Three Inflatable Spheres as a Theoretical Basis for a Parallel Manipulator

Robert L. Read *email: read.robert@gmail.com
Megan Cadena †email: megancad@gmail.com

June 17, 2022

Abstract

A Stewart Platform[5] is a fundamental mechanism for varying the angle between two objects. A soft Stewart Platform can be made of two discs and three inflatable spheres. Soft robots are meant to deform under force, but it is useful to have a closed-form description of a plane in contact with three spheres of changing size. This papers gives closed-form expressions for the plane in contact with three adjacent spheres from known radii, and closed-form expressions for the much harder inverse problem of finding the radii to match a plane inclination. An interactive, browser-based web page that dynamically solves the problem[3] using JavaScript implementations of this math.

1 Introduction

A parallel manipulator varies the angle between two planes. The best known parallel manipulator is a Stewart Platform[5] which as 6 degrees or freedom. Having a soft parallel manipulator analogous to a Stewart Platform would allow soft, gentle positioning, and might be particularly valuable *in vivo*[4] or in some space applications[1]. Varying angular displacement is a composable building block of more complicated systems, such as tentacles.

One theoretical way to build such a manipulator is to have three enlargeable spheres sandwiched between two planes and constrained to always be in contact with each other. As these spheres are enlarged, perhaps by pneumatically inflating them changing their size, the top plane changes its orientation relative to the bottom plane. This creates 3 degrees of rotational freedom (ignoring the slight translation the spheres are capable of by consistently enlarging), which is slightly more constrained than the 6DOF Stewart Platform. In 1881, the problem of computing the plane tangent to three spheres was set as an exercise in a textbook, *Practical Solid Geometry*[2], but not solved.

*read.robert@gmail.com
†megancad@gmail.com

The forward problem is given three radii, to determine the orientation of the top plane. The inverse problem is much harder: given a desired orientation and the radius of one sphere, find the radius of the other two spheres that achieves it. This paper gives closed-form expressions of both the forward and inverse problem of three inflatable spheres as a parallel manipulator as pure solid geometry.

Additionally, we have created an online, interactive, browser based real-time simulation[3] that implements the math in this paper in JavaScript, both verifying it and making it easy to reuse. All of the code is released under the GNU Public License[GPL].

Soft robotics presume to operate under some deforming forces, so in any deformable robot the precise mathematics of positioning must always be corrected by a control system with feedback on the position. Nonetheless, having a closed-form expression of both the inverse and forward problem allow an initial Jacobian to be computed effortlessly, which is likely to assist any soft robotic control algorithm.

2 The Center Plane

As kinematicists, our interest is in the slope of the plane of the tops of these spheres as if they were resting on a fixed-frame such as a table. Then by inflating or deflating spheres, we would be able to control the direction of the top plane or platform.

Choosing the coordinate system of the XZ plane through the center of the spheres greatly simplifies the derivations, because the center of the spheres always form tangent circles in this plane. Call this plane the *center plane.* Following computer graphics convention, we think of the Y dimension as vertical and a right-handed set of axes. The position of A is fixed at the origin, B is constrained to the x-axis touching A, and C is constrained to the positive xz-plane. The center of these circles in the xz plane can be calculated from the radii independent of the tilt they induce. In this coordinate system, the y-coordinate of the center of all spheres is 0. Furthermore, a cone tangent to two spheres has its axis and apex in the xz-plane. The projection of all three spheres into this plane produces three touching circles. We seek an expression for the normal of the plane of the tops of these spheres as a function purely of the three radii. Call this plane the *top plane.* We can imagine the spheres resting on a fixed surface called the *bottom plane.* The tilt of the top and bottom plane relative to the center of the spheres is alway a mirror image of each other across the coordinate XZ-plane.

