UNIVERSITY OF COLOMBO, SRI LANKA

FACULTY OF SCIENCE

FIRST YEAR EXAMINATION IN SCIENCE (SEMESTER I) – 2009/2010

AM 1001 – ORDINARY DIFFERENTIAL EQUATIONS

(Two Hours)

Answer all questions

uoctamil

No. of questions: 04
No. of pages: 07

Important Instructions to the Candidates

- If a page or a part of this question paper is not printed, please inform the Supervisor immediately.
- MCQ TYPE: The MCQ answer sheet is provided with the question paper.
- **Check whether the same CODE is printed on all pages of the exam paper including the answer sheet, except the cover page. If not, please inform the Supervisor immediately.**

  Enter your Index Number and the Code Number in the space provided in the MCQ answer sheet. .

  Choose the correct response to MCQ question and cross the appropriate box with a pen. (Each question has only one correct response.)
- ESSAY TYPE: Write the answers on the answer booklets provided.
- Calculators may be used.
- Electronic devices capable of storing and retrieving text, including electronic dictionaries and mobile phones are not allowed.
- At the end of the examination, attach your written answers together with the MCQ answer sheet and hand it over to the supervisor.

1. i.) Which of the following tables is correct regarding differential equation models and the number of variables therein ?

(a)

DEM	NIV	NDV
ODE	$=1$	$=1$
SODE	>1	$=1$
PDE	$=1$	>1
SPDE	$=1$	>1

(b)

DEM	NIV	NDV
ODE	$=1$	$=1$
SODE	>1	$=1$
PDE	$=1$	>1
SPDE	>1	>1

(c)

DEM	NIV	NDV
ODE	$=1$	$=1$
SODE	$=1$	>1
PDE	>1	$=1$
SPDE	>1	>1

(d)

DEM	NIV	NDV
ODE	$=1$	$=1$
SODE	>1	>1
PDE	$=1$	>1
SPDE	>1	>1

(e)

DEM	NIV	NDV
ODE	$=1$	$=1$
SODE	>1	$=1$
PDE	$=1$	>1
SPDE	>1	$=1$

The above abbreviations stands for DEM=Differential Equation Model, NIV=Number of Independent Variables, NDV=Number of Dependent Variables, ODE=Ordinary Differential Equation, SODE=System of Ordinary Differential Equations, PDE=Partial Differential Equations, SPDE=System of Partial Differential Equations

ii.) The order and the degree of the ordinary differential equation

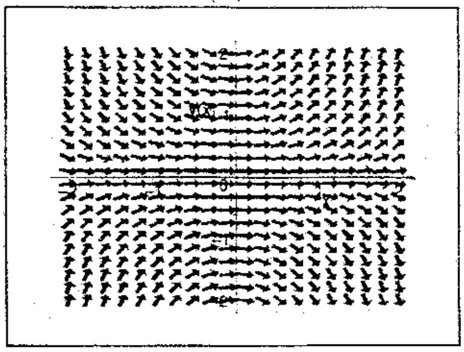
$$4\left(\frac{d^2y}{dx^2}\right)^2 = \left(1+\left(\frac{dy}{dx}\right)^2\right)^3$$

are respectively

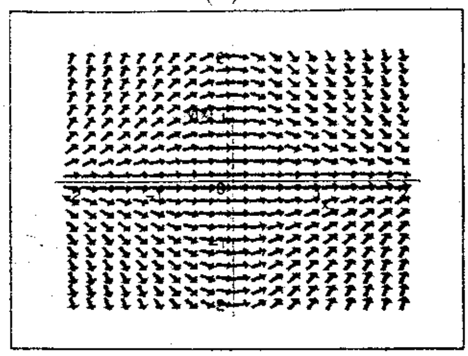
(a) 2 and 3
(b) 2 and 6
(c) 2 and 2
(d) 4 and 2
(e) 1 and 2

iii.) The direction field of the differential equation $y' = xy$ is given by

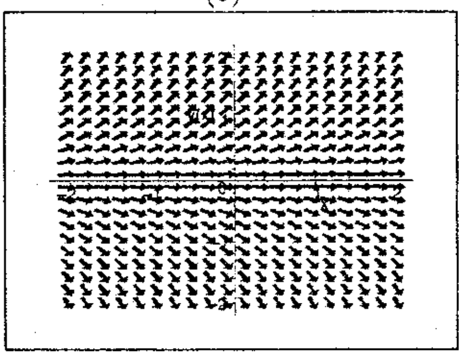
(a)



