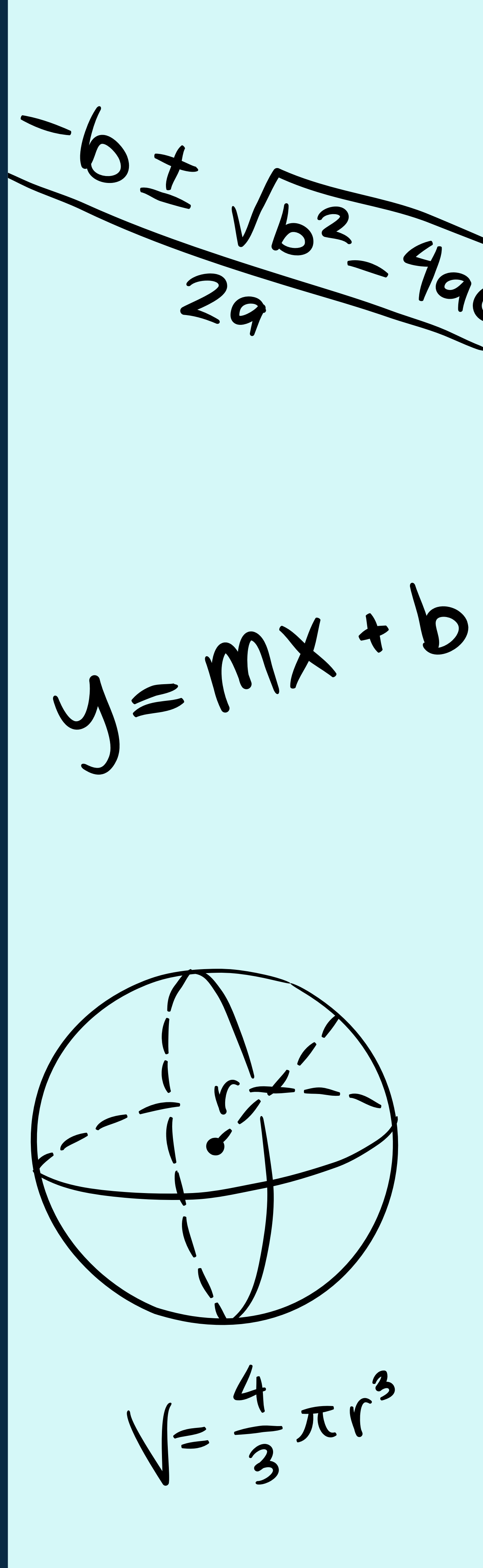
Linear Algebra

1-1 ~ 1-3

BIG PICTURE

1. linear system 怎麼解？
2. matrix 運算



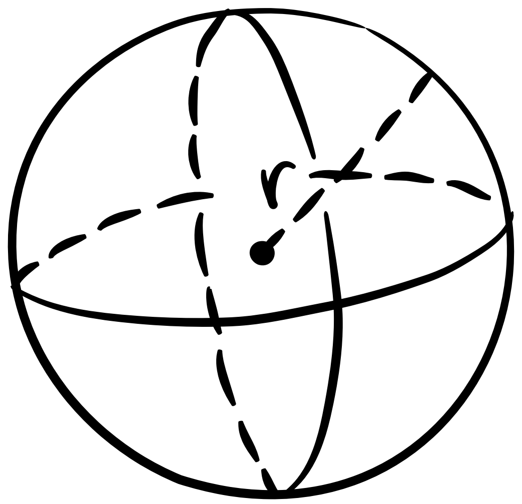
LINEAR SYSTEM

Linear system:

