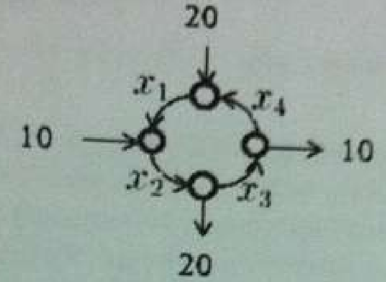
線性代數　（1至10題複選題請以電腦答案卡作答，否則不予計分）

1. (5 分; 複選, 答對每個選項得 1 分, 答錯每個選項扣 1 分; 本題合計得分爲負時, 以 0 分計; 未作答亦以 0 分計) Consider a system of linear equations for solving the flows of traffic (in vehicles per minute) through a one-way circle shown below, where $x_i \geq 0$ denotes the traffic flow of the i-th segment of the circle, $i = 1, \ldots, 4$. Which of the followings are true?



A. This system of equations can be reduced to a strictly triangular form.

B. The row echelon form of the augmented matrix representing this system of equations is $\left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 20 \\ 0 & 1 & 0 & -1 & 20 \\ 0 & 0 & 1 & -1 & 10 \\ 0 & 0 & 0 & 1 & 10 \end{array}\right]$.

C. This linear system is consistent.

D. The largest traffic flow is x_1.

E. When solving this traffic flow problem where the flow directions in the above figure are all reversed, we can obtain a system of equations equivalent to the original one (when the flow directions are as indicated in the above figure).

2. (5 分; 複選, 答對每個選項得 1 分, 答錯每個選項扣 1 分; 本題合計得分爲負時, 以 0 分計; 未作答亦以 0 分計) Which of the followings are true?

A. There exists a matrix **A** such that $\mathbf{A}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$.

B. A system of three linear equations in two unknowns is always inconsistent.

C. If **A** and **B** are nonsingular $n \times n$ matrices, then **AB** is also nonsingular.

D. If **A** and **B** are nonsingular $n \times n$ matrices, then **A+B** is also nonsingular.

E. If **A**, **B**, and **A+B** are nonsingular $n \times n$ matrices, then $\mathbf{A}^{-1}+\mathbf{B}^{-1}$ is also nonsingular.

3. (5 分; 複選, 答對每個選項得 1.25 分, 答錯每個選項扣 1.25 分; 本題合計得分爲負時, 以 0 分計; 未作答亦以 0 分計) Which of the followings are true?

A. The adjoint of the matrix $\begin{bmatrix} 0 & 1 \\ -2 & 1 \end{bmatrix}$ is $\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$.

B. $\det\begin{bmatrix} a & 1 & 1 & 1 \\ 1 & a & 1 & 1 \\ 1 & 1 & a & 1 \\ 1 & 1 & 1 & a \end{bmatrix} = (a+3)(a-1)^3$.

C. For a nonsingular matrix $\mathbf{A}_{3\times 3} = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3]$, $\det(\mathbf{A}^{-1}\mathbf{B}) = 1$ where $\mathbf{B} = \begin{bmatrix} \mathbf{a}_2^T \\ 2\mathbf{a}_1^T - \mathbf{a}_2^T + \mathbf{a}_3^T \\ \mathbf{a}_1^T \end{bmatrix}$.

D. If $\det\begin{bmatrix} a & b & c \\ d & e & f \\ 1 & 3 & 5 \end{bmatrix} = 7$ and $\det\begin{bmatrix} a & b & c \\ d & e & f \\ 1 & 0 & 1 \end{bmatrix} = 9$, then $\det\begin{bmatrix} a & b & c \\ d & e & f \\ 1 & 6 & 9 \end{bmatrix} = 5$.

4. (5 分; 複選, 答對每個選項得 1 分, 答錯每個選項扣 1 分; 本題合計得分爲負時, 以 0 分計; 未作答亦以 0 分計) Which of the followings are true?
   A. The formula $\det\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} = (\det \mathbf{A})(\det \mathbf{D}) - (\det \mathbf{C})(\det \mathbf{B})$ always hold for square matrices **A**, **B**, **C**, and **D**.
   B. The formula $\det(\mathbf{A}^k) = (\det \mathbf{A})^k$ always hold for positive integer k and square matrix **A**.
   C. The formula $\det(\mathbf{A}) = \det(-\mathbf{A})$ always hold for square matrix **A**.
   D. The formula $\det(\mathbf{A}^T\mathbf{A}) > 0$ always hold for nonsingular matrix **A**.
   E. The formula $\det(\mathbf{A}) = 1$ always hold for orthogonal matrix **A**.

