Model Predictive Control, Project 3 v2: Stochastic MPC

Optimizing for an economic goal while dealing with uncertainties

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I. Introduction

A common problem in practical control applications is dealing with uncertainties which are unavoidable when using mathematical models to abstract from reality or measuring quantities. Not accounting for them may have unforeseen and sometimes severe consequences. Examples include the speed control of a vehicle in changing road conditions, the position control of an aircraft under turbulence or the energy management in buildings at the grid level. This is especially relevant for Model Predictive Control (MPC) where these models are used to predict future behavior of a given system to derive optimal control inputs. Thus methods to calculate robust control systems for MPC are needed. A promising approach is the use of Stochastic MPC as introduced by Calafiore & Campi [1].

In this project we dealt with an energy management control system for a single house. This presented many challenges due to a number of influencing factors related to grid occupancy, weather and additional energy generation by a solar roof as well as temperature and battery decay. To deal with these uncertainties we implemented a number of approaches based on the concept of Stochastic MPC.

A. Problem

The system of our task (see figure 1) is a house with a battery energy storage, access to a grid, a solar panel for electricity generation and a heating-, ventilation-, and air conditioning unit (HVAC). The control goal is to maintain the internal temperature as well as a reference state of charge (SoC) of the battery within predefined ranges. A controller should achieve this by setting appropriate levels of power consumption for the battery, the HVAC and the grid access while external disturbances, i.e. solar radiation, internal gains and external temperature, influence the house.

For a given discrete time step k the state of the system is denoted by $x_k = \begin{bmatrix} T_k & E_{\text{bat},k} \end{bmatrix}^T$, the control inputs by $u_k = \begin{bmatrix} P_{\text{hvac}} & P_{\text{bat}} & P_{\text{grid}} \end{bmatrix}^T$ and the external disturbances by $d_k = \begin{bmatrix} d_{\text{T}} & d_{\text{sr}} & d_{\text{int}} \end{bmatrix}^T$ using the notation given in table I.

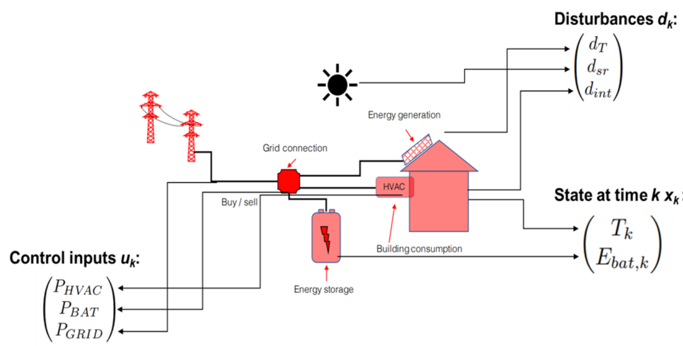


Fig. 1. The system

The discrete time behavior of the system is described by the equation

$$x_{k+1} = A \cdot x_k + B \cdot u_k + E \cdot d_k + w_k$$

where

$$A = \begin{bmatrix} \alpha_1 \cdot 0.8511 & 0 \\ 0 & \alpha_2 \cdot 1 \end{bmatrix},$$

$$B = \begin{bmatrix} 0.0035 & 0 & 0 \\ 0 & -5 & 0 \end{bmatrix}, E = \begin{bmatrix} 22.217 & 1.7912 & 42.212 \\ 0 & 0 & 0 \end{bmatrix}$$

and k describes a half-hour time step.

TABLE I
Notation used in the formalization

Symbol[a]	Meaning	Explanation
T_k	internal temperature of the house in °C	-
$E_{\text{bat},k}$	SoC of the battery	-
P_{hvac}	power consumed by the HVAC	≥ 0: HVAC is heating the house, < 0: HVAC is cooling the house
P_{bat}	power supplied to or taken from the battery	≥ 0: battery is discharging, < 0: battery is charging
P_{grid}	power supplied to or bought from the grid	≥ 0: energy is bought from the grid, < 0: energy is sold to the grid.
d_{T}	external temperature	-
d_{sr}	solar radiation	-
d_{int}	internal gains within the house	-

[a] All quantities aside from T_k are treated as dimensionless.