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}
$$

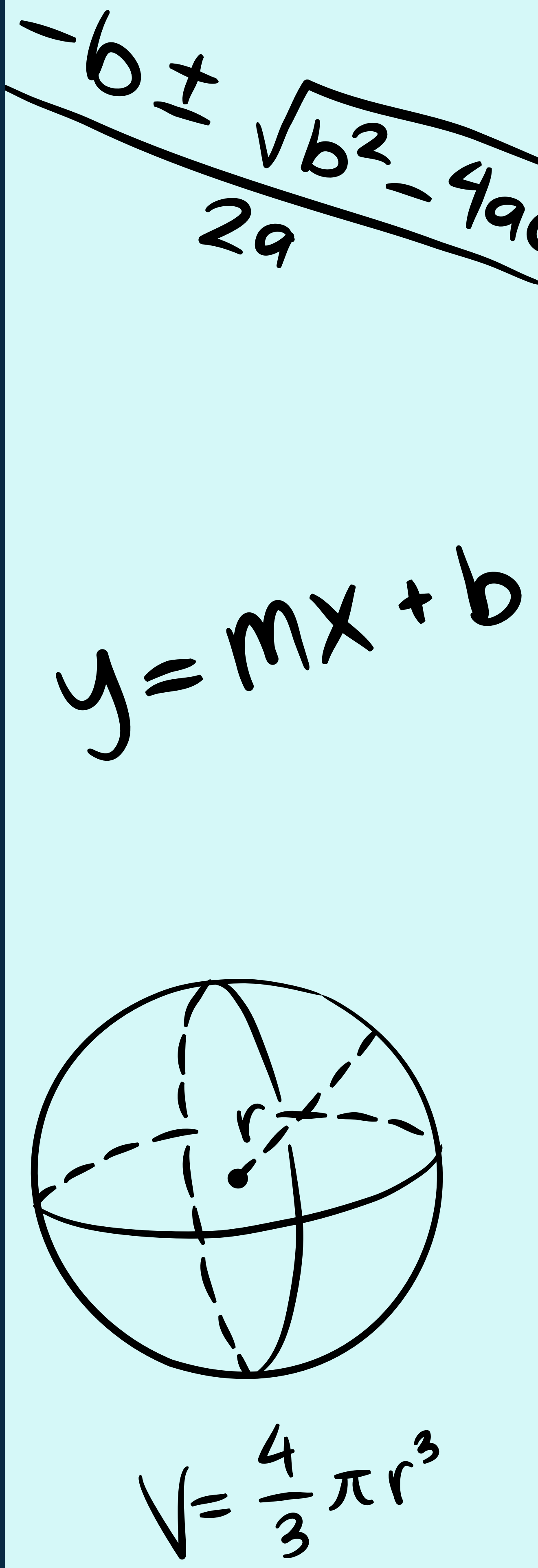
$\frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$y = mx + b$



$V = \frac{4}{3}\pi r^3$

HOW TO SOLVE ?



Examples:

$$\begin{cases} x_1 + x_2 + x_3 = 3 \\ \qquad\;\; x_2 + x_3 = 2 \\ \qquad\qquad\;\;\, x_3 = 1 \end{cases}$$

what are x1, x2, x3 ?

ANSWERS:

back substitute

$$\begin{cases} x_3 = 1 \\ x_2 + x_3 = 2 \\ x_1 + x_2 + x_3 = 3 \end{cases}$$

$$\begin{cases} x_3 = 1 \\ x_2 + 1 = 2 \\ x_1 + x_2 + 1 = 3 \end{cases}$$

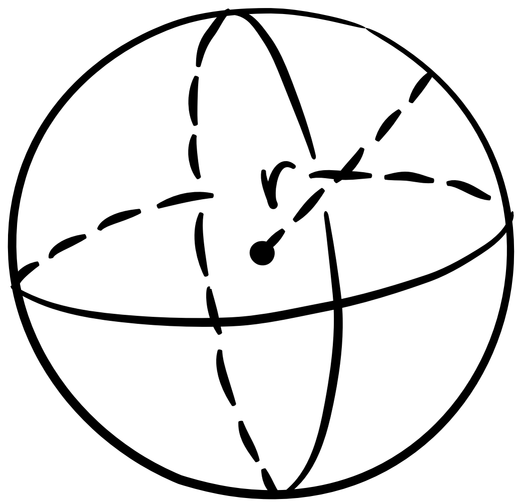
$$\begin{cases} x_3 = 1 \\ x_2 = 1 \\ x_1 + 1 + 1 = 3 \end{cases}$$

$$\begin{cases} x_3 = 1 \\ x_2 = 2 \\ x_1 = 3 \end{cases}$$

other method ?

