

# Three Inflatable Spheres as a Theoretical Basis for a Soft Stewart Platform

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## Abstract

A Stewart Platform[4] is a fundamental mechanism for varying the angle between two objects. A soft Stewart Platform can be made of two discs and three inflatable spheres. Stacking such devices might make a soft tentacle. Soft robots are meant to deform under force, but it is useful to have an analytic description of a plane in contact with three spheres before deforming force is applied. In 1881, the problem of computing the plane tangent to three spheres was set as an exercise in a textbook, *em Practical Solid Geometry*[1], but not solved. We have solved this problem in JavaScript, producing an interactive, browser-based web page that dynamically solves the problem[2]. All of the code is released under the GNU Public License.

## 1 Introduction

## 2 Prelude: Three Touching Circles

Our goal is to be able to determine the orientation of plane resting on top of three spheres of different radii which are touching each other. Because there is a plane through any three points and we have three spheres, we can construct the plane through the center of these points. The projection of the edges of the spheres onto this plane form three touching circles. Knowing the position of these circles is a valuable prelude to solving the three dimensional problem.

To solve this problem most conveniently, we place the first circle on the negative  $x$ -axis, and the second circle on the positive  $x$ -axis, with the circles intersecting at the origin. The third circle is placed in the positive  $y$  direction. Its center will not be on the  $y$ -axis itself unless the radii of the first two circles are equal.

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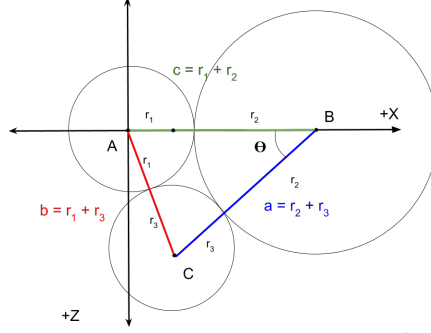


Figure 1: Three Touching Circles

We seek a formula for the coordinates of the third circle in terms of three input radii  $r_1, r_2, r_3$ .

Because the distance between adjacent circles is the sum of their radii, define:

$$a = r_2 + r_3 \quad (1)$$

$$b = r_1 + r_3 \quad (2)$$

$$c = r_1 + r_2 \quad (3)$$

Then we can use the cosine law to compute the angle  $\angle ABC = \theta$ :

$$\theta = \arccos \frac{a^2 + b^2 - c^2}{2bc} \quad (4)$$

$$\theta = \arccos \frac{(r_2 + r_3)^2 + (r_1 + r_3)^2 - (r_1 + r_2)^2}{2(r_1 + r_3)(r_1 + r_2)} \quad (5)$$

$$(6)$$

It is clear that once  $\theta$  has been calculated:

$$C_z = a \sin \theta \quad (7)$$

Allowing us to form a right triangle  $\triangle ACD$  and use the Pythagorean theorem:

$$b^2 = C_z^2 + C_x^2 \quad (8)$$

$$C_x = \sqrt{b^2 - C_z^2} \quad (9)$$

$$(10)$$

### 3 Three Touching Spheres

Our fundamental goal now is to describe three touching spheres. As robotocists, our interest is in the slope of the plane of the tops of these spheres as if they were

resting on a table. Then by inflating or deflating spheres, we would be able to control the direction of a plane or platform. Such a device is sometimes called a parallel manipulator, of which a Stewart Platform[4] ([https://en.wikipedia.org/w/index.php?title=Stewart\\_platform&oldid=898429010](https://en.wikipedia.org/w/index.php?title=Stewart_platform&oldid=898429010)) is the best-known example.

The fundamental action of a parallel manipulator is to tilt a plane or platform in a desired direction based on changes in the radii of the spheres.

The problem finding the planes tangent to three touching spheres is given as an exercise in an advanced textbook on solid geometry from 1881[1]. It does not give a solution, but it gives a hint: to consider the cones enveloping three spheres taken two at a time.

