

Untwisting the Boerdijk-Coxeter Helix

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Abstract. The Boerdijk-Coxeter helix (BC helix, or tetrahelix) is a face-to-face stack of regular tetrahedra forming a helical column. Considering the edges of these tetrahedra as structural members, the resulting structure is attractive and inherently rigid, and therefore interesting to architects, mechanical engineers, and roboticists. A formula is developed that matches the visually apparent helices forming the outer rails of the BC helix. This formula is generalized to a formula convenient to designers. Formulae for computing the parameters that give edge-length minimax-optimal tetrahelices are given, defining a continuum of optimum tetrahelices of varying curvature. The endpoints of this continuum are the BC helix and a structure of zero curvature, the *equitetrabeam*. Numerically finding the rail angle from the equation for pitch allows optimal tetrahelices of any pitch to be designed. An interactive tool for such design and experimentation is provided: <https://pubinv.github.io/tetrahelix/>. A formula for the inradius of optimal tetrahelices is given. The continuum allows a regular Tetrobot in the BC configuration to untwist into a hexapodal or n -podal robot to use standard gaits.

Key words. Boerdijk-Coxeter helix, tetrahelix, robotics, tetrobot, unconventional robots, structural engineering, mechanical engineering, tensegrity, variable-geometry truss

18 AMS subject classifications. 51M15

1. Introduction. The Boerdijk-Coxeter helix[2] (BC helix) (see Figures 1 and 2), is a face-to-face stack of tetrahedra that winds about a straight axis. Because architects, structural engineers, and roboticists are inspired by and follow such regular mathematical models but can also build structures and machines of differing or even dynamically changing length, it is useful to develop the mathematics of structures formed from tetrahedra where we relax regularity.

The vertices of the tetrahedra lie upon three helices about the central axis. The Tetrobot[12, 7] uses the regularity of this geometry to make a tentacle-like robot that can crawl like a slug or mollusk. These modular robotic systems use mechanical actuators which can change their length, connected by special joints, such as the 3D printable Song-Kwon-Kim[16] joint or the CMS joint[6] used in the original Tetrobot, which allow many members to meet in a single point. Such machines can follow purely regular mathematical models such as the Boerdijk-Coxeter helix or the Octet Truss[3].

32 Buckminster Fuller called the BC helix a *tetrahelix*[4], a term now commonly used. In this
 33 paper we reserve *BC helix* to mean the purely regular structure and use *tetrahelix* to refer to
 34 any structure isomorphic to the BC helix.

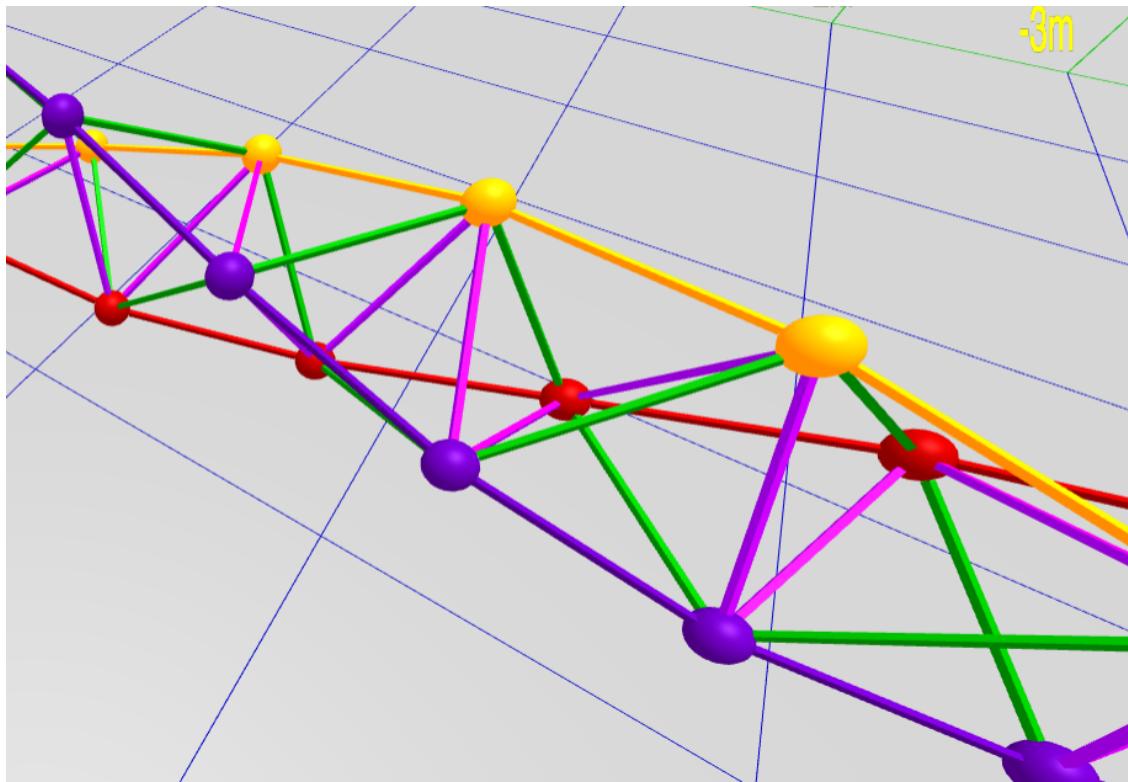


Figure 1. BC Helix Close-up (partly along axis)

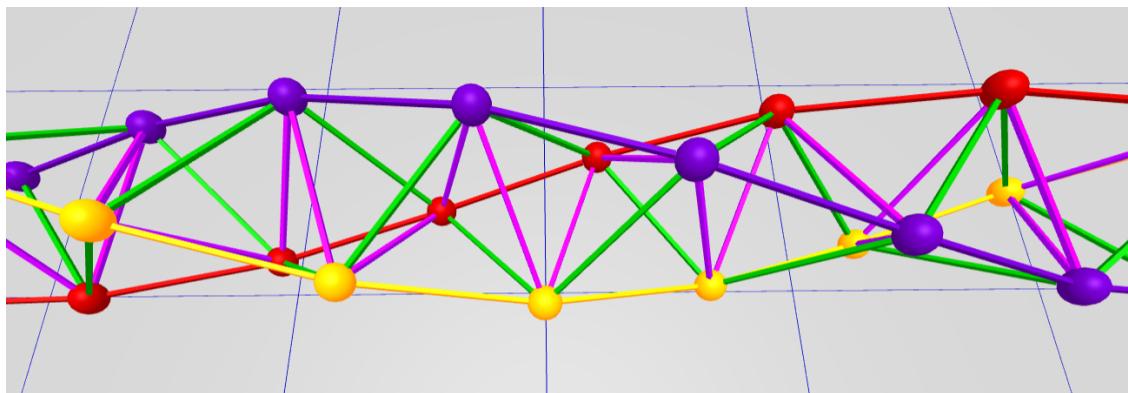


Figure 2. BC Helix Close-up (orthogonal)

35 Imagining Figures 1 and 2 as a static mechanical structure, we observe that it is useful
 36 to the mechanical engineer because the structure is a mechanically strong, inherently rigid,
 37 omni-triangulated space frame. Then we can imagine that each static edge is replaced with
 38 an actuator that can dynamically become shorter or longer in response to electronic control,
 39 and the vertices are joints that support sufficient angular displacement for this to be possible.

