國立交通大學102學年度碩士班考試入學試題

【不可使用計算機】*作答前請先核對試題、答案卷(試卷)與准考證之所組別與考科是否相符！！

請在答案紙上按題號依序作答，並清楚標示題號。Please write down your answers with the question numbers on the ANSWER PAPER in order.

PART 1:

第1題至第15題為複選題（正確的選項可能有一個或多個），每題配分5分，每題中有5個選項，其中至少有一選項為真，請在答案紙上寫出為真的選項編號，不需說明及計算過程，答對一個選項得1分，答錯一個選項倒扣1分，倒扣至該題分數扣完為止，整題未作答亦不給分。

For problems 1-15, each problem is worth 5 points. Each problem has five statements, and at least one is TRUE. Please choose the TRUE statements, and write down your answers on the answer paper. **Explanations and/or calculation processes are not needed.** Each statement with correct answer gets 1 point, and each statement with wrong answer deducts 1 point. If the total points obtained in a problem are negative, it will be treated as 0. If a problem is not answered, no point is obtained from that problem.

1. (5 points) If a nonzero matrix is said to be in reduced row echelon form, the properties of this matrix are
   (A) the first nonzero entry in each nonzero row is 1.
   (B) if row *m* does not consist entirely of zeros, the number of leading zero entries in row *m+1* is greater than the number of leading zero entries in row *m*.
   (C) if there are rows whose entries are all zero, those rows are below the rows having nonzero entries.
   (D) the first nonzero entry in each column is the only nonzero entry in its row.
   (E) all of the above are true.

2. (5 points) An *nxn* matrix **A** is said to be nonsingular,
   (A) if **A** is non-invertible.
   (B) if there exists a matrix **B** such that **AB**=**BA**=**I**.
   (C) if the determinant of **A** is nonzero.
   (D) if **Ax**=**0** where **x** is nonzero vector.
   (E) all of the above are true.

3. (5 points) There are many types of elementary matrices. Which of the following definitions are elementary matrices? (I: identity matrix)
   (A) An *nxn* matrix is obtained by interchanging two rows of **I**.
   (B) An *nxn* matrix is obtained by multiplying a row of I by a nonzero constant.
   (C) An *nxn* matrix is obtained from **I** by adding a multiple of one row to another row.
   (D) An *nxn* matrix is formed from **I** by reordering its columns.
   (E) All of the above are elementary matrices.

4. (5 points) If there are two *nxn* matrices **A**, **B** and one scalar *r*,
   (A) det(**A**) = det(**B**) implies **A** = **B**.
   (B) det(*r***A**+**B**) = *r* det(**A**)+det(**B**).
   (C) det((**AB**)^T) = det(**A**)det(**B**).
   (D) det(**A**) = det(**B**) where **A** and **B** are row equivalent matrices.
   (E) none of the above is true.

(接續下一頁)

【不可使用計算機】*作答前請先核對試題、答案卷(試卷)與准考證之所組別與考科是否相符！！

5. (5 points) The $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ be n column vectors in R^n and let $\mathbf{A} = [\mathbf{a}_1\ \mathbf{a}_2 \ldots \mathbf{a}_n]$.
   (A) The vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ will be linearly independent if and only if $\mathbf{A}$ is singular.
   (B) The vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ will be linearly independent if and only if $\mathbf{Ax} = \mathbf{0}$ has only the trivial solution $\mathbf{x} = [0\ 0 \ldots 0]^T$.
   (C) The rank of $\mathbf{A}$ plus the nullity of $\mathbf{A}$ equals n.
   (D) The rank of $\mathbf{A}$ equals the rank of $\mathbf{A}^T$.
   (E) All of the above are true.