This equation is influenced by three uncertain quantities:

- α_1, α_2, which describe model uncertainties following a uniform distribution between 0.8 and 1.0
- $w_k = \begin{bmatrix} w_k^1 & w_k^2 \end{bmatrix}^T$, which describe additive disturbances, following some distribution between $\begin{bmatrix} -0.1 & -100 \end{bmatrix}^T$ and $\begin{bmatrix} 0.1 & 100 \end{bmatrix}^T$

The control goal is to minimize spending on grid power and to achieve a reference SoC $E_{\text{bat}}^{\text{ref}}$ in the battery while maintaining the state x_k in an acceptable range between $\begin{bmatrix} 20 & 0 \end{bmatrix}^T$ and $\begin{bmatrix} 23 & 200000 \end{bmatrix}^T$. Also, all power for the HVAC must be provided either by the grid, the battery or the solar panel such that the constraint

$$|P_{\text{hvac}}| \leq P_{\text{bat}} + P_{\text{grid}} + \underbrace{0.5 \cdot d_{\text{sr}}}_{\text{solar power}}$$

is fulfilled. While the control inputs must also obey certain bounds, those will be discussed later on in section III as they have greater implications on the viability of the solution.

Under these constraints the controller will try to minimize the equation

$$\sum_{k=0}^{N_{\text{pred}}-1} (P_{\text{grid}}^k + \gamma(E_{\text{bat}}^k - E_{\text{bat}}^{\text{ref}})^2)$$

for every single MPC time step where N_{pred} is the prediction horizon and γ is a parameter to weight deviation from the reference SoC.

To emphasize the need for a controller we simulated the behavior of the uncontrolled system for different $\alpha-$values (figure 2). The best case $\alpha_{1,2} = 1$ enables the system to retain its initial SoC while it is unable to achieve (nor retain) the minimal required temperature. If we set $\alpha_{1,2} = 0.8$ the temperature drops even further (to roughly 6°C). Additionally the system is not able to keep the reference SoC anymore and instead loses its charge entirely.

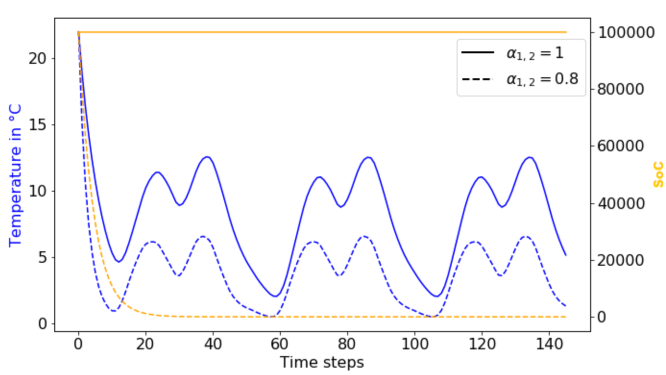


Fig. 2. The unctrolled system for different α values

II. Methodology

The main goals can be summarized as follows:

- The economic control goal is to maximize the energy sold to the grid.
- The state of charge in the battery should stay close to a desired energy level to ensure flexibility.

Given the behavior of the system, we need to make use of a method that can generate control sequences and that will achieve the stated goals while also being able to deal with the large variance inherent in the model uncertainties. To deal with this complex problem, we implemented and evaluated three approaches:

- a **Nominal MPC approach**,
- a **Stochastic MPC approach** for *convex* optimization problems and
- a **Stochastic MPC approach** for *non-convex* optimization problems.

For each of these approaches we implemented a controller using MPC for a simulation of the system consisting of 145 time steps (as this is the amount of time steps data for the external disturbances was available) with a prediction horizon $N_{\text{pred}} = 4$. We analyzed the performance of all controllers by subjecting them to a number of different scenarios, i.e. different values for $\alpha_{1,2}$ and sequences of w_k. The quality of the approaches was evaluated by their performance in terms of achieving the economic goal as defined above and satisfying the given system constraints. The associated optimization and MPC problems were solved using the Python package `CasADi`.

A. Nominal approach

For the purposes of this report we define the Nominal approach as calculating an optimal control sequence using MPC for a single system out of the set of all possible ones, typically a particularly representative one (e.g. a system whose uncertain parameters lie at the center of the associated probability distribution), and assuming no further random disturbances.

