Partial Derivatives

$$f'_x(a,b) = \lim_{h \to 0} \frac{f(a+h,b) - f(a,b)}{h}$$
$$f'_y(a,b) = \lim_{h \to 0} \frac{f(a,b+h) - f(a,b)}{h}$$

Take the derivative with respect to one variable, while the other variables are treated like constants.

$$(f_x)_y = f_{xy} = \frac{\partial}{\partial y}(\frac{\partial f}{\partial x}) = \frac{\partial^2 f}{\partial y \partial x}$$

Multivariate Limits

If a function is "well behaved" at (a,b), then you can solve the limit by plug and chug.

It is sometimes easier to prove that a limit does not exist than to prove that it exists. You can usually* do this by showing that the limits of the partial derivatives at (a,b) do not equal each other.

What may also help is converting a function into polar coordinates (instead of x and y, it's just r), and solving like that.

Differentiability

If the partial derivatives $f_x(x,y)$ and $f_y(x,y)$ exist and are continuous on a neighborhood of (a,b), then f is differentiable at (a,b).

Tangent plane to graph $z = f(x,y)$ at the point (a,b):

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

$$z = f(a,b) + \nabla f(a,b) \cdot < x-a, y-b)$$

Gradient/normal vector formula:

$$\nabla f(a,b) = < f_x(a,b), f_y(a,b) >$$

Directional Derivatives

If function f is differentiable at (a, b), the directional derivative of f in direction u (a vector) is:

$$D_u f(a,b) = \nabla f(a,b) \cdot u = ||\nabla f(a,b)|| \cos\theta$$

The maximum rate of change a function f has in the direction $\nabla f(a,b)$ is $||\nabla f(a,b)||$

The Chain Rule

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Let $r(t)$ be a curve in the xy-plane, $r(t) =< x(t), y(t) >$

$$\frac{d}{dt}(f(r(t))) = \nabla f(r(t)) \cdot r'(t)$$

Implicit Differentiation Example

$$x^2 \ln z + z^2 y = 2$$

Finding $\frac{\partial z}{\partial x}$ at point $(3, 2, 1)$:

$$\frac{\partial}{\partial x}(x^2 \ln z + z^2 y) = \frac{\partial}{\partial x}(2)$$

$$2x \ln z + x^2(\frac{1}{z})\frac{\partial z}{\partial x} + 2zy\frac{\partial z}{\partial x}$$

Plug in points... $9\frac{\partial z}{\partial x} + 4\frac{\partial z}{\partial x} = 0 \Rightarrow \frac{\partial z}{\partial x} = 0$

The Tangent Plane and Linear Approximation

Tangent plane to graph $z = f(x, y)$ at the point (a, b):

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

Tangent plant to the surface $F(x, y, z) = k$ at the point (a, b, c):

$$0 = F_x(a,b,c)(x-a) + F_y(a,b,c)(y-b) + F_z(a,b,c)(z-c)$$

$$0 = \nabla F_x(a,b,c) \cdot < x-a, y-b, z-c >$$

Differentials

Differential equation for function $z = f(x, y)$ at point (a, b):

$$dz = f_x(a, b)dx + f_y(a, b)dy$$

Example: Circular cone with base radius of 1in, height of 2in, has error in the radius of at most 0.01in and error in the height of at most 0.03in. Estimate the maximum error of the volume of the cone.

$$V = \frac{1}{3}\pi r^2 h$$
$$dV = V_r dr + V_h dh = (\frac{2}{3}\pi rh)dr + (\frac{1}{3}\pi r^2)dh$$
$$dV = (\frac{2}{3}\pi(1)(2))(0.03) + (\frac{1}{3}\pi(1)^2)(0.03) = \mathbf{\frac{7\pi}{300} in^3}$$

*If your problem has something like "up to 1% error in the radius" then

Critical Points of Functions Defined on Open Sets / Second Partials Test

Critical points:

1. Stationary points: points $(a, b) \in D$ where $\nabla f(a, b) = 0$
2. Singular points points $(a, b) \in D$ where $\nabla f(a, b)$ is not differentiable

The discriminant can be used to find what type of extrema a stationary point is.

$$D(a, b) = f_{xx}(a, b) f_{yy}(a, b) - (f_{xy}(a, b))^2$$

1. *If $D > 0$ and $f_{xx}(a, b) > 0 \Rightarrow f$ has a local minimum at (a, b).*
2. *If $D > 0$ and $f_{xx}(a, b) < 0 \Rightarrow f$ has a local maximum at (a, b).*
3. *If $D < 0 \Rightarrow f$ has a saddlepoint at (a, b).*
4. *If $D = 0 \Rightarrow$ test is inconclusive.*

Functions on Closed and Bounded Regions

In a closed bounded region D, the points on the boundary ∂D are also critical points. Closed bounded regions must also have a global maximum and global minimum by the Extreme Value Theorem

Lagrange Multipliers

Langrange Multipliers need a function f and a constraint g. There is some value of λ that satifies the following system of equations:

$$\nabla f(a, b) = \lambda \nabla g(a, b)$$

$$g(a, b) = 0$$

Just solve the system and then plug all of the points into f. The biggest value of f is the maximum, and smallest is the minimum.

Optimization Example 1

Example: minimize the distance from the origin to the surface $z^2 = xy - y + 3$

Distance squared from origin: $f(x, y, z) = x^2 + y^2 + z^2$
Getting 2-variable equation: $f(x, y) = x^2 + y^2 + xy - y + 3$
Find critical points: $\nabla f(x, y) =< 2x + y, 2y + x - 1 >= 0 =< 0, 0 >$
Solution at $x = -\frac{1}{3}, y = \frac{2}{3} \Rightarrow f(-\frac{1}{3}, \frac{2}{3}) = \frac{8}{3}$, which is a minimum
Because we used the equation for distance squared, the total distance is $\sqrt{\frac{8}{3}}$.

Optimization Example 2

Example: Find the minimum and maximum of $f(x, y) = x^2 + y$ on the circle $x^2 + y^2 = 1$

$$\nabla f(x, y) = \lambda \nabla g(x, y)$$

Solving system of three equations:

$$\text{Eq(1): } 2x = \lambda 2x \qquad \text{Eq(2): } 1 = \lambda 2y \qquad \text{Eq(3): } x^2 + y^2 = 1$$

$$2x(1 - \lambda) = 0 \Rightarrow x = 0 \; or \; \lambda = 1$$

$$x = 0 \Rightarrow by \; Eq(3), \; y = \pm 1$$

$$\lambda = 1 \Rightarrow by \; Eq(2), \; y = \frac{1}{2}$$

$$y = \frac{1}{2} \Rightarrow by \; Eq(3), \; x = \pm \frac{\sqrt{3}}{2}$$

Four points to check: $(0, 1), (0, -1), (\frac{\sqrt{3}}{2}, \frac{1}{2})$, and $(-\frac{\sqrt{3}}{2}, \frac{1}{2})$

$$f(0, 1) = 1 \qquad f(0, -1) = -1 \qquad f(\frac{\sqrt{3}}{2}, \frac{1}{2}) = f(\frac{-\sqrt{3}}{2}, \frac{1}{2}) = \frac{5}{4}$$

Maximum: $\frac{5}{4}$ **Minimum: -1**