

※ 注意：請於答案卷上依序作答，並應註明作答之題號。

Note that for problems 1-15, we consider only matrices with real values.

1. (3%) What is the determinant of

$$\begin{bmatrix} 2 & 1 & 3 & 4 \\ 1 & 5 & 2 & 0 \\ 1 & 0 & 0 & 3 \\ 2 & 0 & 0 & 1 \end{bmatrix} ?$$

(A) 70 (B) -70 (C) 65 (D) -65 (E) 85

2. (3%) If  $\mathbf{v} = [1, 2, 3, 4, 5]^T$ , what is the rank of  $\mathbf{v}\mathbf{v}^T$ ?

(A) 1 (B) 2 (C) 3 (D) 0 (E) Not well defined

3. (3%) For the same  $\mathbf{v}$  in problem 2, what is the rank of  $\mathbf{v}^T\mathbf{v}$ ?

(A) 1 (B) 2 (C) 3 (D) 0 (E) Not well defined

4. (3%) Which of the following is incorrect?

(A) Any  $n + 1$  different nonzero points in  $R^n$  must be linearly dependent.

(B) In  $R^2$  any two linearly dependent vectors must be on a straight line.

(C) Let  $A$  be an  $m \times n$  matrix with  $m > n$  and  $\mathbf{b}$  be an  $m \times 1$  vector. Then  $A\mathbf{x} = \mathbf{b}$  may have multiple solutions.

(D) Let  $A$  be an  $m \times n$  matrix with  $m < n$  and  $\mathbf{b}$  be an  $m \times 1$  vector. Then  $A\mathbf{x} = \mathbf{b}$  must have at least one solution.

(E) Let  $A$  be any  $5 \times 4$  matrix and  $I$  be the  $5 \times 5$  identity matrix. Then  $I + AA^T$  is invertible.

5. (3%) Which of the following is incorrect?

(A) If  $A, B$  are invertible, then  $(AB)^{-1}$  exists and  $(AB)^{-1} = B^{-1}A^{-1}$

(B)  $\text{trace}((A+B)C) = \text{trace}(AC) + \text{trace}(BC)$

(C)  $\det(AB) = \det(A)\det(B)$

(D)  $\text{trace}(AB) = \text{trace}(A)\text{trace}(B)$

(E) If  $A, B$  are positive definite, then so is  $A + B$

6. (3%)  $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ . Which of the following symmetric matrix  $B$  satisfies

$$x^T A x = x^T B x, \forall x \in R^3?$$

(A)  $\begin{bmatrix} 1 & 4 & 7 \\ 4 & 5 & 6 \\ 7 & 6 & 9 \end{bmatrix}$  (B)  $\begin{bmatrix} 1 & 4 & 6 \\ 4 & 5 & 6 \\ 6 & 6 & 9 \end{bmatrix}$  (C)  $\begin{bmatrix} 1 & 4 & 5 \\ 4 & 5 & 7 \\ 5 & 7 & 9 \end{bmatrix}$  (D)  $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 6 \\ 3 & 6 & 9 \end{bmatrix}$  (E)  $\begin{bmatrix} 1 & 3 & 5 \\ 3 & 5 & 7 \\ 5 & 7 & 9 \end{bmatrix}$

7. (3%) Given an  $n \times n$  matrix  $A$  written as

$$A = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix},$$

where  $B$  is  $n_1 \times n_1$ ,  $C$  is  $n_1 \times n_2$ ,  $D$  is  $n_2 \times n_2$ ,  $0$  is  $n_2 \times n_1$ , and  $n = n_1 + n_2$ . Assume  $B$  and  $D$  are invertible and  $0$  is a zero matrix. Then what is  $A^{-1}$ ?

(A)  $\begin{bmatrix} B^{-1} & C^{-1} \\ 0 & D^{-1} \end{bmatrix}$  (B)  $\begin{bmatrix} B^{-1} & 0 \\ C^{-1} & D^{-1} \end{bmatrix}$  (C)  $\begin{bmatrix} B^{-1} & 0 \\ -B^{-1}CD^{-1} & D^{-1} \end{bmatrix}$   
 (D)  $\begin{bmatrix} B^{-1} & -B^{-1}CD^{-1} \\ 0 & D^{-1} \end{bmatrix}$  (E)  $A$  may not be always invertible.