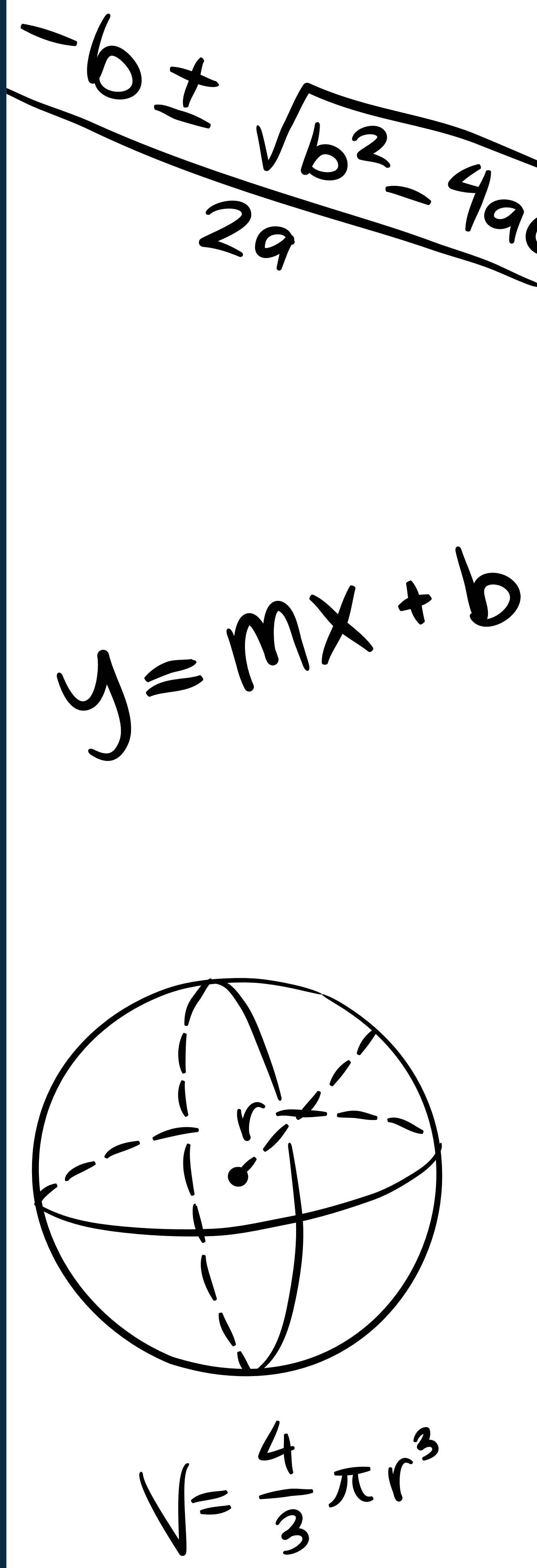
$\frac{-b \pm \sqrt{b^2 - 4a}}{2a}$

$y = mx + b$



$V = \frac{4}{3}\pi r^3$

MATRIX FORM ?



MATRIX:

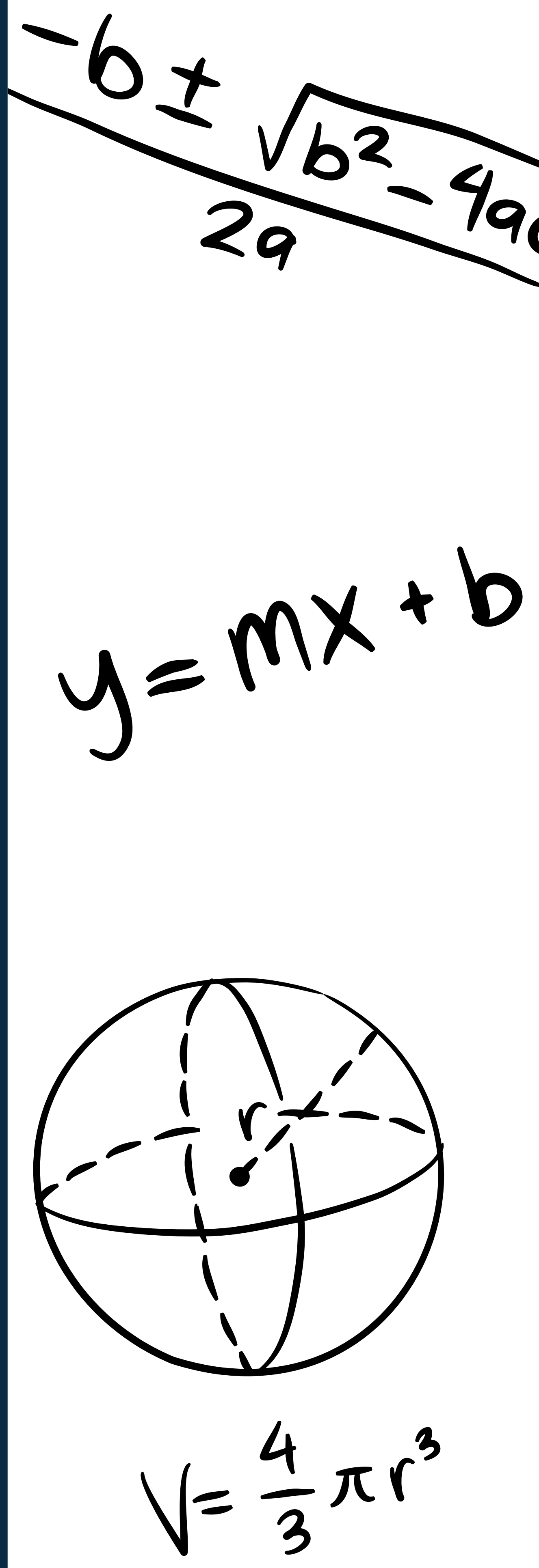
$$\begin{cases} x_1 + x_2 + x_3 = 3 \\ \phantom{x_1 + {}} x_2 + x_3 = 2 \\ \phantom{x_1 + x_2 + {}} x_3 = 1 \end{cases}$$

