LECTURE 5: ERROR PROBABILITY COMPUTATION

INTRODUCTION TO ERROR PROBABILITY COMPUTATION

To measure the goodness of a digital radio link: error probability

SYMBOL ERROR RATE = SER = $P_s(e)$ =

$$P_S(e) = P(\underline{s}_{\underline{R}}[n] \neq \underline{s}_{\underline{T}}\,[n])$$

BIT ERROR RATE = BER = $P_b(e)$ =

$$P(\,u_{\underline{R}}[i] \neq u_{\underline{T}}\,[i])$$

INTRODUCTION TO ERROR PROBABILITY COMPUTATION

R_b
Bit Rate

$T_b = 1/ R_b$
Bit time

$T = k\,T_b$
Symbol time

$R = 1/T$
Symbol Rate

INTRODUCTION TO ERROR PROBABILITY COMPUTATION

E_b
Energy per information bit

E_S
Energy per transmitted signal

$S = E_b R_b = E_S R$
Signal power

INTRODUCTION TO ERROR PROBABILITY COMPUTATION

N_0

Noise spectral density

B

Signal Bandwidth

$N = N_0 B$

Noise power

INTRODUCTION TO ERROR PROBABILITY COMPUTATION

S/N

Signal to Noise ratio

E_b/N_0

Signal to Noise ratio referred to an information bit

Connection:

$$\frac{S}{N} = \frac{E_b}{N_0}\frac{R_b}{B} = \frac{E_b}{N_0}\eta$$

where $\eta = \frac{R_b}{B}$ **spectral efficiency**

INTRODUCTION TO ERROR PROBABILITY COMPUTATION

The system performance are expressed as a function of E_b/N_0

This ratio is proportional to the received power

$$S = \frac{S}{N} N = \frac{E_b}{N_0} \frac{R_b}{B} N_0 B = \frac{E_b}{N_0} R_b N_0$$

SER COMPUTATION

Definition: $$P_S(e) = P(\underline{s}_R \neq \underline{s}_T)$$

We have

$$P_S(e) = \sum_{i=1}^{m} P_S(e \mid \underline{s}_T = \underline{s}_i) P(\underline{s}_T = \underline{s}_i) = \frac{1}{m} \sum_{i=1}^{m} P_S(e \mid \underline{s}_T = \underline{s}_i)$$

We have to compute

$$P_S(e \mid \underline{s}_T = \underline{s}_i) = P(\underline{s}_R \neq \underline{s}_T \mid \underline{s}_T = \underline{s}_i)$$

SER COMPUTATION

First formulation:

$$P_S(e \mid \underline{s}_T = \underline{s}_i) = P(\underline{s}_R \neq \underline{s}_T \mid \underline{s}_T = \underline{s}_i) = 1 - P(\underline{s}_R = \underline{s}_T \mid \underline{s}_T = \underline{s}_i) =$$

$$= 1 - P(\underline{\rho} \in V(\underline{s}_i) \mid \underline{s}_T = \underline{s}_i)$$

Second formulation:

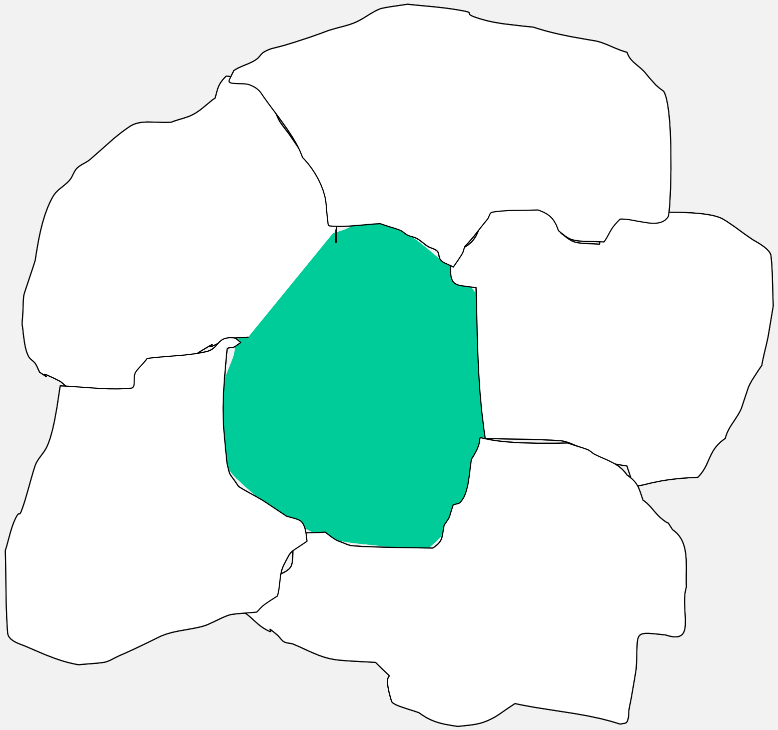
$$P_S(e \mid \underline{s}_T = \underline{s}_i) = P(\underline{s}_R \neq \underline{s}_T \mid \underline{s}_T = \underline{s}_i) = P(\underline{\rho} \notin V(\underline{s}_i) \mid \underline{s}_T = \underline{s}_i) =$$

