Three Inflatable Spheres as a Theoretical Basis for a Parallel Manipulator

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Abstract

A Stewart Platform[5] is a fundamental mechanism for varying the angle between two objects. A soft Stewart Platform can be made of two discs and three inflatable spheres. Soft robots are meant to deform under force, but it is useful to have a closed-form description of a plane in contact with three spheres of changing size. This papers gives closed-form expressions for the plane in contact with three adjacent spheres from known radii, and closed-form expressions for the much harder inverse problem of finding the radii to match a plane inclination. An interactive, browser-based web page that dynamically solves the problem[3] using JavaScript implementations of this math.

1 Introduction

A parallel manipulator varies the angle between two planes. The best known parallel manipulator is a Stewart Platform[5] which as 6 degrees or freedom. Having a soft parallel manipulator analogous to a Stewart Platform would allow soft, gentle positioning, and might be particularly valuable *in vivo*[4] or in some space applications[1]. Varying angular displacemnt is a composable buliding block of more complicated systems, such as tentacles.

One theoretical way to build such a manipulator is to have three enlargable spheres sandwiched between two planes and constrained to always be in contact with each other. As these spheres are enlarged, perhaps by pneumatially inflating them changing their size, the top plane changes its orientation relative to the bottom plane. This creates 3 degress of rotational freedom (ignoring the slight translation the spheres are capable of by consistently enlarging), which is slightly more constrained than the 6DOF Stewart Platform. In 1881, the problem of computing the plane tangent to three spheres was set as an exercise in a textbook, *Practical Solid Geometry*[2], but not solved.

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The forward problem is given three radii, to determine the orientation of the top plane. The inverse problem is much harder: given a desired orientation and the radius of one sphere, find the radius of the other two spheres that acheives it. This paper gives closed-form expressions of both the forward and inverse problem of three inflatable spheres as a parallel manipulator as pure solid geometry.

Additionally, we have created an online, interactive, browser based realtime simulation[3] that implements the math in this paper in JavaScript, both verifying it and making it easy to reuse. All of the code is released under the GNU Public License[GPL].

Soft robotics presume to operate under some deforming forces, so in any deformable robot the precise mathematics of positioning must always be corrected by a control system with feedback on the position. Nonetheless, having a closed-form expression of both the inverse and forward problem allow an initial Jacobian to be computed effortlessly, which is likely to assist any soft robotic control algorithm.

2 The Center Plane

As kinematicists, our interest is in the slope of the plane of the tops of these spheres as if they were resting on a fixed-frame such as a table. Then by inflating or deflating spheres, we would be able to control the direction of the top plane or platform.

Choosing the coordinate system of the XZ plane through the center of the spheres greatly simplifies the derivations, because the center of the spheres always form tangent circles in this plane. Call this plane the *center plane.* Following computer graphics convention, we think of the Y dimension as vertical and a right-handed set of axes. The position of A is fixed at the origin, B is constrained to the x-axis touching A, and C is contrained to the positive xz-plane. The center of these circles in the xz plane can be calculated from the radii independent of the tilt they induce. In this coordinate system, the y-coordinate of the center of all spheres is 0. Furthermore, a cone tangent to two spheres has its axis and apex in the xz-plane. The projection of all three spheres into this plane produces three touching circles. We seek an expression for the normal of the plane of the tops of these spheres as a function purely of the three radii. Call this plane the *top plane.* We can imagine the spheres resting on a fixed surface called the *bottom plane.* The tilt of the top and bottom plane relative to the center of the spheres is alway a mirror image of each other across the coordinate XZ-plane.

Define extrinsic Euler angles θ to be the rotation about the Z-axis and then γ to be the rotaion about the X-axis. The tilt of the top plane relative to the bottom plane is given by the zx extrinsic Euler angles $(2\theta, 2\gamma)$.[1]

Because there is a plane through any three points and we have three spheres, we can construct the plane through the center of these points. The projection

[1] Is this really true? We should be able to verify this with a JavaScript program.