SAME

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

coefficient matrix

arugument matrix



Another Example

original

$$\begin{cases} x_1 + x_2 + x_3 = 3 \\ 2x_1 + 2x_2 + 2x_3 = 6 \\ x_3 = 1 \end{cases}$$

(2) divide 2

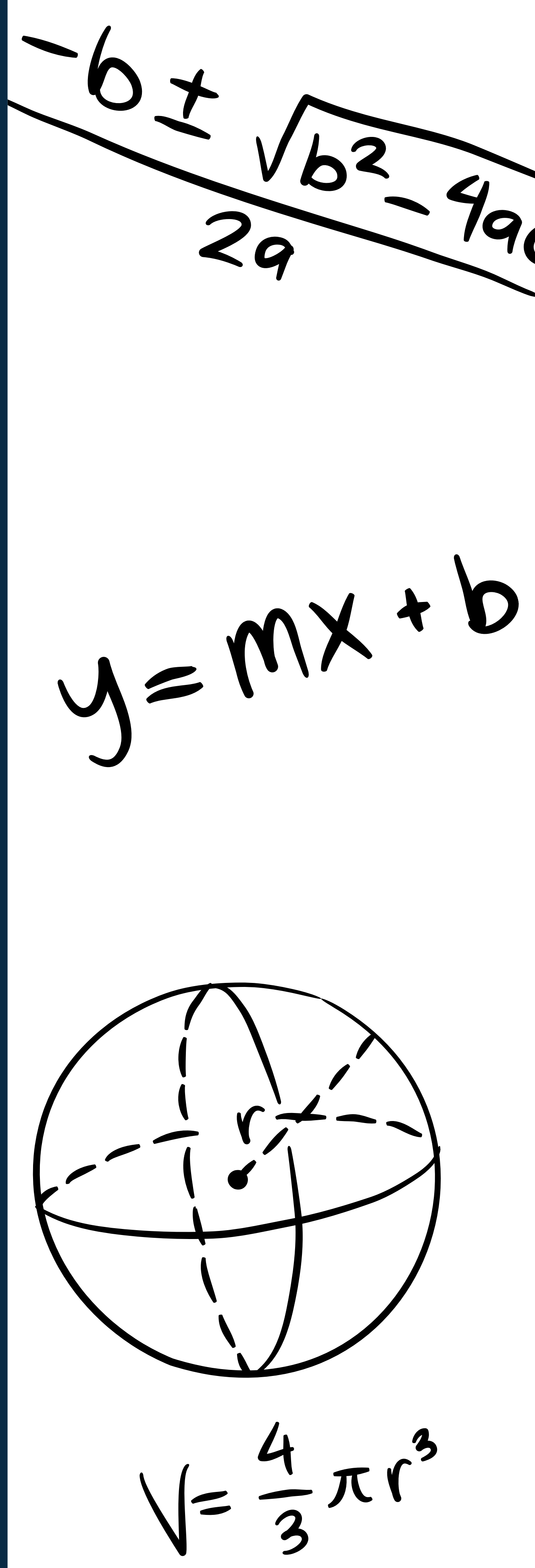
$$\begin{cases} x_1 + x_2 + x_3 = 3 \\ x_1 + x_2 + x_3 = 3 \\ x_3 = 1 \end{cases}$$

(2) = (2) - 1 * (1)

$$\begin{cases} x_1 + x_2 + x_3 = 3 \\ 0 + 0 + 0 = 0 \\ x_3 = 1 \end{cases}$$

exchange((2), (3))

$$\begin{cases} x_1 + x_2 + x_3 = 3 \\ x_3 = 1 \\ 0 + 0 + 0 = 0 \end{cases}$$



MATRIX SOLUTION

original

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 2 & 2 & 2 & 6 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

(2) = (2) - 1 * (1)

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

(2) divide 2

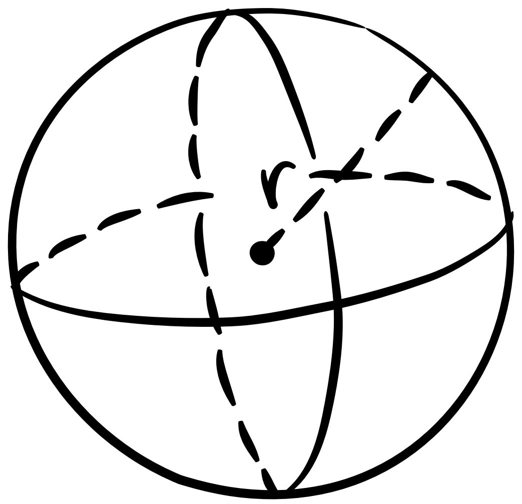
$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 1 & 1 & 3 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

exchange((2), (3))

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\frac{-b \pm \sqrt{b^2 - 4a}}{2a}$$

$$y = mx + b$$



$$V = \frac{4}{3}\pi r^3$$

ROW OPERATION

Elementary Row Operations

I. Interchange two rows.
II. Multiply a row by a nonzero real number.
III. Replace a row by the sum of that row and a multiple of another row.

