres as if they were resting on a table. Then by inflating or deflating spheres, we would be able to control the direction of a plane or platform. Such a device is sometimes called a parallel manipulator, of which a Stewart Platform[4] (`https://en.wikipedia.org/w/index.php?title=Stewart_platform&oldid=898429010`) is the best-known example.

The fundamental action of a parallel manipulator is to tilt a plane or platform in a desired direction. Having developed the math for the three touching circles in the previous section, we now use it to find the normal of the a plane resting on the top of three spheres by choosing our coordinate system to the plane defined by the center points of the three spheres. In this coordinate system, the z-coordinate of the center of all spheres is 0. The projection of all three spheres into this plane produces three touching circles. We seek an expression for the normal of the plane of the tops of these spheres as a function purely of the three radii. Call this plane the *top* plane.

As the top plane tilts, the points tangent to this plane move away from the highest points on each of the three spheres, making the problem more difficult.

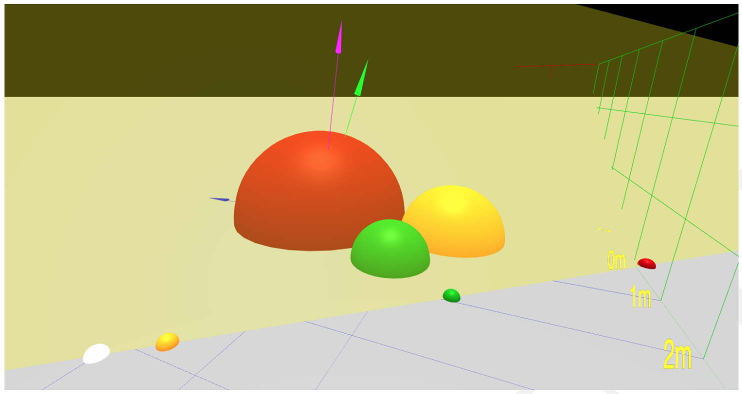

Figure 2: Three Touching Spheres

This problem can clearly be solved quickly using numerical methods, because by guessing a candidate normal to the top plane, it is relatively easy to compute an error function based on the distance between the closest point on each sphere to the top plane and the top plane. We would expect an iterative approach to converge very quickly.

Nonetheless we seek an analytic solution.

4.1 Variable definitions

We assume that the spheres centered at points A, B, and C of radii r_A, r_b, and r_c. We setup our coordinate system as a right- handed coordinate system with XZ plane contaning the center of all spheres. The A sphere is placed at the origin, so that $O = A$, and the B sphere is place on the positive X axis. Without loss of generality assume $r_a \geq r_b \geq r_c$. Following computer graphics convention, we think of the Y dimension as vertical.

Taking any two spheres defines a cone whose apex is in the XZ plane. Call the apex of the AB cone U, the AC cone V, and the BC cone W.

We assert that these three points form a line, which we call the *apex line*, in the XZ plane and that this line is the intersection of the *tangent plane* touching all three spheres at a single point with the XZ plane. We will use points U and V in our calculation. Let α be the half-angle of the AB cone.

It behooves us to name the intersection of the apex line with the Z axis, I_z, so that we can use it in calculations.

4.2 Strategy

We seek to compute the normal of the tangent plane. Observe that this plane is tangent to all three cones. Observe that the AB cone intersects the A sphere in a circle on the surface of the A perpendicular to and centered on the X axis.

A vector of length r_A that begins pointing in the X direction and is rotated counterclockwise about the Z-axis by $\pi/2 - \alpha$.

However, we must rotate this vector about the X axis by an unknown amount θ in order to bring capture the tilt which is not purely a rotation about the Z axis. Since this angle is computed in the ZY plane, we compute a projection of the point V into that plane, forming a triangle in the ZY plane. Call this point I_z, the intersection of the apex line with the Z-axis.

$$\overrightarrow{AB} = \overrightarrow{A} - \overrightarrow{B} \tag{12}$$
$$\overrightarrow{BC} = \overrightarrow{B} - \overrightarrow{C} \tag{13}$$
$$\overrightarrow{CA} = \overrightarrow{C} - \overrightarrow{A} \tag{14}$$
$$\theta_{ab} = \arcsin \frac{r_a - r_b}{r_a + r_b} \tag{15}$$
$$\theta_{bc} = \arcsin \frac{r_b - r_c}{r_b + r_c} \tag{16}$$
$$\theta_{ca} = \arcsin \frac{r_a - r_c}{r_a + r_c} \tag{17}$$
$$\overrightarrow{AB_{apex}} = \overrightarrow{A} + \hat{\overrightarrow{AB}} \frac{r_a}{\sin \theta_{ab}} \tag{18}$$
$$\overrightarrow{BC_{apex}} = \overrightarrow{B} + \hat{\overrightarrow{BC}} \frac{r_b}{\sin \theta_{bc}} \tag{19}$$
$$\overrightarrow{CA_{apex}} = \overrightarrow{C} + \hat{\overrightarrow{CA}} \frac{r_c}{\sin \theta_{ca}} \tag{20}$$
$$\overrightarrow{U} = \overrightarrow{AB_{apex}} \tag{21}$$
$$\overrightarrow{V} = \overrightarrow{CA_{apex}} \tag{22}$$
$$\overrightarrow{W} = \overrightarrow{BC_{apex}} \tag{23}$$