6. (5 points) Which of the following statements are TRUE?
   (A) Given a square matrix $\mathbf{A}$, its eigenvectors associated with non-zero eigenvalues must be in its column space.
   (B) The shortest (minimal 2-norm) least-squares solution to a problem $\mathbf{Ax} = \mathbf{b}$ must be in the row space of $\mathbf{A}$.
   (C) For any non-zero real symmetric matrix $\mathbf{A}$, its singular value decomposition ($\mathbf{A} = \mathbf{U\Sigma V^T}$) can be the same as its eigenvalue decomposition ($\mathbf{A} = \mathbf{S\Lambda S^{-1}}$).
   (D) For any non-zero real matrix $\mathbf{A}$, the singular value decomposition of $\mathbf{A^TA}$ can be the same as its eigenvalue decomposition.
   (E) If $\mathbf{A}$ is a full row rank matrix, then $\mathbf{AA^T}$ must be positive definite.

7. (5 points) Suppose $\mathbf{A}_{4x4} = \mathbf{xy^T}$ is a rank-1 real matrix with $\mathbf{x^Ty} = 0$ and $\|\mathbf{x}\|_2 = \|\mathbf{y}\|_2 = 1$. Which of the following statements are TRUE?
   (A) $\mathbf{A}$ must have a non-zero eigenvalue.
   (B) There must be an invertible eigenvector matrix $\mathbf{S}$ ($\mathbf{AS} = \mathbf{S\Lambda}$).
   (C) The nullspace of $\mathbf{A}$ contains its column space, i.e., $\mathrm{N}(\mathbf{A}) \supset \mathrm{C}(\mathbf{A})$.
   (D) $\mathbf{v} = \mathbf{x} + \mathbf{y}$ can be the shortest (minimal 2-norm) least-squares solution to some problem of the form $\mathbf{Av} = \mathbf{b}$.
   (E) When $\mathbf{A}$ is factorized by singular value decomposition (SVD) into $\mathbf{U\Sigma V^T}$, the only non-zero singular value is 1.

8. (5 points) Given a matrix $\mathbf{A} = \begin{bmatrix} 2 & 3 & 1 & 5 \\ 2 & 3 & 0 & 4 \\ 0 & 0 & 1 & 1 \end{bmatrix}$, which of the following statements are TRUE?
   (A) $\mathbf{A}$ has a column space of dimension 3.
   (B) The column space of $\mathbf{A}$ is identical to the column space of matrix $\begin{bmatrix} 1 & 1 & 5 \\ 1 & 0 & 4 \\ 0 & 1 & 1 \end{bmatrix}$.
   (C) The left nullspace of $\mathbf{A}$ is a line in R^3.
   (D) $\mathbf{A^TA}$ is invertible.
   (E) If the two systems $\mathbf{Ax} = \mathbf{b}$ and $\mathbf{Ax} = \mathbf{c}$ have the same shortest (minimal 2-norm) least-squares solution, then the projections of $\mathbf{b}$ and $\mathbf{c}$ onto the column space of $\mathbf{A}$ must be identical.

(接續下一頁)

【不可使用計算機】＊作答前請先核對試題、答案卷(試卷)與准考證之所組別與考科是否相符！！

9. (5 points) Consider a matrix $\mathbf{A}=\begin{bmatrix}2&3&1&5\\2&3&0&4\\0&0&1&1\end{bmatrix}$ and a system of linear equations $\mathbf{Ax}=\mathbf{b}=(0,3,3)^{\mathrm{T}}$. Which of the following vectors $\mathbf{c}$ will make $\mathbf{Ax}=\mathbf{c}$ have a least-squares error (i.e., $\min_{x\in R^4}\|\mathbf{Ax}-\mathbf{c}\|_2^2$) smaller than that of $\mathbf{Ax}=\mathbf{b}$ (i.e., $\min_{x\in R^4}\|\mathbf{Ax}-\mathbf{b}\|_2^2$)?
   (A) $\mathbf{c}=(3,3,3)^{\mathrm{T}}$.
   (B) $\mathbf{c}=(2,1,1)^{\mathrm{T}}$.
   (C) $\mathbf{c}=(6,1,1)^{\mathrm{T}}$.
   (D) $\mathbf{c}=(10,1,2)^{\mathrm{T}}$.
   (E) $\mathbf{c}=(1,4,7)^{\mathrm{T}}$.