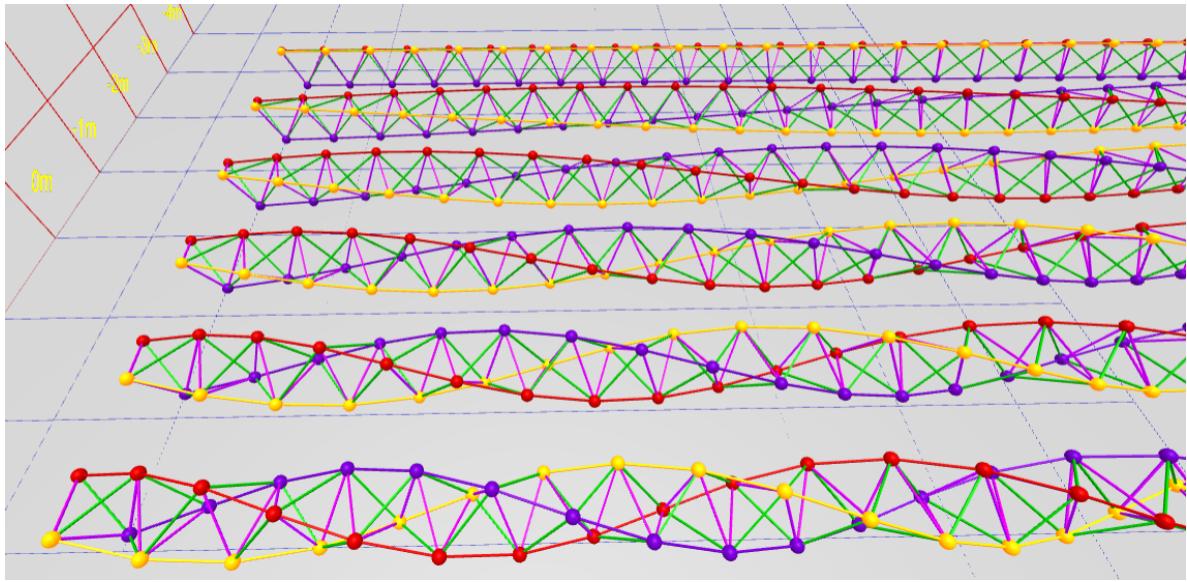


Figure 3. A Continuum of Tetrahelices

40 An example of such a machine is a Tetrobot, shown in Figure 12.

41 A BC helix does not rest stably on a plane. It is convenient to be able to “untwist” it and
 42 to form a tetrahelix space frame that has a flat planar surface. By making length changes in a
 43 certain way, we can untwist a tetrahelix to form a *tetrabeam* which has planar faces and has,
 44 for example, an equilateral triangular profile. This paper develops the equations needed to
 45 untwist the tetrahelix. All math developed here is available in JavaScript and demonstrated by
 46 an interactive design website <https://pubinv.github.io/tetrahelix/>[13], from which Figures 1
 47 to 3, 7 and 8 are taken.

48 Figure 3 displays a continuum of tetrahelices optimal in a certain sense, which is the main
 49 result of this paper. The closest helix is the BC helix, and the furthest is the equitetrabeam,
 50 defined in section 6 and Figures 7 and 8.

51 **2. A Designer’s Formulation of the BC Helix.** We would like to design nearly regular
 52 tetrahelices with a formula that gives the vertices in space. Eventually we would like to design
 53 them by choosing the lengths of a small set of members. In a space frame, this is a static
 54 design choice; in a tetrobot, it is a dynamic choice that can be used to twist the robot and/or
 55 exert linear or angular force on the environment.

56 Ideally we would have a simple formula for defining the vertices based on any curvature
 57 or pitch we choose. It is a goal of this paper to relate the Cartesian coordinate approach and
 58 the member-length approach to generating a tetrahelix continuum.

59 H.S.M Coxeter constructs the BC helix[2] as a repeated rotation and translation of the
 60 tetrahedra by showing the rotation is:

$$61 \quad \theta_{bc} = \arccos(-2/3)$$

62 and the translation:

$$63 \quad h_{bc} = 1/\sqrt{10}.$$

64 Note that θ_{bc} is approximately $0.37 \cdot 2\pi$ radians or 131.81 degrees. The angle θ_{bc} is the
65 rotation of *each* tetrahedron, not the tetrahedra along a rail. In [Figure 1](#), each tetrahedron
66 has either a yellow, blue, or red outer edge or rail. That is, a blue-rail tetrahedron is rotated
67 slightly more than a 1/3 of a revolution to match the face of the yellow tetrahedron.

68 R.W. Gray's website[\[5\]](#), repeating a formula by Coxeter[\[2\]](#) in a more accessible form, gives

69 the Cartesian coordinates $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ for a counter-clockwise BC Helix in a right-handed coordinate
70 system:

$$71 \quad (1) \quad \mathbf{V}(n) = \begin{bmatrix} r_{bc} \cos n\theta_{bc} \\ r_{bc} \sin n\theta_{bc} \\ nh_{bc} \end{bmatrix}, \text{ where: } \begin{aligned} r_{bc} &= \frac{3\sqrt{3}}{10} \approx 0.5196, \\ h_{bc} &= 1/\sqrt{10} \approx 0.3162, \text{ and} \\ \theta_{bc} &= \arccos(-2/3), \end{aligned}$$

72 and where n represents each integer numbered vertex in succession on the z -axis.

73 The apparent rotation of a vertex on an outer-edge, that is $\mathbf{V}(n)$ relative from $\mathbf{V}(n+3)$
74 for any integer n in (1), is $3\theta_{bc} - 2\pi$.

75 This formula defines a helix, but it is not any of the apparent helices, or *rail* helices, of the
76 BC helix, but rather one that winds much more rapidly through all vertices. To a designer of
77 tetrahelices, it is more natural to think of the three helices which are visually apparent, that
78 is, those three which are closely approximated by the outer edges or rails of the BC helix. We
79 think of each of these three rails as being a different color: red, blue, or yellow. This situation
80 is illustrated in [Figure 4](#), wherein the black helix represents that generated by (1), and the
81 colored helices are generated by (2).

82 In order to develop the continuum of slightly irregular tetrahelices described in [section 7](#),
83 we need a formula that gives us the vertices of just one rail helix, denoted by color c and
84 integer vertex number n :

$$85 \quad (\forall n \in \mathbb{Z}, \forall c \in \{0, 1, 2\} : \mathbf{H}_{BCcolored}(n, c) = \mathbf{V}(3n + c)).$$

86 Such a helix can be written:

$$87 \quad (2) \quad \mathbf{H}_{BCcolored}(n, c) = \begin{bmatrix} r_{bc} \cos ((3\theta_{bc} - 2\pi)n + c\theta_{bc}) \\ r_{bc} \sin ((3\theta_{bc} - 2\pi)n + c\theta_{bc}) \\ 3h_{bc}(n + c/3) \end{bmatrix}, \text{ where } \begin{aligned} r_{bc} &= \frac{3\sqrt{3}}{10}, \\ h_{bc} &= 1/\sqrt{10}, \text{ and} \\ \theta_{bc} &= \arccos(-2/3). \end{aligned}$$

88 In this formula, integral values of n may be taken as a vertex number for one rail and
89 used to compute its Cartesian coordinates. Allowing n to take non-integer values defines a
90 continuous helix in space which is close to the segmented polyline of the outer tetrahedra
91 edges, and equals them at integer values.

92 [Figure 4](#) illustrates this difference with a 7-tetrahedra BC helix, which is in fact the same
93 geometry as the robot illustrated in [Figure 12](#). Although the vertices coincide, (1) evaluated
94 at real values generates the black helix which runs through every vertex, and (2) defines the

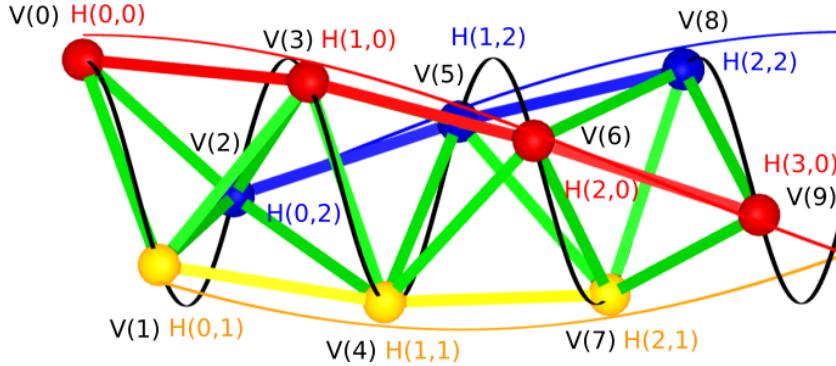


Figure 4. Rail helices (H) vs. Coxeter/Gray helix (V)

95 red, yellow, and blue helices. (In this figure these rail helices have been rendered at a slightly
96 higher radius than the vertices for clarity; in actuality the maximum distance between the
97 continuous, curved helix and the straight edges between vertices is much smaller than can be
98 clearly rendered.)

99 The quantity $(3\theta_{bc} - 2\pi) \approx 35.43^\circ$ is the angular shift between adjacent vertices on the
100 same rail: $\mathbf{V}(3n + c) = \mathbf{H}_{BCcolored}(n, c)$ and $\mathbf{V}(3(n + 1) + c) = \mathbf{H}_{BCcolored}(n + 1, c)$. This
101 quantity appears so often that we call it the “rail angle ρ ”. For the BC helix, $\rho_{bc} = (3\theta_{bc} - 2\pi)$.

