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## Solutions to exercises and problems

from

### Introduction to Algorithms *Fourth Edition*

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## **Disclaimer**

This document is incomplete and was pre-released early to make easier access to the material for beta readers. Please be aware that the solutions in this document may contain bugs, be untested or badly formatted. Until the material reaches a mature state, I hope to receive feedback via the project's GitHub. There you can also track progress and read about the project's history and future:

<https://github.com/wojtask/clrs4e-solutions>

## Contents

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***Part I      Foundations***

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## 1.1 Algorithms

**1.1-1**

One real-world example that requires sorting is online marketplace search results. When users search for a specific item on e-commerce websites, the search results need to be sorted in a meaningful way to provide the most relevant and helpful information to the users.

An example that requires finding the shortest distance between two points is route planning or navigation systems. These systems are widely used in various applications, such as GPS devices, online mapping services, ride-sharing apps, and logistics management.

**1.1-2**

In a real-world setting, there are several measures of algorithm efficiency that go beyond just speed. Some of these include:

**Memory Usage:** The amount of memory required by an algorithm can significantly impact its efficiency. Algorithms that consume excessive memory may not scale well when dealing with large data sets or on systems with limited memory resources. It's important to consider the memory requirements of an algorithm to ensure it can run efficiently in different environments.

**Scalability:** An algorithm's ability to handle increasingly larger input sizes without a significant degradation in performance is crucial in real-world scenarios. Scalability is particularly important when dealing with big data or high-volume systems where the algorithm needs to process and analyze vast amounts of information.

**Parallelizability:** In modern computing environments, the ability to parallelize an algorithm and leverage multiple processors or distributed systems can greatly enhance efficiency. Algorithms that can be effectively parallelized can take advantage of hardware resources and potentially reduce overall processing time.

**I/O Efficiency:** Algorithms that involve reading from or writing to external storage, such as databases or files, need to be mindful of I/O efficiency. Minimizing disk accesses or network transfers and optimizing data retrieval and storage techniques can significantly impact the overall performance of an algorithm.

**Energy Consumption:** With the increasing emphasis on energy-efficient computing, algorithms that consume less power are desirable in many real-world settings. Optimizing an algorithm to reduce unnecessary computations or limit resource-intensive operations can help conserve energy and improve efficiency, particularly in resource-constrained environments like mobile devices or IoT devices.

**Robustness and Reliability:** Efficiency should not come at the cost of algorithmic robustness and reliability. In a real-world setting, algorithms need to handle various edge cases, exceptions, and error conditions gracefully. An algorithm that is efficient but lacks robustness may result in incorrect outputs, system failures, or security vulnerabilities.

**Maintainability and Modularity:** In many real-world scenarios, algorithms need to be maintained, extended, or integrated into existing systems. The efficiency of an algorithm can be influenced by factors like code readability, ease of understanding, and modularity, which impact the ease of maintenance and future enhancements.

### 1.1-3

We'll discuss the properties of **arrays** — one of the simplest non-trivial data structures widely used in computer programming. The strengths of arrays include:

**Simplicity:** Arrays are a straightforward and easy-to-understand data structure. They have a simple interface and operations like insertion and deletion are well-defined. This simplicity makes arrays a popular choice for beginners and for scenarios where the data size is known in advance.

**Random Access:** Arrays provide constant-time access to individual elements. Each element in an array can be accessed directly by its index, allowing for efficient random access operations. This makes arrays suitable for scenarios where quick access to elements is required.

**Efficiency:** Arrays have a fixed size and provide efficient memory allocation. The elements in an array are stored contiguously in memory, allowing for efficient traversal and manipulation of the array. This makes arrays well-suited for operations that involve iterating over elements or performing arithmetic computations.

Arrays also have some limitations that may make them less useful or efficient in certain problems. These are:

**Fixed Size:** Arrays have a fixed size, meaning they cannot be dynamically resized once created. This can be a limitation when the number of elements in the array needs to change dynamically. Resizing an array typically involves creating a new, larger array and copying the existing elements, which can be inefficient.