In our case this means a control sequence was calculated assuming $\alpha_1 = 0.95, \alpha_2 = 0.9$ and setting the additive disturbances $w_k = 0$ for every time step k. The external disturbances d_k were still accounted for as these are not subject to stochastic variance.

B. Stochastic MPC approach for convex optimization problems

A common method for dealing with uncertainties are worst-case approaches in which a controller is designed such that the worst possible scenarios for all uncertainties will not lead the controller to violate system constraints. However, this usually introduces large amounts of conservatism in terms of performance, especially when worst cases are not very probable.

The Stochastic MPC approach for convex systems proposed by Calafiore & Campi in [1] promises better performance by loosening the need to satisfy the system constraints for every possible scenario. Instead this method provides a probabilistic guarantee of a system's robustness. This is achieved by calculating a control sequence that satisfies all system constraints for a finite amount of sample scenarios N from the uncertainty space. For a predetermined probability ϵ of constraint violation

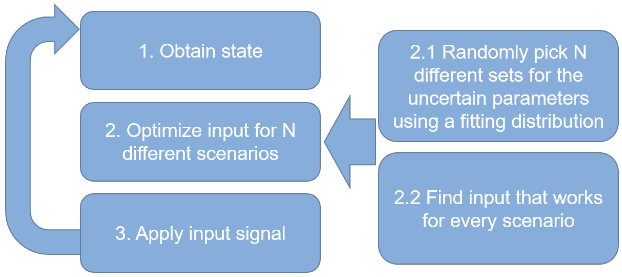


Fig. 3. Stochastic approach for the control loop

and an associated confidence level β, [1] gives a formula for obtaining a lower bound for N:

$$N \geq N_{log}(\epsilon, \beta) \doteq \left\lceil \frac{2}{\epsilon} \ln \frac{1}{\beta} + 2n_\theta + \frac{2n_\theta}{\epsilon} + \ln \frac{2}{\epsilon} \right\rceil, \quad (1)$$

where n_θ is the amount of free variables used in the associated MPC optimization problem, which in our case is $N_{\text{pred}} \cdot 3 = 12$ as there are three control inputs for each time step. As a consequence, at every time step of the MPC problem N scenarios are sampled from the underlying random distribution of the uncertainties (figure 3).

For our simulations, we set $\epsilon = 0.2$ and $\beta = 10^{-6}$, resulting in a sample size of $N = 439$. In practice, following the argumentation of [1] this means we would expect with a probability of $1 - \beta = 99.9999\%$ at most 20% of all control attempts calculated in this way to violate the system constraints.

Closed loop attempt: The approach described above works on an open loop paradigm as the method yields only one control sequence that has to work with all of the N sampled scenarios. Closed loop approaches tend to perform better and achieve higher rates of feasibility. This is because of the usage of a multitude of control sequences which incorporate feedback from the system including the uncertainties. The first control input is the same for every scenario, after that the sequence branches out as different scenarios provide different feedback. The amount of branches is determined by the *robust* prediction horizon $N_{\text{pred,R}}$. Since increasing $N_{\text{pred,R}}$ grows the computational cost exponentially we set $N_{\text{pred,R}} = 1$, resulting in N (as determined above) different control sequences with the same starting element. This single control input is then used at the respective MPC time step.

While performance and feasibility gains are desirable this method lacks the theoretical grounding from above: the lower bound given in formula 1 only to applies to open loop implementations.[1] For the simulations we used the same sample size as above, however, no guarantees about ϵ and β can be given this way.

C. Stochastic MPC approach for non-convex optimization problems

The above method delivers an a priori probabilistic guarantee of robustness for convex optimization problems. These results do not hold for non-convex optimization problems. In [2], Campi, Garatti & Ramponi introduce a similar method for achieving robustness which also works for non-convex problems. While an a priori calculation of a fitting sample size can not be given, instead a posteriori probabilities of constraint violation ϵ can be calculated for a set sample size N and confidence level β. For practical applications, [2] provides an algorithm for obtaining those:

1) Sample a set L of N scenarios, calculate optimal control input u_k for all of these
2) For all scenarios $l \in L$:
   - Calculate control input u_l for $L \setminus l$
   - If $u_k = u_l$:
     - $L := L \setminus l$
3) $s_k := |L|$

With s_k, which is the size of the minimal-length support sub-sample (see [2]), ϵ can be obtained with:

$$\epsilon = 1 - \sqrt[N-s_k]{\frac{\beta}{\binom{N}{s_k}}} \quad (2)$$

For our simulations, we chose $\beta = 10^{-6}$ and $N = 439$.