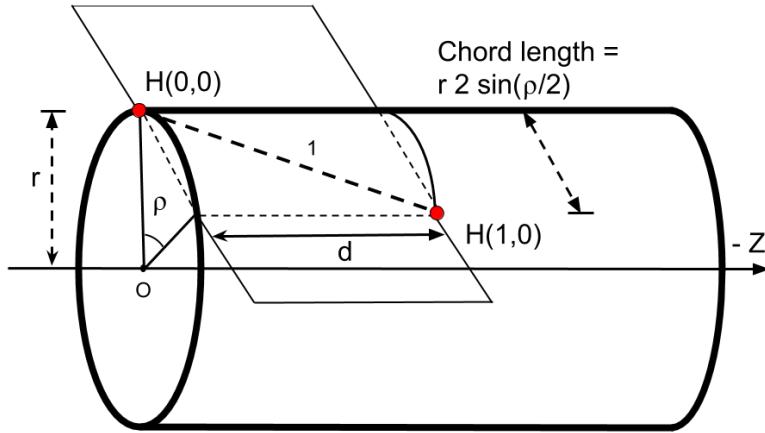


Figure 5. Rail Angle Geometry

102 Note in Figure 5 the z -axis travel for one rail edge is denoted by d . In (1) and (2),
103 the variable h is used for one third of the distance we name d . We will later justify that
104 $d = 3h$. In this paper we assume the length of a rail is always 1 as a simplification, except in

105 proofs concerning rail length. (We make the rail length a parameter in our JavaScript code
 106 in https://github.com/PubInv/tetrahelix/blob/master/js/tetrahelix_math.js [13].)

107 The $\mathbf{H}_{BCcolored}(n, c)$ formulation can be further clarified by rewriting directly in terms of
 108 the rail angle ρ_{bc} rather than θ_{bc} . Intuitively we seek an expression where $c/3$ is multiplied by
 109 a $1/3$ rotation plus the rail angle ρ . We expand the expressions θ_{bc} and ρ_{bc} in (2) and seek to
 110 isolate the term $c2\pi/3$.

111 Thus, starting with the expression:

$$113 \quad c\theta_{bc}$$

114 we introduce 3 into the denominator...

$$115 \quad (c/3)(3\theta_{bc})$$

117 we want 2π in numerator, so add canceling terms...

$$119 \quad (c/3)((3\theta_{bc} - 2\pi) + 2\pi)$$

120 ...and then use the definition of ρ_{bc}

$$121 \quad (c/3)\rho_{bc} + c2\pi/3$$

123 finally we obtain:

$$124 \quad c(\rho_{bc} + 2\pi)/3.$$

126 This allows us to redefine:

$$127 \quad (3) \quad \mathbf{H}_{BCcolored}(n, c) = \begin{bmatrix} r \cos \rho_{bc} n + c(\rho_{bc} + 2\pi)/3 \\ r \sin \rho_{bc} n + c(\rho_{bc} + 2\pi)/3 \\ (n + c/3)h_{bc} \end{bmatrix}, \text{ where } \begin{aligned} \rho_{bc} &= (3\theta_{bc} - 2\pi), \text{ and} \\ h_{bc} &= 1/\sqrt{10}. \end{aligned}$$

128 Recall that $c \in \{0, 1, 2\}$, but n is continuous (rational or real-valued.) We can now assert
 129 that in Figure 4 the black helix winds at $\frac{3\theta_{bc}}{\rho_{bc}} \approx 11.16$ times the rate of a rail helix.

130 From this formulation it is easy to see that moving one vertex on a rail ($\mathbf{H}_{BCcolored}(n, c)$
 131 to $\mathbf{H}_{BCcolored}(n + 1, c)$ for any n and c) moves us ρ_{bc} radians around a circle. Since:

$$132 \quad \frac{2\pi}{\rho_{bc}} \approx 10.16$$

133 we can see that there are approximately 10.16 red, blue or yellow tetrahedra on one rail in a
 134 complete revolution of the tetrahelix.

135 The *pitch* of any tetrahelix, defined as the axial length of a complete revolution where
 136 $\rho \neq 0$ is:

$$137 \quad (4) \quad p(\rho) = \frac{2\pi d}{\rho}.$$

138 The pitch of the Boerdijk-Coxeter helix of edge length 1 is the length of three tetrahedra
139 times $p(\rho_{bc})$:

140
$$\frac{3h_{bc}2\pi}{\rho_{bc}} = \frac{6\pi}{\sqrt{10}\rho_{bc}} \approx 9.64.$$

142 The pitch is less than the number of tetrahedra because the tetrahedra edges are not
143 parallel to the axis of the tetrahelix. It is a famous and interesting result that the pitch is
144 irrational. A BC helix never has two tetrahedra at precisely the same orientation around the
145 z -axis. However, this is inconvenient to designers, who might prefer a rational pitch. The
146 idea of developing a rational period by arranging solid tetrahedra by relaxing the face-to-
147 face matching has been explored[14]. We develop below slightly irregular edge lengths that
148 support, for example, a pitch of precisely 12 tetrahedra in one revolution which would allow
149 an architect to design a pleasing column having the top and bottom tetrahedra in the same
150 relationship to the capital and the basis to the viewer.

151 **3. Optimal Tetrahelices are Triple Helices.** We use the term *tetrahelix* to mean any
152 structure physically constructible of vertices and finite edges which is isomorphic to the BC
153 helix and in which the vertices lie on three helices. By isomorphic we mean there is a one-
154 to-one mapping between both vertices and edges in the two tetrahelices. One could consider
155 various definitions of optimality for a tetrahelix, but the most useful to us as roboticists
156 working with the Tetrobot concept is to minimize the maximum ratio between any two edge
157 lengths, because the Tetrobot uses mechanical linear actuators with limited range of extension.

158 A *triple helix* is three congruent helices that share an axis. We show that optimal tetra-
159 helices are in fact triple helices with the same radius, so that all vertices are on a cylinder. In
160 stages, we demonstrate that optimal tetrahelices:

- 161 1. have the same pitch,
- 162 2. have parallel axes,
- 163 3. share the same axis,
- 164 4. have the same radius,
- 165 5. have the same rail lengths,
- 166 6. have axially equidistant vertices, and therefore
- 167 7. are in fact triple helices.

168 Suppose that all three rails do not have the same pitch. If we start at any shortest
169 edge between two rails, as we move from vertex to vertex away from our start edge the edge
170 lengths between rails must always lengthen without bound, which cannot be optimal. So we
171 are justified in talking about the *pitch* of the optimal tetrahelix as the pitch of its three rail
172 helices, even though there are three such helices of equivalent pitch.

173 Similarly, if the axes are not parallel, there is an edge of unbounded length in the structure,
174 so we do not consider such cases.

175 Define a *minimax edge-length optimal tetrahelix* or just an *optimal tetrahelix* to be a
176 tetrahelix for which there exists no other tetrahelix with lower ratio of longest edge length to
177 shortest edge length.

178 We wish to show that in an optimal tetrahelix, all vertices lie on the cylinder of radius r ,
179 regardless of where they lie on the z -axis.

180 As a little lemma for the proof below, observe that a tetrahelix of zero radius, where all
 181 points lie on the same line, is not as optimal as a tetrahelix of a small radius. The edges
 182 between rails will be shorter than the rail edges, and moving them apart slightly lengthens
 183 the between-edge rails, improving the ratios.

184 In the proof below we find it useful to consider projection diagrams that are the axial
 185 projection of a tetrahelix onto the XY -plane. [Figure 10](#) is an example of such a diagram.