8. (3%) Given

$$A = \begin{bmatrix} 0.6 & 0.4 \\ 0.4 & 0.6 \end{bmatrix}.$$

Let  $\lambda_1 > \lambda_2$  be  $A$ 's eigenvalues. What is  $\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}$ ?

(A)  $\begin{bmatrix} 0.5 \\ 0.3 \end{bmatrix}$  (B)  $\begin{bmatrix} 0.9 \\ 0.3 \end{bmatrix}$  (C)  $\begin{bmatrix} 1 \\ 0.2 \end{bmatrix}$  (D)  $\begin{bmatrix} 2 \\ 0.3 \end{bmatrix}$  (E)  $\begin{bmatrix} 3 \\ 0.5 \end{bmatrix}$

9. (3%) Following problem 8, what are  $A$ 's eigenvectors associated with  $\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}$ ? Each eigenvector is scaled to have length one.

(A)  $\begin{bmatrix} \frac{\sqrt{2}}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}, \begin{bmatrix} \frac{-1}{\sqrt{3}} \\ \frac{\sqrt{2}}{\sqrt{3}} \end{bmatrix}$  (B)  $\begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{\sqrt{2}}{\sqrt{3}} \end{bmatrix}, \begin{bmatrix} \frac{\sqrt{2}}{\sqrt{3}} \\ \frac{-1}{\sqrt{3}} \end{bmatrix}$  (C)  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$  (D)  $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$   
 (E)  $\begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}, \begin{bmatrix} \frac{-2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}$

10. (3%) Assume the answer of problem 9 are  $v_1, v_2$ . If  $x = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$  and we represent it as

$$x = \alpha_1 v_1 + \alpha_2 v_2,$$

then what are  $\alpha_1$  and  $\alpha_2$ ?

(A)  $(\sqrt{3}, -\sqrt{3})$  (B)  $(2\sqrt{2}, -2\sqrt{2})$  (C)  $(2\sqrt{2}, 2\sqrt{2})$  (D)  $(\sqrt{3}, \sqrt{3})$  (E)  $(\sqrt{3}, 2\sqrt{3})$

11. (4%) Following problems 8-10, what is

$$\lim_{n \rightarrow \infty} A^n x?$$

(A)  $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$  (B)  $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$  (C)  $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$  (D)  $\begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{\sqrt{2}}{\sqrt{3}} \end{bmatrix}$  (E)  $\begin{bmatrix} \frac{\sqrt{2}}{\sqrt{3}} \\ \frac{-1}{\sqrt{3}} \end{bmatrix}$

12. (4%) Consider an  $n \times n$  symmetric matrix  $A$ . If  $A$  is positive definite, then which of the following four properties is wrong? If you think all are correct, select E.

- (A)  $A_{ii} > 0, \forall i$   
 (B)  $A$  is invertible  
 (C)  $A_{ii} + A_{jj} - 2A_{ij} > 0, \forall i \neq j$   
 (D)  $A_{ii}A_{jj} - A_{ij}^2 > 0, \forall i \neq j$   
 (E) All are correct.

13. (4%) If

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 3 & 2 & 1 & 0 & 0 \\ 1 & 2 & 3 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

what is the solution of

$$A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1.5 \\ 7.5 \\ 15.5 \\ 14.0 \\ 4.0 \end{bmatrix} ?$$

$$(A) \begin{bmatrix} 0.5 \\ 0.5 \\ 1.5 \\ 2.5 \\ -0.5 \end{bmatrix} \quad (B) \begin{bmatrix} -0.5 \\ -1.5 \\ 0.5 \\ 1.5 \\ 2.5 \end{bmatrix} \quad (C) \begin{bmatrix} 1.5 \\ -0.5 \\ -1.5 \\ 0.5 \\ 1.5 \end{bmatrix} \quad (D) \begin{bmatrix} 1.5 \\ 2.5 \\ -0.5 \\ -1.5 \\ 0.5 \end{bmatrix} \quad (E) \begin{bmatrix} 0.5 \\ 1.5 \\ 2.5 \\ -0.5 \\ -1.5 \end{bmatrix}$$

14. (4%) Given any vectors  $x_1, \dots, x_l \in R^n$ . Define an  $l \times l$  matrix  $A$  with  $A_{ij} \equiv x_i^T x_j, i, j = 1, \dots, l$ . Which of the following is incorrect?