Choosing the coordinate system of the  $XZ$  plane through the center of the spheres greatly simplifies the derivations, because the center of the spheres always form tangent circles in this plane. Call this plane the *center plane*. The position of  $A$  is fixed,  $B$  is constrained to the  $x$ -axis, and  $C$  is constrained to the  $xz$ -plane. The center of these circles in the  $xz$  plane can be calculated from the radii independent of the tilt they induce. In this coordinate system, the  $y$ -coordinate of the center of all spheres is 0. Furthermore, a cone tangent to two spheres has its axis and apex in the  $xz$ -plane. The projection of all three spheres into this plane produces three touching circles. We seek an expression for the normal of the plane of the tops of these spheres as a function purely of the three radii. Call this plane the *top plane*. We can imagine the spheres resting on a fixed surface called the *bottom plane*. The tilt of the top and bottom plane relative to the center of the spheres is always a mirror image of each other across the coordinate  $XZ$ -plane.

If the tilt of the top plane relative to the coordinate plane is given by a rotation about the  $z$  axis of  $\theta$  and then a rotation about  $x$  axis of  $\gamma$ , the tilt of the top plane relative to the bottom plane is given by the  $zx$  extrinsic Euler angles  $(2\theta, 2\gamma)$ .

### 3.1 Physical Embodiment

To embody this system precisely, the sphere  $A$  could be pinned to the origin of the bottom plane. The contact point of the sphere  $B$  with the bottom plane could be placed on a linear slider on the  $x$ -axis. The contact point  $C$  could be constrained to rest on the bottom plane by gravity or some elastic cables. As the spheres change, the origin of the center plane will experience minor translation relative to the bottom plane, but will not experience rotation about the  $y$ -axis. If needed the coordinates of the spheres in the bottom plane can be easily calculated from the radii and the computed Euler angles  $\theta$  and  $\gamma$ .

### 3.2 Variable definitions

We assume that the spheres centered at points  $A, B$ , and  $C$  of radii  $r_A, r_b$ , and  $r_c$ . We setup our coordinate system as a right-handed coordinate system with  $XZ$  plane containing the center of all spheres. The  $A$  sphere is placed at the

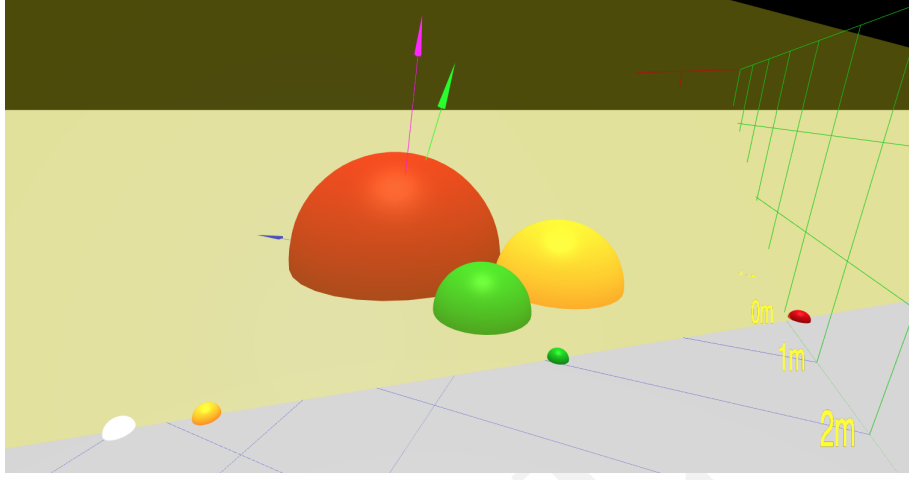


Figure 2: Three Touching Spheres

origin, so that  $O = A$ , and the  $B$  sphere is placed on the positive  $X$  axis. Without loss of generality assume  $r_a \geq r_b \geq r_c$ . Following computer graphics convention, we think of the  $Y$  dimension as vertical.

Taking any two spheres defines a cone whose apex is in the  $XZ$  plane. Call the apex of the  $AB$  cone  $U$ , the  $AC$  cone  $V$ , and the  $BC$  cone  $W$ .

It is a beautiful fact that the three apices  $U, V$  and  $W$  are colinear on a line we call the *apex line*, depicted in both Figure 2 or Figure 3. Further, by your choice of coordinates, this line is in the  $XZ$  plane. Finally, the top and bottom planes intersect at this line, because those planes are tangent to the three cones.

We assert that these three points form a line, which we call the *apex line*, in the  $XZ$  plane and that this line is the intersection of the *tangent plane* touching all three spheres at a single point with the  $XZ$  plane. We will use points  $U$  and  $V$  in our calculation.