ROW ECHELON FORM

A matrix is said to be in **row echelon form** if

(i) The first nonzero entry in each nonzero row is 1.

(ii) If row k does not consist entirely of zeros, the number of leading zero entries in row $k + 1$ is greater than the number of leading zero entries in row k.

(iii) If there are rows whose entries are all zero, they are below the rows having nonzero entries.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Gauss-reduction

ROW REDUCED ECHELON FORM (RREF)

A matrix is said to be in **reduced row echelon form** if

(i) The matrix is in row echelon form.

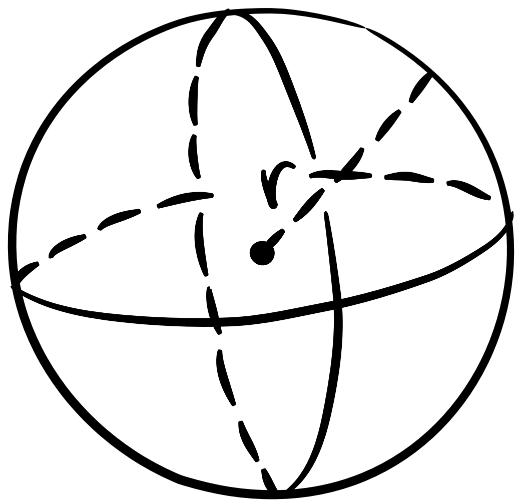
(ii) The first nonzero entry in each row is the only nonzero entry in its column.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Gauss-Jordan-reduction

$$\frac{-b \pm \sqrt{b^2 - 4a}}{2a}$$

$$y = mx + b$$



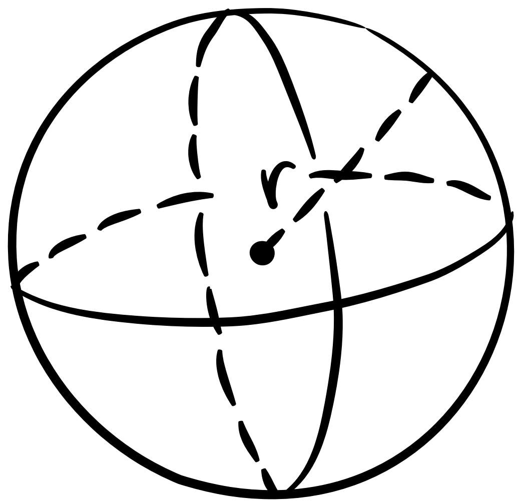
$$V = \frac{4}{3}\pi r^3$$

RREF → UNIQUE SOLUTION ?

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

$\frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$y = mx + b$



$V = \frac{4}{3}\pi r^3$

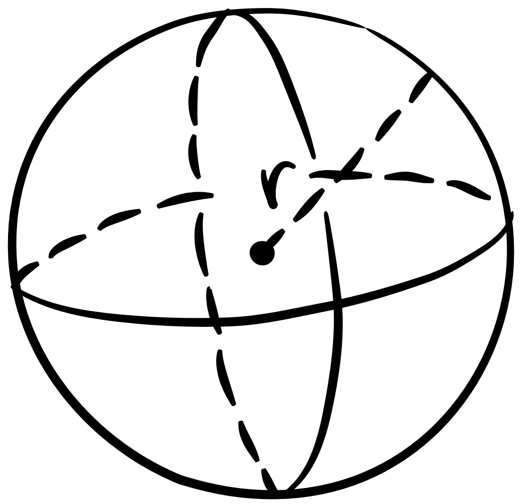
SO, WHEN UNIQUE SOLUTION ?

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

strickly triangular matrix

$\frac{-b \pm \sqrt{b^2 - 4a}}{2a}$

$y = mx + b$



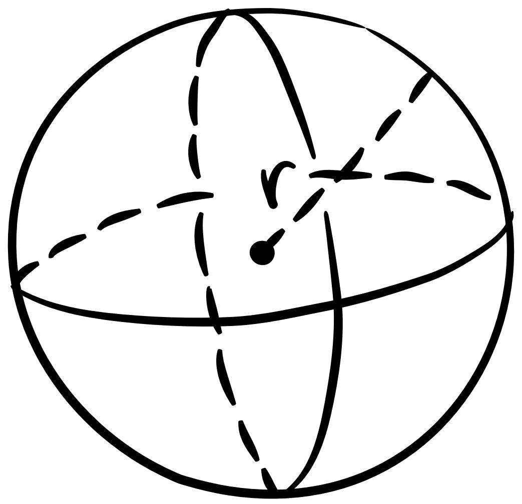
$V = \frac{4}{3}\pi r^3$

SOLUTION CASE

1. consistent
   a. unique solution
   b. infinite solution
2. non-consistent (no solution)