**Wasted Space:** Arrays require contiguous memory allocation, which can lead to wasted space when the array is not fully occupied. If an array is created with a larger capacity than needed, the unused space remains allocated, resulting in wasted memory. This is especially problematic when dealing with large arrays or in situations where memory is a scarce resource.

**Insertion and Deletion:** Inserting or deleting elements in an array can be inefficient. If an element is inserted or deleted in the middle of the array, all the subsequent elements need to be shifted to accommodate the change. This operation takes the time proportional to the number of elements in the array. Therefore, arrays are not the best choice when frequent insertions or deletions at arbitrary positions are required.

**Homogeneousness:** Arrays have a fixed data type, which means they can only store elements of a single type. This can be limiting when working with heterogeneous data.

#### 1.1-4

The problems are similar in terms of optimizing routes in a graph, but differ in objectives, constraints, and computational complexity:

- The shortest-path problem finds the minimum cost path between two nodes, while the traveling-salesperson problem finds the shortest route that visits all nodes exactly once and returns to the starting node.
- The shortest-path problem has no constraints, while the traveling-salesperson problem has the constraint of visiting all nodes once.
- The shortest-path problem can be solved efficiently, while the traveling-salesperson problem is computationally challenging and has no known polynomial-time solution.

#### 1.1-5

When providing critical medical diagnoses, we are interested in finding only the best solutions. For instance, in complex cases where a patient's life is at stake, accurate and precise diagnoses are crucial. The healthcare professionals need to identify the exact underlying condition to provide the appropriate treatment plan. In such situations, the highest level of accuracy is required to ensure the patient's well-being and potentially save lives.

A real-world problem where approximately optimal solution is sufficient is travel planning. When planning a vacation or business trip, finding the absolute best itinerary

might not always be necessary. For example, when searching for flights, accommodations, and activities, there are often multiple viable options that meet the basic criteria and provide an enjoyable experience. While it's desirable to find the most cost-effective or time-efficient solution, a reasonably good itinerary that meets most of the traveler's preferences can still lead to a satisfying trip. The goal is to strike a balance between the available options and the constraints, without necessitating an exhaustive search for the absolute best solution.

**1.1-6**

A real-world problem that fits this description is real-time stock market analysis and trading. Sometimes traders have access to the entire input data before making decisions, allowing them to analyze historical data and market trends. Other times, the input data arrives gradually over time, requiring traders to adapt and make decisions based on evolving real-time information such as news releases and market updates. In both cases, the goal is to make profitable trades based on the available information, but the difference lies in the availability of the data.

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## 1.2 Algorithms as a technology

**1.2-1**

One example of an application that requires algorithmic content at the application level is a recommendation system for an online video or music streaming platform. These platforms use algorithms to analyze user preferences, historical data, and content metadata in order to provide personalized recommendations to their users. The algorithms involved in these recommendation systems serve several functions:

**Content-Based Filtering:** Content-based filtering algorithms analyze the attributes or characteristics of the content itself, such as genre, director, actors, or music genre, artist, and lyrics. By building a profile for each user based on their preferences, these algorithms recommend similar content that matches the user's preferences.

**Collaborative Filtering:** Collaborative filtering algorithms analyze user behavior and preferences to find similarities between users and their viewing or listening habits. This helps identify users with similar tastes and enables the system to recommend content based on what similar users have liked or consumed. Collaborative filtering algorithms can be based on user-based or item-based approaches.

**Reinforcement Learning:** Some advanced recommendation systems employ reinforcement learning techniques. These algorithms continuously learn and adapt based on user feedback and interactions. They optimize the recommendation process by dynamically adjusting the weights assigned to different features and learning from the user's explicit or implicit feedback (e.g., ratings, likes, skips).

**Contextual Factors:** Algorithms in recommendation systems also consider contextual factors such as time of day, day of the week, or a user's location. These factors help tailor recommendations to specific situations and user contexts, ensuring that the recommended content is relevant and timely.