186 **Lemma 1.** *If the rail angle $0 < \rho < \pi$ is a rational multiple of π , then the projection of
 187 edges in a helix of that rail angle along the z -axis onto the XY -plane form a regular polygon
 188 of 3 or more sides, or else they fill in a complete circle.*

189 *Proof.* All points lying on a helix projected along the axis lie on a circle in the XY -plane.
 190 Helices are periodic in the z dimension modulo 2π . If $2\pi/\rho$ is irrational, the projection onto
 191 the XY -plane will contain an unbounded number of points on a circle. If and only if $2\pi/\rho$
 192 is rational, the projection onto the XY -plane will contain a finite number of points. Because
 193 π is transcendental and irrational, $2\pi/\rho$ is rational if and only $\rho = a\pi/b$, where a and b are
 194 integers and without loss of generality a and b are coprime. Since $\rho < \pi$, therefore $a < b$.
 195 Also, $\rho > 0$, therefore $a > 0$. The number of points in the projection is $2b$ if a is odd, and b if
 196 a is even. This polygon has at least 3 sides, since either ρ is irrational or $b > a$, and therefore
 197 $b \geq 2$. If $a/b = 1/2$, the projection is a square, which has four sides. ■

198 **Theorem 2.** *Any optimal tetrahelix with a rail angle of magnitude less than π has all three
 199 axes coincident.*

200 *Proof.* Case 1: Suppose that ρ is zero. Each helix has zero curvature, that is, it is a
 201 straight line. These lines are equivalent to some three degenerate helices with coincident axes,
 202 possibly with different radii, so long as there is a phase term in the definition of the helix, as
 203 in [\(2\)](#). We later show the radii must be equivalent.

204 Case 2: Suppose that ρ is positive but less than π . In this case each rail helix has curvature.
 205 The projection of points in the XY plane creates a figure guaranteed to have points on either
 206 side of any line through the axis of such a helix, because the figure is either an n -gon or a
 207 circle by [Lemma 1](#). We show that the three helices share a common axis.

208 Without loss of generality define the Red helix to have its axis on the z -axis. Since there
 209 must be at least one Red-to-Yellow or a Red-to-Blue edge that is either a minimum or a
 210 maximum, without loss of generality define the Blue helix to be a helix that has an edge
 211 connection to the Red helix that is either a maximum or a minimum. Let B' be a translation
 212 in the XY -plane of the blue helix B so that its axis is the z -axis and coincident with the red
 213 helix R . Let D be the distance between the axis of the Blue helix B and B' . We will show
 214 that if $D > 0$ then B “wobbles” in a way that cannot be optimal. Define a wobble vector by:

$$215 \quad \mathbf{W}(n) = \mathbf{B}(n) - \mathbf{B}'(n) .$$

216 where $\mathbf{B}(n)$ and $\mathbf{B}'(n)$ is the vector $\begin{bmatrix} x \\ y \end{bmatrix}$ for the projection of the n th vertex of B and B' .
 217 Note that $\|\mathbf{R}(n) - \mathbf{B}'(n+k)\|$ (the Euclidean distance of the vertices) is a constant for any k ,
 218 because R and B' have the same pitch and the same axis, even if they do not have the same
 219 radius.

220 Figure 6 illustrates this situation. Like most diagrams, it is over specific, in that the two
 221 circles are drawn of the same radius but we do not depend upon that in this proof. The
 222 diagram represents the projection along the z axis of a few points into the XY -plane.

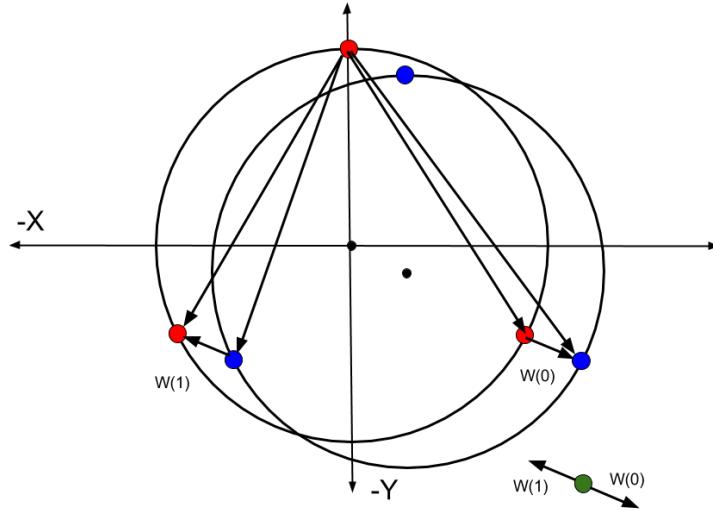


Figure 6. Wobble Vectors from Non-Coincident Axes

223 Since $\rho < \pi$ by assumption, by [Lemma 1](#), the set of wobbles $\{\mathbf{W}(n)\}$ for any n contains
 224 at least three vectors, at least two of which pointing in different directions. For any point not
 225 at the origin, at least one of these vectors moves closer to the point and at least one moves
 226 further away.

227 The set of all lengths in the tetrahelix is a superset of: $L = \{||\mathbf{R}(n) - \mathbf{B}(n)||\}$, which by
 228 our choice has at least one longest or shortest length. $L = \{||\mathbf{R}(n) - (\mathbf{B}'(n) + \mathbf{W}(n))||\}$ and
 229 so $L = \{||(\mathbf{R}(n) - \mathbf{B}'(n)) - \mathbf{W}(n)||\}$. But $\mathbf{R}(n) - \mathbf{B}'(n)$ is a constant, so the minimax value
 230 of L is improved as $||\mathbf{W}(n)||$ decreases. By our choice that there is a Blue-to-Red edge that is
 231 either a maximum or a minimum, this improves the minimax value of the total tetrahelix.

232 This process can be carried out on both the Blue and Yellow helices (perhaps simultane-
 233 ously) until $\mathbf{W}(n)$ is zero for both, finding a tetrahelix of improved overall minimax value at
 234 each step. So a tetrahelix is optimal only when $\mathbf{W}(n) = 0$, and therefore when $D = 0$ and
 235 $\mathbf{B}(n) = \mathbf{B}'(n)$, and all three axes are coincident. ■

236 Now that we have shown that axes are coincident and parallel and that the pitches are
 237 the same for all helices, we can assert that any optimum tetrahelix can be generated with an
 238 equation for helices:

239 (5)
$$\mathbf{V}_{\text{triple}}(n, c) = \begin{bmatrix} r_c \cos(n\alpha + c2\pi/3 + \phi_c) \\ r_c \sin(n\alpha + c2\pi/3 + \phi_c) \\ \frac{d(n+c/3)}{3} \end{bmatrix}, \text{ where: } c \in \{0, 1, 2\}$$

240 which would be much more complicated if the axes where not coincident. Note that we have
 241 not yet shown that the relationships of the radius r_c or the phase ϕ_c for the three helices, so we
 242 denoted them with a c subscript to show they are dependent on the color. We have not yet
 243 investigated in the general case the relationships between α , r , ϕ and d in (5). In section 4 we
 244 give a more specific version of this formula which generates optimal tetrahelices. We observe
 245 that when $\alpha = 0$, the helices are degenerate, having curvature of 0, but because of the ϕ_c
 246 term, they are not collinear.