$$\phi = \arcsin \frac{r_c}{I_z} \tag{24}$$
$$\theta = \frac{\pi}{2} - (\frac{\pi}{2} - \phi) \tag{25}$$
$$\theta = \phi \tag{26}$$
$$I_z = U_x (\frac{V_z}{(U_x - V_x)}) \tag{27}$$

4.3 Axis Angle of Cone Enveloping Two Spheres

The apex angle of a cone which envlopes (by being tangent to) two tangent circles of radii r_x, r_y, is:

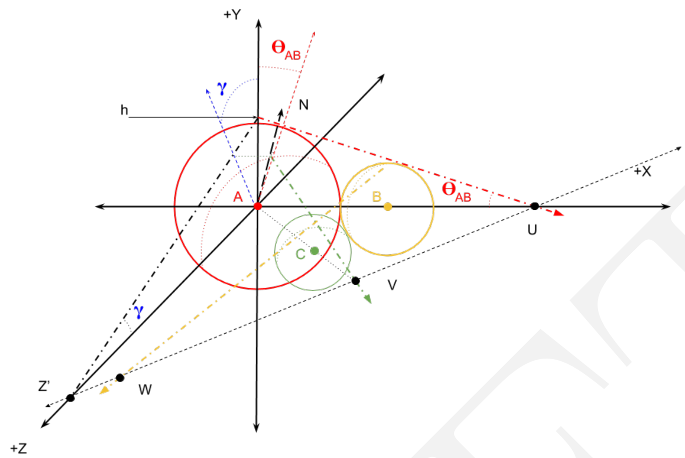

Figure 3: Rotation Math

$$z = -2\frac{r_x^2}{r_x - r_y} \tag{28}$$

$$\psi = 2 \arcsin \frac{r_x}{z + r_y} \tag{29}$$

$$\psi = 2 \arcsin \frac{r_y - r_x}{r_x + r_y} \tag{30}$$

where $r1 < r2$ without loss of generality.

4.4 Computation of γ and θ

Compute these by considering the three cones that envelope the three spheres taken in a pair-wise fashion. Let θ_{xy} be the half-angle of the aperture of the cone that envelopes sphere x and sphere y.

Compute the apices of the three cones, naming the apex of the cone that envelopes sphere X and sphere Y $\overrightarrow{XY_{apex}}$. It is interesting to note that the three apices lie on a line, which is where the normal plane intersects the XZ-plane where the center of all the spheres lie.

Let Z' be the z value of the intersection of this line with z-axis.

Let the normal of the plane tangent to the three spheres be generated by a (negative) rotation θ_{AB} of the $[0, 1, 0]$ vector about the Z-axis followed by a rotation γ about the X-axis, as depicted in Figure 3.

$$\psi = \arccos \hat{\overrightarrow{CA}}_z (\text{angle of CA against the z axis}) \tag{31}$$
$$L = \|A - CA_{apex}\| \tag{32}$$
$$Z' = L \cos\psi \frac{U_x}{U_x - V_x} \tag{33}$$
$$\gamma = \frac{\pi}{2} - \arccos \frac{r_a}{Z'}; \gamma = \arcsin \frac{r_a}{Z'}; \tag{34}$$

5 Thing to Research

Idea: It may be possible to solve for the support points as the intersection of cones[3].

Note problem CXLVIII (148). Page 195 Joseph Payne (of the Charterhouse) "Practical solid geometry; or, Orthographic and Isometric projection.

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[1] Payne, J. *Practical Solid Geometry*, 4 ed. Murby's "Science and Art Departemnt" Series of Text-Books. Thomsas Murby, 32 Bouverie Street, Fleet Street, E.C.;, 1 1881. Available free as an electronic book, see Problem CXLVIII (148), Page 195.

[2] Read, R. L., and Cadena, M. Plane Tanget to 3 Spheres, 2019. [Online; accessed 13-November-2019].

[3] Shene, C.-K., and Johnstone, J. K. On the lower degree intersections of two natural quadrics. *ACM Transactions on Graphics (TOG) 13*, 4 (1994), 400–424.

[4] Wikipedia contributors. Stewart platform — Wikipedia, the free encyclopedia, 2019. [Online; accessed 9-October-2019].