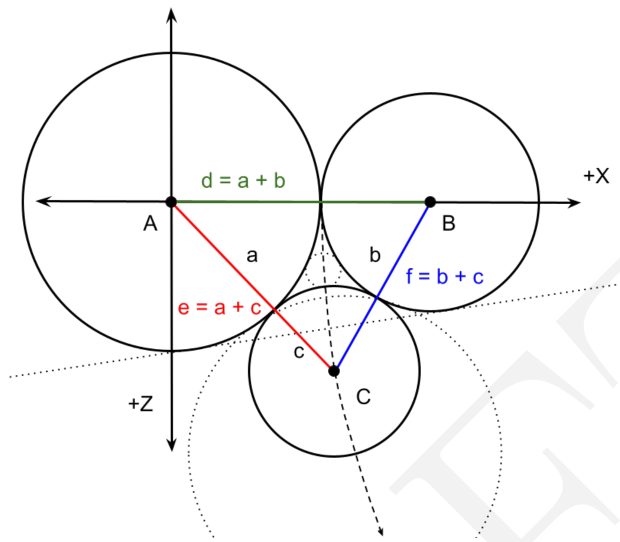

Figure 1: Three Touching Circles

of the edges of the spheres onto this plane form three touching circles.

To solve this problem most conveniently, we place the first circle at the orign, and the second circle on the positive x-axis, with the circles intersecting at the origin. The third circle is place in the positive z direction touching both other circles.

We seek a formula for the coordinates of the third circle in terms of three input radii a, b, c.

Because the distance between adjacent circles is the sum of their radii, define:

$$d = a + b \tag{1}$$
$$e = a + c \tag{2}$$
$$f = b + c \tag{3}$$
$$\tag{4}$$

Considering Fig. 1, varying the radius c without constraint will cause the center of the circle C to move, as shown by the dotted circles, describing a gentle curve depicted by the dashed curve which at very large c is perpendicular to a line tangent to both A and B, indicated by the dotted line. By considering the desired tilt of the top plane, a particular point on this curve is chosen.

2.1 The half-angle of a cone eneveloping two tangent spheres

The apex half-angle ψ of a cone which envelopes (by being tangent to) two tangent spheres of radii r, s, is purely a function of r and s. Luckily, to develop

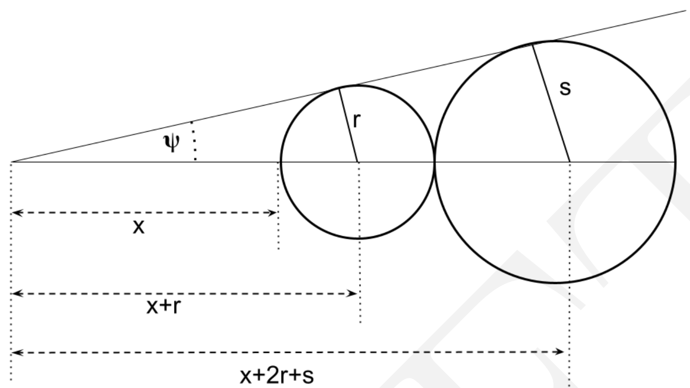

Figure 2: Half-angle of Cone Tangent to Two Tangent Spheres

a formula for ψ, we can treat the problem as a two-dimenstional problem, as shown in Fig. 2

