UNIVERSITY OF COLOMBO, SRI LANKA

FACULTY OF SCIENCE

LEVEL I EXAMINATION IN SCIENCE (SEMESTER I) – 2010/2011

AM 1001

DIFFERENTIAL EQUATIONS

(Two Hours)

Answer all questions

No. of pages: 06

Important Instructions to the Candidates

- Check the number of question and number of pages. If a page or a part of this question paper is not printed, please inform the Supervisor immediately.
- Enter your Index Number on all pages of the answer scripts and also in the box provided in the MCQ answer sheet.
- MCQ TYPE: In each of these multiple choice questions mark the correct response on the given MCQ answer sheet with a pen. Write down the question paper code number in the space provided on the answer sheet.
- **ESSAY TYPE**: Write the answers to these questions on the writing paper that is provided.
- Calculators may be used.
- Attach the MCQ answer sheet to the answer script and hand it over to the supervisor. **Do not** attach the question paper to the answer script.

1. i.) Which of the following statements is true ?

(a) Any first order first degree ordinary differential equation has a solution.

(b) Any first order first degree ordinary differential equation is separable.

(c) Any first order first degree ordinary differential equation is linear.

(d) Any first order first degree ordinary differential equation is exact.

(e) None of the above.

ii.) The initial value problem

$$(x^2-1)\frac{dy}{dx} = 4xy,\ y(-2) = 2, x \in \mathbb{R}$$

(a) has a unique solution.

(b) has no solution at $(1,0)$

(c) has infinitely many solutions.

(d) has infinitely many solutions for $x < -1$.

(e) has no solution at $(-1,0)$

iii.) Which of the following statements is true ?

(a) The general solution of the n^{th} order ordinary differential equation contains n arbitrary constants.

(b) Any singular solution can be obtained by substituting suitable values for the arbitrary constants in the general solution.

(c) If $y_1(x)$ and $y_2(x)$ are any two solutions of any given differential equation, then
$y_1(x) + y_2(x)$ is also a solution of the same equation.

(d) A singular solution is a solution obtained by setting all the arbitrary constants of the general solution to zero.

(e) The general solution of the n^{th} degree ordinary differential equation contains n arbitrary constants.

iv.) The order and the degree of the differential equation $(y'')^3 - 7x(y')^4 = 9x^4$ are respectively

(a) 3 and 4
(b) 3 and 3
(c) 2 and 3
(d) 4 and 3
(e) undefined.

v.) $y = cx - c^2$, where c is an arbitrary constant is a solution of the differential equation

(a) $y' - xy'^2 + y = 0$
(b) $y'^2 - xy' - y = 0$
(c) $y'^2 - xy' + y = 0$
(d) $y'^2 - xy' - y^2 = 0$
(e) $xy'^2 - x^2y' - 4xy = 0$

vi.) If $y_1(x)$ and $y_2(x)$ are solutions of the differential equations $y' - p(x)y = q(x)$ and $y' - p(x)y = 0$ respectively, then

(a) $y_2(x) - y_1(x)$ is a solution of $y' - p(x)y = 0$
(b) $y_2(x) - y_1(x)$ is a solution of $y' - p(x)y = q(x)$
(c) $y_2(x) + y_1(x)$ is a solution of $y' - p(x)y = 0$
(d) $y_2(x) + y_1(x)$ is a solution of $y' - p(x)y = q(x)$
(e) $C_1y_2(x) + C_2y_1(x)$ is a solution of $y' - p(x)y = 0$, where C_1, C_2 are arbitrary constants

vii.) Consider a tank with 200 liters of salt-water solution, 30 grams of which is salt. Pouring into the tank is a brine solution at a rate of 4 liters/minute and with a concentration of 1 gram per liter. The well-mixed solution pours out at a rate of 5 liters/minute. The amount of salt in the tank at time t is

(a) $200 - t + 170\left(\frac{t}{200} - 1\right)^5$
(b) $200 + t + 170\left(\frac{t}{200} - 1\right)^5$
(c) $200 - t + 170\left(\frac{t}{200} + 1\right)^5$
(d) $200 - t - 170\left(\frac{t}{200} - 1\right)^5$
(e) $200 - t - 170\left(\frac{t}{200} + 1\right)^5$

viii.) Consider the differential equation $\frac{dy}{dx} = \sqrt{|y|}$, $y(0) = 0$, $x, y \in \{(x, y) \mid |x| < 2, |y| < 2\}$

(a) It has no solution.
(b) It has a solution.
(c) It has a unique solution.