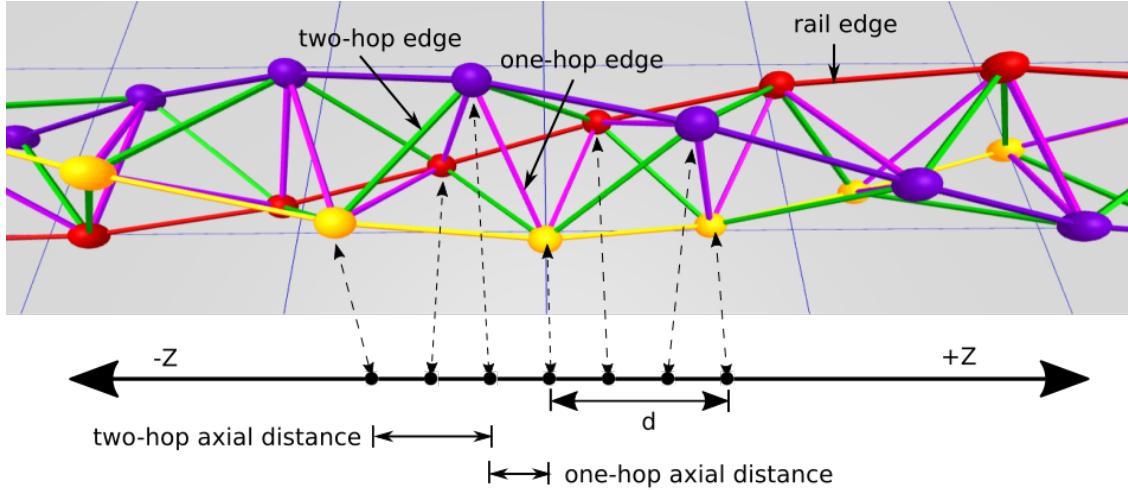


Figure 7. Edge Naming

247 In principle any three helices generated with (5) has at most nine distinct edge length
 248 classes. Each edge that connects two rails potentially has a longer length and shorter length
 249 we denote with a + or -. So the classes are $\{RR, BB, YY, RB_+, RB_-, BY_+, BY_-, RY_+, RY_-\}$.
 250 If when projecting all vertices onto the z -axis (dropping the x and y coordinates), the interval
 251 defined by the z axis value of its endpoints contains no other vertices, we call it a *one-hop*
 252 edge, and if it does contain another vertex we call it a *two-hop* edge, as illustrated in Figure 7.
 253 Then there are 3 rail edges $\{RR, BB, YY\}$, 3 one-hop lengths $\{RB_-, BY_-, RY_-\}$ between
 254 each pair of 3 rails, and 3 two-hop lengths $\{RB_+, BY_+, RY_+\}$ between each pair of 3 rails,
 255 where the two-hop length is at least the one-hop length. However, if we generate the three
 256 helices symmetrically with (5), many of these lengths will be the same. In fact, it is possible
 257 that there will be only two distinct such classes. In the purely regular BC helix there is only
 258 one length.

259 **Theorem 3.** *Optimal tetrahelices have the same radii for all three helices.*

260 **Proof.** To prove this we exhibit a symmetric tetrahelix (not yet shown to be optimal)
 261 which happens to be a triple helix, that has the property that all rail edges are equal to
 262 all one-hop edges and all two-hop edges are equal to each other. Observe that although we
 263 have not yet given the formula for the radii of such a triple helix, there are some values for
 264 r and α , and ϕ in (5) for which all the three helices are symmetrically and evenly spaced.

265 Furthermore, we can choose these values such that the three rail edges are of length 1 and so
266 that the one-hop lengths are also all of length 1, and the two-hop lengths are slightly longer.

267 Now consider a tetrahelix in which the radius of one of the helices is different. By the
268 connections made in a tetrahelix, any increase to a radius increases both a one-hop and two-
269 hop distance, and any decrease likewise decreases two. Since there exists a tetrahelix which
270 has only two distinct classes of edge lengths, (the smaller being one-hop = rail, the larger
271 being the two-hop distance), the helix with a larger radius increases a longest edge without
272 increasing a shortest edges. Likewise, a helix with a smaller radius decreases a one-hop edge
273 without decreasing a two-hop edge. Therefore, a tetrahelix with different radii is not as
274 optimal as some two-class tetrahelix generated by (5), and so it not optimal. We have not
275 yet proved that a two-class tetrahelix is optimal, but it suffices to show that there exist such
276 a better tetrahelix to show that different radii imply a suboptimal tetrahelix. ■

277 Because an optimal tetrhelix has equivalent radii and equivalent pitch for all three helices,
278 it has equivalent rail edge lengths. Likewise, there is a single rail angle ρ that represents the
279 rotation of two vertices connected by a single rail edge, and it is the same for all three rails.

280 Now that we have shown that on any optimal tetrahelix the vertices are on helices of the
281 same axes and pitch, we see that the vertices of any optimal tetrahelix will lie on a cylinder,
282 or a circle when the axis dimension is projected out. Therefore it is reasonable to now speak
283 of the singular *radius r* of a tetrahelix as the radius of the cylinder. We can now go on to the
284 harder proof about where vertices occur along the z -axis.

285 We show that in fact the vertices must be distributed in even thirds along the z -axis, as
286 in fact they are in the regular BC helix.

287 However, we have already shown the rail lengths are equal in any optimal tetrahelix.

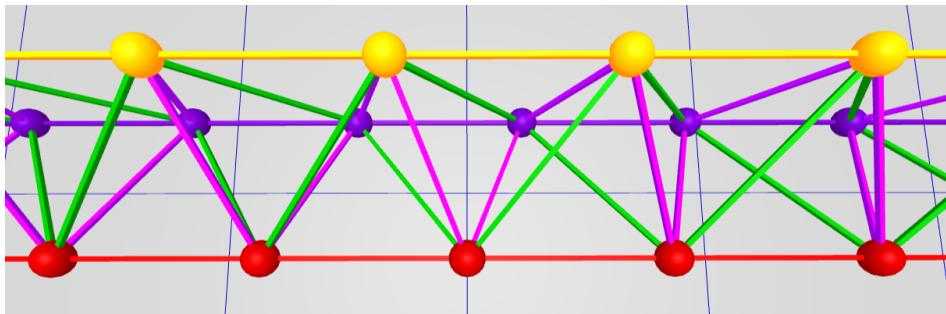


Figure 8. Equitetrabeam

288 Figure 8 shows the equitetrabeam, which is defined in section 6, but also conveniently
289 illustrates the one-hop and two-hop edge definitions. The green edges are the two-hop edges
290 and the purple edges are the one-hop edges. Note that the green edges are slightly longer than
291 the purple edges. In Figure 7, which depicts the BC helix, the two-hop and one-hop edges are
292 of equal length (but the projection onto the z -axis, the axial length, of the two-hop edge is
293 longer than the axial one-hop length.)

294 **Theorem 4.** *An optimal tetrahelix of any rail angle $0 \leq \rho < \pi$ is a triple helix with all
295 vertices evenly spaced at $d/3$ intervals on the z axis. Any one tetrahedron in a tetrahelix has*

296 1 rail edge, 2 one-hop edges connected to the rail and 2 two-hop edges connected to the rail.
 297 The sixth edge is opposite of the rail edge and is a one-hop edge.

298 *Proof.* Consider a tetrahelix in which the vertices are evenly spaced at $d/3$ intervals on
 299 the z axis. Every edge is either a rail edge, or it makes one hop, or two hops. All of the
 300 one-hop edges are equal length. All of the two-hop edges are equal length.

301 Every vertex is connected to 4 non-rail edges. There is a one-hop edge in both the positive
 302 and negative z direction. Likewise there is a two-hop edge in both the positive and negative
 303 z direction. Let A be the set of edge lengths, which has only 3 members, represented by
 304 $A = \{o, t, r\}$ for the one-hop, two-hop, and rail edge lengths.

305 Any attempt to perturb any rail in either z direction lengthens one two-hop edge to t' ,
 306 where $t' > t$ and shortens one one-hop edge $o' < o$. Let $B = \{o', t'\} \cup A$ be the edge lengths of
 307 such a perturbed tetrahelix. The minimax of B is greater than the minimax of A since there
 308 is a single rail length which cannot be both greater than t' and o' and less than t' and o' .
 309 Therefore, any optimal tetrahelix has all one-hop edges between all rails equal to each other,
 310 and all two-hop edges equal to each other, the z distances between rails equal. Therefore
 311 vertices are $d/3$ from each other on the z -axis. ■

312 Note that based on [Theorem 4](#), there are only 3 possible lengths in an optimal tethrahelix,
 313 and we are justified in classifying edge lengths as *rail*, *one-hop*, or *two-hop*. The one-hop edges
 314 are the edges between rails that are closest on the z -axis, and the two-hop edges are those
 315 that skip over a vertex.

316 Taking all of these results together, each helix in an optimal tetrahelix is congruent to the
 317 others, shares an axis, is the same radius, and are evenly spaced axially. An optimal tetrahelix
 318 is therefore a *triple helix*, of a radius we have not yet demonstrated.

319 **4. Parameterizing Tetrahelices via Rail Angle.** We seek a formula to generate optimal
 320 tetrahelices that accepts a parameter that allows us to design the tetrahelix conveniently.
 321 Please refer back to [Figure 5](#). The pitch of the helix is an obvious choice, but is not defined
 322 when the curvature is 0, an important special case. The radius or the axial distance between
 323 two vertices on the same rail are possible choices, but perhaps the clearest choice is to build
 324 formulae that takes as their input the “rail angle” ρ . We define ρ to be the angle formed in
 325 the X,Y plane with origin O : $\angle \mathbf{H}(0,0)O\mathbf{H}(0,1)$ projecting out the z axis and sighting along
 326 the positive z axis. In other words, ρ controls how far a rail edge of a tetrahelix deviates from
 327 being parallel with the axis, or the “twistiness” of the tetrahelix. We use the parameter $\chi = 1$
 328 to indicate a chirality of counter-clockwise, and $\chi = -1$ for clockwise. We take our coordinate
 329 system to be right-handed.