10. (5 points) Given the matrix $\mathbf{A}$ and the vector $\mathbf{b}$ in the above problem, which of the following vectors $\mathbf{c}$ will make $\mathbf{Ax}=\mathbf{c}$ have the same shortest (minimal 2-norm) least-squares solution as the system $\mathbf{Ax}=\mathbf{b}$?
   (A) $\mathbf{c}=(2,1,1)^{\mathrm{T}}$.
   (B) $\mathbf{c}=(2-\sqrt{3},1+\sqrt{3},\sqrt{3})^{\mathrm{T}}$.
   (C) $\mathbf{c}=(2.5,0.5,0.5)^{\mathrm{T}}$.
   (D) $\mathbf{c}=(1,2,2)^{\mathrm{T}}$.
   (E) $\mathbf{c}=(2/\sqrt{2},1/\sqrt{2},1/\sqrt{2})^{\mathrm{T}}$.

11. (5 points) In the following statements, which are TRUE?
   (A) If n is an integer that is not a multiple of 4, then $n^2\equiv 1 \bmod 4$.
   (B) A positive integer congruent to 1 modulo 4 cannot have a prime factor congruent to 3 modulo 4.
   (C) There is no such integer x such that $x\equiv 2 \pmod 6$ and $x\equiv 3 \pmod 9$
   (D) Assume that a, b and m are integers with $m>1$. If $a\equiv b \pmod{2m}$, then $a\equiv b \pmod m$.
   (E) If n is an integer that is not a multiple of 3, then $n^2\equiv 1 \bmod 3$.

12. (5 points) In the following statements, which are TRUE?
   (A) For all integers a,b, if $a|b$ and $b|a$, then $a=b$.
   (B) Suppose that the only currency were 3-dollar bills and 10-dollar bills. Then any dollar amount greater than 17 dollars could be made from a combination of these bills.
   (C) A set with 50 elements has (2^{50} – C(50,1) – C(50,2) – C(50,3)) subsets with more than three elements.
   (D) A computer is programmed to print subsets of {1,2,3,4,5} at random. If the computer prints 65 subsets, then some subset must have been printed at least three times.
   (E) 30! ends in exactly six 0s.

13. (5 points) In the following statements, which are TRUE?
   (A) The coefficient of x^9 in the expansion of $(2+3x^3)^{10}$ is 2^7C(10,3).
   (B) The number of terms in the expansion of $(5a+8b)^{16}$ is 17.
   (C) The largest coefficient in the expansion of $(x+1)^6$ is 15.
   (D) There are 6 permutations of 12345 that leave 3 in the third position but leave no other integer in its own position.
   (E) If the permutations of 1,2,3,4,5,6 are written in lexicographic order, with 123456 in position #1, 123465 in position #2, etc., then the permutation in position #483 is 512436.

(接續下一頁)

【不可使用計算機】＊作答前請先核對試題、答案卷(試卷)與准考證之所組別與考科是否相符！！

14. (5 points) In the following statements, which are TRUE?
    (A) If 1+1 = 2 or 1+1 = 3, then 2+2 = 3 and 2+2 = 4.
    (B) $p\to(q\to r)$ and $p\to(q\wedge r)$ are equivalent.
    (C) $p\to(\neg q\wedge r)$ and $\neg p\vee\neg(r\to q)$ are logically equivalent.
    (D) The proposition $((p\to q)\wedge\neg p)\to\neg q$ is a tautology.
    (E) The proposition $((p\to\neg q)\wedge q)\to\neg p$ is a tautology.