- (A)  $A$  is symmetric  
 (B)  $A$  is positive semi-definite (i.e.,  $A$ 's eigenvalues are  $\geq 0$ )  
 (C)  $A$  is invertible  
 (D)  $A$ 's diagonal elements are nonnegative  
 (E)  $A$  is a square matrix

15. (4%) Define

$$f(x) \equiv \frac{1}{2} \begin{bmatrix} x_1 & \dots & x_5 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 6 & 7 & 8 \\ 5 & 6 & 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_5 \end{bmatrix}$$

and

$$\nabla f(x) \equiv \begin{bmatrix} \frac{\partial f(x_1, \dots, x_5)}{\partial x_1} \\ \vdots \\ \frac{\partial f(x_1, \dots, x_5)}{\partial x_5} \end{bmatrix}.$$

What is

$$\nabla f \left( \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \\ 3 \end{bmatrix} \right) ?$$

$$(A) \begin{bmatrix} 22 \\ 25 \\ 33 \\ 39 \\ 45 \end{bmatrix} \quad (B) \begin{bmatrix} 22 \\ 28 \\ 36 \\ 40 \\ 51 \end{bmatrix} \quad (C) \begin{bmatrix} 22 \\ 25 \\ 33 \\ 40 \\ 46 \end{bmatrix} \quad (D) \begin{bmatrix} 22 \\ 28 \\ 34 \\ 40 \\ 46 \end{bmatrix} \quad (E) \begin{bmatrix} 22 \\ 24 \\ 35 \\ 39 \\ 45 \end{bmatrix}$$

16. (5%) The maximum height of a binary tree with  $n$  nodes is \_\_\_\_\_.
17. (5%) A nonempty subset  $I$  of a ring  $(R, +, \cdot)$  is called an ideal of  $R$  if for all  $a, b \in I$  and all  $r \in R$ :  $a - b \in I$ ,  $a \cdot r \in I$ , and  $r \cdot a \in I$ . If  $(R, +, \cdot)$  is furthermore a field, then it has \_\_\_\_\_ ideals.
18. (10%) If  $a \in \mathbb{Z}_n$  has a multiplicative inverse (i.e., there exists a  $b$  such that  $a \cdot b = b \cdot a = 1$ ), then  $a$  is called a unit. For any positive integer  $n > 1$ , there are \_\_\_\_\_ units in  $\mathbb{Z}_n$ .
19. (5%) Let  $K_n^*$  be a directed graph with  $n$  nodes. If for each distinct pair  $x, y$  of nodes, either  $(x, y) \in K_n^*$  or  $(y, x) \in K_n^*$  but not both, then  $K_n^*$  is called a tournament. A directed graph  $(V, E)$  is transitive if  $(a, b) \in E \wedge (b, c) \in E$  imply  $(a, c) \in E$ . There are \_\_\_\_\_ transitive tournaments on  $n$  players.
20. (10%) Consider a binary string  $x_1 x_2 \cdots x_n$ . The weight of  $x_1 x_2 \cdots x_n$  is defined as  $\sum_i x_i$ . There are  $2^n$  strings. Among them, \_\_\_\_\_ have even weight.
21. (5%) Consider  $x_1 + x_2 + \cdots + x_n < r$ , where  $x_i \geq 0$  for  $1 \leq i \leq n$ . the number of nonnegative integer solutions is \_\_\_\_\_.
22. (5%) The generating function for  $1, 2, 3, 4, \dots$  is \_\_\_\_\_.
23. (5%) There are \_\_\_\_\_ functions from  $\{0, 1\}^m$  to  $\{0, 1\}^n$ .