5. (5 分; 複選, 答對每個選項得 1 分, 答錯每個選項扣 1 分; 本題合計得分爲負時, 以 0 分計; 未作答亦以 0 分計) Which of the followings are true?
   A. Let R^+ denote the set of positive real numbers. Define the operation of scalar multiplication, denoted $\circ$, by $\alpha \circ x = x^\alpha$ for each $x \in R^+$ and for any real number α. Define the operation of addition, denoted $\oplus$, by $x \oplus y = x \cdot y$ for all $x, y \in R^+$. Then R^+ is a vector space with these operations.
   B. Let S be the set of all ordered pairs of real numbers. Define scalar multiplication and addition on S by $\alpha \circ (x_1, x_2) = (\alpha x_1, \alpha x_2)$ and $(x_1, x_2) \oplus (y_1, y_2) = (x_1 + y_1, 0)$. Then S is a vector space with these operations.
   C. If A is an $r \times s$ matrix, then the rank of A plus the nullity of A equals r.
   D. Let S_1 and S_2 be two subspaces of R^4 consisting of all vectors of the form $(a+b, a-b+2c, b, c)^T$ and $(a+b, a-b+2c, a+b, b-c)^T$ respectively, where a, b and c are real numbers. Then the dimension of S_1 plus the dimension of S_2 equals 5.
   E. Any two finite dimensional vector spaces with the same dimension are isomorphic.

6. (5 分; 複選, 答對每個選項得 1 分, 答錯每個選項扣 1 分; 本題合計得分爲負時, 以 0 分計; 未作答亦以 0 分計) Let A, B and C be three matrices, and AB=C. Which of the followings are true?
   A. The row space of C is a subspace of the row space of A.
   B. The column space of C is a subspace of the column space of A.
   C. The row space of C is a subspace of the row space of B.
   D. If the column vectors of A are linearly dependent, then the column vectors of C are linearly dependent.
   E. If the row vectors of A are linearly dependent, then the row vectors of C are linearly dependent.

7. (5 分; 複選, 答對每個選項得 1 分, 答錯每個選項扣 1 分; 本題合計得分爲負時, 以 0 分計; 未作答亦以 0 分計) Which of the followings are true?
   A. Let $L: R^n \to R^m$ be a linear transformation. If A is the standard matrix representation of L, then an $n \times n$ matrix B will also be a matrix representation of L if and only if B is similar to A.
   B. Let $L: V \to W$ be a linear transformation. $v_1, v_2, \ldots, v_k$ are linearly dependent in V, if and only if $L(v_1), L(v_2), \ldots, L(v_k)$ are linearly dependent in W.
   C. The transition matrix from one basis to another must be nonsingular, and a matrix representation of

a linear transformation can be singular.

D. Any two matrices have the same trace if and only if they are similar.

E. Let $[u_1, u_2]$ and $[v_1, v_2]$ be ordered bases for R^2, where $u_1=\begin{bmatrix}1\\3\end{bmatrix}$, $u_2=\begin{bmatrix}2\\7\end{bmatrix}$ and $v_1=\begin{bmatrix}5\\2\end{bmatrix}$, $v_2=\begin{bmatrix}4\\9\end{bmatrix}$. Let L be a linear transformation from R^2 to R^2 whose matrix representation with respect to the ordered basis $[u_1, u_2]$ is $A=\begin{bmatrix}2&1\\3&2\end{bmatrix}$. Then the matrix representation of L with respect to the ordered basis $[v_1, v_2]$ is $\begin{bmatrix}5&4\\2&9\end{bmatrix}^{-1}\begin{bmatrix}1&2\\3&7\end{bmatrix}\begin{bmatrix}2&1\\3&2\end{bmatrix}\begin{bmatrix}1&2\\3&7\end{bmatrix}^{-1}\begin{bmatrix}5&4\\2&9\end{bmatrix}$.

8. (5分; 複選，答對每個選項得1分，答錯每個選項扣1分；本題合計得分爲負時，以0分計；未作答亦以0分計) Consider a matrix $\mathbf{A}=\begin{bmatrix}1&2&1&2\\1&2&2&5\\0&0&1&3\end{bmatrix}$ and a system of linear equations $\mathbf{Ax}=\mathbf{b}=[-1,3,1]^T$.

Which of the following vectors $\mathbf{c}$ will make the two systems $\mathbf{Ax}=\mathbf{b}+\mathbf{c}$ and $\mathbf{Ax}=\mathbf{b}$ have the same least-squares error.

A. $\mathbf{c}=[0,-1,-1]^T$.

B. $\mathbf{c}=[3,2,2]^T$.

C. $\mathbf{c}=[1,0,-1]^T$.

D. $\mathbf{c}=[2,1,2]^T$.

E. $\mathbf{c}=[1,2,1]^T$.

9. (5分; 複選，答對每個選項得1.25分，答錯每個選項扣1.25分；本題合計得分爲負時，以0分計；未作答亦以0分計) Suppose $\mathbf{A}_{3\times 3}$ has three distinct eigenvalues 0, 1, 3 with corresponding eigenvectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$. Which of the following statements are true?

