Exercises for Introduction to Quantum Computing

Name: Pugazharasu Anancia Devaneyan (s6puanan)
Matriculation number: 3300280

1 Mapping 1+1D Quantum Electrodynamics to Qubits

a) We are to show that the mass term of staggered fermions contains a factor of . We know that the mass term of the lattice Hamiltonian reads as,

$$\sum_{n=0,2,4,...}^{N-2} \bar{\psi}(n)\psi(n)$$

We can transform the fermions into staggered fermions via the transform,

$$\psi(n) = \sqrt{a}\begin{pmatrix} \phi(n) \\ \phi(n+1) \end{pmatrix}$$

Thus, for a single term we have

$$\bar{\psi}(n)\psi(n) = a\left[\phi^{\dagger}(n)\phi(n) - \phi^{\dagger}(n+1)\phi(n+1)\right]$$

Therefore, we can rewrite the sum to be

$$\sum_{n=0,2,4,...}^{N-2} \bar{\psi}(n)\psi(n) = a\sum_{n=0}^{N-1}(-1)^n \phi^{\dagger}(n)\phi(n)$$

b) We will now show that the Jordan Wigner transformation given by,

$$\phi(n) = \left(\prod_{l<n} i\sigma_3(l)\right)\sigma^{-}(n)$$

$$\phi^{\dagger}(n) = \left(\prod_{l<n} -i\sigma_3(l)\right)\sigma^{+}(n)$$

is a faithful transformation i.e. it preserves the Fermionic algebra.

$$\left\{\phi^{\dagger}(n), \phi(m)\right\} = \delta_{nm}$$

$$\{\phi(n), \phi(m)\} = 0$$

Let's look at the first anti-commutator relation,

$$\left\{\phi^{\dagger}(n), \phi(m)\right\} = \left\{\left(\prod_{l<n} -i\sigma_3(l)\right)\sigma^{+}(n), \left(\prod_{l<m} i\sigma_3(l)\right)\sigma^{-}(m)\right\}$$

The Pauli strings cancel if an only if $n \leq m$, thus we have,

$$\left\{\phi^{\dagger}(n), \phi(m)\right\} = \delta_{nm}\left\{\sigma^{+}(n), \sigma^{-}(m)\right\}$$

We know that,

$$\left\{\sigma^{+}(n), \sigma^{-}(m)\right\} = \mathbb{I}$$

Thus,

$$\left\{\phi^{\dagger}(n), \phi(m)\right\} = \delta_{nm}$$

QED. Now to prove the other Fermionic anti-commutator relation, we have,

$$\{\phi(n), \phi(m)\} = \left\{\left(\prod_{l<n} i\sigma_3(l)\right)\sigma^{-}(n), \left(\prod_{l<m} i\sigma_3(l)\right)\sigma^{-}(m)\right\}$$

Once again, due to the cancellation of the Jordan-Wigner strings, we have,

$$\{\phi(n), \phi(m)\} = \delta_{nm}\left\{\sigma^{-}(n), \sigma^{-}(m)\right\}$$

We know that,

$$\left\{\sigma^{-}(n), \sigma^{-}(m)\right\} = 0$$

$\forall n, m$, thus,

$$\{\phi(n), \phi(m)\} = 0$$

QED.

c) We shall now rewrite the in terms of spin operators by means of the Jordan Wigner transformation which we explored in (b). Our full lattice Hamiltonian reads as,

$$H=-ix\sum_{n=0}^{N-1}\left[\phi^{\dagger}(n)e^{i\theta(n)}\phi(n+1)-\text{ h.c. }\right]+\mu\sum_{n=0}^{N-1}(-1)^{n}\phi^{\dagger}(n)\phi(n)+\sum_{n=0}^{N-1}L^{2}(n)$$

substituting the spin operators we have,

$$H=-ix\sum_{n=0}^{N-1}\left[\left(\prod_{l<n}-i\sigma_{3}(l)\right)\sigma^{+}(n)e^{i\theta(n)}\left(\prod_{l<n}i\sigma_{3}(l)\right)\sigma^{-}(n+1)-\text{ h.c. }\right]+\mu\sum_{n=0}^{N-1}(-1)^{n}\left(\prod_{l<n}-i\sigma_{3}(l)\right)\sigma^{+}(n)\left(\prod_{l<n}i\sigma_{3}(l)\right)\sigma^{-}(n)+\sum_{n=0}^{N-1}L^{2}(n$$