Noting the lengths of the hypoteneuses as shown in the figure, we can calulate x in terms of r and s:

$$\sin\psi = \frac{r}{x+r} \tag{5}$$

$$\sin\psi = \frac{s}{x+2r+s} \tag{6}$$

$$\frac{r}{x+r} = \frac{s}{x+2r+s} \tag{7}$$

$$s(x+r) = r(x+2r+s) \tag{8}$$

$$x = \frac{2r^2}{s-r} \tag{9}$$

We can then eliminate x:

$$\sin\psi = \frac{r}{x+r} \tag{10}$$

$$\sin\psi = \frac{r}{\frac{2r^2}{s-r}+r} \tag{11}$$

$$\sin\psi = \frac{1}{\frac{2r}{s-r}+1} \tag{12}$$

$$\sin\psi = \frac{s-r}{2r+s-r} \tag{13}$$

$$\sin\psi = \frac{s-r}{r+s} \tag{14}$$

$$\psi = \arcsin\frac{s-r}{s+r} \tag{15}$$

where $r < s$ without loss of generality. Note that when $r = s$ there is a special case, the cone tangent to both is degenerate (that is, a cylinder, or a cone of $\psi = 0$ apex angle.)

3 The Tilt from the Radii

Given a, b and c, we seek the angles θ and γ. We seek the point C first. The cosine law to compute the angle $\angle ABC = \alpha$:

$$\alpha = \arccos\frac{a^2+b^2-c^2}{2bc} \tag{16}$$

It is clear that once α has been calculated:

$$C_z = a\sin\alpha \tag{17}$$

Allowing us to form a right triangle $\triangle ACD$ and use the Pythagorean theorem:

$$C_x = \sqrt{b^2 - C_z^2} \tag{18}$$

The problem of finding the planes tangent to three touching spheres is given as an exercise in an advanaced textbook on solid geometry from 1881[2] which does not give a solution, but it gives a hint: to consider the cones enveloping three spheres taken two at a time.

Taking two adjacent spheres defines a cone tangent to both spheres whose apex is in the XZ plane. Call the apex of the AB cone U, the AC cone V, and the BC cone W.

It is a beautiful fact that the the apices U, V and W are colinear on the *apex line* which is in the XZ plane, depicted in both Figure 3 or Figure 4. Finally, the top and bottom planes intersect at this line, because those planes are tangent to the three cones.

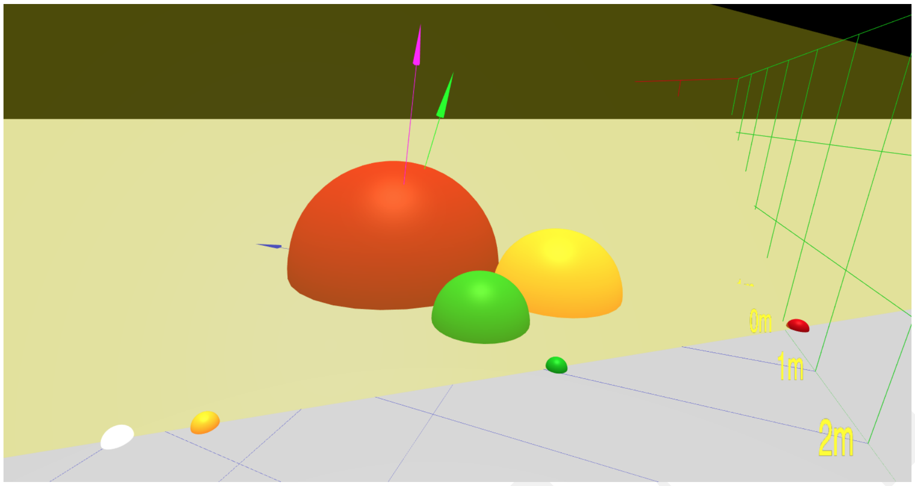

Figure 3: Three Touching Spheres

We seek to compute the normal of the tangent plane. Observe that this plane is tangent to all three cones. Observe that the AB cone intersects the A sphere in a circle on the surface of the A perpendicular to and centered on the X axis.