The primary goal of these algorithms is to enhance user experience by providing personalized content recommendations. By leveraging historical data, user behavior, and content attributes, the algorithms aim to surface relevant and engaging content that matches the user's preferences and interests.

- 1.2-2** For natural  $n$  the inequality  $8n^2 < 64n \lg n$  holds when  $2 \leq n \leq 43$ . Insertion sort beats merge sort for arrays of such sizes.

- 1.2-3** The smallest positive integer  $n$  satisfying the inequality  $100n^2 < 2^n$  is  $n = 15$ .

## Problems

**1-1** *Comparison of running times*

The results are collected in the table in Figure ???. Numbers greater than  $10^6$  are approximate. A month is assumed to have 30 days and a year is assumed to have 365 days.

	1 second	1 minute	1 hour	1 day	1 month	1 year	1 century
$\lg n$	$2^{10^6}$	$2^{6 \cdot 10^7}$	$2^{3.6 \cdot 10^9}$	$2^{8.64 \cdot 10^{10}}$	$2^{2.59 \cdot 10^{12}}$	$2^{3.15 \cdot 10^{13}}$	$2^{3.15 \cdot 10^{15}}$
$\sqrt{n}$	$10^{12}$	$3.6 \cdot 10^{15}$	$1.3 \cdot 10^{19}$	$7.46 \cdot 10^{21}$	$6.72 \cdot 10^{24}$	$9.95 \cdot 10^{26}$	$9.95 \cdot 10^{30}$
$n$	$10^6$	$6 \cdot 10^7$	$3.6 \cdot 10^9$	$8.64 \cdot 10^{10}$	$2.59 \cdot 10^{12}$	$3.15 \cdot 10^{13}$	$3.15 \cdot 10^{15}$
$n \lg n$	62,746	$2.8 \cdot 10^6$	$1.33 \cdot 10^8$	$2.76 \cdot 10^9$	$7.19 \cdot 10^{10}$	$7.98 \cdot 10^{11}$	$6.86 \cdot 10^{13}$
$n^2$	1,000	7,745	60,000	293,938	$1.61 \cdot 10^6$	$5.62 \cdot 10^6$	$5.62 \cdot 10^7$
$n^3$	100	391	1,532	4,420	13,736	31,593	146,645
$2^n$	19	25	31	36	41	44	51
$n!$	9	11	12	13	15	16	17

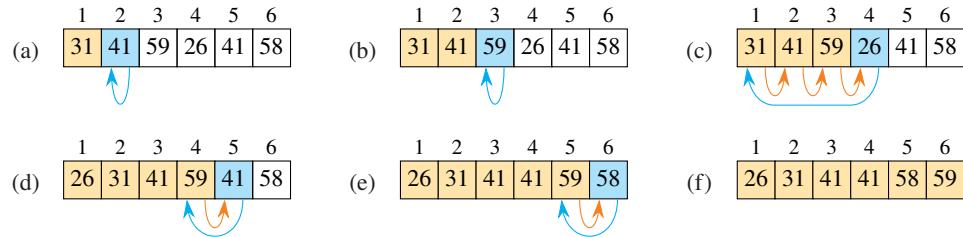
**Figure 1-1** Largest sizes of problems solvable in a given time by an algorithm of a given time efficiency.

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## 2.1 Insertion sort

**2.1-1**

See Figure ??.



**Figure 2.1-1** The operation of  $\text{INSERTION-SORT}(A, n)$  where  $A$  initially contains the sequence  $\langle 31, 41, 59, 26, 41, 58 \rangle$  and  $n = 6$ . (a)–(e) The iterations of the **for** loop of lines 1–8. (f) The final sorted array.

**2.1-2**

We formulate the loop invariant as follows:

At the start of each iteration of the **for** loop of lines 2–3,  $sum$  is the sum of the numbers in the subarray  $A[1:i - 1]$ .