III. Results

In this section, we present the results obtained by applying the methods described in section II. For all runs described below, we used a γ-value of 0.5, a reference SoC level $E_{\text{bat}}^{\text{ref}} = 50000$ and an initial state $x_0 = \begin{bmatrix} 22 & 200000 \end{bmatrix}^T$. The additive disturbances were assumed to be uniformly distributed.

General considerations about the system constraints

Thus far, we avoided giving concrete bounds for the control inputs even though these are crucial parameters for the optimization problem. In the original task description, the bounds for the control inputs were given as

$$\begin{bmatrix} -1000 & -500 & -500 \end{bmatrix}^T \leq u_k \leq \begin{bmatrix} 1000 & 500 & 500 \end{bmatrix}^T.$$

With these bounds it is actually impossible to keep the system within the range of valid states for a large share of possible scenarios. Assuming $\alpha_1 = 0.8$, an initial temperature $T_0 = 20$ and a worst case additive disturbance $w_0^1 = -0.1$ one can derive a minimum P_{hvac} with which the system is able to avoid constraint violation from the temperature component of the state equation:

$$20 \overset{!}{\leq} 0.8511 \cdot 0.8 \cdot 20 + 0.0035 \cdot P_{\text{hvac,min}} - 0.1$$
$$\Rightarrow P_{\text{hvac,min}} \geq 1852.11429$$

To account for this we settled on increasing the given control input bounds, such that

$$\begin{bmatrix} -2000 & -2000 & -2000 \end{bmatrix}^T \leq u_k \leq \begin{bmatrix} 2000 & 1000 & 2000 \end{bmatrix}^T.$$

In particular, the lower bounds were adjusted downwards to account for scenarios where the α_2 parameter is near its lowest value and the battery charge is decaying rapidly.

[1] The calculation of N runs into an endless loop for the closed loop approach as n_θ influences N which in turn influences n_θ and so on.