Define extrinsic Euler angles θ to be the rotation about the Z-axis and then γ to be the rotation about the X-axis. The tilt of the top plane relative to the bottom plane is given by the zx extrinsic Euler angles $(2\theta, 2\gamma)$.[1]

Because there is a plane through any three points and we have three spheres, we can construct the plane through the center of these points. The projection

[1] Is this really true? We should be able to verify this with a JavaScript program.

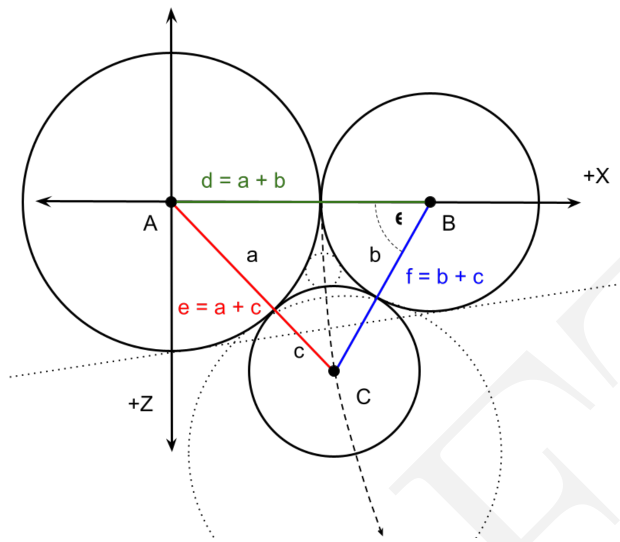

Figure 1: Three Touching Circles

of the edges of the spheres onto this plane form three touching circles.

To solve this problem most conveniently, we place the first circle at the origin, and the second circle on the positive x-axis, with the circles intersecting at the origin. The third circle is place in the positive z direction touching both other circles.

We seek a formula for the coordinates of the third circle in terms of three input radii a, b, c.

Because the distance between adjacent circles is the sum of their radii, define:

$$d = a + b \tag{1}$$
$$e = a + c \tag{2}$$
$$f = b + c \tag{3}$$
$$\tag{4}$$

Considering Fig. 1, varying the radius c without constraint will cause the center of the circle C to move, as shown by the dotted circles, describing a gentle curve depicted by the dashed curve which at very large c is perpendicular to a line tangent to both A and B, indicated by the dotted line. By considering the desired tilt of the top plane, a particular point on this curve is chosen.

2.1 The half-angle of a cone enveloping two tangent spheres

The apex half-angle ψ of a cone which envelopes (by being tangent to) two tangent spheres of radii r, s, is purely a function of r and s. Luckily, to develop

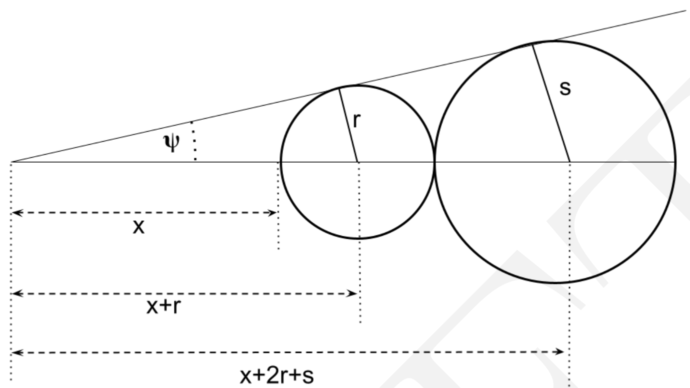

Figure 2: Half-angle of Cone Tangent to Two Tangent Spheres

a formula for ψ, we can treat the problem as a two-dimensional problem, as shown in Fig. 2