**Initialization:** Before the first loop iteration,  $i = 1$ . The subarray  $A[1:i - 1]$  is empty, so the sum of its elements is 0, by definition. It is thus equal to the current value of  $sum$ , just after its initialization in line 1.

**Maintenance:** Suppose that just before the  $i$ th iteration  $sum$  holds the sum of the numbers in the subarray  $A[1:i - 1]$ . The only operation performed in the body of the loop is increasing  $sum$  by the value  $A[i]$ , so before the next iteration  $sum$  contains the sum of the numbers in the subarray  $A[1:i]$ . Incrementing  $i$  for the next iteration of the loop then preserves the loop invariant.

**Termination:** The loop terminates once  $i$  equals  $n + 1$ . The invariant says that  $sum$  is now equal to the sum of numbers in the subarray  $A[1 : n]$ . Therefore, the procedure is correct, since it immediately returns  $sum$  in line 4.

**2.1-3**

In the INSERTION-SORT procedure we can just reverse the second inequality in the **while** loop condition. In other words, replace line 5 by:

```
while  $j > 0$  and  $A[j] < key$ 
```

**2.1-4**

The description leads to the following pseudocode:

```
LINEAR-SEARCH( $A, n, x$ )
1 for  $i = 1$  to  $n$ 
2     if  $A[i] == x$ 
3         return  $i$ 
4 return NIL
```

We'll prove the loop invariant:

At the start of each iteration of the **for** loop of lines ??–??, the value  $x$  does not appear in the subarray  $A[1 : i - 1]$ .

**Initialization:** Prior to the first iteration of the loop,  $i = 1$ . Then, the subarray  $A[1 : i - 1]$  is empty and as such does not contain  $x$ .

**Maintenance:** Suppose that just before the  $i$ th iteration the value  $x$  does not appear in the subarray  $A[1 : i - 1]$ . The loop will terminate in this iteration, if  $A[i]$  equals  $x$ . Otherwise, we increment  $i$  and move on to the next iteration, so the loop invariant remains true.

**Termination:** The loop terminates once it encounters  $i$  such that  $A[i]$  equals  $x$ , or after running for  $n$  iterations. In the first case the procedure returns the index  $i$ . In the second case  $i = n + 1$  and the procedure returns NIL, which is correct, since by the loop invariant, the value  $x$  does not appear in the subarray  $A[1 : n]$ .

**2.1-5**

**ADD-BINARY-INTEGERS( $A, B, n$ )**

```

1 let  $C[0:n]$  be a new array
2  $carry = 0$ 
3 for  $i = 0$  to  $n - 1$ 
4    $C[i] = (A[i] + B[i] + carry) \text{ mod } 2$ 
5    $carry = \lfloor (A[i] + B[i] + carry)/2 \rfloor$ 
6  $C[n] = carry$ 
7 return  $C$ 
```

Once the array  $C$  has been created, the **for** loop of lines ??–?? keeps adding the individual bits of numbers  $a$  and  $b$ . More specifically, it fills array  $C$  by summing up the bits in arrays  $A$  and  $B$  at the same positions, using modular arithmetic and a carry kept in the variable  $carry$ . Just before returning array  $C$  the final carry is put to its last position.

**2.2 Analyzing algorithms****2.2-1**

$$n^3/1000 + 100n^2 - 100n + 3 = \Theta(n^3)$$

**2.2-2**

**SELECTION-SORT( $A, n$ )**

```

1 for  $i = 1$  to  $n - 1$ 
2    $min = i$ 
3   for  $j = i + 1$  to  $n$ 
4     if  $A[j] < A[min]$ 
5        $min = j$ 
6   exchange  $A[min]$  with  $A[i]$ 
```

The outer **for** loop maintains the following invariant:

Just prior to the  $i$ th iteration of the **for** loop of lines ??–??, the subarray  $A[1:i-1]$  contains the  $i-1$  smallest elements of array  $A$  in sorted order.

