A Consideration of Inflatable Circles

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1 Introduction

This is a study of the basic math of inflatable spheres as a tool for soft robotics. We begin with a study in two dimension to simplify the problem. Our final goal is to analyze three dimensional soft robots composed of inflatable spheres.

2 Problem I: Circles in Fixed Postions

A very simplified version of the problem is to imagine that a two-dimensional plane. Instead of spheres, we will assume we have circles of changable radius. This is in fact realistic of a soft robot constrained to a plane.

Eventually we hope to have circles pressing against each other, or tangent or "kissing". However, the problem is a bit simpler if we assume we have two circles, each of which is constrained to have its center on the a vertical line (see Figure 1.) We place the circle C_1 with radius r_1 on the $x = -1$ line, and assume that it rests on a shelf or plane on the x-axis. Assume the C_2 circle whose radius if r_2 is on the $x = 1$ line.

Let A the intersection of the tangent line supported by the inflatable circles with the x-axis. Call the distance of A on the x-axis x. Let ψ be the angle formed by the circle centers with the x axis (measured counterclockwise).

If the radii are less than 1 so that no issues of intersection of the circles arise, we have:

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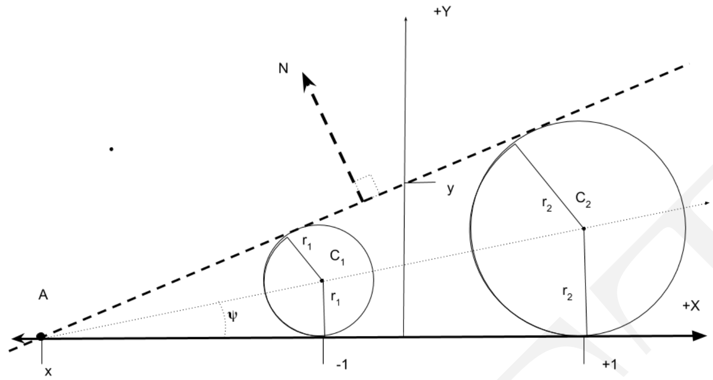

Figure 1: Problem I: Fixed Circles Centers

$$\tan\psi = \frac{r_1}{x-1} = \frac{r_2}{x+1} \tag{1}$$

$$r_2(x-1) = r_1(x+1) \tag{2}$$

$$r_2 x - r_2 = r_1 x + r_1 \tag{3}$$

$$r_2 x - r_1 x = r_1 + r_2 \tag{4}$$

$$x(r_2 - r_1) = r_1 + r_2 \tag{5}$$

$$x = \frac{r_1 + r_2}{r_2 - r_1} \tag{6}$$

$$\tan\psi = \frac{r_1}{x-1} \tag{7}$$

$$\psi = \arctan\frac{r_1}{x-1} \tag{8}$$

$$N = \begin{bmatrix} -\sin\psi \\ \cos\psi \end{bmatrix} \tag{9}$$

$$\tag{10}$$

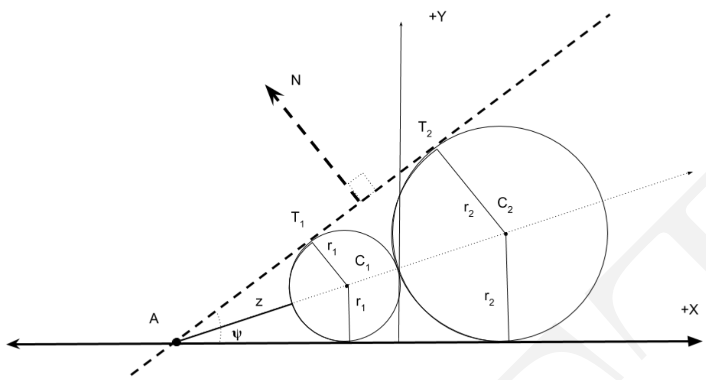

Figure 2: Problem II: Tangent Circles

3 Problem II: Tangent Circles

$$\frac{r_2}{||(C2 - A)||} = \frac{r_1}{||(C_1 - A)||} \tag{11}$$
$$||C2 - A|| = r_2 + 2r_1 + z \tag{12}$$
$$||C1 - A|| = r_1 + z \tag{13}$$
$$\frac{r_2}{r_2 + 2r_1 + z} = \frac{r_1}{r_1 + z} \tag{14}$$
$$z = -\frac{2r_1^2}{r_1 - r_2} \text{ and } r_1 \neq r_2 \text{ and } r_1 r_2 (r_1 + r_2) \neq 0 \tag{15}$$
$$\sin\psi = \frac{r_1}{z + r_1} \tag{16}$$
$$\theta = 2 \arcsin \frac{r_1}{z + r_1} \tag{17}$$
$$N_x = -\sin \pi/2 + \theta \tag{18}$$
$$N_y = -\cos \pi/2 + \theta \tag{19}$$
$$\tag{20}$$

