Exercises for Introduction to Quantum Computing

Name: Pugazharasu Anancia Devaneyan (s6puanan)

Matriculation number: 3300280

In [ ]:
```
import numpy as np
import qiskit
import qiskit.quantum_info as qi
import qiskit.quantum_info as quantu_info
from qiskit import IBMQ, Aer
from qiskit.providers.ibmq import least_busy
from qiskit import QuantumCircuit, transpile, execute
import math
from qiskit.visualization import plot_histogram
from qiskit.circuit.library import QFT
from qiskit.providers.aer import Aer
```

1. Addition by Quantum Fourier Transform

In this exercise sheet we would like to implement a Quantum adder i.e. a Quantum circuit that could add two numbers stored in two quantum registers upto modulo 2^n where n is the number of Qubits in the Quantum register

a) The action of the $R_j(y)$ on a qubit $|\psi\rangle = |0\rangle + e^{2\pi i x 2^{-j}}|1\rangle$ is given by,

$$R_j(y)|\psi\rangle = |0\rangle + e^{2\pi i\left(x2^{-j}+y2^{-j}\right)}|1\rangle$$

This resembles the action of a phase gate and is indeed, a phase rotation gate of the form

$$R_j(y) = \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i y.2^{-j}} \end{pmatrix}$$

Such a matrix can be implemented via the $U_1(\lambda)$ gate,

$$U_1(\lambda) = \begin{pmatrix} 1 & o \\ 0 & e^{i\lambda} \end{pmatrix}$$

with $\lambda = \frac{2\pi y}{2^j}$

b) We are to evaluate

$$A(y) = \mathrm{qFT}^\dagger \left(R_n(y)R_{n-1}(y)\ldots R_1(y)\right) \mathrm{qFT}$$

acting on an n-qubit register $|x\rangle = |x_1 x_2 \ldots x_n\rangle$.

$$A(y)|x\rangle = \mathrm{qFT}^\dagger \left(R_n(y)R_{n-1}(y)\ldots R_1(y)\right) \mathrm{qFT}|x\rangle$$

Evaluating the action of the qFT on $|x\rangle$, we get,

$$A(y)|x\rangle = \frac{1}{2^{\frac{n}{2}}} \mathrm{qFT}^\dagger \left(R_n(y)R_{n-1}(y)\ldots R_1(y)\right) \sum_{k=0}^{2^n-1} e^{\frac{2\pi i k x}{2^n}} |k\rangle$$

the action of the rotation gates, adds a phase to the exponential,

$$A(y)|x\rangle = \frac{1}{2^{\frac{n}{2}}} \mathrm{qFT}^\dagger \sum_{k=0}^{2^n-1} e^{\frac{2\pi i k(x+y)}{2^n}} |k\rangle$$

the inverse-qFT

$$A(y)|x\rangle = \frac{1}{2^n} \sum_{k,m=0}^{2^n-1} e^{\frac{2\pi i k[(x+y)-m]}{2^n}} |m\rangle$$

From the theory of discrete-Fourier transforms, we have the identity,

$$\delta_{b,c} = \frac{1}{N} \sum_{j=1}^{N} e^{\frac{2\pi i j.(b-c)}{N}}$$

where,

$$\delta_{b,c} = \begin{cases} 1 & \text{if, } b = c \\ 0 & \text{if, } b \neq c \end{cases}$$

