Exercises for Introduction to Quantum Computing

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In [ ]:
```
import numpy as np
import qiskit
import qiskit.circuit.library as circuit
import qiskit.quantum_info as qi
from qiskit import QuantumCircuit, QuantumRegister
from qiskit import IBMQ, Aer
from qiskit.providers.ibmq import least_busy
from qiskit import QuantumCircuit, transpile, execute
import math
from qiskit.visualization import plot_histogram
from qiskit.circuit.library import QFT
from qiskit.providers.aer import Aer
```

1. Multiplication by Quantum Fourier Transform

In this exercise sheet we would like to implement a Quantum multiplier i.e. a Quantum circuit that could multiply two numbers stored in two quantum registers upto modulo 2^n where n is the number of Qubits in either of the Quantum registers.

a) The position operator acting on the first quantum register is defined as,

$$\hat{x} = \sum_x x|x\rangle\langle x|$$

and the corresponding momentum operator acting on the first is defined to be,

$$\hat{p}_x = F_x^\dagger \hat{x} F_x$$

where $F_x \equiv \mathrm{qFT}_x$. We have two quantum registers,

$$|x\rangle|y\rangle = |x, y\rangle$$

We are to prove that,

$$e^{\mathrm{i}\hat{x}\hat{p}_y} = F_y^\dagger e^{\mathrm{i}\hat{x}\hat{y}} F_y$$

We shall first expand out the momentum operator in the L.H.S,

$$e^{\mathrm{i}\hat{x}\hat{p}_y} = e^{\mathrm{i}\hat{x}F_y^\dagger \hat{y} F_y}$$

Since does not act on , we can rewrite this to be,

$$e^{\mathrm{i}\hat{x}\hat{p}_y} = e^{F_y^\dagger (\mathrm{i}\hat{x}\hat{y}) F_y}$$

We know that,

$$e^{P.A.P^{-1}} = Pe^A P^{-1}$$

where A is some square matrix and P is an invertible matrix. We see that this identity is exactly what we need, substituting $A = \mathrm{i}\hat{x}\hat{y}$ and $P = F_y^\dagger$, we have

$$e^{\mathrm{i}\hat{x}\hat{p}_y} = F_y^\dagger e^{\mathrm{i}\hat{x}\hat{y}} F_y$$

Hence, proved.

b) We are to show that,

$$e^{ixy} = \prod_{k,l=1}^{n} e^{ix_k y_l 2^{n-k-l}}$$

Let's begin by writing down x and y in their binary representations

$$x = \sum_{n}^{k=1} x_k 2^{n-k}$$

$$y = \sum_{n}^{l=1} y_l 2^{n-l}$$

thus, we have,

$$e^{ixy} = e^{i\sum_{k,l=1}^{n} x_k 2^{n-k} . y_l 2^{n-l}}$$

$$e^{ixy} = e^{i\sum_{l,k}^{n} x_k y_l 2^{n-k-l}}$$

$$e^{ixy} = \prod_{k,l=1}^{n} e^{ix_k y_l 2^{n-k-l}}$$

Hence, proven.

c) Using the previous results, we are to show that,

$$e^{\frac{2\pi \mathrm{i}\hat{x}\hat{p}_y}{2^n}}|x,y\rangle = |x, x+y \mod 2^n\rangle$$

using the result from (a) let us rewrite the L.H.S,

$$e^{\frac{2\pi \mathrm{i}\hat{x}\hat{p}_y}{2^n}}|x,y\rangle = F_y^{\dagger} e^{\frac{2\pi \mathrm{i}\hat{x}\hat{y}}{2^n}} F_y|x,y\rangle$$

acting with F_y on $|x,y\rangle$, we get,

$$e^{\frac{2\pi \mathrm{i}\hat{x}\hat{p}_y}{2^n}}|x,y\rangle = \frac{F_y^{\dagger}}{2^{\frac{n}{2}}}\sum_{k=1}^{n} e^{\frac{2\pi \mathrm{i}\hat{x}\hat{k}}{2^n}} e^{\frac{2\pi iky}{2^n}}|x,k\rangle$$

from (b) we know that this can then be written as,

$$e^{\frac{2\pi \mathrm{i}\hat{x}\hat{p}_y}{2^n}}|x,y\rangle = \frac{F_y^{\dagger}}{2^{\frac{n}{2}}}\sum_{k=1}^{n} e^{\frac{2\pi \mathrm{i}\hat{x}\hat{k}}{2^n}} e^{\frac{2\pi ik(x+y)}{2^n}}|x,k\rangle$$

now we apply the inverse Fourier transform to get,

$$e^{\frac{2\pi \mathrm{i}\hat{x}\hat{p}_y}{2^n}}|x,y\rangle = \frac{1}{2^n}\sum_{k,m=0}^{2^n-1} e^{\frac{2\pi ik[(x+y)-m]}{2^n}}|m\rangle$$

we use the identity,

$$\delta_{b,c} = \frac{1}{N}\sum_{j=1}^{N} e^{\frac{2\pi ij.(b-c)}{N}}$$

Thus we have,

$$e^{\frac{2\pi \mathrm{i}\hat{x}\hat{p}_y}{2^n}}|x,y\rangle = |x, x+y \mod 2^n\rangle$$

