CISC 102 (Fall 21)
Homework #3: Proofs (24 Points)

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Question 1

If we take the premise $\neg p \longrightarrow \neg q$ to be true, then saying $q \longrightarrow p$ is an application of modus tollens, which is valid.

Question 2

If x is any man, then the statement could be written as $P(x) \longrightarrow M(x)$, and if $\forall x P(x)$, then $True \longrightarrow M(x)$, $\therefore M(x)$ is an application of, universal instantiation, general instantiation, and modus ponens.

Question 3

Let r be rain, f be fog, s be the sailing competition, l be the lifesaving demonstration, and t be the trophy;
$\neg(r \wedge f) \longrightarrow (s \wedge l)$ *Premise
$s \longrightarrow t$ *Premise
$\neg t$ *Premise
$\neg t \longrightarrow \neg s$ *Modus Tollens (2)
$\neg s$ *Modus Ponens (4)
$\neg(s \wedge l) \longrightarrow \neg(\neg r \vee \neg f)$ *Modus Tollens (1)
$\neg s \vee \neg l \longrightarrow r \vee f$ *DeMorgan's (7)
$T \vee \neg l \longrightarrow r \wedge f$ *Sub in T for s (5)
$T \longrightarrow r \wedge f$ *Dominance Law (8)
$r \wedge f$ *Modus Ponens (9)
$\therefore r$ *Simplification (10)

Question 4

Let $x \in \mathbb{Z}$
This can be written as: $x^2 + (x+1)^2$
$= x^2 + x^2 + 2x + 1$

$= 2x^2 + 2x + 1$
$= 2\left(x^2 + x\right) + 1$
Since x is an integer, $\left(x^2 + x\right)$ will always yield some integer k.
$= 2\left(k\right) + 1$ is the definition of what it means to be an odd number.
$\therefore x^2 + \left(x + 1\right)^2$ will always yield an odd number.

Question 5

Case 1: n is even, let $n = 2k, k \in \mathbb{Z}$
$\left(2k\right)^3 + 2\left(\left(2k\right)^2\right) + 2k + 4$
$= 8k^3 + 8k^2 + 2k + 4$
$= 2\left(4k^3 + 4k^2 + k + 2\right)$
Since $k \in \mathbb{Z}, \left(4k^3 + 4k^2 + k + 2\right)$ must also yield some integer p.
$= 2\left(p\right)$
This is the definition of an even number, $\therefore$ when n is even, $n^3 + 2n^2 + n + 4$ is even.

Case 2: n is odd, let $n = 2k + 1, k \in \mathbb{Z}$
$\left(2k + 1\right)^3 + 2\left(\left(2k + 1\right)^2\right) + 2\left(k + 1\right) + 4$
$= 8k^3 + 12k^2 + 6k + 1 + 8k^2 + 8k + 2 + 2k + 1 + 4$
$= 8k^3 + 20k^2 + 16k + 8$
$= 2\left(4k^3 + 10k^2 + 8k + 4\right)$
Since $k \in \mathbb{Z}, \left(4k^3 + 10k^2 + 8k + 4\right)$ will always yield some integer p.
$= 2\left(p\right)$
This is the definition of an even number, $\therefore$ when n is odd, $n^3 + 2n^2 + n + 4$ is even.

Since the only 2 cases both yield even integers, we can conclude that $\forall n \in \mathbb{Z}$,
$n^3 + 2n^2 + n + 4$ is even.
QED

Question 6

Let p be the proposition 'For all rational number x and irrational number y, the sum of x and y is irrational.'
If we assume $\neg p$:
$x + y$ is rational.
$\longrightarrow \frac{a}{b} + y = \frac{c}{d}, a, b, c, d \in \mathbb{R}$
Definition of rational numbers, y is irrational (proposition q)
$\longrightarrow y = \frac{c}{d} - \frac{a}{b} \longrightarrow y = \frac{bc-ad}{bd}$
$\longrightarrow$ y can be expressed through fraction of real numbers $\longrightarrow$ (y is not irrational) $\longrightarrow \neg q$
$\longrightarrow \neg p \longrightarrow q \wedge \neg q$, this is a contradiction.
$\longrightarrow$ ($\neg p$ is false) $\longrightarrow$ (p is true), which means that x + y is irrational.