Noting the lengths of the hypotenuses as shown in the figure, we can calculate x in terms of r and s:

$$\sin\psi = \frac{r}{x+r} \tag{5}$$

$$\sin\psi = \frac{s}{x+2r+s} \tag{6}$$

$$\frac{r}{x+r} = \frac{s}{x+2r+s} \tag{7}$$

$$s(x+r) = r(x+2r+s) \tag{8}$$

$$x = \frac{2r^2}{s-r} \tag{9}$$

We can then eliminate x:

$$\sin\psi = \frac{r}{x+r} \tag{10}$$
$$\sin\psi = \frac{r}{\frac{2r^2}{s-r}+r} \tag{11}$$
$$\sin\psi = \frac{1}{\frac{2r}{s-r}+1} \tag{12}$$
$$\sin\psi = \frac{s-r}{2r+s-r} \tag{13}$$
$$\sin\psi = \frac{s-r}{r+s} \tag{14}$$
$$\psi = \arcsin\frac{s-r}{s+r} \tag{15}$$

where $r < s$ without loss of generality. Note that when $r = s$ there is a special case, the cone tangent to both is degenerate (that is, a cylinder, or a cone of $\psi = 0$ apex angle.)

3 The Euler Anglers from the Radii

3.1 Finding the Point C

Given a, b and c, we seek the angles θ and γ representing the Euler anglers of rotation around the Z axis (θ), and then the X axis (γ).

We seek the point C first. Call the angle $\angle ABC = \epsilon$, since the opposite side is labeled e. The cosine law to compute the angle ϵ:

$$\epsilon = \arccos\frac{d^2+f^2-e^2}{2fd} \tag{16}$$

It is clear that once ϵ has been calculated we can calculate C_z:[2]

$$C_z = f\sin\epsilon \tag{17}$$

Noting that $B_x = d = a + b$,

$$C_x = d - \cos\epsilon \tag{18}$$

3.2 The Apex Line

The problem of finding the planes tangent to three touching spheres is given as an exercise in an advanced textbook on solid geometry from 1881[2] which does

[2]We use the common convention that $C_x, C_y, andC_z$ represent the x, y and z coordinate of the point or vector C.

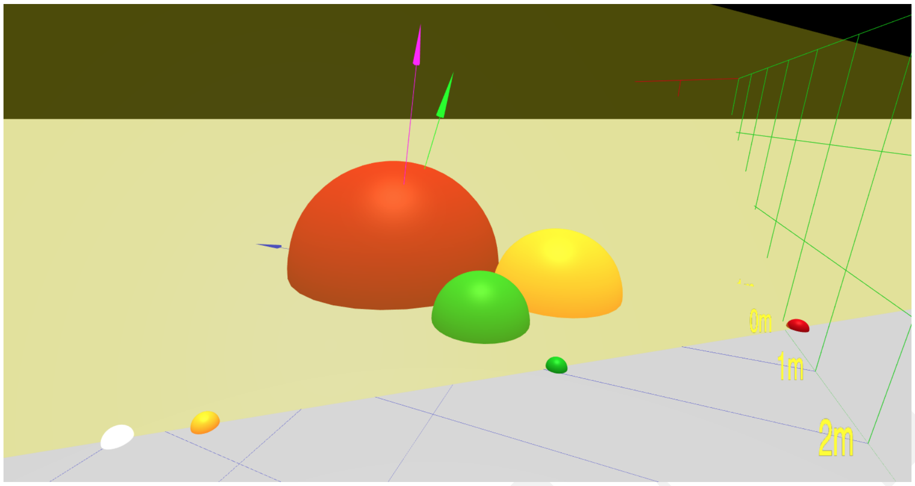

Figure 3: Three Touching Spheres

not give a solution, but it gives a hint: to consider the cones enveloping three spheres taken two at a time.

Taking two adjacent spheres defines a cone tangent to both spheres whose apex is in the XZ plane. Call the apex of the AB cone U, the AC cone V, and the BC cone W.

It is a beautiful fact that the the apices U, V and W are collinear on the *apex line* which is in the XZ plane, depicted in both Figure 3 or Figure 4. Finally, the top and bottom planes intersect at this line, because those planes are tangent to the three cones.