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$y = mx + b$$



$$V = \frac{4}{3}\pi r^3$$

MATRIX ARITHMETIC

OPERATION

Addition

Subtraction

Multiplication

Transpose

MATRIX MULTIPLICAITON

$$A\mathbf{x} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix}$$

$$= x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

row vector

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n$$

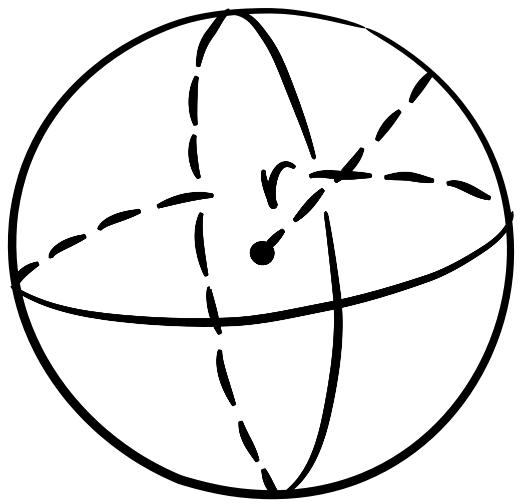
column vector

If $A = (a_{ij})$ is an $m \times n$ matrix and $B = (b_{ij})$ is an $n \times r$ matrix, then the product $AB = C = (c_{ij})$ is the $m \times r$ matrix whose entries are defined by

$$c_{ij} = \vec{\mathbf{a}}_i \mathbf{b}_j = \sum_{k=1}^{n} a_{ik} b_{kj}$$

$$\frac{-b \pm \sqrt{b^2 - 4a}}{2a}$$

$$y = mx + b$$



$$V = \frac{4}{3}\pi r^3$$

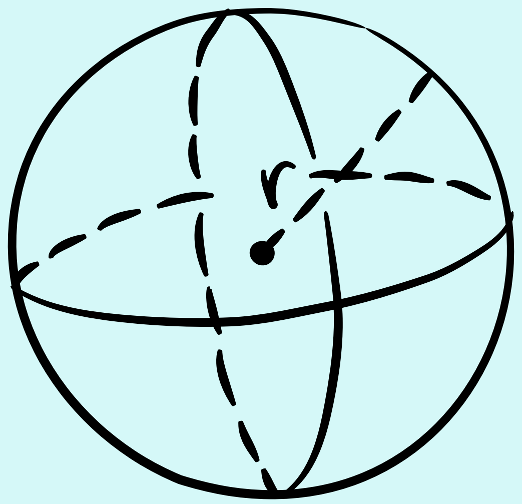
VECTOR:

1. column vector
    a.
2. row vector

$$A\mathbf{x} = \begin{pmatrix} \vec{\mathbf{a}}_1\mathbf{x} \\ \vec{\mathbf{a}}_2\mathbf{x} \\ \vdots \\ \vec{\mathbf{a}}_n\mathbf{x} \end{pmatrix}$$

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$y = mx + b$$



$$V = \frac{4}{3}\pi r^3$$

LINEAR COMBINATION:

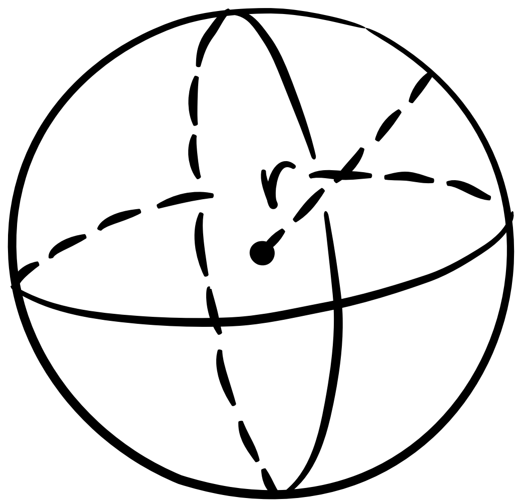
If $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ are vectors in $\mathbb{R}^m$ and $c_1, c_2, \ldots, c_n$ are scalars, then a sum of the form

$$c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + \cdots + c_n\mathbf{a}_n$$

is said to be a **linear combination** of the vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$.

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$y = mx + b$$



$$V = \frac{4}{3}\pi r^3$$

Ax=b consistent

Theorem 1.3.1 Consistency Theorem for Linear Systems

A linear system $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $\mathbf{b}$ can be written as a linear combination of the column vectors of A.