The apex half-angle ψ of a cone which envelopes (by being tangent to) two tangent spheres of radii r, s, is:

$$\psi_{abs} = \arcsin \frac{s-r}{s+r} \tag{19}$$

where $r < s$ without loss of generality. Note that when $r = s$ there is a special case, the cone tangent to both is degenerate (that is, a cylinder, or a cone of $\psi = 0$ apex angle.) Because we use θ as a rotation angle, it must be signed, and is negative if $a > b$.

$$\theta = \begin{cases} \theta_{abs}, & \text{if } a > b, \\ -\theta_{abs}, & \text{otherwise} \end{cases} \tag{20}$$

$$\phi_{abs} = \arcsin \frac{b-c}{b+c} \tag{21}$$

A vector of length a that is always rotated about the origin is always a point on the sphere. The first operation is to move this vector perpendicular to the AB cone. A vector N in the Y direction and rotated counterclockwise about the Z-axis by θ is perpendicular to the AB cone.

However, we must rotate this vector N about the X axis by an unknown amount, γ, in order to orient the vector to line up properly with the desired tilt of the tangent plane despite not being a pure rotation about the Z axis. Since this angle is computed in the YZ plane, we compute a projection of the point V into that plane, forming a triangle in the YZ plane.

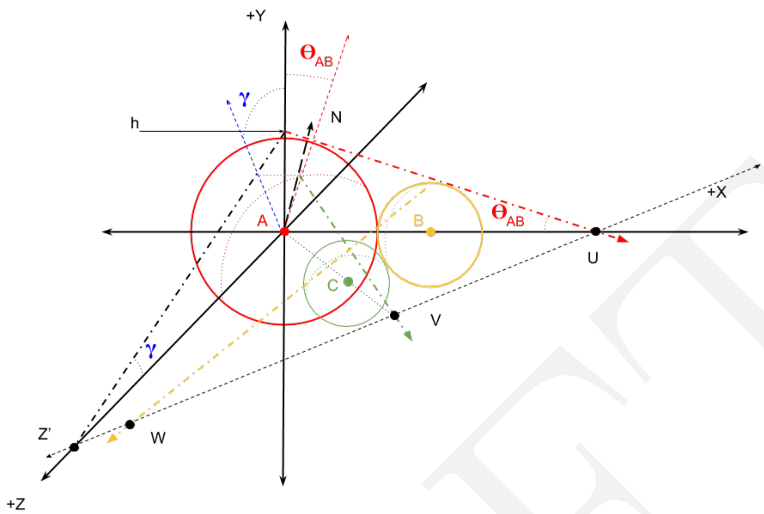

Figure 4: Rotation Math

Call this point I_z, the intersection of the apex line with the Z-axis. Let h be the height of the intersection of the y-axis with the AB cone. Then γ is the angle we need to rotate the plane by so that it intersects with the Z-axis at a point, referred to as Z'. By basic geometry of simialr triangles, this is computed from the right triangle of the origin with h and I_z. Points U, V, and W represent the apices of the apices of the cones tangent to A and B, B and C, and C and A respectively. Recall that $A = [0, 0, 0]$.

$$U = A + (B - A)\frac{a}{\sin\theta} \tag{22}$$

$$V = B + (C - B)\frac{b}{\sin\phi_{abs}} \tag{23}$$

$$Z' = \begin{cases} V_z, & \text{if } a = b \\ \frac{U_z V_x}{U_x - V_x}, & \text{if } a \neq b \end{cases} \tag{24}$$

$$\tag{25}$$

The point H is the intesection of the AB cone with the y axis (and therefore the intersection of the top plane with the y axis.

$$H_y = U_x \tan\theta \tag{26}$$

This is right triangle of the origin with the points U and H. (The hypotenuse of this triangle is $\overline{HU}$, and side adjacent to θ is $\overline{AU}$.) It is computed by projecting the contact point of the top plane in the xy-plane onto the xz-plane. This allows γ to be computed as a pure rotation about the x-axis.

$$\gamma = \arcsin \frac{Z'}{H_y} \tag{27}$$

Equations 20 and 27 give the Euler angles θ and γ for the tilt of the tangent plane as a function of the three radii a, b, c.