$$= \sum_{j \neq i} P(\underline{s}_R = \underline{s}_i \mid \underline{s}_T = \underline{s}_i) = \sum_{j \neq i} P(\underline{\rho} \in V(\underline{s}_j) \mid \underline{s}_T = \underline{s}_i)$$

SER COMPUTATION

First formulation

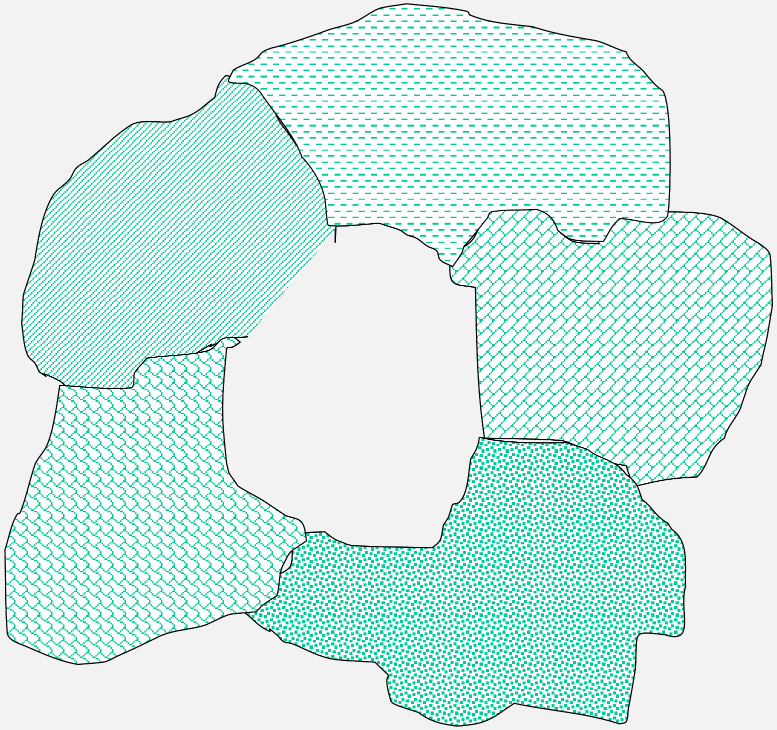
$$\boxed{P_S(e \mid \underline{s_T} = \underline{s_i}) = 1 - P(\underline{\rho} \in V(\underline{s_i}) \mid \underline{s_T} = \underline{s_i})}$$



SER COMPUTATION

Second formulation

$$P_S(e\,|\,\underline{s_T}=\underline{s_i})=P(\underline{\rho}\notin V(\underline{s_i})\,|\,\underline{s_T}=\underline{s_i})=\sum_{j\neq i}P(\underline{\rho}\in V(\underline{s_j})\,|\,\underline{s_T}=\underline{s_i})$$



BER COMPUTATION

When the received signal is correct ($\underline{s}_R = \underline{s}_T$), also the information vector is correct ($\underline{v}_R = \underline{v}_T$).

When the received signal is wrong ($\underline{s}_R \neq \underline{s}_T$), the binary information vector is certainly wrong ($\underline{v}_R \neq \underline{v}_T$), but the number of information bits depends on the labeling and is given by

$$\frac{d_H(\underline{v}_R, \underline{v}_T)}{k}$$

BER COMPUTATION

We have

$$P_b(e) = \frac{1}{m}\sum_{i=1}^{m} P_b(e \mid \underline{s}_T = \underline{s}_i)$$

where

$$P_b(e \mid \underline{s}_T = \underline{s}_i) = \sum_{j \neq i} P_b(e, \underline{s}_R = \underline{s}_j \mid \underline{s}_T = \underline{s}_i) =$$

$$= \sum_{j \neq i} \frac{d_H(\underline{v}_j, \underline{v}_i)}{k} P(\underline{s}_R = \underline{s}_j \mid \underline{s}_T = \underline{s}_i) =$$