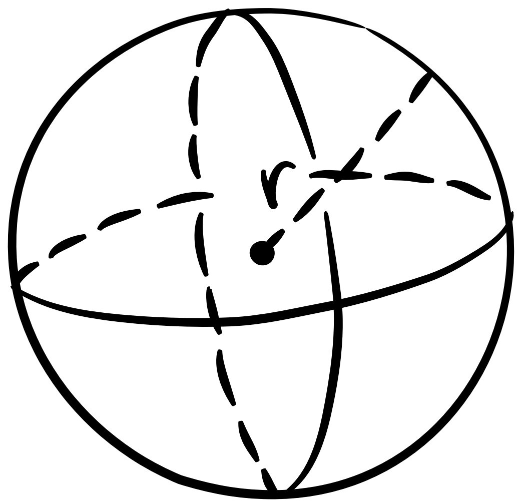
$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$A\mathbf{x} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n$$

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$$

$\frac{-b \pm \sqrt{b^2-4a}}{2a}$

$y = mx + b$



$V = \frac{4}{3}\pi r^3$

TRANSPOSE

The **transpose** of an $m \times n$ matrix A is the $n \times m$ matrix B defined by

$$b_{ji} = a_{ij} \tag{8}$$

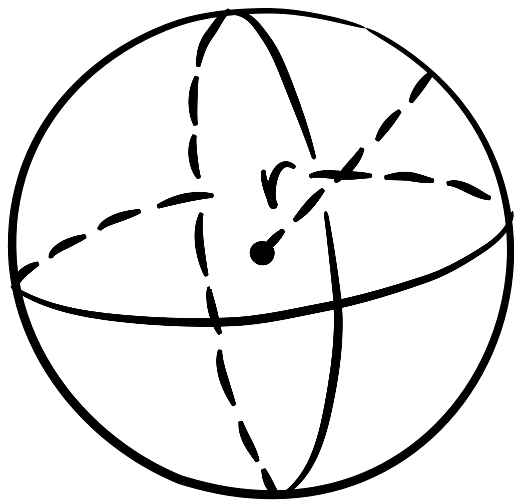
for $j = 1, \ldots, n$ and $i = 1, \ldots, m$. The transpose of A is denoted by A^T.

SYMMETRIC

An $n \times n$ matrix A is said to be **symmetric** if $A^T = A$.

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$y = mx + b$$



$$V = \frac{4}{3}\pi r^3$$

END

KEY WORDS:

Addition

- add
- altogether
- and
- both
- in all
- sum of
- total
- increase

Subtraction

- take away
- difference
- fewer
- gave away
- less
- how much more
- change
- decrease

Multiplication

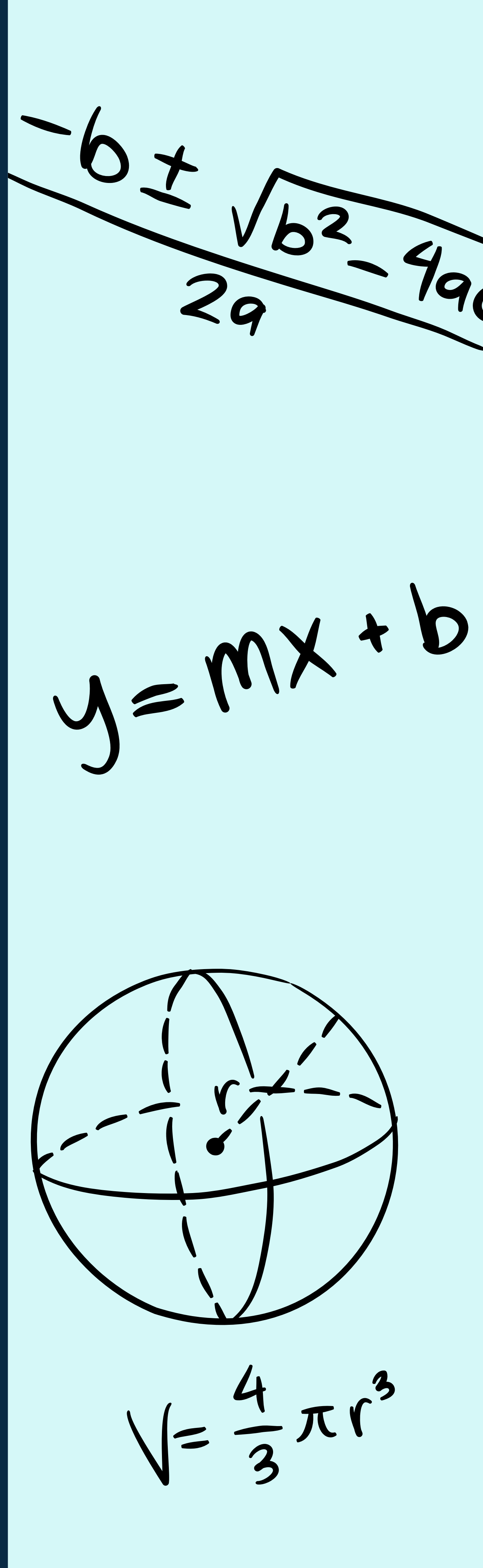
- multiply
- each
- twice
- product
- in all
- double