4 The Inverse Problem

To build a useful robotic platform in by controlling the radii of the spheres by inflating and deflating them, we wish to solve the inverse problem. That is, given input θ and γ as desired Euler angles, we want to compute a, b, c which acheive these angles.

Since these are smooth functions, the problem could be solved easily by a Newton-Raphson style solver based on 20 and 27. However, a closed-form solution is always superior, and allows derivatives, and hence Jacobians, to be expressed symbolically as closed-form solutions.

The radius b can be computed directly from θ because of our choice of coordinates. By considering to proportionality of the similar triangles formed by the contacts points of sphere A and B and their centers with U_x in the XY plane, we obtain:

$$U_x = \frac{a}{\sin\theta} \tag{28}$$

$$b = \frac{(U_x - a)\sin\theta}{\sin\theta + 1} \tag{29}$$

Given the plane defined by a known radius a, θ and γ, the point C is constrained by asserting that the perpendicular distance to this plane of C is c. Additionally, in the central plane we have two other constraints, that C is placed so that it touches the circle centered on A and the circle centered on B.

Then we know that the point C but have distance $a + c$ from the point A (the origin) and distance $b + c$ from the known point B. These two constraints can be generated from the Pythagorean Theorem and can be expressed:

$$a + c = \sqrt{C_x^2 + C_z^2} \tag{30}$$

$$b + c = \sqrt{(a + b - C_x)^2 + C_z^2} \tag{31}$$

If we were to plot the point C as c increases, we would find a gentle curve moving away from the intersection of the circle A and B. An additional constraint will allow us to select a point on this curve.

This gives us three equations, which should be more than enough to solve for the point C.

Let $H = [0, H_y, 0]$.

Let N be the normal to the plane (pointing up in our diagram when $a >= b >= c$.)

$$\overrightarrow{N} = \overrightarrow{Z' - U} \times \overrightarrow{Z' - H} \tag{32}$$

Since we have the normal of the plane three points (U, Z, H_y) in the plane, computing the distance from the point C to the plane has a simple formula. We'll use the point $U = [Ux, 0, 0]$ as our main point. In our situation, because the sphere A is at the origin, the equation of the top plane is $\overrightarrow{N} \cdot \overrightarrow{U} = a$.

$$a = \frac{N \cdot U}{N_y} \tag{33}$$

$$c = \frac{|N \cdot P - a|}{|N|} \tag{34}$$

$$c = \frac{|C_x N_x + C_z N_z - a|}{1} \tag{35}$$

By substituting 35 into equations 30 and 31, we obtain:

$$a + |C_x N_x + C_z N_z - a| = \sqrt{C_x^2 + C_z^2} \tag{36}$$

$$b + |C_x N_x + C_z N_z - a| = \sqrt{(a + b - C_x)^2 + C_z^2} \tag{37}$$

If the interior expression $C_x N_x + C_z N_z - a$ is positive, then upon solving the system we find only imaginary solutions.

So assuming it is negative, we obtain:

$$2a + -C_x N_x + -C_z N_z = \sqrt{C_x^2 + C_z^2} \tag{38}$$

$$2b + -C_x N_x + -C_z N_z = \sqrt{(a + b - C_x)^2 + C_z^2} \tag{39}$$

This is a system of two nonlinear equations with unkowns C_x and C_z. Solving this system is beyond the capacities of the authors, but not Mathematica, although the answer that Mathematica produces in its raw form is half a page long. Furthermore, there are actually two solutions, corresponding to the two sides of the AB line which the circle C could be on and satsify these constratints. We have selected the solution that place C in the positive Z direction conformant to our problem set-up.