Observe that this plane is tangent to all three cones. Observe that the AB cone intersects the A sphere in a circle on the surface of the A perpendicular to and centered on the X axis.

The apex half-angle ψ of a cone which envelopes (by being tangent to) two tangent spheres of radii r, s, is:

$$\psi_{abs} = \arcsin \frac{s-r}{s+r} \tag{19}$$

where $r < s$ without loss of generality. Note that when $r = s$ there is a special case, the cone tangent to both is degenerate (that is, a cylinder, or a cone of $\psi = 0$ apex angle.)

Let θ be the rotation of the ground plane (the XZ plane in this system) need to make it rest on the A and B spheres and intersect the AB cone. Because we use θ as a rotation angle, it must be signed, and is negative if $a > b$.

$$\theta = \begin{cases} \theta, & \text{if } a > b, \\ -\theta, & \text{otherwise} \end{cases} \tag{20}$$

$$\tag{21}$$

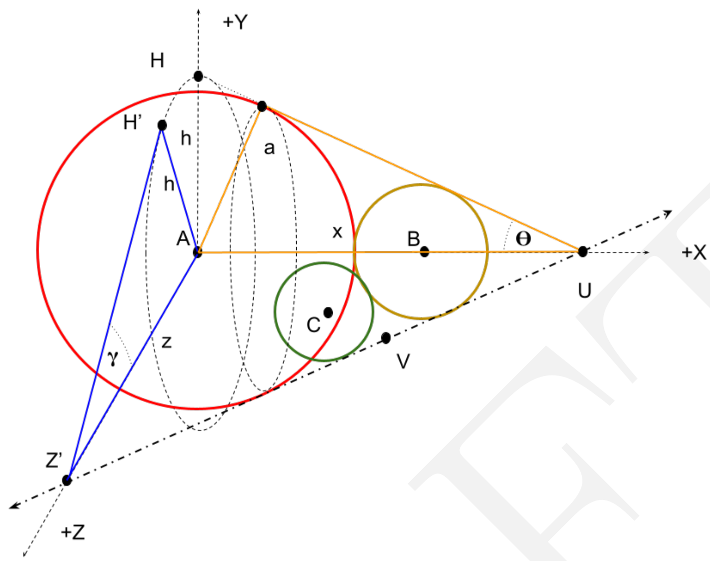

Figure 4: Rotation Math

3.3 Computing γ

Note that in this paper, like THREE.js and many computer graphics systems, we use a right-handed coordinate system and describe angles of rotation as positive when counter-clockwise looking from a positive axis as the origin.

In our main view (see Fig. 4) and in the initial view of our software, thus has a positive γ and negative θ, when $a > b$ and $a > c$.

In order to compute γ, we seek to form a triangle in the YZ plane. Because Euler angles are performed in a certain order, it is necessary to imagine the ground plane after it has been rotated by θ but before it has been rotated by γ. Call this the "Z-rotated" plane.

After a rotation of θ about the Z axis, the Z-rotated plane intersects the X axis at a point $U = [U_x, 0, 0]$, the apex of the AB cone:

$$U_x = \frac{a}{\sin(-\theta)} \tag{22}$$

The y coordinate of the point H where where the top plane intersects the Y axis is:

$$H_y = U_x \tan\theta \tag{23}$$

Any rotation about the X axis of the Z-rotated plane (and H) will keep that point in the YZ plane and a distance $h = H_y$ from the origin.

Call the intersection of the Z-axis apex line with apex line Z', and let $z = Z_z$. Then γ is the angle we need to rotate the plane already rotated by θ so that it intersects with the Z-axis at Z'. Points U and V represent the apices of the apices of the cones tangent to A and B, A and C, respectively. The apex of the BC cone, W, is also on the apex line but not used in our calculations.