According to the invariant, once the loop terminates after performing the  $n-1$  iterations, the subarray  $A[1:n-1]$  contains the  $n-1$  smallest elements of  $A$  in sorted order. The last element then must be greater than or equal to each element of the subarray  $A[1:n-1]$ , which means that the  $n-1$  iterations suffice to sort the entire array.

In the  $i$ th iteration of the outer **for** loop we determine the smallest element of  $A[i : n]$ . After initializing the position of a potential minimum in line ??, we need to examine the remaining  $n - i$  positions in the inner **for** loop. Let's assign statement costs for each line of the algorithm, so that line  $k$  has cost  $c_k$ . For each  $i = 1, 2, \dots, n - 1$ , let  $t_i$  denote the number of times line ?? is executed during the  $i$ th iteration of the outer **for** loop. Then, the running time of the algorithm on an  $n$ -element array is

$$T(n) = c_1n + c_2(n - 1) + c_3 \sum_{i=1}^{n-1} (n - i + 1) + c_4 \sum_{i=1}^{n-1} (n - i) + c_5 \sum_{i=1}^{n-1} t_i + c_6(n - 1).$$

In the worst case  $t_i = n - i$  for each  $i = 1, 2, \dots, n - 1$ , and by the fact that

$$\begin{aligned} \sum_{i=1}^{n-1} (n - i + 1) &= \sum_{i=2}^n i \\ &= \frac{n(n + 1)}{2} - 1 \end{aligned}$$

and

$$\begin{aligned} \sum_{i=1}^{n-1} (n - i) &= \sum_{i=1}^{n-1} i \\ &= \frac{n(n - 1)}{2}, \end{aligned}$$

the running time of **SELECTION-SORT** is

$$\begin{aligned} T(n) &= c_1n + c_2(n - 1) + c_3 \left( \frac{n(n + 1)}{2} - 1 \right) + c_4 \left( \frac{n(n - 1)}{2} \right) \\ &\quad + c_5 \left( \frac{n(n - 1)}{2} \right) + c_6(n - 1) \\ &= \left( \frac{c_3}{2} + \frac{c_4}{2} + \frac{c_5}{2} \right) n^2 + \left( c_1 + c_2 + \frac{c_3}{2} - \frac{c_4}{2} - \frac{c_5}{2} + c_6 \right) n - (c_2 + c_3 + c_6). \end{aligned}$$

Therefore,  $T(n)$  can be expressed as  $an^2 + bn + c$  for constants  $a$ ,  $b$ , and  $c$  that depend on the statement costs  $c_k$ , and so  $T(n) = \Theta(n^2)$ .

The best case differs from the worst case in the values  $t_i$ , which all are 0. But this difference only cancels out the terms involving  $c_5$  in the above formula, and so it doesn't affect the rate of growth of the running time of the algorithm for the best case.

### 2.2-3

By Exercise ??, linear search will check  $(n + 1)/2$  elements on average. In the worst case all  $n$  elements will need to be examined. Therefore, since the algorithm terminates once either the searched for value has been found, or all elements have been examined, both the average-case and the worst-case running times of linear search are  $\Theta(n)$ .

**2.2-4**

A sorted array does not need to be sorted. Any sorting algorithm can be modified so that at the beginning it checks whether the input array is sorted, and immediately exits if it is. The extra step of detecting the ordering of the input elements requires a complete scan of the input array. Thus, the best-case running time of the algorithm for an input array of length  $n$  is  $\Theta(n)$ .

## 2.3 Designing algorithms

**2.3-1**

See Figure ??.