However, by introducing new variables to represent repeated portions of this long expression, the expressions be made tractable and easily programmed.

$$M = (a + b + aN_x - bN_x)^2 - 4abN_z^2 \tag{40}$$
$$L = a(a + b + aN_x - bN_x) - b(a + b)N_z^2 \tag{41}$$
$$G = -a^2(a - b)^2 bN_z^2 \tag{42}$$
$$H = 2a(-1 + N_x) - b((-1 + N_x)^2 + N_z^2) \tag{43}$$
$$F = \sqrt{GH} \tag{44}$$
$$K = 4a^2b^2N_z^2 + 2a^3b(-3 + N_x)N_z^2 \tag{45}$$
$$J = b^3(-1 + N_x)N_z^2 \tag{46}$$
$$C_{x0} = \frac{2(F + aL)}{M} \tag{47}$$
$$C_{z0} = \frac{K + (2bF(-1 + N_x) - 2a(F(1 + N_x) + b^3(-1 + N_x)N_z^2))}{(a - b)MN_z} \tag{48}$$
$$C_{x1} = \frac{-2F + 2aL}{M} \tag{49}$$
$$C_{z1} = \frac{K - (2bF(-1 + N_x) - 2a(F(1 + N_x) + b^3(-1 + N_x)N_z^2))}{(a - b)MN_z} \tag{50}$$

One solution ((C_{x0}, C_{z0}) or (C_{x1}, C_{z1})) will be on the positive side of the X axis, but which one is highly dependendent on the input variables, so both are computed and the positive z value selected.

The variable c can be computed from a, C_x and C_z form Eq. 30.

This math can be found in the JavaScript which is licensed under the GPL[GPL] at our website[3].

5 Special Cases of the Inverse Problem

When both $\theta = 0$ and $\gamma = 0$, then $c = b = a$.

5.1 Special case when γ is 0

In the above math when $\gamma = 0$, then N_z is zero, creating a division by zero. In this case we can use the proportionality of similar triangles to assert:

$$a = c + \frac{aC_x}{U_x} \tag{51}$$

Combined with our previous equations Eqn. 30 and Eqn. 31 in the plane relating C_x and C_z to c, we obtain the following equation:

$$c = \frac{a(a + b)(a - U_x)}{a^2 - ab + aU_x + bU_x} \tag{52}$$

5.2 Special case when θ is 0

When $\theta = 0$, the point U is at infinity or cannot be found. In thise case $b = a$. In this case the sphere C is symmetrically positioned in contact with A and B because they have the same radius; therefore $C_x = a$ and:

$$(a + c)^2 = a^2 + C_z^2 \tag{53}$$

In this case,

$$Z' = \frac{a}{\sin \gamma} \tag{54}$$

Projecting out the X dimension and forming similar triangles in the YZ plane, we find that

$$c = \frac{a(Z' - C_z)}{Z'} \tag{55}$$

Asking Mathematica to solve these two non-linear equations, we obtain, after again discarding the solution producing negative z values:

$$C_z = \frac{2a^2 Z' - \sqrt{a^4 (Z')^2 + 3a^2 (Z')^4}}{a^2 - (Z')^2}, \tag{56}$$

which allows us to obtain obtain c from 55.

6 Usage

These equations can be used to compute the radii b and c from a given radius a and either a desired normal or euler angles. When implemented as a soft robot, it must be understand that the center plane is conceptual. The roboticists probably want to think of moving the top plane relative to the bottom plane, which can be implemented as a simple rigid disc. The angles and normals between the top and bottom and will be twice the angles[2] measured against the center plane.

Furthermore, an implementation based on soft inflatable spheres will not be really acheive perfect spheres at any pressure. Nonetheless, because these equations are trivial to implement in a micro-controller, they make a good basis for a control algorithm because it allows the Jacobian to be expressed as a closed from solution. A Stewart platform implemented in this would be expected to be have "softly" in response to pressure, and yet have a predictable shape in the absence of external forces.

A series of such Stewart platforms stacked together would make a soft tentacle.

[2] Megan, is this really true? Can we simply double the angles? I think so, but that might not be true.

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