The point V is computable as the apex of the AC cone:

$$\phi = \arcsin \frac{a-c}{a+c} \tag{24}$$

$$V = A + (C - A)\frac{b}{\sin \phi} \tag{25}$$

$$V = C(\frac{b(a+c)}{a-c}) \tag{26}$$

$$\tag{27}$$

The z coordinate of the point Z' can be calculated from Z intercept fact that it is collinear with U and V, which has a slope in the XZ plane of $\frac{V_z}{U_x - V_x}$:

$$z = \begin{cases} V_z, & \text{if } a = b \\ \frac{V_z U_x}{U_x - V_x}, & \text{if } a \neq b \end{cases} \tag{28}$$

$$\tag{29}$$

The line $Z'H'$ is tangent to a circle of radius h around the point A in the YZ plane, so $\angle Z'H'A$ is a right angle. The point H' is in the top plane by definition and Z' is in the top plane because it is on the apex line. The hypotenuse of the $\triangle Z'H'A$ is the line AZ' on the Z axis of length z, and the angle $\gamma = \angle AZ'H$ has an opposite side of length h, so:

$$\gamma = \arcsin \frac{h}{z} \tag{30}$$

Equations 20 and 30 give the Euler angles as a rotation θ about the Z axis followed by a rotation γ about the X axis for the tangent plane as a function of the three radii a, b, c

4 The Inverse Problem

To build a theoretic robotic platform by controlling the radii of the spheres by inflating and deflating them, we wish to solve the inverse problem. That is, given input θ and γ as desired Euler angles, we want to compute a, b, c which achieve these angles.

Since these are smooth functions, the problem could be solved easily by a Newton-Raphson style solver based on 20 and 30. However, a closed-form solution is always superior, and allows derivatives, and hence Jacobians, to be expressed symbolically as closed-form solutions.

The radius b can be computed directly from θ because of our choice of coordinates. By considering to proportionality of the similar triangles formed by the contacts points of sphere A and B and their centers with U_x in the XY plane, we obtain:

$$U_x = \frac{a}{\sin\theta} \tag{31}$$

$$b = \frac{(U_x - a)\sin\theta}{\sin\theta + 1} \tag{32}$$

Given the plane defined by a known radius a, θ and γ, the point C is constrained by asserting that the perpendicular distance to this plane of C is c. Additionally, in the central plane we have two other constraints, that C is placed so that it touches the circle centered on A and the circle centered on B.

Then we do not know the point C but know the distance $a + c$ from the point A (the origin) and the distance $b + c$ from the known point B. These two constraints can be generated from the Pythagorean Theorem and can be expressed:

$$a + c = \sqrt{C_x^2 + C_z^2} \tag{33}$$

$$b + c = \sqrt{(a + b - C_x)^2 + C_z^2} \tag{34}$$

If we were to plot the point C as c increases, we would find a gentle curve moving away from the intersection of the circle A and B, curling in the direction of the smaller of A or B. An additional constraint will designate a point on this curve. This gives us three equations, which should be more than enough to solve for the point C.

Let $H = [0, H_y, 0]$.

Let N be the normal to the plane (pointing up in our diagram when $a >= b >= c$.)

$$\overrightarrow{N} = \overrightarrow{Z' - U} \times \overrightarrow{Z' - H} \tag{35}$$

Since we have the normal of the plane and three points (U, Z, H_y) in the plane, computing the distance from the point C to the plane has a simple formula. We'll use the point $U = [Ux, 0, 0]$ as our main point. In our situation, because the sphere A is at the origin, the equation of the top plane is $\overrightarrow{N} \cdot \overrightarrow{U} = a$.

$$a = \frac{N \cdot U}{N_y} \tag{36}$$

$$c = \frac{|N \cdot P - a|}{|N|} \tag{37}$$

$$c = \frac{|C_x N_x + C_z N_z - a|}{1} \tag{38}$$

By substituting 38 into equations 33 and 34, we obtain:

$$a + |C_x N_x + C_z N_z - a| = \sqrt{C_x^2 + C_z^2} \tag{39}$$
$$b + |C_x N_x + C_z N_z - a| = \sqrt{(a + b - C_x)^2 + C_z^2} \tag{40}$$

If the interior expression $C_x N_x + C_z N_z - a$ is positive, then upon solving the system we find only imaginary solutions.