### 3.3 Axis Angle of Cone Enveloping Two Spheres

In order to compute the apex line, it is helpful to note that the apex angle  $\psi$  of a cone which envelopes (by being tangent to) two tangent spheres of radii  $r, s$ , is:

$$\psi = \arcsin \frac{rs - r^2}{2r^2 - rs + s^2} \quad (11)$$

$$(12)$$

where  $r < s$  without loss of generality. Note that when  $r = s$  there is a special case, the cone tangent to both is degenerate (that is, a cylinder, or a cone of  $\psi = 0$  apex angle.)

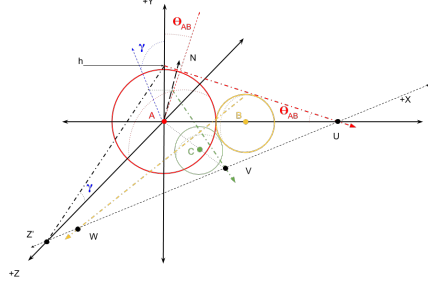


Figure 3: Rotation Math

### 3.4 Strategy

We seek to compute the normal of the tangent plane. Observe that this plane is tangent to all three cones. Observe that the  $AB$  cone intersects the  $A$  sphere in a circle on the surface of the  $A$  perpendicular to and centered on the  $X$  axis.

The half-angle of a cone tangent to two tangent spheres is computed from the radii directly:

$$\theta_{ab} = \arcsin \frac{r_a - r_b}{r_a + r_b} \quad (13)$$

$$\theta_{bc} = \arcsin \frac{r_b - r_c}{r_b + r_c} \quad (14)$$

A vector of length  $r_A$  that is always rotated about the origin is always a point on the sphere. The first operation is move this vector perpendicular to the  $AB$  cone. A vector  $N$  in the  $YZ$  plane and rotated counterclockwise about the  $Z$ -axis by  $\theta_{ab}$  is perpendicular to the  $AB$  cone.

However, we must rotate this vector  $N$  about the  $X$  axis by an unknown amount,  $\gamma$ , in order to orient the vector to line up properly with the desired tilt of the tangent plane despite not being a pure rotation about the  $Z$  axis. Since this angle is computed in the  $YZ$  plane, we compute a projection of the point  $V$  into that plane, forming a triangle in the  $YZ$  plane. Call this point  $I_z$ , the intersection of the apex line with the  $Z$ -axis. Let  $h$  be the height of the intersection of the  $y$ -axis with the  $AB$  cone. Then  $\gamma$  is the angle we need to rotate the plane by so that it intersects with the  $Z$ -axis at a point, referred to as  $I_z$ . By basic geometry of similar triangles, this is computed from the right triangle of the origin with  $h$  and  $I_z$ . Having computed  $\theta_{ab}$  and  $\gamma$ , these two degrees of rotation give us the plane tangent to all three spheres, and its corresponding tilt for our desired soft Stewart Platform.

Vectors  $\vec{U}$ ,  $\vec{V}$ , and  $\vec{W}$  represent the vectors from the center of the spheres  $A$ ,  $B$ , and  $C$  to the apices of the cones tangent to  $A$  and  $B$ ,  $B$  and  $C$ , and  $C$  and  $A$  respectively.

$$\vec{U} = \vec{A} + \vec{AB} \frac{r_a}{\sin \theta_{ab}} \quad (15)$$

$$\vec{V} = \vec{B} + \vec{BC} \frac{r_b}{\sin \theta_{bc}} \quad (16)$$

$$(17)$$

The point  $H$  is the intesection of the  $AB$  cone with the  $y$  axis. This is right triangle of the origin with the points  $U$  and  $H$ . (The hypotenuse of this triangle is  $\overline{HU}$ , and side adjacent to  $\theta_{AB}$  is  $\overline{OU}$ .) It is computed by projecting the contact point of the top plane in the  $xy$ -plane onto the  $xz$ -plane. This allows  $\gamma$  to be computed as a pure rotation about the  $x$ -axis.

$$H_y = U_x \tan \theta_{ab} \quad (18)$$

$$z' = \frac{U_x V_z}{U_x - V_x} \quad (19)$$

$$\gamma = \arcsin \frac{H_y}{z'} \quad (20)$$

## References

- [1] PAYNE, J. *Practical Solid Geometry*, 4 ed. Murby's "Science and Art Departemnt" Series of Text-Books. Thomsas Murby, 32 Bouverie Street, Fleet Street, E.C.;, 1 1881. Available free as an electronic book, see Problem CXLVIII (148), Page 195.
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