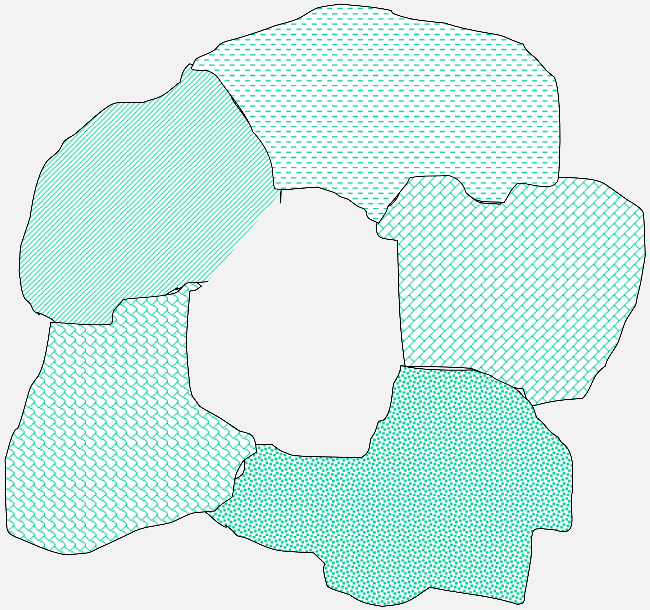
$$= \sum_{j \neq i} \frac{d_H(\underline{v}_j, \underline{v}_i)}{k} P(\underline{\rho} \in V(\underline{s}_j) \mid \underline{s}_T = \underline{s}_i)$$

$$\left[\text{where } \underline{v}_i = e^{-1}\left(\underline{s}_i\right) \text{ and } \underline{v}_j = e^{-1}\left(\underline{s}_j\right)\right]$$

BER COMPUTATION

$$P_b(e) = \frac{1}{m}\sum_{i=1}^{m} P_b(e \mid \underline{s_T} = \underline{s_i})$$

$$P_b(e \mid \underline{s_T} = \underline{s_i}) = \sum_{j \neq i} \frac{d_H(\underline{v_j}, \underline{v_i})}{k} \; P(\underline{\rho} \in V(\underline{s_j}) \mid \underline{s_T} = \underline{s_i})$$



ERFC

Given a Gaussian random variable n with

- mean μ
- variance σ^2
- density

$$f_n(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-\frac{(x-\mu)^2}{2\sigma^2})$$

We have

$$P(n > x) = \int_{x}^{+\infty} f_n(x)\, dx = \frac{1}{2} erfc\left(\frac{x-\mu}{\sqrt{2}\sigma}\right)$$

ERFC

where

$$erfc(x) = \frac{2}{\sqrt{\pi}} \int_x^{+\infty} e^{-t^2} dt$$

In fact

$$P(n > x) = \int_x^{+\infty} f_n(x)\, dx = \int_x^{+\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-\frac{(x-\mu)^2}{2\sigma^2}) dx =$$

$$\frac{1}{\sqrt{\pi}} \int_{\frac{(x-\mu)}{\sqrt{2}\sigma}}^{+\infty} e^{-t^2} dt = \frac{1}{2} erfc\left(\frac{x-\mu}{\sqrt{2}\sigma}\right)$$

In case of zero mean and variance $N_0/2$

$$P(n > x) = \frac{1}{2} erfc\left(\frac{x-\mu}{\sqrt{2}\sigma}\right) = \frac{1}{2} erfc\left(\frac{x}{\sqrt{N_0}}\right)$$

SER/BER COMPUTATION FOR BINARY ANTIPODAL SIGNALS

Let us consider a one-dimensional constellation (d=1) composed by two signals (m=2), symmetrical with respect to the origin:

$$M = \{\underline{s}_1 = (+A) \;\; \underline{s}_2 = (-A)\;\}$$

The Voronoi regions are:

$$V(\underline{s}_1) = \{\underline{\rho} = (\rho_1) \;,\; \rho_1 \geq 0\,\}$$

$$V(\underline{s}_2) = \{\underline{\rho} = (\rho_1) \;,\; \rho_1 \leq 0\,\}$$

SER/BER COMPUTATION FOR BINARY ANTIPODAL SIGNALS

We have:

$$P_S(e) = \frac{1}{m}\sum_{i=1}^{m} P_S(e \mid \underline{s_T} = \underline{s_i}) = \frac{1}{2}\left[P_S(e \mid \underline{s_T} = \underline{s_1}) + P_S(e \mid \underline{s_T} = \underline{s_2}) \right]$$

