UNIVERSITY OF COLOMBO, SRI LANKA

FACULTY OF SCIENCE

LEVEL I EXAMINATION IN SCIENCE (SEMESTER I) – 2011/2012

AM 1001

DIFFERENTIAL EQUATIONS I

(Two Hours)

uoctamil

Answer all questions

No. of pages: 08

Important Instructions to the Candidates

- Check the number of question and number of pages. If a page or a part of this question paper is not printed, please inform the Supervisor immediately.
- Enter your Index Number on all pages of the answer scripts and also in the box provided in the MCQ answer sheet.
- MCQ TYPE: In each of these multiple choice questions mark the correct response on the given MCQ answer sheet with a pen. Write down the question paper code number in the space provided on the MCQ answer sheet. Check whether the **same code** has been printed on the question paper as well as on the MCQ answer sheet. If not pls inform the supervisor immediately.
- STRUCTURED TYPE: Write the answers in the space provided in the question paper.
- ESSAY TYPE: Write the answers to these questions on the writing paper that is provided.
- Attach the MCQ answer sheet and the structured type question together with the answers to the essay type question and hand it over to the supervisor. **Do not attach the MCQ question paper and essay type question to the answer scripts.**

1. i. $Pdx + Qdy = 0$ is said to be exact if

   (a) $P_x = Q_y$
   (b) there exists F such that $dF = Pdx + Qdy$ ✓
   (c) there exists F such that $dF = Qdx + Pdy$
   (d) $P_{xy} = Q_{yx}$
   (e) $P_{xx} = Q_{yy}$

   ii. The order and the degree of the ordinary differential equation $x^3\frac{d^2y}{dx^2} - x^2(\frac{dy}{dx})^2 - 3y^2 = 4xy$ are respectively

   ✓(a) 2 and 1
   (b) 2 and 2
   (c) 1 and 2
   (d) 1 and 1
   (e) none of the above.

   iii. The singular points of the differential equation $(x^2-4)\frac{dy}{dx} = 3xy$ are

   (a) (0,0) and (2,-2)
   ✓(b) (-2,0) and (2,0)
   (c) (0,-2) and (0,2)
   (d) (0,0) and (2,2)
   (e) (0,0) and (-2,-2)

   iv. A particular solution of a second order second degree ordinary differential equation contains

   (a) no arbitrary constant.
   (b) one arbitrary constant.
   ✓(c) two arbitrary constants.
   (d) three arbitrary constants.
   (e) four arbitrary constants.

   v. The general solution of $y' = \frac{y^2-x^2}{2xy}$ is

   ✓(a) $\left(x - \frac{c}{2}\right)^2 + y^2 = \frac{c^2}{2}$ $\quad x^2 + \frac{c^2}{4} - cx + y^2 = \frac{c^2}{2}$
   (b) $\left(x + \frac{c}{2}\right)^2 + y^2 = \frac{c^2}{2}$ $\quad x^2 + \frac{c^2}{4} + cx + y^2 - c^2/2$
   (c) $\left(2x - \frac{c}{2}\right)^2 + y^2 = \frac{c^2}{4}$ $\quad 4x^2 + \frac{c^2}{4} - 2cx + y^2$
   (d) $\left(x + \frac{c}{2}\right)^2 + y^2 = \frac{c^2}{4}$
   (e) $\left(x - \frac{c}{2}\right)^2 - y^2 = \frac{c^2}{4}$

   where c is an arbitrary constant.

vi. The initial value problem $(x^2-1)\frac{dy}{dx} = 4xy,\ y(1) = 0$ has

(a) no solution.
(b) a unique solution.
(c) infinitely many solutions.
(d) a solution defined only for $-1 < x < 1$.
(e) none of the above.

vii. An integrating factor of $\frac{dy}{dx} - \frac{y}{x} = x^2$ is given by

(a) $\frac{1}{x}$
(b) $\frac{1}{x^2}$
(c) e^x
(d) $e^{\frac{1}{x}}$
(e) $e^{\frac{1}{x^2}}$

viii. The orthogonal trajectory of the family of curves $y = cx^2$ is given by

(a) $y^2 - x^2 = c$
(b) $y^2 + x^2 = c^2$
(c) $y^2 - x^2/2 = c$
(d) $2y^2 + x^2 = c^2$
(e) none of the above

Here c is an arbitrary constant.

ix. Which of the following equations is exact ?