330 The quantities ρ, r, d (see [Figure 5](#)) are related by the expression:

$$331 \quad 1^2 = d^2 + (2r \sin \rho/2)^2, \text{ or} \\ 332 \quad (6) \quad d^2 = 1 - 4r^2(\sin \rho/2)^2.$$

333

335 Checking the important special case of the BC helix, we find that this equation indeed
 336 holds true, treating d in this equation as $3h_{bc}$ as defined by Gray and Coxeter, that is, $d_{bc} =$

337 $3h_{bc}$, where they are using h for the axial height from one vertex to the next of a different
 338 color, but we use d to mean distance between vertices of the same color.

339 The rail angle ρ also has the meaning that $2\pi/\rho$ is the number of tetrahedra in a full
 340 revolution of the helix.

341 In choosing ρ , one greatly constrains r and d , but does not completely determine both of
 342 them together, so we treat both as additional parameters.

343 Rewriting our formulation in terms of ρ :

$$344 \quad (7) \quad \mathbf{H}_{general}(\chi, n, c, \rho, d_\rho, r_\rho) = \begin{bmatrix} r_\rho \cos(\chi \cdot (n\rho + c(\rho + 2\pi)/3)) \\ r_\rho \sin(\chi \cdot (n\rho + c(\rho + 2\pi)/3)) \\ d_\rho(n + c/3) \end{bmatrix}$$

345 where: $1 = d_\rho^2 + 4r_\rho^2(\sin \rho/2)^2$, and
 346 $\chi \in \{-1, 1\}$.

347 $\mathbf{H}_{general}$ forces the user to select three values: ρ , r_ρ , and d_ρ satisfying (6).
 348 Note that when $\rho = 0$ then $d_\rho = 1$, but r_ρ is not determined by (6).

349 **Theorem 5.** For rail angles of magnitude at most ρ_{bc} , tetrahelices generated by $\mathbf{H}_{general}$
 350 are optimal in terms of minimum maximum ratio of member length when radius is chosen so
 351 that the length of the one-hop edge is equal to the rail length.

352 *Proof.* By Theorem 4, we can compute the (at most) three edge-lengths of an optimal
 353 tetrahelix by formula universally quantified by n and c :

$$354 \quad \text{rail} = \|\mathbf{H}_{general}(n, c, \rho, d_\rho, r) - \mathbf{H}_{general}(n+1, c, \rho, d_\rho, r)\| = 1 ,$$

$$355 \quad \text{one-hop} = \|\mathbf{H}_{general}(n, c, \rho, d_\rho, r) - \mathbf{H}_{general}(n, c+1, \rho, d_\rho, r)\| \text{ and,}$$

$$356 \quad \text{two-hop} = \|\mathbf{H}_{general}(n, c, \rho, d_\rho, r) - \mathbf{H}_{general}(n, c+2, \rho, d_\rho, r)\| .$$

357

359 This syntax just represents the Euclidean distance between vertices. Thus:

$$360 \quad \text{one-hop} = \sqrt{\frac{d_\rho^2}{9} + r^2(\sin^2(\rho/3 + \frac{2\pi}{3}) + (1 - \cos(\rho/3 + \frac{2\pi}{3}))^2)}$$

362

363 but: $d_\rho^2 = 1 - 4r^2(\sin(\rho/2)^2)$, so we substitute:

$$364 \quad \text{one-hop} = \sqrt{\frac{1}{9} + r^2(-\frac{4(\sin^2(\rho/2))}{9} + \sin^2(\rho/3 + \frac{2\pi}{3}) + (1 - \cos(\rho/3 + \frac{2\pi}{3}))^2)}.$$

365

367 By similar algebra and trigonometry:

$$368 \quad \text{two-hop} = \sqrt{\frac{4}{9} + r^2(-\frac{16(\sin^2(\rho/2))}{9} + \sin^2(2\rho/3 + \frac{4\pi}{3}) + (1 - \cos(2\rho/3 + \frac{4\pi}{3}))^2)}.$$

369

371 By definition of minimax edge length optimality, we are trying to minimize:

372

$$\frac{\max \{1, \text{one-hop}(r), \text{two-hop}(r)\}}{\min \{1, \text{one-hop}(r), \text{two-hop}(r)\}}.$$

373 But since $\text{two-hop}(r) \geq \text{one-hop}(r)$, this is equivalent to:

374

$$\frac{\max\{1, \text{two-hop}(r)\}}{\min\{1, \text{one-hop}(r)\}}.$$

375 This quantity will be equal to one of:

376 (8)

$$\frac{\text{two-hop}(r)}{1}, \frac{1}{\text{one-hop}(r)}, \frac{\text{two-hop}(r)}{\text{one-hop}(r)}.$$

377 We know that both $\text{one-hop}(r)$ and $\text{two-hop}(r)$ increase monotonically and continuously
378 with increasing r . By inspection it seems likely that we will minimize this set by equating
379 $\text{one-hop}(r)$ or $\text{two-hop}(r)$ to 1, but to be absolutely sure and to decide which one, we must
380 examine the partial derivative of the ratio $\frac{\text{two-hop}(r)}{\text{one-hop}(r)}$ in this range.

381 Although complicated, we can use Mathematica to investigate the partial derivative of
382 $\frac{\text{two-hop}(r)}{\text{one-hop}(r)}$ with respect to the radius to be able to understand how to choose the radius to form
383 the minimax optimum.

384 Let:

385

$$f_\rho = -\frac{4(\sin^2(\rho/2))}{9},$$

386

387

$$g_\rho = -\frac{16(\sin^2(\rho/2))}{9},$$

388

389

$$j_\rho = \sin^2(\rho/3 + \frac{2\pi}{3}) + (1 - \cos(\rho/3 + \frac{2\pi}{3}))^2,$$

390 and:

391

$$k_\rho = \sin^2(2\rho/3 + \frac{4\pi}{3}) + (1 - \cos(2\rho/3 + \frac{4\pi}{3}))^2.$$

392 Then:

393

$$\frac{\text{two-hop}(r)}{\text{one-hop}(r)} = \frac{\sqrt{\frac{4}{9} + r^2(g_\rho + j_\rho)}}{\sqrt{\frac{1}{9} + r^2(f_\rho + k_\rho)}}.$$

394

395 By graph inspection using Mathematica (<https://github.com/PubInv/tetrahelix/blob/master/tetrahelix.nb>), we see the partial derivative of this with respect to radius r is al-
396 ways negative, for any $\rho \leq \rho_{bc}$. (When the rail angle approaches π , corresponding to going
397 almost to the other side of the tetrahelix, this is not necessarily true, hence the limitation in
398

399 our statement of the theorem is meaningful.) Since the partial derivative of $\frac{\text{two-hop}(r)}{\text{one-hop}(r)}$ with
 400 respect to the radius r is negative for all ρ up until ρ_{bc} , this ratio goes down as the radius goes
 401 up, and we minimize the maximum edge-length ratio by choosing the largest radius up until
 402 one-hop = 1, the rail-edge length. If we attempted to increase the radius further we would
 403 not be optimal, because the ratio $\frac{\text{two-hop}(r)}{1}$ would become the largest ratio in our set of ratios
 404 (8).

