Green's Theorem

Let C be closed curve surrounding a region S in the xy-plane.Let function $\mathbf{F}$ be a continuous vector field where $\mathbf{F}(x,y) = M(x,y)\mathbf{i} + N(x,y)\mathbf{j}$, and has continuous partial derivatives on S. Then

$$\int_C \mathbf{F} \cdot dr = \iint_S (\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y})\, dA = \iint_S \operatorname{curl} \mathbf{F} \cdot \mathbf{k}\, dA$$

$$\text{sometimes seen as: } \int_C M dx + N dy$$

Abbreviated Example: Compute $\int_C \mathbf{F} \cdot dr$ where C is the positively oriented triangle with vertices $(0,0)$, $(1,0)$, and $(1,2)$ and $\mathbf{F}$ is the vector field $\mathbf{F}(x,y) = (xy + y^2)\mathbf{i} + (xy + x^3)\mathbf{j}$

C encloses the triangular area T: $(x,y)|0 \le x \le 1, 0 \le y \le 2x$

$$\int_C \mathbf{F} \cdot dr = \iint_T (\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y})\, dA$$

$$\int_0^1 \int_0^{2x} (y + 3x^2) - (x + 2y)\, dA = \text{...solve from here}$$

Plane Divergence Theorem

Pretty much Green's Theorem but for Flux Integrals

$$\int_C \mathbf{F} \cdot \mathbf{n}\, ds = \iint_S (\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y})\, dA = \iint_S \operatorname{div} \mathbf{F}\, dA$$

Abbreviated Example: Compute $\int_C \mathbf{F} \cdot \mathbf{n}\, ds$ where C is the positively oriented unit circle and $\mathbf{F}$ is the vector field $\mathbf{F}(x,y) =< xy + e^x - 3y, y^2 - ye^x >$

$$\operatorname{div} \mathbf{F} = y + e^x + 2y - e^x = 3y$$

$$\iint_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_S 3y\, dA = \int_0^{2\pi} \int_0^1 3r \sin\theta\, r dr d\theta = \text{...solve from here}$$

Curl & Divergence Revisited

Let C_r be a circle of radius r centered at (a, b), $\mathbf{F}$ is a continuous vector field with continuous partials, etc...

$$\operatorname{div}\mathbf{F}(a, b) = \lim_{r\to 0} \frac{1}{\pi r^2} \int_{C_r} \mathbf{F} \cdot \mathbf{n}\, ds$$

$$\operatorname{curl}\mathbf{F}(a, b) \cdot \mathbf{k} = \lim_{r\to 0} \frac{1}{\pi r^2} \int_{C_r} \mathbf{F} \cdot dr$$

Divergence at a point is a measure of how much the vector field is emanating from a given point, Curl is a measure of the tendency of the vector field to rotate around the point

Parameterized Surfacs

Jacobian for a parametrization is the magnitude of the cross product of the two partial derivatives of the parametrization:

$$||r_u(u,v) \times r_v(u,v)||$$

Function	Parametrization	Jacobian
$z = f(x,y)$	$r(x,y) =< x, y, f(x,y) >$	$\sqrt{f_x^2 + f_y^2 + 1}$
Unit sphere	$r(\theta,\phi) =< \sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi >$	$\sin\phi$
Cylinder $x^2 + y^2 = a^2$	$r(\theta,z) =< a\cos\theta, a\sin\theta, z >$	a
$z = f(r,\theta)$	$r(r,\theta) =< r\cos\theta, r\sin\theta, f(r,\theta) >$	$\sqrt{f_\theta + f_r^2 r^2 + r^2}$

Surface Integrals

Let G be a parameterized curve and f be a function with points defined at G. Let R be a subset of the uv-plane. Surface integral of f over G:

$$\iint_G f(x,y,z)\, dS = \iint_R f(x(u,v), y(u,v), z(u,v))\, ||r_u(u,v) \times r_v(u,v)||\, dA$$

When $f = 1$, the surface integral just gives you the surface area of the surface:

$$A(G) = \iint_G 1\, dS = \iint_R ||r_u(u,v) \times r_v(u,v)||\, dA$$

A surface flux integral measures how much the vector field $\mathbf{F}$ is flowing out of the surface G

$$\iint_G \mathbf{F} \cdot \mathbf{n}\, dS$$

The above integral is > 0 if $\mathbf{F}$ is flowing through G in the direction of $\mathbf{n}$, < 0 if flowing opposite, or 0 if there's no net flow through G

Abbreviated Example: Compute $\iint_G \mathbf{F} \cdot \mathbf{n}\, dS$ if G is the surface determined by $z = 1 - x^2 - y^2$, $\mathbf{n}$ is the outward pointing normal vector, and $\mathbf{F}(x,y,z) =< x, y, 0 >$

$$f(x,y,z) = z + x^2 + y^2 = 1 \quad \Rightarrow \quad \nabla f =< 2x, 2y, 1 >$$

$$\mathbf{n} = \frac{\nabla f}{||\nabla f||} = \frac{< 2x, 2y, 1 >}{\sqrt{1 + 4x^2 + 4y^2}}$$

$$\iint_G \mathbf{F} \cdot \mathbf{n}\, dS = \iint_D < x, y, 0 > \cdot \frac{< 2x, 2y, 1 >}{\sqrt{1 + 4x^2 + 4y^2}} \sqrt{1 + 4x^2 + 4y^2}\, dA$$

The Divergence Theorem

Let F be a continuous vector field with continuous partials. Let G be a closed surface with outward pointing normal vector $\mathbf{n}$. Let R be the interior of G. Then

$$\iint_G \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_R \operatorname{div} \mathbf{F}\, dV$$

The flux through any surface containing the origin is the same

Stokes' Theorem

Using the same definitions of $\mathbf{F}$ and $\mathbf{n}$ as before, and G is now any surface, let C be boundary around G positively oriented. Then

$$\iint_G \operatorname{curl} \mathbf{F} \cdot \mathbf{n}\, dS = \int_C \mathbf{F} \cdot dr$$

For all closed surfaces S:

$$\iint_S \operatorname{curl} \mathbf{F} \cdot \mathbf{n}\, dS = 0$$

Abbreviated Example: Let S be the surface determined by $z = 1 - x^2 - y^2$ for $z \geq 0$. $\mathbf{F}(x, y, z) =< -y + xz, x - z^2, 1 + x >$. Find $\iint_S \operatorname{curl} \mathbf{F} \cdot \mathbf{n}\, dS$

At $z = 0$, $x^2 + y^2 = 1$, or is a unit circle. There parametrization for this boundary C is

$$r(t) =< cos(t), sin(t), 0 > \quad 0 \leq t \leq 2\pi$$

$$r'(t) =< -sin(t), cos(t), 0 >$$

$$\int_C \mathbf{F} \cdot dr = \int_0^{2\pi} < -sin(t), cos(t), 1 + cos(t) > \cdot < -sin(t), cos(t), 0 > \; dt$$

Above, we parameterized $\mathbf{F}$ by substituting x, y, and z with the components of $r(t)$