國立臺灣大學 103 學年度碩士班招生考試試題

1. (10%) How many ways are there to arrange TALLAHA with no adjacent As?

2. (10%) The number of positive-integer solutions to $x_1 + x_2 + \cdots + x_n = r$, where $r > 0$, is __________. (That is, all x_i must be positive integers to qualify as one solution.)

3. (5%) How many functions from $\{0,1\}^m$ (an m-dimensional boolean vector) to $\{0,1\}^2$ (a 2-dimensional boolean vector) are there?

4. (10%) The function

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots = \sum_{i=0}^{\infty} a_i x^i$$

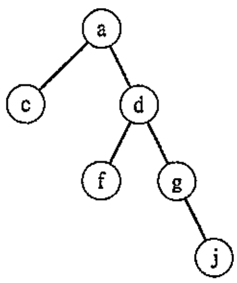
is the generating function for the sequence $\{a_i\}_{i=0,1,\ldots}$. The harmonic numbers $\{H_i\}_{i=0,1,2,\ldots}$ are defined by

$$\begin{aligned} H_0 &= 0, \\ H_i &= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{i} \quad (i \geq 1). \end{aligned}$$

Derive the closed-form generating function for the harmonic numbers.

5. (10%) Solve the recurrence equation $a_{n+2} = a_{n+1} + 2a_n$ with $a_0 = 0$ and $a_1 = 1$.

6. (5%) The postorder traversal of the following rooted binary tree is __________.



7. (10%) Let V be a vector space over a scalar field F. For any subset S of V, let span(S) consist of the vectors of V that can be written as $a_1x_1 + a_2x_2 + \cdots + a_nx_n$ with $a_1, a_2, \ldots, a_n \in F$ and $x_1, x_2, \ldots, x_n \in S$ for some positive integer n. Prove that span(S) is the unique subspace of V such that any subspace of V containing S has to contain span(S). Specifically, you have to show that (a) span(S) is a subspace of V, (b) if U is a subspace of V with $S \subseteq U$, then span(S) $\subseteq U$, and (c) there is no other subspace of V satisfying property (b).

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8. (10%)

   (a) Let T be a linear transformation from V to W, where V and W are finite-dimensional vector spaces over a comon scalar field F. We have nullity(T) + rank(T) = dim(__________).

   (b) Let R consist of the real numbers. Let function $f : R^3 \to R^3$ be defined as
   $$f(x, y, z) = (x + y + z, x - y, y - z).$$
   If $g : R^3 \to R^3$ is a linear function with $g(1, 1, 0) = (2, 0, 1)$, $g(1, 0, 1) = (2, 1, -1)$, and $g(0, 1, 1) = (2, -1, 0)$, then $g(5, 3, 0) =$ __________.

   (c) If the dimension of the vector space $M_{7\times4}(C)$ of matrices with seven rows and four columns over the field C of complex numbers equals the dimension of the vector space R^n of n-tuples over the field R of real numbers, then $n =$ __________.

   (d) Let A be an $m \times n$ matrix over the field R of real numbers. If the m rows of A are linearly independent, then the dimension of the vector space spanned by the n rows of A is __________.

   (e) If U and V are two distinct subspaces of a vector space W with $\dim(W) = 6$, then $\dim(U \cap V)$ is either __________ or __________.

9. (10%) Let
$$A = \begin{bmatrix} -1 & -2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}.$$
Find A^{-100} and A^{101}.

10. (10%) Consider the following system of linear equations:
$$\begin{aligned} 2x + y + z &= 4 \\ 4x + 2y + 2z &= 8 \\ 5x + y &= 19. \end{aligned}$$
Find the solution (x, y, z) to the above system of linear equations that minimizes $x^2 + y^2 + z^2$.

11. (10%) Find the eigenvalues of the following matrix:
$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$