Let us compute

$$P_S(e \mid \underline{s_T} = \underline{s_1})$$

and

$$P_S(e \mid \underline{s_T} = \underline{s_2})$$

SER/BER COMPUTATION FOR BINARY ANTIPODAL SIGNALS

$$P_S(e\,|\,\underline{s_T}=\underline{s_1})=P(\underline{\rho}\in V(\underline{s_2})\,|\,\underline{s_T}=\underline{s_1})=P(\rho_1<0\,|\,\underline{s_T}=\underline{s_1})$$

We have:

$$\boxed{\underline{r}=\underline{s_T}+\underline{n}\qquad \underline{r}=\underline{\rho}\qquad \underline{s_T}=\underline{s_1}}$$

where $\underline{\rho}=(\rho_1)$ $\quad \underline{s_1}=(s_{11})=(+A)$ $\quad \underline{n}=(n_1)$

It follows:

$$\rho_1=A+n_1$$

SER/BER COMPUTATION FOR BINARY ANTIPODAL SIGNALS

$$P_S(e\,|\,\underline{s_T} = \underline{s_1}) = P(\rho_1 < 0\,|\,\underline{s_T} = \underline{s_1}) = P(A + n_1 < 0) = P(n_1 < -A)$$

The random variable n_1 is Gaussian, with mean zero and variance $N_0/2$

$$P_S(e\,|\,\underline{s_T} = \underline{s_1}) = P(n_1 < -A) = P(n_1 > A) = \frac{1}{2} erfc\left(\frac{A}{\sqrt{N_0}}\right)$$

SER/BER COMPUTATION FOR BINARY ANTIPODAL SIGNALS

Compute now the error probbaility for $\underline{s}_T = \underline{s}_2$

$$P_S(e \mid \underline{s}_T = \underline{s}_2) = P(\underline{\rho} \in V(\underline{s}_1) \mid \underline{s}_T = \underline{s}_2) = P(\rho_1 > 0 \mid \underline{s}_T = \underline{s}_2)$$

We have:

$$\underline{r} = \underline{s}_T + \underline{n} \qquad \underline{r} = \underline{\rho} \qquad \underline{s}_T = \underline{s}_2$$

then

$$\underline{\rho} = (\rho_1) \qquad \underline{s}_2 = (s_{21}) = (-A) \qquad \underline{n} = (n_1)$$

$$\rho_1 = -A + n_1$$

SER/BER COMPUTATION FOR BINARY ANTIPODAL SIGNALS

$$P_S(e \mid \underline{s_T} = \underline{s_2}) = P(-A + n_1 > 0) = P(n_1 > A)$$

$$P_S(e \mid \underline{s_T} = \underline{s_2}) = \frac{1}{2} erfc\left(\frac{A}{\sqrt{N_0}}\right)$$

SER/BER COMPUTATION FOR BINARY ANTIPODAL SIGNALS

We have

$$P_S(e \mid \underline{s_T} = \underline{s_1}) = P_S(e \mid \underline{s_T} = \underline{s_2})$$

It follows:

$$P_S(e) = \frac{1}{2}\left[P_S(e \mid \underline{s_T} = \underline{s_1}) + P_S(e \mid \underline{s_T} = \underline{s_2}) \right] = P_S(e \mid \underline{s_T} = \underline{s_1})$$

then

$$P_S(e) = P_S(e \mid \underline{s_T} = \underline{s_1}) = \frac{1}{2} erfc\left(\frac{A}{\sqrt{N_0}} \right)$$

[note that

$$P_S(e) = P_S(e \mid \underline{s_T} = \underline{s_1}) = \frac{1}{2} erfc\left(\frac{d}{2\sqrt{N_0}} \right)$$

SER/BER COMPUTATION FOR BINARY ANTIPODAL SIGNALS

We have:

$$P_S(e) = P_S(e \mid \underline{s_T} = \underline{s_1}) = \frac{1}{2} erfc\left(\frac{A}{\sqrt{N_0}} \right)$$

We want to write it as a function of E_b/N_0.