405 Therefore we decrease the minimax length of the whole system as we increase the radius
 406 up to the point that the shorter, one-hop distance is equal to the rail-length, 1. In order to
 407 optimize the whole system so long as $\rho \leq \rho_{bc}$, we equate one-hop to 1 to find the optimum
 408 radius:

$$409 \quad 1 = \sqrt{\frac{1}{9} + r_{opt}^2 \left(-\frac{4(\sin^2(\rho/2))}{9} + \sin^2(\rho/3 + \frac{2\pi}{3}) + (1 - \cos(\rho/3 + \frac{2\pi}{3}))^2 \right)}, \text{ so...}$$

$$410 \quad (9) \quad r_{opt} = \frac{2}{\sqrt{\frac{9}{2} \cdot (\sqrt{3}\sin(\rho/3) + \cos(\rho/3)) + \cos(\rho) + 8}}.$$

411

413 We can now give a formula for d_{opt} computed from ρ, r_{opt} via the rail angle equation (6):

$$414 \quad d_{opt}^2 = 1 - 4 \left(\frac{2}{\sqrt{\frac{9}{2} \cdot (\sqrt{3}\sin(\rho/3) + \cos(\rho/3)) + \cos(\rho) + 8}} \right)^2 (\sin \rho/2)^2$$

$$415 \quad d_{opt}^2 = 1 - \frac{16(\sin \rho/2)^2}{9(\sqrt{3}\sin(\rho/3) + \cos(\rho/3)) + \cos(\rho) + 8}$$

$$416 \quad (10) \quad d_{opt} = \sqrt{1 - \frac{16 \sin^2(\rho/2)}{\cos(\rho) + 9(\sqrt{3}\sin(\rho/3) + \cos(\rho/3)) + 8}}.$$

417

419 Thus, by computing r_{opt} and d_{opt} as a function of ρ from this equation, we can construct
 420 minimax optimal tetrahelix for an $0 \leq \rho \leq \rho_{bc}$. ■

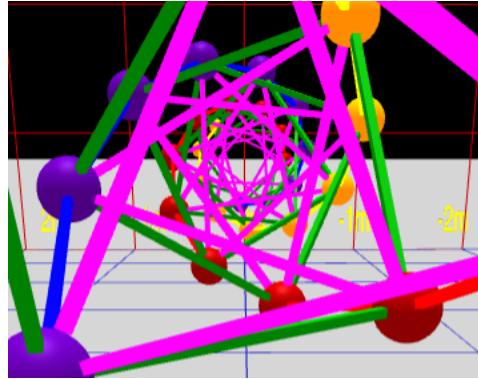


Figure 9. Axial view of a BC-Helix

5. The Inradius. Since the axes are parallel, we may define the *inradius*, represented by the letter i , of a tetrahelix to be the radius of the largest cylinder parallel to this axis that is surrounded by each tetrahelix and penetrated by no edge.

If we look down the axis of an optimal tetrahelix as shown in Figure 9, it happens that only the one-hop edges (rendered in purple in our software) comes closest to the axis. In other words, they define the radius of the incircle of the projection, or the radius of a cylinder that would just fit inside the tetrahelix. A formula for the inradius of the tetrahelix is useful if you are designing it as a structure that bears something internally, such as a firehose, a pipe, or a ladder for a human. The inradius $r_{in}(\rho)$ of an optimal tetrahelix is a remarkably simple function of the radius r and the rail angle ρ :

$$431 \quad (11) \qquad r_{in}(\rho) = r \sin \frac{\pi - \rho}{6} ,$$

which can be seen from the trigonometry of a diagram of the projected one-hop edges connecting four sequentially numbered vertices:

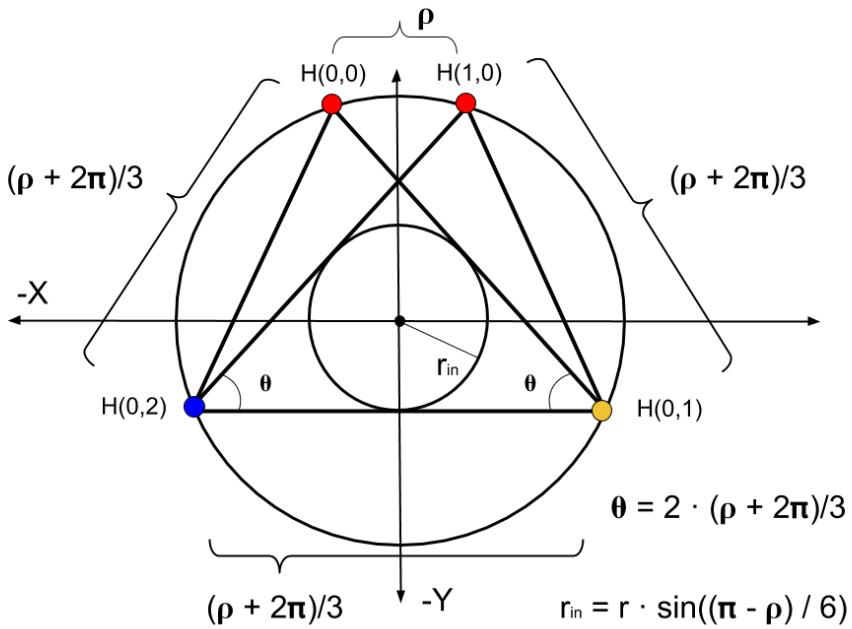


Figure 10. General One-hop Projection Diagram

From this equation with the help of symbolic computation we observe that inradius of the BC helix of unit rail length is $r_{in(\rho_{bc})} = \frac{3}{10\sqrt{2}} \approx 0.21$.

6. The Equitetrabeam. Just as $\mathbf{H}_{general}$ constructs the BC helix (with careful and non-obvious choices of parameters) which is an important special case due to its regularity, it constructs an additional special (degenerate) case when the rail angle $\rho = 0$ and $d = 1$ (the edge-length), where the cross sectional area is an equilateral triangle of unchanging orientation, as shown in Figure 8 and at the rear of Figure 3. We call this the *equitetrabeam*. It is not possible to generate an equitetrabeam from (1) without the split into three rails introduced by (2) and completed in (7).

Corollary 6. *The equitetrabeam with minimal maximal edge ratio is produced by $\mathbf{H}_{general}$ when $r = \sqrt{\frac{8}{27}}$ and the longest member is $\frac{2}{\sqrt{3}}$, which is less than 16% longer than the shortest member.*

Proof. Choosing $d = 1$ and $\rho = 0$ we use (9) to find the radius of optimal minimax difference. Substituting into (7) and setting one-hop¹ to 1 gives: $1 = \sqrt{\frac{1}{9} + 3r^2}$, or $r = \sqrt{\frac{8}{27}}$. This radius produces a two-hop edge length of $\frac{2}{\sqrt{3}} \approx 1.1547$. ■

¹Before developing the optimal solution (9) we found another interesting but non-optimal tetrabeam of zero curvature by setting the mean of one-hop and two-hop to 1 ((one-hop + two-hop)/2 = 1), which gives $r = \sqrt{35}/4$ and three length classes of $\frac{11}{12}, \frac{12}{12}, \frac{13}{12}$.

449 The inradius of the equitetrabeam of unit rail length from both (11) and the fact that the
 450 inradius of an equilateral triangle is half the circumradius is $\sqrt{\frac{8}{27}}/2$, or $\frac{\sqrt{6}}{9}$.

451 In Figure 3, the furthest tetrahelix is the optimal equitetrabeam. Figure 8 is a closeup of
 452 an equitetrabeam.

453 To the extent that we value tetrabeams (that is, tetrahelices with a rail angle of 0, and
 454 therefore zero curvature) as mathematical or engineering objects, we have motivated the
 455 development of $\mathbf{H}_{\text{general}}$ as a transformation of $\mathbf{V}(n)$ defined by (1) from Gray and Coxeter.
 456 It is difficult to see how the $\mathbf{V}(n)$ formulation could ever give rise to a continuum producing
 457 the tetrabeam, since setting the angle in that equation to zero can produce only collinear
 458 points.

459 The equitetrabeam may possibly be a novel construction. The fact that 6 members meet
 460 in a single point would have been a manufacturing disadvantage that may have dissuaded
 461 structural engineers from using this geometry. However, the advent of additive manufacturing,
 462 such a 3D printing, and the invention of two distinct concentric multimember joints[16, 6] has
 463 improved that situation.

464 Note that the equitetrabeam has chirality, which becomes important in our attempt to
 465 build a continuum of tetrahelices.

466 **7. An Untwisted Continuum.** We observe that (9) and (10) compute r_{opt} and d_{opt} which
 467 create an optimal tetrahelix for any rail angle ρ between 0, which gives the equitetrabeam
 468 and $\rho_{bc} \approx 35.43^\circ$, which gives the BC helix.

469 Because the equitetrabeam which has a rail angle of 0 still has chirality, that is, one still
 470 must decide to connect the one-hop edge to the clockwise or the counter-clockwise vertex,
 471 it is not possible to build a smooth continuum where ρ transitions from positive to negative
 472 which remains optimal. One can use a negative ρ in $\mathbf{H}_{\text{general}}$ but it does not produce minimax
 473 optimal tetrahelices. In other words, untwisting a counter-clockwise tetrahelix to rail angle 0
 474 and then going even further does produce a clockwise tetrahelix, but one in which the one-
 475 hop and two-hop lengths in the wrong places, that is, two-hop becomes shorter than one-hop.
 476 Likewise, $\rho > \rho_{bc}$ generates a tetrahelix, but we have not proved or investigated optimality in
 477 that range.