the Jordan-Wigner strings i.e. the product of the Pauli Z operators that are non-local, cancel out almost completely to leave,

$$H=-ix\sum_{n=0}^{N-1}\left[\sigma^{+}(n)e^{i\theta(n)}\sigma_{3}(n)\sigma^{-}(n+1)-\text{ h.c. }\right]+\mu\sum_{n=0}^{N-1}(-1)^{n}\sigma^{+}(n)\sigma^{-}(n)+\sum_{n=0}^{N-1}L^{2}(n)$$

We know that,

$$\left[\sigma^{+}(n),\sigma^{-}(m)\right]=2\delta_{nm}\sigma_{3}(n)$$

$$\left[\sigma^{\pm}(n),\sigma_{3}(n)\right]=0$$

and we can commute the Z Pauli through the exponential since it is just a scalar, thus we have,

$$H=-x\sum_{n=0}^{N-1}\left[\sigma^{+}(n)e^{i\theta(n)}\sigma^{-}(n+1)+\text{ h.c. }\right]+\frac{\mu}{2}\sum_{n=0}^{N-1}(-1)^{n}\sigma_{3}(n)+\sum_{n=0}^{N-1}L^{2}(n)$$

d) We are given the transformations,

$$s^{-}(n)=\prod_{l<n}e^{-i\theta(l)}\sigma^{-}(n)$$

$$s^{+}(n)=\prod_{l<n}e^{i\theta^{\dagger}(l)}\sigma^{+}(n)$$

we need to invert these in order to substitute them into the Hamiltonian,

$$\sigma^{-}(n)=\prod_{l<n}e^{i\theta(l)}s^{-}(n)$$

$$\sigma^{+}(n)=\prod_{l<n}e^{-i\theta^{\dagger}(l)}s^{+}(n)$$

Thus, inserting these into the Hamiltonian we get,

$$H=-x\sum_{n=0}^{N-1}\left[s^{+}(n)s^{-}(n+1)-h.c\right]+\frac{\mu}{2}\sum_{n=0}^{N-1}(-1)^{n}\sigma_{3}(n)+\sum_{n=0}^{N-1}L^{2}(n)$$

From Gauss' law, we know that,

$$L(n)-L(n-1)=q(n)$$

where the Fermionic charge is given by,

$$q(n)=\phi^{\dagger}(n)\phi(n)-\frac{1}{2}[1-(-1)^{n}]$$

Rewriting this in terms of spins, we have,

$$q(n)=\frac{1}{2}[(-1)^{n}+-i\sigma_{3}]$$

Applying Gauss' law recursively and the spin reformulation of the Fermionic charge to the final term of the Hamiltonian we get,

$$+\sum_{n=0}^{N-1}L^{2}(n)=\sum_{n=1}^{N-2}\left[\frac{1}{2}\sum_{m=1}^{n}\left(\sigma_{3}(m)+(-1)^{m}\right)\right]^{2}$$

Thus, the full Hamiltonian reads as,

$$H=-x\sum_{n=0}^{N-1}\left[s^{+}(n)s^{-}(n+1)-h.c\right]+\frac{\mu}{2}\sum_{n=0}^{N-1}(-1)^{n}\sigma_{3}(n)+\sum_{n=1}^{N-2}\left[\frac{1}{2}\sum_{m=1}^{n}\left(\sigma_{3}(m)+(-1)^{m}\right)\right]^{2}$$

e) If we set periodic boundary conditions i.e.

$$L(N) = L(0)$$

the topology of the lattice reduced to that of a circle. From the definition of the gauge field, we would have,

$$L(N) = L(N-1) + q(N)$$

Doing this recursively, we have,

$$L(N) = \sum_{i=1}^{N} q(i) + L(0)$$

From the boundary conditions we would have,

$$\sum_{i=1}^{N} q(i) = 0$$

Thus, the gauge field cannot be eliminated.