$$E(\underline{s_1}) = E(\underline{s_2}) = A^2$$

$$E_S = \frac{E(\underline{s_1}) + E(\underline{s_2})}{2} = A^2$$

$$E_b = \frac{E_S}{k} = E_S = A^2$$

SER/BER COMPUTATION FOR BINARY ANTIPODAL SIGNALS

Fundamental result

$$P_S(e) = \frac{1}{2} erfc\left(\sqrt{\frac{E_b}{N_0}}\right)$$

SER/BER COMPUTATION FOR BINARY ANTIPODAL SIGNALS

For this constellation we can establish this binary labeling:

$$e : H_1 \Leftrightarrow M$$

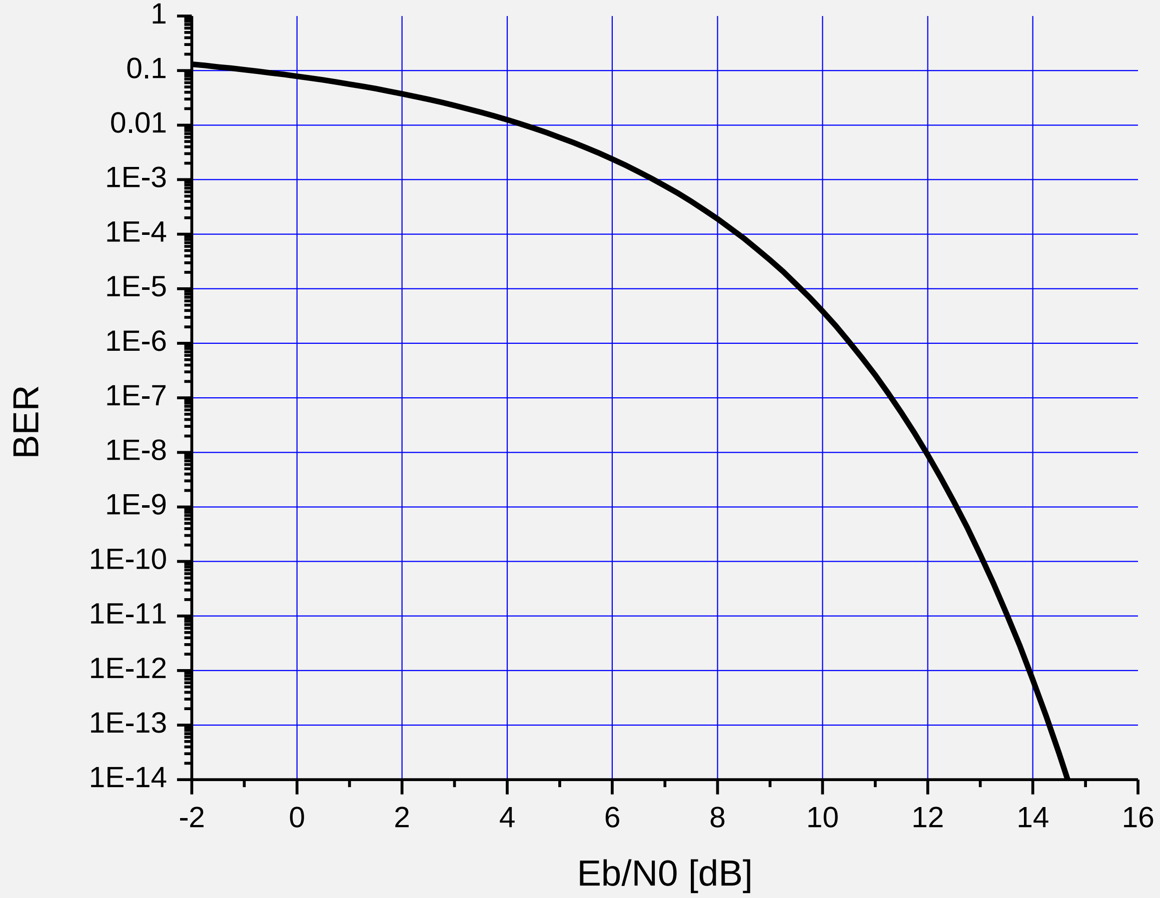
$$\underline{v}_1 = (0) \Leftrightarrow \underline{s}_1$$

$$\underline{v}_2 = (1) \Leftrightarrow \underline{s}_2$$

In this case, if the received signal is wrong, the information bit is wrong, too.

$$P_b(e) = P_S(e) = \frac{1}{2} erfc\left(\sqrt{\frac{E_b}{N_0}}\right)$$

SER/BER computation for binary antipodal signals



SER/BER COMPUTATION FOR BINARY ANTIPODAL SIGNALS

Different signal constellations (different transmitted waveforms) with the same vector constellations achieve equal BER performance!

As an example, the BER performance of a binary antipodal constellation does not depend on the versor

$$b_1(t) = \frac{1}{\sqrt{T}} P_T(t)$$

$$b_1(t) = \sqrt{\frac{2}{T}} P_T(t) \cos(2\pi f_0 t)$$