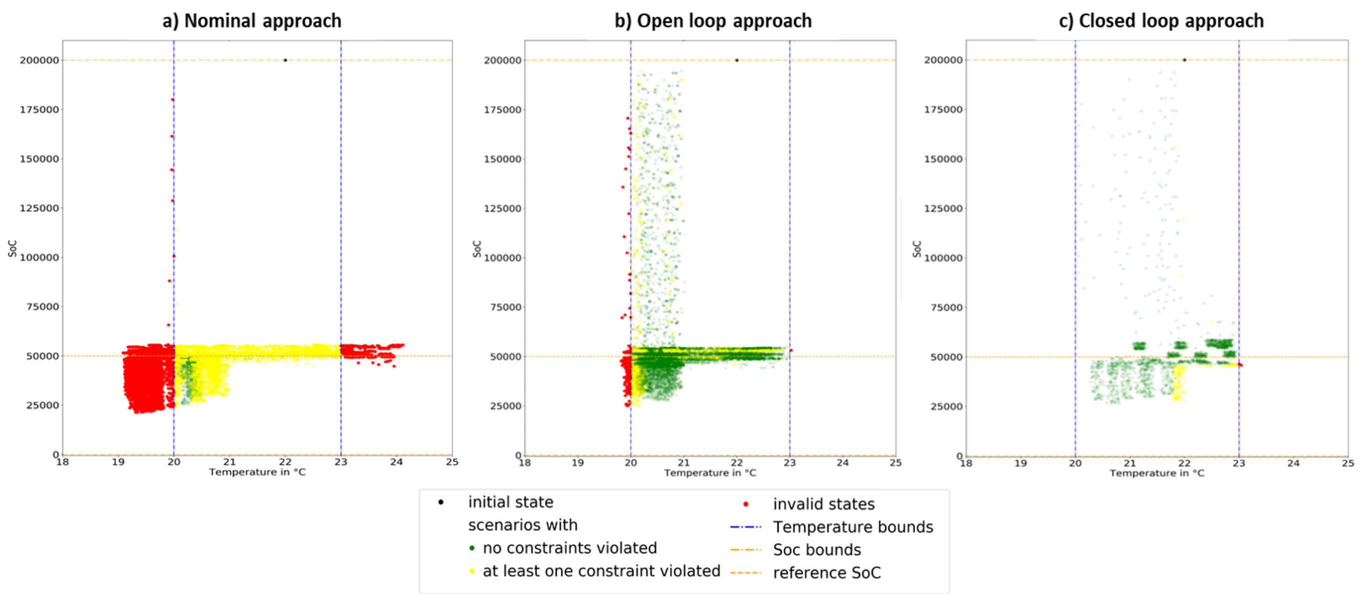


Fig. 4. Plots of all states from all time steps throughout 100 scenarios for the different approaches.

Even with this relaxation, in our experiments the solver often found the underlying optimization problem to be infeasible. To counteract this we further restricted the possible range of the α_1 parameter depending on the approach.

A. Nominal approach

In an experiment of running Nominal approach MPC the system delivered highly undesirable results (figure 4, a): in 94% of all scenarios the controller was unable to prevent the system states from leaving the allowed range. Those scenarios that did satisfy all constraints were those near the middle of the uncertainty range, i.e. very near to our chosen nominal system itself.

B. Stochastic MPC for convex optimization problems

Applying the Stochastic MPC approach on 100 trial runs yielded a result of a 21% failure rate (figure 4, b), which is almost exactly the expected value derived earlier.[2] As already mentioned before, we had to decrease the range of possible uncertainties for this approach to

$$\begin{bmatrix}0.95\\0.8\end{bmatrix} \leq \begin{bmatrix}\alpha_1\\\alpha_2\end{bmatrix} \leq \begin{bmatrix}1\\1\end{bmatrix}$$

to still achieve feasibility in most scenarios.

In terms of economic performance, the approach achieved a net grid power intake of ≈ 1818.12 for 145 time steps ($\equiv 3$ days) averaged over all runs, which translates to an expense of 41.35€ .[3]

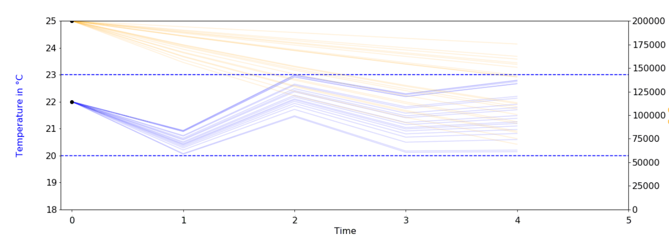
[2] As our sample size was only 100 due to time constraints, slight deviations from 20 as calculated earlier are to be expected.

[3] This is assuming an electricity price of 0.3137 $\frac{€}{\text{kWh}}$ (accurate as of January 2020 in Germany) and that P_{grid} is given in W.

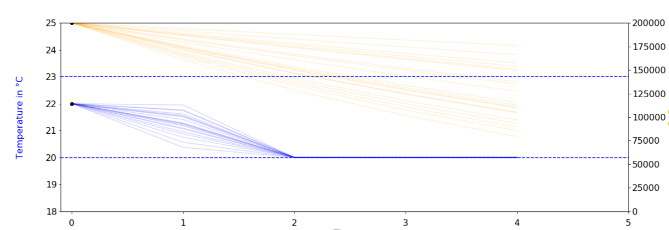
Closed loop: Using the closed loop approach as introduced earlier, we observed a failure rate of 6% in 50 trial runs (figure 4, c). This is a significantly lower result than that of the open loop. Additionally, this approach was able to achieve this on a larger range for the uncertainties:

$$\begin{bmatrix}0.9\\0.8\end{bmatrix} \leq \begin{bmatrix}\alpha_1\\\alpha_2\end{bmatrix} \leq \begin{bmatrix}1\\1\end{bmatrix}.$$

When rerunning this approach on the easier range of uncertainties as earlier, it performs similarly but slightly better for the economic goal: the net grid intake was ≈ 1812.41 (41.22€).