Applying this to the previous equation,

$$A(y)|x\rangle = \sum_{m=0}^{2^n-1} \delta_{x+y,m} |m\rangle$$

thus we have,

$$A(y)|x\rangle = |x + y \pmod{2^n}\rangle \tag{1}$$

c) For the case of $x + y \geq 2^n$, we have

$$x + y \neq m$$

as m lies between 0 and $2^n - 1$, thus we would have

$$A(y)|x\rangle = 0$$

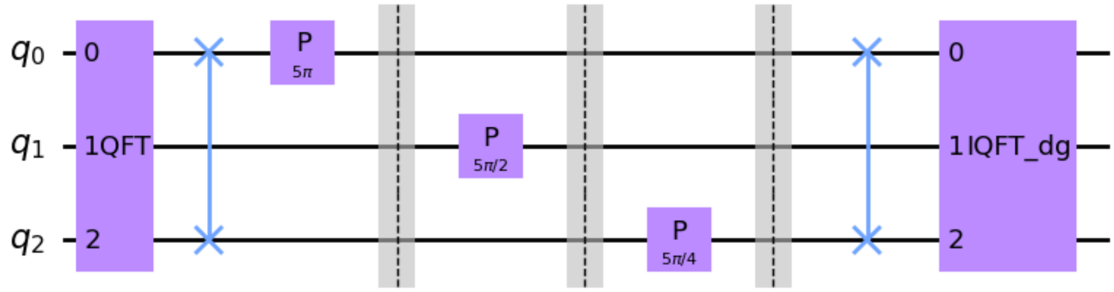
$\forall x + y \geq 2^n$

d) Let's now implement the circuit for the case of 3-Qubits,

In [ ]:
```
# Defining the swap function for the action of SWAP gates over all Qubits
def swap_registers(circuit, n):
    """Swaps all the Qubits in the circuit by means of applying appropriate SWAP ga

    Args:
        circuit (QuantumCircuit):
        n (int): Number  of Qubits

    Returns:
        QuantumCircuit : Returns the swapped circuit
    """
    for qubit in range(n//2):
        circuit.swap(qubit, n-qubit-1)
    return circuit


def A(y,qubits):
    """ Performs an addition of x (stored in the qubits) and y provided x + y < 2^{

    Args:
        y (int): the integer to be added  to x
        qubits (int): number of qubits

    Returns:
        QuantumCircuit : Returns the Quantum adder circuit
    """
    # Create and set up circuit
    qFT_adder = QuantumCircuit(qubits)
    qFT_adder = qFT_adder.compose(QFT(qubits, inverse=False), np.arange(0,qubits,1)
    swap_registers(qFT_adder,3)
    for k in range(qubits):
        param = (2*np.pi*y)/(2**(k+1))
        qFT_adder.p(param,k)
        qFT_adder.barrier()
    swap_registers(qFT_adder,3)
    qFT_adder = qFT_adder.compose(QFT(qubits, inverse=True), np.arange(0,qubits,1))
    return qFT_adder
```

In [ ]:
```
A(5,3).draw('mpl')
```

Out[ ]:



In [ ]:
```
qFT_adder = A(5,3)
```

Now let's check if the circuit gives the outputs we expect

In [ ]:
```
state_2 = qi.Statevector.from_label('000') # initialising the state as |101>
state_2 = state_2.evolve(qFT_adder)
(state_2.draw(output='latex'))
```

Out[ ]:

$$|101\rangle$$

In [ ]:
```
state_2 = qi.Statevector.from_label('101') # initialising the state as |101>
state_2 = state_2.evolve(qFT_adder)
(state_2.draw(output='latex'))
```

Out[ ]:

$$|010\rangle$$

We see that indeed we get the expected results. We can thus conclude that we have sucessfully implemented the Quantum adder circuit for the case of 3 Qubits.

e) We are to compute,

$$cA_{y,x}|y\rangle|x\rangle = \mathrm{qFT}_x^\dagger \left(\prod_{\ell_n=1}^{n} cR_{y_{\ell_n}}x_n\left(2^{n-\ell_n}\right)\prod_{\ell_{n-1}=2}^{n} cR_{y_{\ell_{n-1}}x_{n-1}}\left(2^{n-\ell_{n-1}}\right)\dots\prod_{\ell_1=n}^{n} cR_{y_{\ell_1}x_1}\left(2^{n-\ell_1}\right.$$

Let us begin by computing the action of the qFT.

$$qFT_x|y\rangle|x\rangle = \frac{1}{2^{\frac{n}{2}}}|y\rangle \otimes_{l=1}^{n} |0\rangle + e^{\frac{2\pi ikx}{2^l}}|1\rangle$$

The product of the controlled rotation gates $\prod_{l_a=n-a+1}^{n} cR_{y_{l_a x_{l_a}}(2^{n-l_a})}$ only acts on $|x_a\rangle$, thus it's action on the states after the action of the qFT_x reads as:

$$\left(\prod_{\ell_n=1}^{n} cR_{y_{\ell_n}}x_n\left(2^{n-\ell_n}\right)\prod_{\ell_{n-1}=2}^{n} cR_{y_{\ell_{n-1}}x_{n-1}}\left(2^{n-\ell_{n-1}}\right)\dots\prod_{\ell_1=n}^{n} cR_{y_{\ell_1}x_1}\left(2^{n-\ell_1}\right)\right)\mathrm{qFT}_x|y\rangle|x\rangle = \frac{\cdot}{2}$$

Now acting on this state with a $qFT^\dagger$ will give us

$$cA_{y,x}|y\rangle|x\rangle = \frac{1}{2^n}\sum_{k=0}^{2^n-1} e^{\frac{2\pi i(x+y-m)k}{2^n}}|y\rangle|m\rangle$$

using the same identity we employed for (b), we see that

$$cA_{y,x}|y\rangle|x\rangle = |y\rangle|x+y \pmod{2^n}\rangle$$