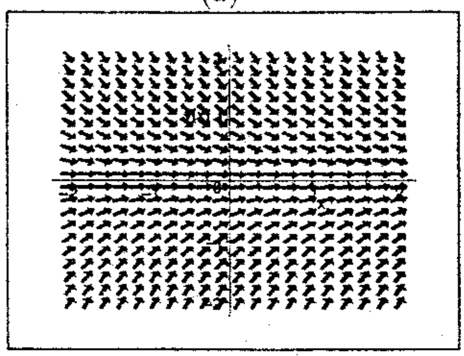
(b)



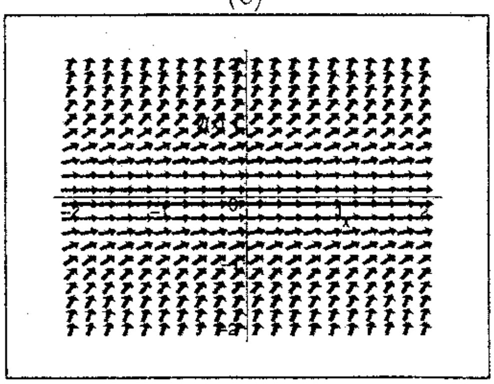
(c)



(d)



(e)



iv.) The general solution of $(D^2 + 2D + 1)y = 2x + x^2$ is given by

(a) $Ae^{x} + Be^{x} + x^2 - 2x + 2$
(b) $Ae^{x} + Be^{-x} + x^2 - 2x + 2$
(c) $Axe^{-x} + Be^{-x} + x^2 - 2x + 2$
(d) $Axe^{x} + Be^{x} + x^2 + 2x + 2$
(e) none of the above. Here A, B are arbitrary constants.

v.) The singular point/s of the differential equation $x(y-1)y' = y^2 - 1$ is/are given by

(a) $(0,1)$
(b) $(-1,0)$
(c) $(0,-1)$
(d) $(1,0)$
(e) all of the above

vi.) Consider the differential equation $\frac{y}{x}\frac{dy}{dx} = \frac{x^2+2y^2-3}{2x^2+y^2-3}$. Which of the following statements is true ?

(a) This equation is equivalent to $\frac{dY}{dX} = \frac{X+2Y}{2X+Y}$
(b) This equation is equivalent to $\frac{dY}{dX} = \frac{2X-Y}{X-2Y}$
(c) This equation is equivalent to $X\frac{dU}{dX} = \frac{1+U^2}{U+2}$
(d) This equation is equivalent to $X\frac{dU}{dX} = \frac{1-U^2}{U+2}$
(e) none of the above

$X = x^2 + h$
$Y = y^2 + k$

vii.) The general solution of the differential equation $\frac{dy}{dx} = \frac{x-y+3}{2x-2y+5}$ is given by

$x - y = z$
$\frac{dz}{dx} = 1 - \frac{dy}{dx}$

(a) $2(x-y+3) + \ln|x-y+2| = C$
(b) $2(x+y+3) + \ln|x+y+2| = C$
(c) $(x-2y) + \ln|x-y+2| = C$
(d) $(x+2y) + \ln|x+y+2| = C$
(e) none of the above

Here C is an arbitrary constant

Q

viii.) $P(x,y)dx + Q(x,y)dy = 0$ is said to be exact if
(a) $\frac{\partial P}{\partial x} = \frac{\partial Q}{\partial y}$
(b) $\frac{\partial^2 P}{\partial y \partial x} = \frac{\partial^2 Q}{\partial x \partial y}$
(c) $\frac{1}{P}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$ is only a function of x
(d) $\frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$ is only a function of y
(e) $P(x,y)dx + Q(x,y)dy = df$ for some $f(x,y)$

may be

ix.) Which of the following equations is exact ?
(a) $xdx + ydy + \frac{xdy - ydx}{x^2+y^2} = 0$
(b) $xdx + ydy + \frac{xdy + ydx}{x^2+y^2} = 0$
(c) $xdx + ydy + \frac{ydx - xdy}{x^2+y^2} = 0$
(d) $xdx + ydy + \frac{xdy - 2ydx}{x^2+y^2} = 0$
(e) $xdx + ydy + \frac{2xdy - ydx}{x^2+y^2} = 0$

x.) An integrating factor of $(x^2 + y^2)dx - 2xydy = 0$ is

(a) $\frac{2}{x}$
✓(b) $\frac{1}{x^2}$
(c) x^2
(d) $-\frac{2}{x}$
(e) $\frac{2}{x^2}$