(a) Open loop



(b) Closed loop

Fig. 5. Initial state for both approaches with state predictions

Figure 5 shows the crucial difference between the open loop and the closed loop approach: In (a), the predictions for the future states diverge quickly as only one control sequence is considered while in (b), after an initial smaller divergence the states are predicted to converge to a point that is optimal in terms of the economic goal. This is a result of the multiple control sequences this approach takes into account.

C. Stochastic MPC for non-convex optimization problems

For this approach we reused the algorithm for solving MPC problems we used for the other approaches and inserted the a posteriori calculation of ϵ, meaning that this should yield similar results in terms of economic performance and feasibility. However, due to the large computational cost incurred by running the algorithm for determining ϵ as introduced in (2) it was not feasible for us to run the MPC loop for the entire simulation time of 145 steps. At each time step of the MPC loop, calculating s_k means that a much larger number of optimization problems than before needs to be solved. Instead, we settled on running only four iterations, which gave us a rough estimate of the computation time needed per iteration of $\sim$30 min (resulting in a total expected run time of $\sim$72 h).

In each of these iterations, the algorithm was able to find a minimal-length support sub-sample from all scenarios of sizes 2 or 3. This then translates to ϵ values of $5-6\%$. While these are significantly lower than the bounds derived earlier, they also only pertain to those iterations they were calculated for.

The time needed for each method discussed above is given in table II.

TABLE II

TIME NEEDED FOR EACH METHOD

Method	num runs	time/run	time total[b]
Nominal	100	$\sim$3.67 sec	$\sim$5 min
Stochastic	100	$\sim$5.24 min	$\sim$9 h
Stochastic (closed loop)	50	$\sim$3.36 min	$\sim$3 h
Stochastic (non-convex)	1	———	$\sim$2 h

[b]Processing was completed on an Intel Core i7-7700k CPU with 16 GB RAM available.

Influence of design parameters on results

We investigated the influence of the various design parameters and variables on the results of our system:

- We found that the γ-parameter had little influence on the behavior of our controllers. Only for very low values ($\leq$ 0.01) it became apparent that the controller deprioritized maintaining the reference SoC, completely discharging the battery for $\gamma = 0$. Figure 6 illustrates this phenomenon.
- N_{pred}, the prediction horizon, seemed to also have little influence on the economic goal, but had a significant impact on the feasibility of the underlying optimization problems, with larger prediction horizons causing more single optimizations to be infeasible.
- The reference SoC mainly determines how much energy is taken from the battery before most power is bought from the grid. As a consequence, lower values for

$$E_{\text{bat}}^{\text{ref}}$$

tend to cause better economic performance.
- As discussed above, the range of possible uncertainties for the parameter α has a significant impact on the feasibility of the MPC problems. The different approaches have different limits with regards to how big of an uncertainty they can handle while still delivering valid solutions.
- The control input bounds are a determining factor for the feasibility of the MPC problem. Using bounds which are too strict[4] causes infeasibility, as the state constraints simply cannot be satisfied with those control inputs.
- While the additive disturbances w_k in their given range have a comparatively small influence on the problem, increasing the value of $|w_{k,max}|$ has different consequences depending on the approach. As can be seen in figure 7 and table III, in our experiments the open loop Stochastic MPC approach (using the same inital state, reference SoC etc. as in section III) is able to handle a doubling of the maximum additive disturbance, while disturbances four or ten times the original cause a near 100% failure rate. The closed loop approach on the other hand achieves only slightly degraded performance even for very large disturbances.

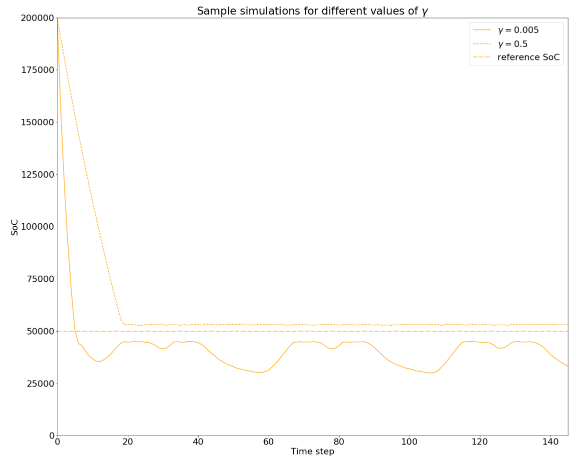


Fig. 6. Comparison of SoC over time for two sample simulations with different γ values