478 The pitch of a helix for a known z -axis travel d is trivial (see (4)). If one is varying ρ
 479 to obtain a desired pitch from (10) and (4) it is not so simple. However, the pitch increases
 480 monotonically and smoothly with decreasing ρ , so (4) can be easily solved numerically with a
 481 Newton-Raphson solver, as we do on our website. For a pitch at least $p \geq \frac{3\sqrt{2}\pi}{\sqrt{5}\rho_{bc}} \approx 9.64$, using
 482 (10) produces minimax optimal tetrahelices.

483 In this way a rail angle can be chosen for any desired (sufficiently large) pitch, yielding
 484 the optimum radius, the one-hop length, and the two-hop length that an engineer needs to
 485 construct a physical structure.

486 The curvature of a rail helix is formally given by:

487 (12)
$$\frac{|r_\rho|}{r_\rho^2 + (d_\rho/\rho)^2} .$$

488 which goes to 0 as ρ approaches 0 (the equitetrabeam.) As ρ increase up to ρ_{bc} the curvature
 489 increases smoothly until the BC Helix is reached.

490 Perhaps surprisingly, the optimal untwisting is accomplished only by changing the length
491 of the two-hop member, leaving the one-hop member and rail length equivalent within this
492 continuum.² However, it should be noted that an engineer or architect may also use $\mathbf{H}_{general}$
493 directly and interactively via <https://pubinv.github.io/tetrahelix/>, and that minimax length
494 optimality is a mathematical starting point rather than the final word on the beauty and
495 utility of physical structures. For example, a structural engineer might increase radius past
496 optimality in order to resist buckling.

497 If an equitetrabeam were actually used as a beam, an engineer might start with the optimal
498 tetrabeam and dilate it in one dimension to stiffen the beam by deepening it. Similarly, simple
499 length changes curve the equitetrabeam into an arch. The “colored” approach of (7) exposes
500 these possibilities more than the approach of (1).

501 Trusses and space frames remain an important design field in mechanical and structural
502 engineering[9], including deployable and moving trusses[1].

²Before deriving (9), we created a continuum by using a linear interpolation between the optimal radius for the Equitetrabeam and the BC Helix. This minimax optimum of this simpler approach was at most 1% worse than the optimum computed by (9).

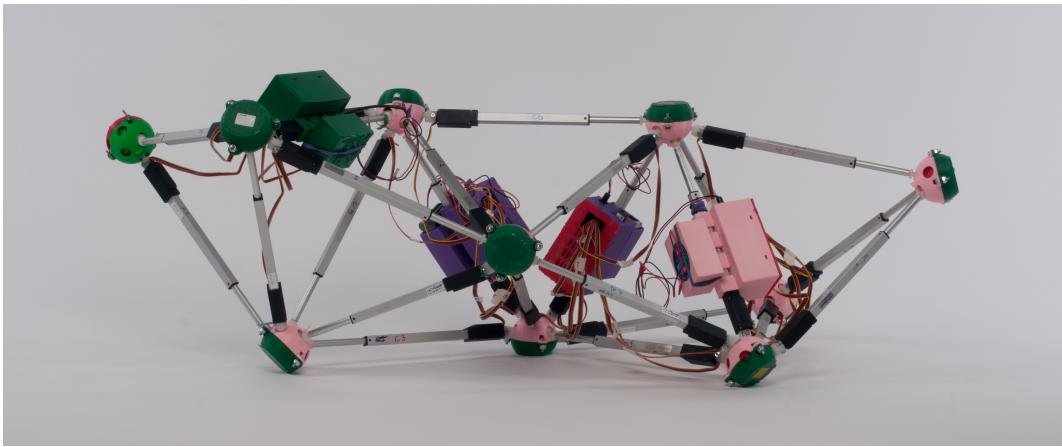


Figure 11. 7-Tet Tetrobot in relaxed, or BC helix configuration

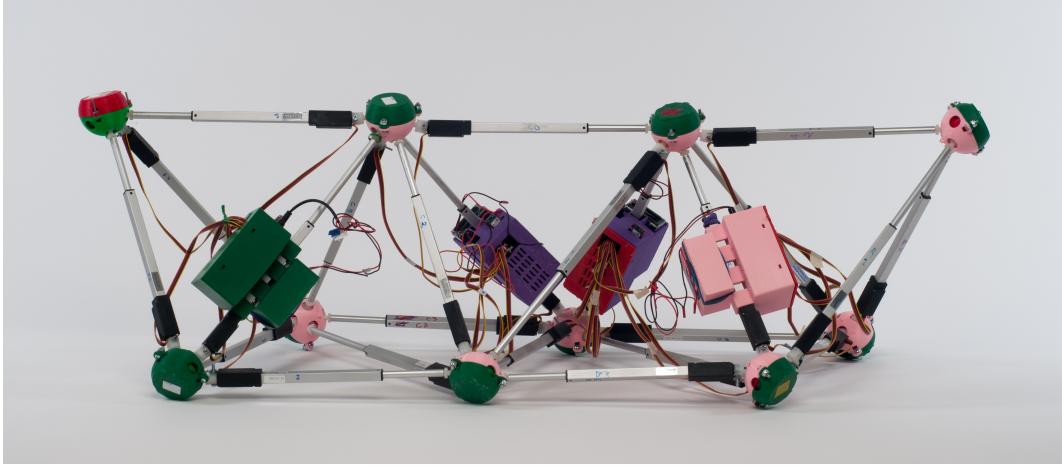


Figure 12. The Equitetrabeam: Fully Untwisted 7-Tet Tetrobot in Hexapod Configuration

503 **8. Utility for Robotics.** Starting twenty years ago, Sanderson[15], Hamlin,[7], Lee[8], and
 504 others created a style of robotics based on changing the lengths of members joined at the
 505 center of a joint, thereby creating a connection to pure geometry. More recently NASA has
 506 experimented with tensegrities[11], a different point in the same design spectrum. In particular
 507 the tetrahelix is related to the tetraspline[10] concept used as a physical concept of a tensegrity
 508 robot. Although tensegrities are not member-for-member isomorphic to the tetrahelix as the
 509 tetrobot is, they often used repeated tetrahedral cells, so this work of pure geometry has some
 510 applied relevance to them.

511 As suggested by Buckminster Fuller, the most convenient geometries to consider are those
 512 that have regular member lengths, in order to facilitate the inexpensive manufacture and
 513 construction of the robot. In a plane, the octet truss[3] is such a geometry, but in a line, the
 514 Boerdijk-Coxeter helix is a regular structure.

515 However, a robot must move, and so it is interesting to consider the transmutations of
 516 these regular geometries. Developing a functional gait for our physical Tetrobot was in fact the

517 motivation for this work. By changing only the length of the longer members that connect two
518 distinct rails (the two-hop members), we can dynamically untwist a physical Tetrobot forming
519 the Boerdijk-Coxeter configuration into the equitetrabeam which rests flat on the plane, as
520 shown in [Figure 12](#).

521 By untwisting the tetrahelix so that it has a planar surface resting on the ground, we may
522 consider each vertex touching the ground a foot or pseudopod. A robot can thus become a
523 hexapod or n -pod tetrobot, and the already well-developed approaches to hexapod gaits may
524 be applied to make the robot walk or crawl.

525 **9. Conclusion.** The BC Helix is the end point of a continuum of tetrahelices, the other end
526 point being an untwisted tetrahelix with equilateral cross section, constructed by changing the
527 length of only those members crossing the outside rails after hopping over the nearest vertex.
528 Under the condition of minimum maximum length ratios of all members in the system, all
529 such tetrahelices have vertices evenly spaced along the axis generated by a simple equation
530 and are in fact triple helices. A machine, such as a robot or a variable-geometry truss, that
531 can change the length of its members can thus twist and untwist itself by changing the length
532 of the appropriate members to achieve any point in the continuum. With a numeric solution,
533 a designer may choose a rotation angle and member lengths to obtain a desired pitch.

534 **10. Contact and Getting Involved.** The Tetrobot Project:
535 <http://pubinv.github.io/tetrobot/> is part of Public Invention, a free-libre, open-source re-
536 search, hardware, and software project that welcomes volunteers. To assist, contact:
537 read.robert@gmail.com.

538

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