Double Integrals Over Rectangles

$$\iint_R f(x,y)dA = \lim_{n,m\to\infty} \sum_{i=1}^{n}\sum_{j=1}^{m} f(x_i, y_j)\Delta x \Delta y$$

Since it's over a rectangular area, you can switch the order of integration without changing the bounds

$$\iint_R f(x,y)dA = \int_a^b \int_c^d f(x,y)dydx = \int_c^d \int_a^b f(x,y)dxdy$$

Double Integrals Over Nonrectangular Regions

You can reverse the order of integration, but this changes the bounds.

$$\int_0^2 \int_0^{x^2} f(x,y)dydx = \int_0^4 \int_{\sqrt{y}}^2 f(x,y)dxdy$$

In the above example, the "range" of this portion of the function is [0,4], while the "domain" is [0,2]. y is bounded at the top by x^2, and x is bounded at the left by $\sqrt{y}$

Double Integrals in Polar Coordinates

$$\iint_R f(x,y)dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\mathbf{r}\, dr\, d\theta$$

where α and β are radians; and a and b represent the radius. dA in polar coordinates is $r\,dr\,d\theta$. Don't forget the r!

Also remember that $x^2 + y^2 = r^2$. If you know the bounds for $x^2 + y^2$, then you have to square root them to find the bounds for r^2

Surface Area of Graphs

Surface area S of the function $z = f(x, y)$ over the region R can be given by:

$$S = \iint_R \sqrt{1 + f_x(x, y)^2 + f_y x, y^2} dA$$

Remember that f_x and f_y are the partial derivatives with respect to x and y (respectively)

Triple Integrals in Cartesian Coordinates

$$\iiint_R f(x, y, z) dV = \lim_{n,m,p \to \infty} \sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{k=1}^{p} f(x_i, y_j, z_k) \Delta x \Delta y \Delta z$$

$$\iiint_R f(x, y, z) dV = \int_a^b \int_c^d \int_e^f f(x, y, z) dz\, dy\, dx$$

the above integral can be reordered, but you still might have to change the bounds if it's not rectangular

Example: Integrate $f(x, y, z) = 4x^2y$ over the region E which lies above the rectangle $[0, 1] \times [0, 1]$ in the xy plane and under the graph of $z = 2 - y^2$

$$\iiint_E = 4x^2y\; dV = \int_0^1 \int_0^1 \int_0^{2-y^2} 4x^2y\; dz\, dx\, dy = \int_0^1 \int_0^1 4x^2y(z - y^2)\; dx\, dy$$

you got it from here :)

Triple Integrals in Cylindrical & Spherical Coordinates

Cylindrical Coordinates	**Polar Coordinates**
$x = r \cos\theta$	$x = \rho \sin\phi \cos\theta$
$y = r \sin\theta$	$y = \rho \sin\phi \sin\theta$
$dV = r\; dr\, d\theta\, dz$	$dV = \rho^2 \sin\phi\; d\rho\, d\theta\, d\phi$

Remember that $x^2 + y^2 + z^2 = \rho^2$. Remember that θ and ϕ for a full sphere are $[0, 2\pi]$ and $[0, \pi]$ respectively

Example integral where S is the spherical shell $1 \leq \rho \leq 5$

$$\iiint_S \frac{1}{\sqrt{x^2 + y^2 + z^2}} dV = \int_0^{\pi} \int_0^{2\pi} \int_1^5 \frac{1}{\rho} \rho^2 \sin\phi\; d\rho\, d\theta\, d\phi$$

Change of Variables

For when you want to integrate a function $f(x, y)$ over some non-rectangular region S in the xy-plane, so you use another coordinate system, u and v, where S is more easily integrate-able

$$\iint_S f(x,y)\, dA = \iint_R f(x(u,v), y(u,v))\, |J(u,v)|\, du\, dv$$

where J is the Jacobian

$$J(u,v) = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

When solving for the new bounds for the region R, you have to use the change of variables.

Example: if S had a vertex at $(2, 4)$, and you had to use the change of variables $x = 3u + v$ and $y = -u + 2v$, to get the new vertex for R, you do:

$$3u + v = 2 \quad -u + 2v = 4$$

Solve for u and v, and you get $u = 0$ and $v = 2$, so new vertex at $(0, 2)$

Vector Fields

Let function $\mathbf{F}$ be a vector field with n dimensions. $\mathbf{F}$ is *conservative* if there exists a function f where $\nabla f = \mathbf{F}$

To find f, you take n integrals of $\mathbf{F}$ with respect to each dimension (x, y, z, etc). If the integrals could be equal, then $\mathbf{F}$ is conservative.

$$\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k} \qquad \nabla = \tfrac{\partial}{\partial x}\mathbf{i} + \tfrac{\partial}{\partial y}\mathbf{j} + \tfrac{\partial}{\partial z}\mathbf{k}$$

Divergence of F (gives a scalar function)

$$div\ \mathbf{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z} = \nabla \cdot \mathbf{F}$$

Curl of F (gives a vector field)

$$curl\ \mathbf{F} = (\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z})\mathbf{i} + (\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x})\mathbf{j} + (\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y})\mathbf{k} = \nabla \times \mathbf{F}$$

Line Integrals

Line integral of f along oriented curve C with respect to arclength:

$$\int_C f\ ds = \int_a^b f(r(t))||r'(t)||\ dt$$

$$\int_C \mathbf{F} \cdot T\ ds = \int_a^b \mathbf{F}(r(t)) \cdot r'(t)\ dt = \int_C \mathbf{F} \cdot dr$$

This integral is positive when $\mathbf{F}$ goes in the direction of C, negative when it opposes C, and 0 when perpendicular/no movement. a and b are the bounds for the curve C with respect to time.

If you see a flux integral, it uses the same formula above, but replaces $r'(t)$ with $n(t)$, the normal vector, and measures how much $\mathbf{F}$ is growing into C.

Fundamental Theorem of Line Integrals

If a vector field $\mathbf{F}$ is conservative ($\mathbf{F} = \nabla f$), then it's line integral for a curve C with parametrization $r(t)$ for $a \leq t \leq b$ is

$$\int_C \mathbf{F} \cdot dr = f(r(b)) - f(r(a))$$

Conservative vector fields are also *path independent*, which means that any path connecting the same two points gives the same value for its line integral.

<u>A conservative vector field will have a curl of 0</u>