(d) the given information is not enough to determine the existence of a solution

(e) None of the above is true.

ix.) The initial value problem

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0, \; x, y \in D = \{(x, y) \| x - x_0| < a, |y - y_0| < b\},$$

$f(x, y)$ continuous on D, has a solution if

(a) for all $(x, y) \in D$ there exists $K \in \mathbb{R}$ such that $|f(x, y)| < K$.

(b) for all $(x, y) \in D$ there exists $K \in \mathbb{R}$ such that $|f(x, y)| > K$.

(c) $f(x, y)$ has an asymptote at $x = x_0$

(d) (x_0, y_0) is a singular point.

(e) $f(x, y)$ is unbounded on D.

x.) The initial value problem

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0, \; x, y \in D = \{(x, y) \| x - x_0| < a, |y - y_0| < b\},$$

$\frac{\partial f}{\partial y}$ and $f(x, y)$ are continuous on D, has a unique solution if

(a) $f(x, y)$ is bounded on D.

(b) $f(x, y)$ is continuous on D.

(c) $f(x, y)$ has an asymptote at $x = x_0$

(d) for all $(x, y) \in D$ there exists $K, M \in \mathbb{R}$ such that $|f(x, y)| < K$ and $|\frac{\partial f}{\partial y}| < M$.

(e) $f(x, y)$ is unbounded on D.

i.) The solution of the initial value problem $\frac{dy}{dx} = -\frac{y}{x}$, $y(1) = 1$ is

(a) $\ln|y| = -\ln|x| - C$

(b) $\ln|y| = -2\ln|x| - 1$

(c) $\ln|y| = -x + 1$

(d) $xy = C$

(e) $y = 1/x$

ii.) Which of the following equations can be reduced to a separable equation

(a) $\frac{dy}{dx} = \frac{x^3y - 2xy - 1}{y^3 - 3x^2y + 3}$

(b) $\frac{dy}{dx} = \frac{(x-1)^2 + y - 1}{(y-1)^2 + x - 1}$

(c) $(x^3 + 3xy^2)dx + (y^3 + 3x^2y)dy = 0$

(d) $(x^2 + 2y + 3)dx + (y^2 - 3x - 1)dy = 0$

(e) $(x^2y - xy + 9)dx - (y^2x + 3yx^2 + 1)dy = 0$

ii.) The equation $P(x,y)dx + Q(x,y)dy = 0$ is said to be exact if

(a) $\frac{\partial P}{\partial x} = \frac{\partial Q}{\partial y}$
(b) there exists $u(x,y)$ such that $du = P(x,y)dx + Q(x,y)dy$
(c) $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$
(d) $\frac{\partial^2 P}{\partial x \partial y} = \frac{\partial^2 Q}{\partial y \partial x}$
(e) $\frac{\partial^2 P}{\partial y \partial x} = \frac{\partial^2 Q}{\partial x \partial y}$

v.) $(ax + by)dx + (lx + my)dy = 0$, where a, b, l, m are constants, is exact if

(a) $a = b$
(b) $b = l$
(c) $a = l$
(d) $a = m$
(e) $b = m$

v.) Which of the following equations is exact

(a) $y^2 \sin 2x dx + (y^2 + \cos^2 x)dy = 0$
(b) $(x^3 + 3xy^2)dx + (y^3 + 4x^2 y)dy = 0$
(c) $(x^2 + 2y + 3)dx + (y^2 - 3x - 1)dy = 0$
(d) $y \sin 2x dx - (y^2 + \cos^2 x)dy = 0$
(e) $y^2 \sin 2x dx - (y^2 + \cos^2 x)dy = 0$

vi.) The solution of $\{y(1 + 1/x) + \cos y\}dx + (x + \ln x - x \sin y)dy = 0$ is