So assuming it is negative, we obtain:

$$2a + -C_x N_x + -C_z N_z = \sqrt{C_x^2 + C_z^2} \tag{41}$$
$$2b + -C_x N_x + -C_z N_z = \sqrt{(a + b - C_x)^2 + C_z^2} \tag{42}$$

This is a system of two nonlinear equations with unknowns C_x and C_z. Solving this system is beyond symbolic manipulating capacities of the authors, but not Mathematica. However, Mathematica expends half a page expressing the answer in its raw form.

Furthermore, there are actually two solutions, corresponding to the two sides of the AB line which the circle C could be on and satisfy these constraints. We have selected the solution that place C in the positive Z direction conformant to our problem set-up.

However, by introducing new variables to represent repeated portions of this long expression, the expressions be made tractable and easily programmed by computer.

$$M = (a + b + aN_x - bN_x)^2 - 4abN_z^2 \tag{43}$$
$$L = a(a + b + aN_x - bN_x) - b(a + b)N_z^2 \tag{44}$$
$$G = -a^2(a - b)^2 bN_z^2 \tag{45}$$
$$H = 2a(-1 + N_x) - b((-1 + N_x)^2 + N_z^2) \tag{46}$$
$$F = \sqrt{GH} \tag{47}$$
$$K = 4a^2b^2N_z^2 + 2a^3b(-3 + N_x)N_z^2 \tag{48}$$
$$J = b^3(-1 + N_x)N_z^2 \tag{49}$$
$$C_{x0} = \frac{2(F + aL)}{M} \tag{50}$$
$$C_{z0} = \frac{K + (2bF(-1 + N_x) - 2a(F(1 + N_x) + b^3(-1 + N_x)N_z^2))}{(a - b)MN_z} \tag{51}$$
$$C_{x1} = \frac{-2F + 2aL}{M} \tag{52}$$
$$C_{z1} = \frac{K - (2bF(-1 + N_x) - 2a(F(1 + N_x) + b^3(-1 + N_x)N_z^2))}{(a - b)MN_z} \tag{53}$$

One solution $((C_{x0}, C_{z0})$ or $(C_{x1}, C_{z1}))$ will be on the positive side of the X axis, but which one is highly dependent on the input variables, so both are computed and the positive z value selected.

The variable c can be computed from a, C_x and C_z form Eqn. 33.

This math can be found in the JavaScript which is licensed under the GPL[GPL] at our website[3].

5 Special Cases of the Inverse Problem

When both $\theta = 0$ and $\gamma = 0$, then $c = b = a$.

5.1 Special case when γ is 0

In the above math when $\gamma = 0$, then N_z is zero, creating a division by zero. In this case we can use the proportionality of similar triangles to assert:

$$a = c + \frac{aC_x}{U_x} \tag{54}$$

Combined with our previous equations Eqn. 33 and Eqn. 34 in the plane relating C_x and C_z to c, we obtain the following equation:

$$c = \frac{a(a+b)(a-U_x)}{a^2 - ab + aU_x + bU_x} \tag{55}$$

5.2 Special case when θ is 0

When $\theta = 0$, the point U is at infinity or cannot be found. In this case $b = a$. In this case the sphere C is symmetrically positioned in contact with A and B because they have the same radius; therefore $C_x = a$ and:

$$(a+c)^2 = a^2 + C_z^2 \tag{56}$$

In this case,

$$Z' = \frac{a}{\sin\gamma} \tag{57}$$

Projecting out the X dimension and forming similar triangles in the YZ plane, we find that

$$c = \frac{a(Z' - C_z)}{Z'} \tag{58}$$

Asking Mathematica to solve these two non-linear equations, we obtain, after again discarding the solution producing negative z values:

$$C_z = \frac{2a^2 Z' - \sqrt{a^4 (Z')^2 + 3a^2 (Z')^4}}{a^2 - (Z')^2}, \tag{59}$$

which allows us to obtain obtain c from 58.