$\frac{1}{I}\frac{dI}{dx} = \frac{1}{Q}\left[\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right]$

2. i.) Consider the initial value problem $y' = 2(1 + y^2), y(0) = 0$ in the region $R = \{(x, y)|\ |x| < 5, |y| < 3\}$. It follows from the existence and uniqueness theorems that the solution exits in the interval

✓(a) $-0.15 < x < 0.15$
(b) $-0.3 < x < 0.3$
(c) $-0.5 < x < 0.5$
(d) $-0.4 < x < 0.4$
(e) $-5 < x < 5$

$|x-0| < 5 - a$
$|y-0| < 3 = b$
$K = 20$
$a \quad \frac{b}{K}$
$5 , \frac{3}{20}$

ii.) A homicide victim was found at 6:00PM in an office building that is maintained at 72F. When the victim was found, his body temperature was at 84 F. Three hours later at 9:00PM, his body temperature was recorded to be 78F. Assume the temperature of the body at the time of death is the normal human body temperature of 98.6F.

The governing equation for the temperature θ of the body is

$$\frac{d\theta}{dt} = -k(\theta - \theta_e)$$

where θ is the temperature of the body at time t (in hours), θ_e is room temperature, k is a constant based on thermal properties of the body and air.
The estimated time of death most nearly is

✓(a) 2:11 PM
(b) 2:34 PM
(c) 4:34 PM
(d) 5:12 PM
(e) 5:22 PM

iii.) The orthogonal trajectories to the family of curves $y^2 = cx^3$ is given by

(a) $y^2 = -x^3/c$
✓(b) $y^2 + 2x^2/3 = c$
(c) $y^2 = x^2/3 + c$
(d) $y^2 = cx^2/3$
(e) $y^3 = cx^2/3$

Here c is an arbitrary constant.

$y^2 = cx^3$

$2y \cdot \frac{dy}{dx} = 3x^2 \cdot c$

$\frac{dy}{dx} = \frac{3x^2 c}{2y}$

$\frac{dy}{dx} = \frac{-2y\, x^3}{3x^2 \cdot \cancel{x} y^2}$

$= \frac{-2x}{3y}$

$\frac{3y^2}{2} = \frac{-2x^2}{3} + c_1$

*iv.) An integrating factor of $(x^2y - 2xy^2)dx - (x^3 - 3x^2y)dy = 0$ is

(a) x^2
(b) y^2
(c) x^2y^2
(d) x^2/y^2
(e) $\frac{1}{x^2y^2}$

$|x-1| < 1$ (a)
$|y-0| < 1$ (b)

v.) The initial value problem $(x^2 - 1)y' = 4xy,\ y(1) = 0$ in $R = \{(x, y) | \ |x - 1| < 1, |y| < 1\}$ has

(a) no solution
(b) more than one solution
(c) a unique solution
(d) a singular solution
(e) a particular solution

$y' = \frac{4xy}{(x^2-1)}$

$\ln y = 2\ln|x^2-1| + c$

$\frac{y}{(x^2-1)^2} = (A)$

$|x-1| < 1$ (a)

vi.) The initial value problem $(y^2 - 1)y' = 2xy,\ y(1) = 1$ in $R = \{(x, y) | \ |x - 1| < 1, |y - 1| < 1\}$ has

(a) no solution.
(b) infinitely many solutions.
(c) a unique solution.
(d) exactly two solutions.
(e) none of the above.

$\frac{dy}{dx} = \frac{2xy}{y^2-1}$

$(y - \frac{1}{y})dy = 2x\,dx$

$\frac{y^2}{2} - \ln y = x^2 + c$

$\frac{y^2}{2} - \ln y = x^2 + c$

$\frac{1}{2} - 0 = c + 1$

vii.) Consider the linear differential equations

$$\begin{aligned} (aD^2 + bD + c)y &= f(x) & (1) \\ (aD^2 + bD + c)y &= 0. & (2) \end{aligned}$$

where a, b, c are real constants and $f(x)$ is function of x. Let $c_1(x), c_2(x)$ be any two independent solutions of (2) and $p_1(x), p_2(x)$ be any two independent solutions of (1). Which of the following statements is true ?