**2.3-2**

*Instead of  $p \neq r$  it should be  $p == r$  (in both places). Corrected in fourth printing.*

Lines 3–7 of MERGE-SORT are executed for subarrays  $A[p:r]$  of more than one element, i.e., when  $p < r$ . Note that  $p \leq (p+r)/2 \leq r$ , and since both  $p$  and  $r$  are integers,

$$p \leq \lfloor (p+r)/2 \rfloor \leq r.$$

Therefore, the recursive call in line 4 is for a subarray of at least one element. Suppose that the procedure is called in line 5 for an empty subarray. Then,  $\lfloor (p+r)/2 \rfloor + 1 > r$  and by the above inequalities, there must be  $\lfloor (p+r)/2 \rfloor = r$ . This equality holds only when  $(p+r)/2 = r$ , which implies  $p = r$ , contradicting the assumption that  $p < r$ . Therefore, replacing the test in line 1 by “**if  $p == r$** ” guarantees no recursive call works on an empty subarray.

**2.3-3**

We use the following loop invariant:

Prior to each iteration of the **while** loop of lines 12–18, the subarray  $A[p:k-1]$  contains the elements of  $L[0:i-1]$  and the elements of  $R[0:j-1]$ , all arranged in sorted order.

The **while** loop of lines 12–18 terminates once  $i = n_L$  or  $j = n_R$ . When it happens, the above invariant says that all elements from at least one array,  $L$  or  $R$ , have been copied into the subarray  $A[p:k-1]$ , and that the subarray is sorted. Since both arrays  $L$  and  $R$  are sorted, all elements in  $L[i:n_L-1]$ , as well as all elements in  $R[j:n_R-1]$ , are greater than or equal to all elements in  $A[p:k-1]$ . To finish merging the arrays  $L$  and  $R$  into  $A[p:r]$ , it suffices to copy the subarrays  $L[i:n_L-1]$  and  $R[j:n_R-1]$  to  $A[k:r]$ , which is exactly what the **while** loops of lines 20–23 and 24–27 do. In total, the three **while** loops copy exactly  $n_L + n_R = r - p + 1$  elements from  $L[0:n_L-1]$  (copied from  $A[p:q]$ ) and  $R[0:n_R-1]$  (copied from  $A[q+1:r]$ ) to  $A[p:r]$ , so the MERGE procedure works correctly.

## 2.3-4

Let  $n = 2^k$  for  $k \geq 1$ . We'll do the proof by induction on  $k$ . For  $k = 1$  we have  $n = 2$  and  $T(n) = 2 = 2\lg 2$ , so the base case holds. For the inductive step, assume that  $k > 1$ , or  $n > 2$ , and that  $T(n/2) = (n/2)\lg(n/2)$ . Then,

$$\begin{aligned} T(n) &= 2T(n/2) + n \\ &= 2(n/2)\lg(n/2) + n \\ &= n(\lg n - 1) + n \\ &= n \lg n. \end{aligned}$$

## 2.3-5

**RECURSIVE-INSERTION-SORT( $A, n$ )**

```

1  if  $n > 1$ 
2    RECURSIVE-INSERTION-SORT( $A, n - 1$ )
3  key =  $A[n]$ 
4  j =  $n - 1$ 
5  while  $j > 0$  and  $A[j] > key$ 
6     $A[j + 1] = A[j]$ 
7    j =  $j - 1$ 
8   $A[j + 1] = key$ 
```

Let  $T(n)$  be the worst-case running time of sorting an array  $A[1:n]$  with the above algorithm. Because inserting  $A[n]$  into the sorted subarray  $A[1:n - 1]$  takes at most  $\Theta(n)$  time, we get the following recurrence:

$$T(n) = \begin{cases} \Theta(1), & \text{if } n = 1, \\ T(n - 1) + \Theta(n), & \text{if } n > 1. \end{cases}$$

## 2.3-6

Instead of  $v$  there should be  $x$ . Corrected in third printing.