TABLE III

RESULTS FOR 30 FULL RUNS WITH DIFFERING MAXIMUM ADDITIVE DISTURBANCES

Method	Scale factor for $\lvert w_{k,max}\rvert$	Failure rate of 30 runs
open loop	2	16.67%
	4	96.67%
	10[c]	100%
closed loop	2	3.33%
	4	10%
	10[c]	13.33%

[c]This means that $|w_k^1|$ may be as large as 1°C.

[4]Translating to e.g. a smaller HVAC Unit in a real world scenario.

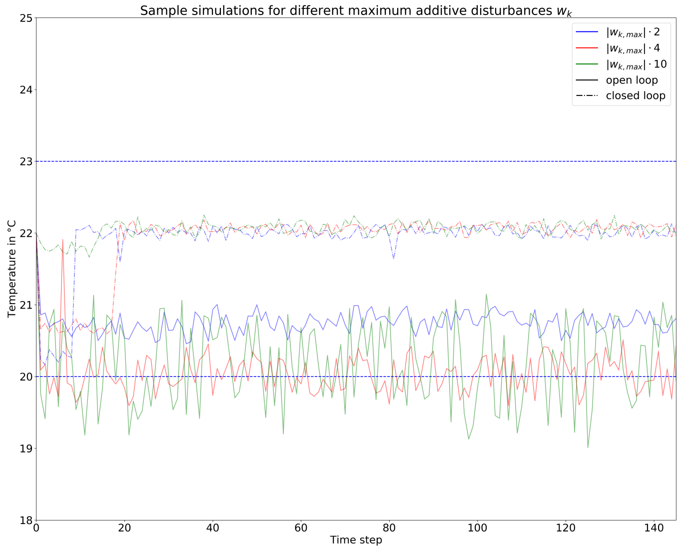


Fig. 7. Temperature time series for a number of representative experiment runs (compare with table III)

IV. Conclusions

In this work we tried to optimize the power consumption of a household from an energy grid. The goals were to minimize the grid spending while achieving a reference state of charge in the battery of the house. This is made difficult by uncertain parameters within the system formalization as well as additive disturbances.

To deal with this complex problem, we implemented several approaches:

- A *Nominal MPC approach* that assumes no disturbances and middle model. While this approach requires very few computational resources, we observed a very high failure rate, rendering this approach practically useless for real world applications.
- A *Stochastic MPC approach* for convex problems as introduced by [1]. This achieved the promises about robustness given by the theoretical groundwork with a failure rate of 21 % while also using computational resources appropriate for the problem scale (approx. 5 min computation time per complete run, which is achievable in the context of the problem given that it is discretized in 30 min steps).
- Based on this, we tried to apply the idea of a closed loop to this, which resulted in a significantly better failure rate of 6% and similar economic performance. However, there is no theoretical groundwork that could give guarantees about robustness, making this a less safe method for now.
- Finally, we implemented the *Stochastic MPC* approach for non-convex problems found in [2]. While our implementation achieved the results promised by the theory behind it, we found it to be too taxing on computational resources, which makes this approach an appropriate choice only if one is restricted to non-convex problems. Otherwise the Stochastic MPC for convex problems achieves similar performance using much less computation time.

Most of the restrictions and limitations discussed in this report, e.g. the above mentioned computation times, do not restrict the real world applicability of these approaches. Nevertheless, it is crucial that the formal description of the uncertainties is accurate, e.g. that a fitting probability distribution is chosen, as all methods of Stochastic MPC rely on sampling the uncertainty space.

Insofar this is true, the approaches discussed in this report seem to be fit for applications to real control problems.

References

[1] Calafiore, G. and Campi, M., 2006. The Scenario Approach to Robust Control Design. *IEEE Transactions on Automatic Control, 51(5)*, pp.742-753.

[2] Campi, M., Garatti, S. and Ramponi, F., 2018. A General Scenario Theory for Nonconvex Optimization and Decision Making. *IEEE Transactions on Automatic Control, 63(12)*, pp.4067-4078.