(a) $(x^2y + 2xy)dx + (x^3/3 - x^2)dy = 0$
(b) $(x^2y + 2xy)dx + (xy^2 + y^2)dy = 0$
(c) $(x^2y - 2xy)dy + (xy^2 + y^2)dx = 0$
(d) $(x^2y + 2xy)dy + (xy^2 - y^2)dx = 0$
(e) $(x^2y + 2xy)dy + (xy^2 + y^2)dx = 0$

x. The general solution of $x + yy' = 1$ is

(a) a circle.
(b) an ellipse.
(c) a set of concentric circles.
(d) a hyperbola.
(e) a set of ellipses.

2. i. The orthogonal trajectory of the family of curves $y = -\frac{1}{2}\ln|x| + c$,where c is an arbitrary constant, is given by

(a) $y^2 = cx$ ✗
✓(b) $y = x^2 + c$ ✗
(c) $xy = c^2$
(d) $x^2 - y^2 = c^2$
(e) $x^2 + y^2 = c^2$

ii. Suppose $h_1(x), h_2(x)$ are two solutions of the second order ordinary differential equation $(aD^2+bD+c)y = 0$, where $a, b, c \in \mathbb{R}$. Which of the following statements is true ?

(a) $h_1(x)/h_2(x)$ is a solution of the given equation.
(b) $h_1(x)h_2(x)$ is a solution of the given equation.
(c) $h_1(x) - h_2(x)$ is a solution of the given equation.
(d) $h_2(x)/h_1(x)$ is a solution of the given equation.
✓(e) None of the above.

iii. Consider the initial value problem $y' = 1-y^2, y(0) = 0, (x, y) \in R$ with $R = \{(x, y)||x| < 5, |y| < 3\}$. The existence and uniqueness theorems guarantee the existence of a unique solution in the interval

(a) $-5 < x < 5$
(b) $-3 < x < 3$
(c) $-0.525 < x < 0.575$
✓(d) $-0.375 < x < 0.375$
(e) $-0.375 < x < 0.525$

iv. The general solution of the differential equation $y'' + \omega^2 y = 0$ can be written as

✓(a) $y = a\cos\omega x + b\sin\omega x$, where a and b are arbitrary constants.
(b) $y = e^{i\omega x}\cos a + e^{-i\omega x}\sin b$, where a and b are arbitrary constants and $i^2 = -1$.
(c) $y = ae^{i\omega x} - be^{i\omega x}$, where a and b are arbitrary constants and $i^2 = -1$.
(d) $y = a\cos(\omega x + b)^2$, where a and b are arbitrary constants.
(e) $y = \cos a\omega x + \sin b\omega x$, where a and b are arbitrary constants.

v. A tank filled with $10m^3$ of pure water. $0.5m^3$ of brine , each cubic meter containing 2kg of dissolved salt flows into the tank per minute, and the mixture kept uniform by stirring, flows out at the same rate. The amount of salt $y(t)kg$ in the tank at any time t is given by

(a) $y(t) = 18(e^{-t} + 2)$
(b) $y(t) = 20(e^{-0.5t} + 1)$
(c) $y(t) = 20(e^{-0.05t} + 2)$
(d) $y(t) = -20(e^{-0.05t} - 1)$
(e) $y(t) = 20$

vi. Two chemicals A and B react in the ratio 2:3 to form the compound Z. 45g of A and 60g of B are mixed together to form Z. If the rate of formation of Z is proportional to the product of the amounts of the two components A and B which have not yet combined together, the differential equation of the reaction can be written as