(a) $xy - \ln|xy| + x\cos y = C$
(b) $xy - x\ln|y| + x\cos y = C$
(c) $xy - y^2 \ln|x| + x\cos y = C$
(d) $x/y - y\ln|x| + x\cos y = C$
(e) $xy + y\ln|x| + x\cos y = C$

vii.) $I(x,y)$ is said to be an integrating factor of $P(x,y)dx + Q(x,y)dy = 0$ if

(a) $\frac{\partial}{\partial x}\{I(x,y)P(x,y)\} = \frac{\partial}{\partial y}\{I(x,y)Q(x,y)\}$
(b) $\frac{\partial}{\partial y}\{I(x,y)P(x,y)\} = \frac{\partial}{\partial x}\{I(x,y)Q(x,y)\}$
(c) $I(x,y) = e^{\int P(x,y)dx}$
(d) $I(x,y) = e^{\int Q(x,y)dx}$
(e) there exists $u(x,y)$ such that $du = I(x,y)P(x,y)dx + I(x,y)Q(x,y)dy$

viii.) The complete primitive of $(D^2 - 4D + 4)y = x^2$ is given by

(a) $y(x) = Ae^{-2x} + Be^{-2x} + \frac{1}{4}\{x^2 + 2x + 1.5\}$
(b) $y(x) = Ae^{2x} + Be^{2x} + \frac{1}{4}\{x^2 + 2x + 1.5\}$
(c) $y(x) = (Ax + B)e^{-2x} + \frac{1}{4}\{x^2 + 2x + 1.5\}$
(d) $y(x) = (Ax - B)e^{2x} + \frac{1}{4}\{x^2 + 2x - 1.5\}$
(e) $y(x) = (Ax - B)e^{2x} + \frac{1}{4}\{x^2 + 2x + 1.5\}$

where A, B are arbitrary constants.

ix.) Consider the initial value problem $y' = 2xy, y(0) = 1, x, y \in D = \{(x, y) \| x| < 2, |y - 1| < 2\}$. The existence theorem guarantees that a solution exists for

(a) $|x| < 1$
(b) $|x| < \frac{1}{2}$
(c) $|x| < \frac{1}{3}$
(d) $|x| < \frac{1}{4}$
(e) $|x| < \frac{1}{6}$

x.) The solution of the differential equation $(D^2 - 2D + 1)y = 0$ is

(a) $y(x) = A\sin(x) + B\cos(x)$
(b) $y(x) = Ae^{x} + Be^{x}$
(c) $y(x) = Axe^{x} + Be^{x}$
(d) $y(x) = Axe^{-x} + Be^{-x}$
(e) $y(x) = Ae^{-x} + Be^{-x}$

where A, B are arbitrary constants.

3. Two chemical substances A and B react in the ratio a : b to form a third substance Z. Z(t) is the amount of the third substance at time t. The rate of formation of Z is proportional to the product of the amounts of the two components A and B which have not yet combined together. Let A_0, B_0 be the initial amounts of the reactants.

   (a) Construct a suitable differential equation to express the rate of formation of Z.

   (b) If the substances A and B combine in the ratio 2 : 3 to form Z, when 45g of A and 60g of B are mixed together, 50g of Z is formed in 5 minutes.

      i.) How long will it take to produce 70g of Z ?

      ii.) How long will it take to produce 100g of Z ?

4. Let us consider a drug administration process. Suppose that a certain medicine diffuses from the blood stream of a patient according to the law

$$\frac{dx(t)}{dt} = -kx$$

and equal doses d of medicine are given at times $0, T, 2T, \ldots .nT, \ldots$, where $x(t)$ is the amount of the medicine in the blood stream at time t and T is a constant time period. k and d are positive constants.

   (a) Find the solution $x(t)$ for $0 \leq t < T$.

   (b) Find the amount of medicine remaining in the blood stream at $t = T$ just before the dose.

   (c) Hence write down a expression for the amount of medicine in the blood stream during the period $[T, 2T)$

   (d) Extend your solution and sketch the graphs of $x(t)$ in each of the time intervals of the drug administration process $[0, T), [T, 2T), [2T, 3T)$ $, \ldots, [(n-1)T, nT)$, where n is a positive integer.

   (e) Let $x_i(t)$ be the amount of the medicine in the blood stream immediately after the i^{th} dose. Show that $x_{i+1} \geq x_i$ for any $i \in \mathbb{N}$.

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