A. The rank of $\mathbf{A}$ must be 2.

B. The matrix $\mathbf{A}^2-\mathbf{I}$ is invertible.

C. $\mathbf{v}$ and $\mathbf{w}$ must span the column space of $\mathbf{A}$.

D. The least-squares error of $\mathbf{Ax}=2\mathbf{v}+3\mathbf{w}+\mathbf{u}$ must be $\|\mathbf{u}\|^2$.

10. (5分; 複選，答對每個選項得1.25分，答錯每個選項扣1.25分；本題合計得分爲負時，以0分計；未作答亦以0分計) Suppose $\mathbf{A}_{3\times 3}=\mathbf{xy}^T$ is a rank-1 matrix. Which of the following statements are true?

A. The eigenvector matrix $\mathbf{S}$ (i.e., $\mathbf{AS}=\Lambda\mathbf{S}$) must be invertible.

B. $\mathbf{A}$ must have one non-zero eigenvalue.

C. When $\mathbf{A}$ is factorized by singular value decomposition (SVD) into $\mathbf{U\Sigma V}^T$, the only non-zero singular value is $\|\mathbf{x}\|\|\mathbf{y}\|$.

D. When $\mathbf{A}$ is factorized by SVD into $\mathbf{U\Sigma V}^T$, $\mathbf{U}$ must include $\frac{\mathbf{x}}{\|\mathbf{x}\|}$ as one of its column vectors.

First you need to write down your answers clearly and then explain how to compute the answers. You also need to answer the questions in order. Do not jump around.

11.(a) (4 points) Consider the following sequence of numbers:

$$1, 2, 2, 3, 3, 3, 4, 4, 4, 4 \ldots$$

The 1st number is 1; the 4th number is 3; etc. Define a function, for $j = 1, 2, 3, \ldots$

$$g(j) =_{def} \text{ the } k\text{-th number in this sequence, where } k = (\lfloor (j+1)^2/2 \rfloor)$$

What is $g(j)$?

11.(b) We say a binary string is *happy* if and only if it does not contain two consecutive 0's. For instance, 101010 and 011010 are happy while 000011 is not. Let $h(n) =_{def}$ the number of n-bit happy strings.

(i) (4 points) What is $h(n)$ (as a recurrence relation)?

(ii) (5 points) Solve this recurrence relation.

11.(c) (4 points) How many functions are there from a finite set A to a finite set B?

11.(d) (4 points) Compute f^{1001}, where f is defined as the following permutation:

$$\begin{bmatrix} a & b & c & d & e & f & g & h \\ d & f & a & c & g & e & h & b \end{bmatrix}$$

11.(e) (4 points) How many integer solutions are there for the following equation:

$$x_1 + x_2 + x_3 + 5 \cdot x_4 = 18$$

Assume that $x_1 \geq 0$, $x_2 \geq 0$, $x_3 \geq 0$, $x_4 \geq 0$.

12.(a) (10 points) Answer True/False to the following statements. Each correct answer gets 2 points, and each wrong answer deducts 2 points. If the number of points obtained in this question set is negative, it will be treated as 0.

(i) If a graph contains a subgraph homeomorphic to $K_{3,3}$ or K_5, the graph is nonplanar.

(ii) The number of regions of a general graph is given by e-v+2 where e is the number of edges of the graph and v is the number of vertices of the graph.

(iii) The number of nodes in a rooted tree with height h is at most $2^{h+1}-1$.

(iv) A simple graph is connected if and only if it has a spanning tree.

(v) The characteristic equation of a linear homogeneous recurrence relation of degree k with constant coefficients is a k-degree polynomial equation.

12.(b) (10 points) Let $S=\{1,2,\ldots,40\}$. For $a,b\in S$, we write $a\mathrm{R}b$ if and only if $a\equiv b$ mod 7. Please answer the following questions, and highlight your answers.

(i) (5 points) Prove that R is an equivalence relation.

(ii) (2 points) If n is in S, let $[n]$ denote the set of elements of S that are equivalent to n. How many elements are in the equivalence class [5]?

(iii) (3 points) We know that $P=\{[1],[2],\ldots,[7]\}$ is a partition of S. For $[a],[b]\in P$, we write $[a]\leq[b]$ if and only if (a mod 7) $\leq$ (b mod 7). Please draw the Hasse diagram for the poset $(P,\leq)$.

12.(c) (5 points) The 7-Day University needs to arrange 7 different courses, C1, C2, ..., C7, in a week. One course is for one day, and all 7 courses must be arranged. C1, C2, ..., and C7 can't be on Monday, Tuesday, Wednesday, Thursday, Friday, Saturday, and Sunday, respectively. How many possible arrangements does the 7-Day University have?