Critical Points of Functions Defined on Open Sets / Second Partials Test

Critical points:

1. Stationary points: points $(a, b) \in D$ where $\nabla f(a, b) = 0$
2. Singular points points $(a, b) \in D$ where $\nabla f(a, b)$ is not differentiable

The discriminant can be used to find what type of extrema a stationary point is.

$$D(a, b) = f_{xx}(a, b) f_{yy}(a, b) - (f_{xy}(a, b))^2$$

1. *If $D > 0$ and $f_{xx}(a, b) > 0 \Rightarrow f$ has a local minimum at (a, b).*
2. *If $D > 0$ and $f_{xx}(a, b) < 0 \Rightarrow f$ has a local maximum at (a, b).*
3. *If $D < 0 \Rightarrow f$ has a saddlepoint at (a, b).*
4. *If $D = 0 \Rightarrow$ test is inconclusive.*

Functions on Closed and Bounded Regions

In a closed bounded region D, the points on the boundary ∂D are also critical points. Closed bounded regions must also have a global maximum and global minimum by the Extreme Value Theorem

Lagrange Multipliers

Langrange Multipliers need a function f and a constraint g. There is some value of λ that satifies the following system of equations:

$$\nabla f(a, b) = \lambda \nabla g(a, b)$$

$$g(a, b) = 0$$

Just solve the system and then plug all of the points into f. The biggest value of f is the maximum, and smallest is the minimum.

Optimization Example 1

Example: minimize the distance from the origin to the surface $z^2 = xy - y + 3$

Distance squared from origin: $f(x, y, z) = x^2 + y^2 + z^2$
Getting 2-variable equation: $f(x, y) = x^2 + y^2 + xy - y + 3$
Find critical points: $\nabla f(x, y) =< 2x + y, 2y + x - 1 >= 0 =< 0, 0 >$
Solution at $x = -\frac{1}{3}, y = \frac{2}{3} \Rightarrow f(-\frac{1}{3}, \frac{2}{3}) = \frac{8}{3}$, which is a minimum
Because we used the equation for distance squared, the total distance is $\sqrt{\frac{8}{3}}$.

Optimization Example 2

Example: Find the minimum and maximum of $f(x, y) = x^2 + y$ on the circle $x^2 + y^2 = 1$

$$\nabla f(x, y) = \lambda \nabla g(x, y)$$

Solving system of three equations:

$$\text{Eq(1): } 2x = \lambda 2x \qquad \text{Eq(2): } 1 = \lambda 2y \qquad \text{Eq(3): } x^2 + y^2 = 1$$

$$2x(1 - \lambda) = 0 \Rightarrow x = 0 \; or \; \lambda = 1$$

$$x = 0 \Rightarrow by \; Eq(3), \; y = \pm 1$$

$$\lambda = 1 \Rightarrow by \; Eq(2), \; y = \frac{1}{2}$$

$$y = \frac{1}{2} \Rightarrow by \; Eq(3), \; x = \pm \frac{\sqrt{3}}{2}$$

Four points to check: $(0, 1), (0, -1), (\frac{\sqrt{3}}{2}, \frac{1}{2})$, and $(-\frac{\sqrt{3}}{2}, \frac{1}{2})$

$$f(0, 1) = 1 \qquad f(0, -1) = -1 \qquad f(\frac{\sqrt{3}}{2}, \frac{1}{2}) = f(\frac{-\sqrt{3}}{2}, \frac{1}{2}) = \frac{5}{4}$$

Maximum: $\frac{5}{4}$ **Minimum: -1**