Division

- divide
- each
- quotient
- share equally
- goes into

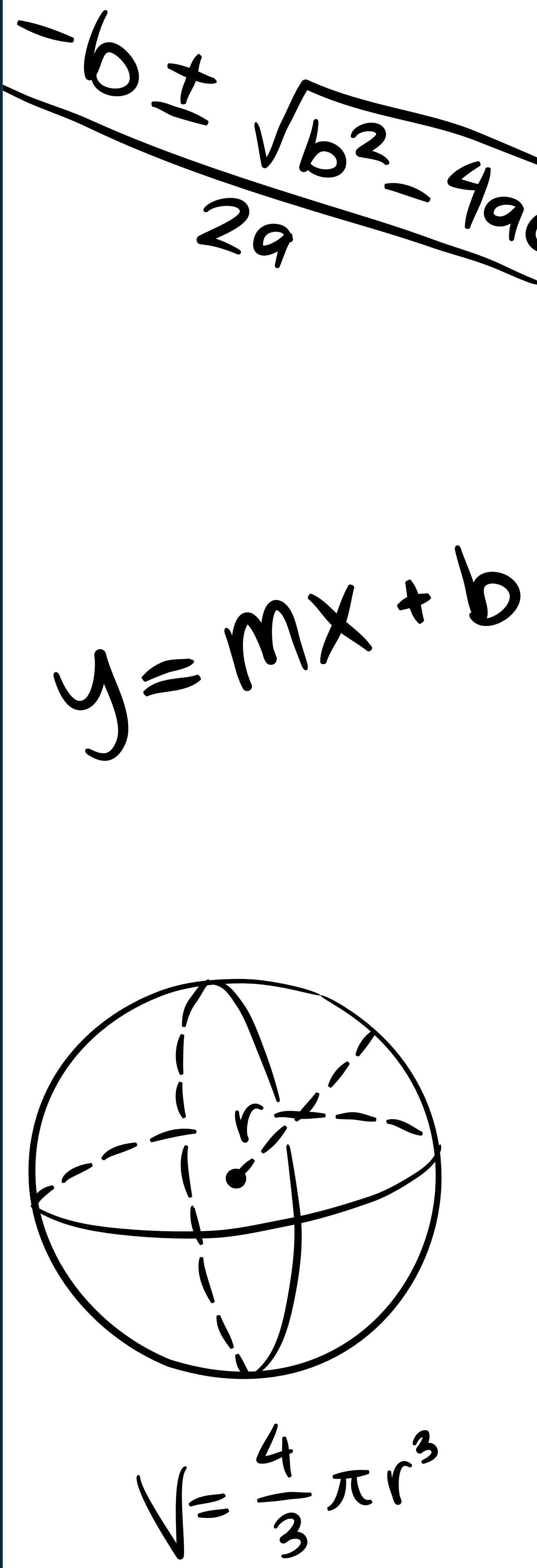
Practice:

1. Twice the sum of 5 and b
2. 12 decreased by n
3. 40 shared equally among v



Answers:

1. $2 (5 + b)$
2. $12 - n$
3. $40 \div v$



Word Problem

Write and expression to solve.

Rachel and her 3 friends each bough lemonade for 2 dollars. How many money was spent?

Examples:

The sum of 65 and p

65 + p

a less than 55

55 - a

Answer

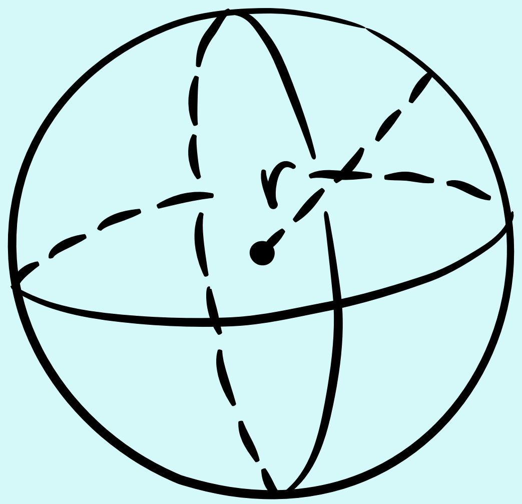
$(1 + 3) \times 2$

Learning Target:

I will be able to write and match simple numerical expressions.

$$\frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$y = mx + b$$



$$V = \frac{4}{3}\pi r^3$$