6 Future Work and Relation to Robotics

Our formulation of the problem with the center of the spheres in the XZ plane has allowed us to obtain a closed-form solution to both the forward and inverse problem.

A Stewart Platform could be constructed of three inflatable spheres. However, our formulation is inconvenient for that problem, because in practice you would know the bottom plane, and not the central plane, as you vary the diameters of the spheres through inflation or deflation. The angle between the top and bottom plane will be approximately but not exactly twice the angles that we have computed here.

We attempted to solve this problem by reformulating the problem in terms of a coordinate system based on fixed bottom plane, but the formulae became intractably long even with the help of Mathematica. However, as a completely deterministic solid geometry system, there must be a closed-form solution, that perhaps mathematicians of greater skill or diligence can find. For example, it might be fruitful to reformulate the problem purely in terms of the normal of the top plane as a linear algebra problem rather than considering Euler angles.

Even if one had such a solution, it would be of limited value to robotocists because the inflation or deflation of spheres in a macroscopic physically constructible Stewart platform would be "soft", and would deform under external pressure. The solutions presented here would likely have value only as approximate or initial solutions.

We conjecture there may be situations in chemistry or astronomy beyond our ability to identify in which tangent spheres remain ideal spheres and these closed-form expressions would remain be accurate.

There are some combinations of radii, when one sphere is much smaller than the other two, in which a plane cannot be tangent to all three spheres. It would be nice to have a mathematical description of this limit, but we have not investigated it.

7 Conclusion

Closed-form expressions of the problem of finding a plane resting on three spheres in contact have been given, and the inverse problem of finding the radii of three spheres necessary to support a plane on threes spheres in contact has been solved. The expressions are in the coordinate system of the the center of the threes spheres, which may be a significant limitation for some applications.

An interactive, browser-based solution that allows real-time manipulation of the mathematical system in both the forward and inverse directions using sliders has been given[3]. The greatest value of this work may be the demonstrating the attractiveness of the solution. The existing of the apex line, which is so visually apparent and pleasing, may not be completely obvious upon first consideration.

References

[1] Samantha Helen Glassner. "Soft Stewart Platform for Robotic In-Space Assembly Applications". thesis. Northeastern University, 2020.

[GPL] *GNU General Public License*. Version 3. Free Software Foundation, June 29, 2007. URL: http://www.gnu.org/licenses/gpl.html.

[2] Joseph Payne. *Practical Solid Geometry. Orthographic and Isometric Projection*. 4th ed. Murby's "Science and Art Departemnt" Series of Text-Books. Available free as an electronic book, see Problem CXLVIII (148), Page 195. 32 Bouverie Street, Fleet Street, E.C.; Thomsas Murby, Jan. 1881. URL: https://play.google.com/store/books/details?id=8TQDAAAAQAAJ&rdid=book-8TQDAAAAQAAJ&rdot=1.

[3] Robert L. Read and Megan Cadena. *Plane Tanget to 3 Spheres*. [Online; accessed 13-November-2019]. 2019. URL: https://pubinv.github.io/softrobotmath/ (visited on 10/13/2019).

[4] Edward L White, Jennifer C Case, and Rebecca Kramer-Bottiglio. "A soft parallel kinematic mechanism". In: *Soft robotics* 5.1 (2018), pp. 36–53.

[5] Wikipedia contributors. *Stewart platform — Wikipedia, The Free Encyclopedia*. [Online; accessed 9-October-2019]. 2019. URL: https://en.wikipedia.org/w/index.php?title=Stewart_platform&oldid=898429010.