Below is recursive pseudocode implementing binary search. The procedure takes as input a sorted array  $A[1:n]$ , indices  $p$  and  $r$ , and a value  $x$ , and searches for  $x$  in the subarray  $A[p:r]$ . It returns an index  $q$  such that  $x$  equals  $A[q]$  or the special value NIL if  $x$  does not appear in  $A$ .

```

BINARY-SEARCH( $A, p, r, x$ )
1 if  $p > r$ 
2   return NIL
3    $q = \lfloor (p + r)/2 \rfloor$ 
4   if  $x == A[q]$ 
5     return  $q$ 
6   if  $x < A[q]$ 
7     return BINARY-SEARCH( $A, p, q - 1, x$ )
8   else return BINARY-SEARCH( $A, q + 1, r, x$ )

```

The procedure finds the midpoint of the subarray  $A[p:r]$  and compares  $x$  to it in line ???. Then it discards approximately half of that subarray, and continues searching for  $x$  in the other half in the recursive call of line ?? or ???. The procedure ends when it finds  $x$  or when the search range becomes empty (i.e.,  $p > r$ ), which means that  $A$  does not contain  $x$ . The recurrence describing the worst-case running time of the algorithm is:

$$T(n) = \begin{cases} \Theta(1), & \text{if } n \leq 1, \\ T(n/2) + \Theta(1), & \text{if } n > 1. \end{cases}$$

In Exercise ?? we show that  $T(n) = \Theta(\lg n)$ .

### 2.3-7

*Instead of  $A[1:j-1]$  there should be  $A[1:i-1]$ . Corrected in third printing.*

Using a binary search we can determine the target position in the subarray  $A[1:i-1]$  for the element initially occupying the position  $i$ . To be able to use here the BINARY-SEARCH procedure presented in Exercise ??, we need to modify it first so it returns  $p$  instead of NIL in line ???. However, moving  $A[i]$  on its target position requires shifting a part of the array to the right, which in the worst case takes the time proportional to the length of the sorted subarray. This approach is therefore no better than running a linear search in the INSERTION-SORT procedure, and would not improve the worst-case running time of insertion sort.

### 2.3-8

Let's represent the set  $S$  as an array  $A[1:n]$ . First, we'll sort  $A$ , so we can use a binary search through it (see Exercise ??). Then we will be scanning  $A$  in the sorted order and for an element  $A[i]$  we will search for another element in the subarray  $A[i+1:n]$ , that added to  $A[i]$  gives  $x$ . That subarray is empty for  $i = n$ , so we can skip searching for the last item. We can also terminate this procedure once  $A[i] \geq x/2$ , since all elements

in the subarray  $A[i + 1 : n]$  are then greater than  $x/2$ , and the search will never succeed. Depending on the search result, we return a boolean value (TRUE or FALSE).

```

SUM-SEARCH( $A, n, x$ )
1  MERGE-SORT( $A, 1, n$ )
2   $i = 1$ 
3  while  $i < n$  and  $A[i] < x/2$ 
4    if BINARY-SEARCH( $A, i + 1, n, x - A[i]$ )  $\neq \text{NIL}$ 
5      return TRUE
6       $i = i + 1$ 
7  return FALSE

```

Sorting the array  $A[1 : n]$  in line ?? takes  $\Theta(n \lg n)$  time. The procedure BINARY-SEARCH is called in line ?? at most  $n - 1$  times, for subarrays of decreasing sizes — from  $n - 1$  down to 1 — so the total time spent in all these calls won't exceed  $n \cdot \Theta(\lg n) = \Theta(n \lg n)$ .

## Problems

### 2-1

#### *Insertion sort on small arrays in merge sort*

a. Insertion sort called for a sublist of length  $k$  runs in  $\Theta(k^2)$  worst-case time. Applied separately to  $n/k$  such sublists takes  $(n/k) \cdot \Theta(k^2) = \Theta(nk)$  in total.

b. In the modified merge sort we'll check at each recursion level if  $r - p + 1 \leq k$ , and if so we'll sort  $A[p : r]$  using insertion sort, ending this recursion branch. The sorted sublists will then be merged as usual by the MERGE procedure. If we considered a recursion tree for the modified algorithm, it would have approximately  $\lg(n/k)$  levels, with each level contributing a cost of  $\Theta(n)$ . The total time spent on merging  $n/k$  sublists is therefore  $\lg(n/k) \cdot \Theta(n) = \Theta(n \lg(n/k))$ .