The slope of the vector $\overrightarrow{T_1 T_2}$ is $\tan\theta$.

$$\overrightarrow{T_1 T_2}_x = \cos\theta \tag{21}$$
$$\overrightarrow{T_1 T_2}_y = \sin\theta \tag{22}$$

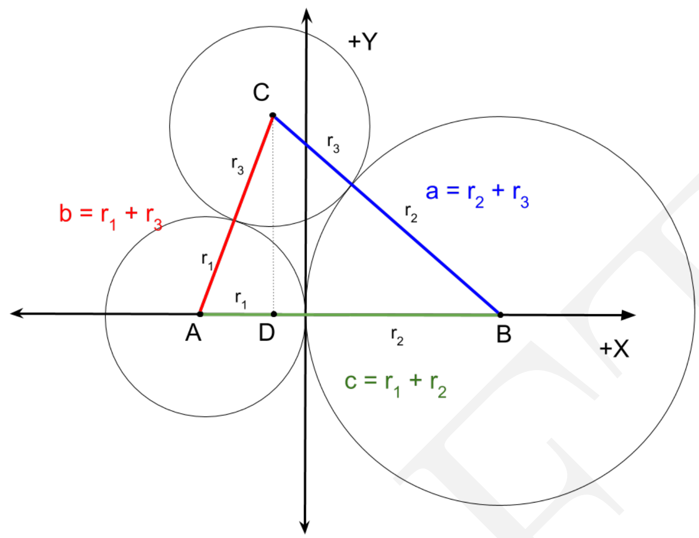

Figure 3: Three Touching Circles

4 Problem: Three Touching Circles

Our goal is to be able to determine the orientation of plane resting on top of three spheres of different radii which are touching each other. Because there is a plane through any three points and we have three spheres, we can construct the plane through the center of these points. The projection of the edges of the spheres onto this plane form three touching circles. Knowing the position of these circles is a valuable prelude to solving the three dimensional problem.

To solve this problem most conveniently, we place the first circle on the negative x-axis, and the second circle on the positive x-axis, with the circles intersecting at the origin. The third circle is place in the positive y direction. It's center will not be on the y-axis itself unless the radii of the first two circles are equal.

We seek a formula for the coordinates of the third circle in terms of three input radii r_1, r_2, r_3.

Because the distance between adjacent circles is the sum of their radii, define:

$$a = r_2 + r_3 \tag{23}$$
$$b = r_1 + r_3 \tag{24}$$
$$c = r_1 + r_2 \tag{25}$$

Then we can use the cosine law to compute the angle $\angle ABC = \theta$:

$$\theta = \arccos \frac{a^2 + b^2 - c^2}{2bc} \tag{26}$$

$$\theta = \arccos \frac{(r_2 + r_3)^2 + (r_1 + r_3)^2 - (r_1 + r_2)^2}{2(r_2 + r_3)(r_1 + r_3)} \tag{27}$$

$$\tag{28}$$

It is clear that once θ has been calculated:

$$C_y = a \sin \theta \tag{29}$$

Allowing us to form a right triangle $\triangle ACD$ and use the Pythagorean theorem:

$$b^2 = C_y^2 + (r_1 - C_x)^2 \tag{30}$$

$$b^2 - C_y^2 = (r_1 - C_x)^2 \tag{31}$$

$$\sqrt{b^2 - C_y^2} = r_1 - C_x \tag{32}$$

$$C_x = r_1 - \sqrt{b^2 - C_y^2} \tag{33}$$

5 Three Touching Spheres

Our fundamental goal now is to describe three touching spheres. As robotocists, our interest is in the slope of the plane of the tops of these spheres as if they were resting on a table. Then by inflating or deflating spheres, we would be able to control the direction of a plane or platform. Such a device is sometimes called a parallel manipulator, of which a Stewart Platform[1] (`https://en.wikipedia.org/w/index.php?title=Stewart_platform&oldid=898429010`) is the best-known example.

The fundamental action of a parallel manipulator is to tilt a plane or platform in a desired direction. Having developed the math for the three touching circles in the previous section, we now use it to find the normal of the a plane resting on the top of three spheres by choosing our coordinate system to the plane defined by the center points of the three spheres. In this coordinate system, the z-coordinate of the center of all spheres is 0. The projection of all three spheres into this plane produces three touching circles. We seek an expression for the normal of the plane of the tops of these spheres as a function purely of the three radii. Call this plane the *top* plane.

As the top plane tilts, the points tangent to this plane move away from the highest points on each of the three spheres, making the problem more difficult.

This problem can clearly be solved quickly using numerical methods, because by guessing a candidate normal to the top plane, it is relatively easy to compute an error function based on the distance between the closest point on each sphere to the top plane and the top plane. We would expect an iterative approach to converge very quickly.

Nonetheless we seek an analytic solution.

References

[1] Wikipedia contributors. Stewart platform — Wikipedia, the free encyclopedia, 2019. [Online; accessed 9-October-2019].