(a) $\frac{dZ}{dt} = k(9 - Z)(36 - Z)$
(b) $\frac{dZ}{dt} = k(18 - Z)(6 - Z)$
(c) $\frac{dZ}{dt} = k(18 - Z)(36 - Z)$
(d) $\frac{dZ}{dt} = k(45 - \frac{2}{5}Z)(60 - \frac{3}{5}Z)$
(e) $\frac{dZ}{dt} = k(6 - Z)(12 - Z)$, where k is a constant.

vii. The solution of $(D^2 + 3D + 2)y = e^{3x}$ is given by

(a) $y = Ae^{2x} + Be^{x} + \frac{1}{11}e^{3x}$
(b) $y = Ae^{-2x} - Be^{-x} + \frac{1}{20}e^{3x}$
(c) $y = Ae^{2x} + Be^{x} + \frac{1}{2}e^{3x}$
(d) $y = Ae^{-2x} + Be^{-x} + \frac{1}{2}e^{3x}$
(e) none of the above.

viii. Consider the linear differential equations

$$\begin{aligned}(aD^2 + bD + c)y &= f(x) \qquad (1)\\ (aD^2 + bD + c)y &= 0 \qquad (2)\end{aligned}$$

where a, b, c are real constants and $f(x)$ a is function of x. Let $c_1(x), c_2(x)$ be any two independent solutions of (2) and $p_1(x), p_2(x)$ be any two independent solutions of (1). Then which of the following statements is true ?

(a) $Ac_1(x) + Bc_2(x) + p_1(x)$ is the general solution of (2)
(b) $Ap_1(x) + Bp_2(x) + p_1(x) + c_1(x)$ is the general solution of (2)
(c) $Ac_1(x) + Bc_2(x) + p_1(x)$ is the general solution of (1)
(d) $Ap_1(x) + Bp_2(x) + p_1(x) + c_2(x)$ is the general solution of (2)
(e) None of the above

Here A, B are arbitrary constants.

ix. The necessary and sufficient condition for the differential equation $P(x,y)dy + Q(x,y)dx = 0$ to be exact is

(a) $\frac{\partial P}{\partial x} = \frac{\partial Q}{\partial y}$
(b) $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$
(c) $\frac{1}{P}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$ is a function of x only
(d) $\frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$ is a function of y only
(e) $P(x,y)dx + Q(x,y)dy = df$ for some $f(x,y)$

x. The general solution of $(D^2 + 2D + 1)y = 2x + x^2$ is given by

(a) $Ae^x + Be^x + x^2 - 2x + 2$
(b) $Ae^x + Be^{-x} + x^2 - 2x + 2$
(c) $Axe^{-x} + Be^{-x} + x^2 - 2x + 2$
(d) $Axe^x + Be^x + x^2 + 2x + 2$
(e) none of the above.

Here A, B are arbitrary constants.

$\frac{1}{(D+1)^2} 2x + x^2$

$\frac{1}{1-(E?)}$ $(1+D+D^2-D^3)(2x+x^2)$ $2x+2$

$(\cancel{2x} + x^2 - \cancel{2} - 2x + 2)$

$x^2 - 2x + 2$

3. **Write the answers in the given booklet.**
Show that the general solution of the differential equation

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x),$$

where $f(x)$ is a given function of x and a, b, c are real constants, is given by

$$\begin{aligned} y(x) &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} + h(x),\ \lambda_1 \neq \lambda_1 \\ &= (c_1 + c_2 x)e^{\lambda x} + h(x),\ \lambda = \lambda_1 = \lambda_1 \end{aligned}$$

where $h(x)$ is a particular solution of the given equation, λ_1, λ_2 are roots of $a\lambda^2 + b\lambda + c = 0$ and a, b, c are real constants.
Hence compute the general solution of

$$(D^2 + 20D + 101)y = e^{-x}$$

and find its behavior as x tends to ∞