d) Now, we shall consider three quantum registers |x⟩, |y⟩, and |z⟩ with n qubits each. We shall attempt to construct a multiply add operation of the form,

$$|x, y, z\rangle \rightarrow |x, y, z + xy \bmod 2^n\rangle$$

First, we need to construct a circuit that multiplies x and y ($mod 2^n$). We know that multiplying x by y simply implies that, we are adding x to itself y times. Thus, a multiplier circuit is simply a quantum adder circuit between the state $|0\rangle$ and $|x\rangle$ performed iteratively y times (this is implemented either by adding a controlled not between the $|y\rangle$ state and the adder for every iteration and decrementing after each iteration or via controlled weighted addition). The addition of z to the product can then be implmented via an adder circuit, let us prove this now, we saw from (c), that the addition operation can be represented via,

$$e^{\frac{2\pi \mathrm{i} \hat{x} \hat{p}_y}{2^n}} |x, y\rangle = |x, x + y \bmod 2^n\rangle$$

to multiply x and y, we simply do,

$$(e^{\frac{2\pi \mathrm{i} \hat{x} \hat{p}_2}{2^n}})^y |x, 0, z\rangle = |x, xy, z\rangle$$

Where the 0 here actual stands for $|0\rangle^{\otimes n}$ i.e. we have initialized the second register i.e. the accumulator to be in the $|0\rangle$ state. The second register now contains the product $xy \bmod 2^n$. The addition with z can then be written in the form,

$$e^{\frac{2\pi \mathrm{i} \hat{y} \hat{p}_z}{2^n}} (e^{\frac{2\pi \mathrm{i} \hat{x} \hat{p}_2}{2^n}})^y |x, 0, z\rangle = |x, xy, z + xy \bmod 2^n\rangle$$

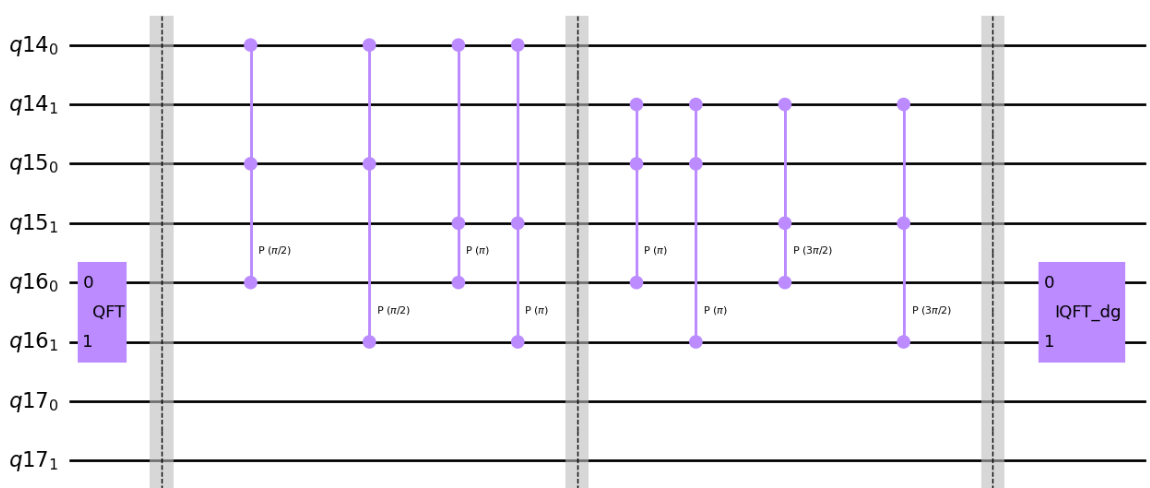
e) We shall now implement this multiply-add operation in Qiskit.

In [ ]:
```
ccp = circuit.UGate(0.1,0,0).control(2, ctrl_state="11")
```

In [ ]:

```
n = 2 #number of qubits in one register

x = QuantumRegister(n)
y = QuantumRegister(n)
ancilla = QuantumRegister(n)
z = QuantumRegister(n)

qc = QuantumCircuit(x,y,ancilla,z)

param = np.pi

ccp = circuit.PhaseGate(param).control(2, ctrl_state="11")



qc.compose(QFT(n, inverse=False, do_swaps=True), [3*n-i for i in range(n,0,-1)], in
qc.barrier()

for i in range(n):
    for j in range(n):
        for k in range(n):
            param = (2*np.pi*((3*n)-3-(int(i)+int(j)+int(k))))/(2**(n))
            ccp = circuit.PhaseGate(param).control(2, ctrl_state="11")
            qc.append(ccp, [x[i],y[j],ancilla[k]])
    qc.barrier()

qc.compose(QFT(n, inverse=True,do_swaps=True), [3*n-i for i in range(n,0,-1)], inpl

qc.draw('mpl')
```

Out[ ]:



In [ ]:

```
state_2 = qi.Statevector.from_label('000101')
state_2 = state_2.evolve(qc)
(state_2.draw(output='latex'))
```

Out[ ]:

$$|110101\rangle$$