(a) $Ac_1(x) + Bc_2(x) + p_1(x)$ is the general solution of (2)
(b) $Ap_1(x) + Bp_2(x) + p_1(x) + c_1(x)$ is the general solution of (2)
(c) $Ac_1(x) + Bc_2(x) + p_1(x)$ is the general solution of (1)
(d) $Ap_1(x) + Bp_2(x) + p_1(x) + c_2(x)$ is the general solution of (2)
(e) None of the above

Here A, B are arbitrary constants.

viii.) A tank with a volume of 10 liters initially contains water and 20 grams of dissolved salt. A solution of water containing 10 grams of salt per liter flows into the tank at a rate of 2 liters per minute and a well-mixed solution of water and salt flows out of the tank at the same rate. The concentration of salt in the tank after 3 minutes is

10 gm/lit

20 → g/l

10 l

20 gm

2l

(a) 4.6 g/l
(b) 5.6 g/l
✓(c) 6.6 g/l
(d) 7.6 g/l
(e) 8.6 g/l

$\frac{dy}{dt} = 20 - \frac{y \times 2}{10}$

$\frac{dy}{dt} = \frac{100 - y}{5}$

$-5 \ln|100-y| = t + c$

$t=0 \quad y = 20$

$-5 \ln 80 = c$

$5 \ln|100-y| = 5\ln 80 - t$

ix.) The necessary and sufficient condition for the differential equation $P(x,y)dy + Q(x,y)dx = 0$ to be exact is that

✓(a) $\frac{\partial P}{\partial x} = \frac{\partial Q}{\partial y}$

(b) $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

(c) $\frac{1}{P}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$ is only a function of x

(d) $\frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$ is only a function of y

(e) $P(x,y)dx + Q(x,y)dy = df$ for some $f(x,y)$

$P(x,y)dx + Q(x,y)dy$

$\frac{1}{I}\frac{dI}{dx} = \frac{1}{Q}\left[\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right]$

x.) The solution of $(1+x)\frac{dy}{dx} - xy = 1 - x$ is given by

(a) $y(1-x) = x + Ce^{x}$
(b) $y(1+x) = x + Ce^{x}$
(c) $y(1-x) = x + Ce^{-x}$
(d) $y = (x + Ce^{x})(1-x)$
(e) none of the above.

Here C is an arbitrary constant.

$\frac{dy}{dx} = \frac{1 - x + xy}{1+x}$

$\frac{dy}{dx} = \frac{1}{1+x} - \frac{x}{1+x} + \frac{xy}{1+x}$

$\frac{dy}{dx}(1+x) + y = 1 + ce^{x}$

$y(1+x) - x$

$= y + xy - x$

$\frac{dy}{dx} + \left(\frac{-x}{1+x}\right)y = \frac{1-x}{1+x}$

$\frac{-x}{1+x}$

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3. (a) Solve the initial value problem

$$\frac{dy}{dx} = \frac{x^2 + y^2 + 1}{2xy}, \quad y(1) = 1$$

removing the singular points. Is it defined for all $x \in \mathbb{R}$?

(b) Consider the differential equation

$$\frac{dy}{dx} = \frac{2(x-2)(y-2)}{(x^2 - 4x + 3)}$$

(i) Find the general solution and draw a rough sketch of the solution. Is it defined for all $x \in \mathbb{R}$? Justify your answer.

(ii) Find a solution passing through the points $(0,0), (2,0)$ and $(4,2)$. Is it unique ? Justify your answer.

4. (a) Show that

(i) the general solution of

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0, \tag{3}$$

where a, b, c are constants, is given by

$$y = \begin{cases} Ae^{\alpha x} + Be^{\beta x}, & if \;\; \alpha \neq \beta \\ (Ax + B)e^{\alpha x}, & if \;\; \alpha = \beta, \end{cases}$$

where A, B are arbitrary constants and α, β are solutions of $a\lambda^2 + b\lambda + c = 0$.

(ii) if $h(x)$ is a solution of

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x) \tag{4}$$

for given $f(x)$, then $y(x) + h(x)$ is the complete primitive of (4).

(b) Find the complete primitive of

$$(D^2 + 3D + 2)y = \sin x$$

$e^{jx} = j\sin x + \cos x$

$e^{-jx} = \cos x - j\sin x$