15. (5 points) In the following statements, which are TRUE?
    (A) $A-(B\cap C) = (A-B)\cup(A-C)$.
    (B) Suppose $g : A\to B$ and $f : B\to C$, where $f\circ g$ is 1-1 and f is 1-1. Then, g is 1-1.
    (C) There is a set A such that $|P(P(A))| = 1024$, where $P(\bullet)$ denotes the power set.
    (D) f is a function from the set of all bit strings to the set of integers if $f(S)$ is the largest integer i such that the i-th bit of S is 0 and $f(S) = 1$ when S is the empty string (the string with no bits).
    (E) Suppose $g : A\to B$ and $f : B\to C$, where $f\circ g$ is 1-1 and g is 1-1. Then, f is 1-1.

PART 2:

第 16 題至第 18 題為問答題，計算過程不給分，每題之配分標示於題目中，共計 25 分。

For problems 16-18, the credit points of each problem are noted in the question sets, and the total points are 25 points. **Calculation processes do not get points.**

16. Little Red needs to solve a linear nonhomogeneous recurrence relation $a_n = 4a_{n-1} - 3a_{n-2} + 4$. Please answer the following questions for him/her.
    (A) (2 points) What is the constant c such that $a_n = cn$ is a solution of $a_n = 4a_{n-1} - 3a_{n-2} + 4$?
    (B) (3 points) What is the characteristic equation and characteristic roots of the homogeneous recurrence relation $a_n = 4a_{n-1} - 3a_{n-2}$?
    (C) (5 points) Given $a_1 = -9$ and $a_2 = -5$, find the solution of $a_n = 4a_{n-1} - 3a_{n-2} + 4$.

17. Let
    A be the set of all rooted binary trees composed of exactly three nodes named a, b and c;
    B be the subset of A composed of all trees whose right subtree (of the root) is not higher than the left subtree (of the root);
    C be the subset of A composed of all trees whose left subtree (of the root) is not higher than the right subtree (of the root); and
    D be the subset of A composed of all trees whose right subtree and left subtree (of the root) are with the same height.
    (A) (3 points) How many different topologies can be found in A? In other words, nodes a, b and c are considered identical.
    (B) (3 points) Choose an element from A, and assume every element has the same probability to be chosen. What is the probability of the event that the element is in B? What is the probability of the event that the element is in C? What is the probability of the event that the element is in D?
    (C) (4 points) According to the similarity of the definitions of B and C, it is obviously that $|B| = |C|$. Besides, we also have $D = B\cap C$ and $A = B\cup C$. Based on the inclusion-exclusion principle $|B\cup C| = |B| + |C| - |B\cap C|$, you can derive an equation to calculate $\Pr(x\in D)$ from only $\Pr(x\in A)$ and $\Pr(x\in B)$. Please write down the equation.

(接續下一頁)

【不可使用計算機】＊作答前請先核對試題、答案卷(試卷)與准考證之所組別與考科是否相符！！

18. (5 points) The transitive closure R^* of a relation R can be calculated by Warshall's algorithm. The pseudocode of Warshall's algorithm is below.

```
Procedure Warshall (M_R: n × n zero-one matrix of R)
W := M_R
for k := 1 to n
begin
    for i := 1 to n
    begin
        for j := 1 to n
            w_ij := w_ij ∨ (w_ik ∧ w_kj)
    end
end {W = [w_ij] is M_R*, the 0-1 matrix of R*}
```

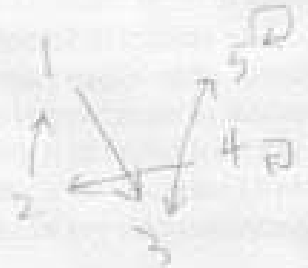
n=5

k=1

Based on the pseudocode, if $\mathbf{M}_R = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$, what is $[w_{31} \quad w_{32} \quad w_{33} \quad w_{34} \quad w_{35}]$ after the execution of the loop of $k := 3$? (You can get the points only if all five entries are correctly answered.)

(考題結束)

$$\begin{matrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{matrix} \qquad \begin{matrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{matrix}$$



0 1 0 1 0