Question 7

P(1) $\equiv 1^2 \geq 1 \equiv 1 \geq 1 \equiv True$
This is a direct proof.

Question 8

Let p be the proposition 'at least three of any 25 days chosen must fall in the same month of the year.'
If we assume $\neg p$:
$\longrightarrow$ (Each month contains at most 2 of the 25 chosen days) Let q be the proposition that 'there are 25 chosen days'.
$\longrightarrow$ (Number of chosen days $\leq 2 \times 12$) $\longrightarrow$ (Number of chosen days ≤ 24) $\longrightarrow \neg q$
$\longrightarrow \neg p \longrightarrow \neg q \wedge q$, this is a contradiction.
($\neg p$ is false) $\longrightarrow$ (p is true), meaning that at least three of any 25 days chosen must fall in the same month of the year.

Question 9

(3x + 2 is even) $\equiv$ ($x + 5$ is odd) $\equiv$ (x^2 is even) *Premise
$\longrightarrow$ ((3x + 2 is even) $\leftrightarrow$ ($x + 5$ is odd) $\leftrightarrow$ (x^2 is even)) Logical Equivalence (1)
$\longrightarrow$ ((x^2 is odd) $\leftrightarrow$ ($x + 5$ is even) $\leftrightarrow$ ((3x + 2 is odd)) Contrapositive (2)

Case 1: x is even, x = 2k, $k \in \mathbb{Z}$
$\longrightarrow$ ((($2k)^2$ is odd) $\leftrightarrow$ (($2k) + 5$ is even) $\leftrightarrow$ ((3(2k) + 2 is odd))
$\longrightarrow$ (($2(2k^2)$ is odd) $\leftrightarrow$ ($2(k + 2) + 1$ is even) $\leftrightarrow$ (2(3k + 1) is odd))
$\longrightarrow$ (False) $\leftrightarrow$ (False) $\leftrightarrow$ (False)

Case 2: x is odd, x = 2k+1, $k \in /zz$
$\longrightarrow$ ((($2k + 1)^2$ is odd) $\leftrightarrow$ (($2k + 1) + 5$ is even) $\leftrightarrow$ ((3(2k+1) + 2 is odd))
$\longrightarrow$ ($2(2k^2 + 2k) + 1$ is odd) $\leftrightarrow$ ($2(k + 3)$ is even) $\leftrightarrow$ (2(3k + 2) + 1 is odd))
$\longrightarrow$ (True) $\leftrightarrow$ (True) $\leftrightarrow$ (True)

Since for both cases, all 3 contrapositive statements produce the same results, the 3 statements are all equivalent.

Question 10

It is easiest to show the the transitivity of equivalences using general predicates a, b, and c.
$(a \leftrightarrow b \wedge b \leftrightarrow c) \longrightarrow a \leftrightarrow c$ * Premise
$\equiv (a \longrightarrow b) \wedge (b \longrightarrow a) \wedge (b \longrightarrow c) \wedge (c \longrightarrow b) \longrightarrow a \leftrightarrow c$ *Logical Equivalence
$\longrightarrow (a \longrightarrow c) \wedge (c \longrightarrow a)$ *Hypothetical Syllogism
$\equiv (a \leftrightarrow c) \longrightarrow (a \leftrightarrow c)$ *Logical Equivalence
$\equiv$ True

With this, we can prove the equivalence across all four predicates, p_1, p_2, p_3, p_4, to show all of their equivalences by applying this previous lemma, we will call this lemma1.
We already know that $p_1 \leftrightarrow p_3, p_1 \leftrightarrow p_4, p_2 \leftrightarrow p_3$
1st Proof: $((p_1 \leftrightarrow p_3 \wedge p_1 \leftrightarrow p_4) \longrightarrow p_3 \leftrightarrow p_4)$ *Application of Lemma1
2nd Proof: $((p_1 \leftrightarrow p_3 \wedge p_2 \leftrightarrow p_3) \longrightarrow p_1 \leftrightarrow p_2)$ *Application of Lemma1
3rd Proof: $((p_2 \leftrightarrow p_3 \wedge p_3 \leftrightarrow p_4) \longrightarrow p_2 \leftrightarrow p_4)$ *Application of Lemma1 using previously found equivalence $(p_3 \leftrightarrow p_4)$
$\therefore p_1 \leftrightarrow p_2 \leftrightarrow p_3 \leftrightarrow p_4$
QED