c. For  $k = c \lg n$ , where  $c$  is a positive constant,

$$\begin{aligned} \Theta(nk + n \lg(n/k)) &= \Theta(cn \lg n + n \lg(n/(c \lg n))) \\ &= \Theta((c + 1) \cdot n \lg n - n \lg c - n \lg \lg n) \\ &= \Theta(n \lg n), \end{aligned}$$

since in  $\Theta$ -notation we can ignore the lower-order terms and the leading term's constant coefficient. Thus, as long as  $k$  grows as fast as  $c \lg n$  for a constant  $c > 0$ , the running

time of the modified merge sort remains the same as the running time of the standard merge sort, in terms of  $\Theta$ -notation.

On the other hand, if  $k$  grows faster than  $c \lg n$  for any constant  $c > 0$ , then the total time spent on sorting sublists by insertion sort, as shown in part (a), is larger than  $n \lg n$  in terms of  $\Theta$ -notation, and so the whole modified algorithm runs slower than the original algorithm.

- d.** In practice,  $k$  should be the largest list length for which a particular implementation of insertion sort runs faster than a particular implementation of merge sort.

## 2-2

### Correctness of bubblesort

- a.** We also need to show that  $A'$  is a permutation of the array  $A$ .

- b.** The inner loop invariant:

At the start of each iteration of the **for** loop of lines 2–4,  $A[j]$  is the smallest element of the subarray  $A[j : n]$ .

**Initialization:** Just before the first loop iteration,  $j = n$ , so the subarray  $A[j : n]$  consists of a single element  $A[j]$  and the invariant trivially holds.

**Maintenance:** By the loop invariant, we assume that just before an iteration,  $A[j]$  is the smallest element of  $A[j : n]$ . Then,  $A[j - 1]$  is the smallest element of  $A[j - 1 : n]$ , unless  $A[j] < A[j - 1]$ , in which case the elements  $A[j]$  and  $A[j - 1]$  are exchanged in line 4. Decrementing  $j$  for the next iteration preserves the invariant.

**Termination:** The loop terminates, since it is a **for** loop iterating from  $n$  down to  $i + 1$ , and  $2 \leq i + 1 \leq n$ . At termination,  $j = i$ , and  $A[i]$  is the smallest element of  $A[i : n]$ .

- c.** The outer loop invariant:

At the start of each iteration of the **for** loop of lines 1–4, the subarray  $A[1 : i - 1]$  consists of the  $i - 1$  smallest elements of  $A$  in sorted order.

**Initialization:** Prior to the first iteration,  $i = 1$ , so the subarray  $A[1 : i - 1]$  is empty and therefore trivially sorted.

**Maintenance:** By the assumption that the subarray  $A[1 : i - 1]$  is sorted, the largest element of that subarray is  $A[i - 1]$ . The inner **for** loop places the smallest element of the subarray  $A[i : n]$  at position  $i$  (which we proved in part (b)). By the loop

invariant holding before the iteration, it follows that in the subarray  $A[i : n]$  there are no elements smaller than  $A[i - 1]$ , so in particular  $A[i - 1] \leq A[i]$ . Hence, the subarray  $A[1 : i]$  contains the  $i$  smallest elements of  $A$  in sorted order. By incrementing  $i$  for the next loop iteration, we preserve the invariant.

**Termination:** The loop makes exactly  $n - 1$  iterations, and so it terminates. When it does,  $i = n$ , and the subarray  $A[1 : i - 1]$  consists of the  $i - 1 = n - 1$  smallest elements of  $A$  in sorted order. Thus, the element at position  $n$  must be the largest element of  $A$ , so the algorithm sorts correctly.

- d. For any input, the **for** loop of lines 2–4 performs  $n - i$  iterations for each  $i = 1, 2, \dots, n - 1$ , and each iteration of that loop takes constant time. The total number of the inner loop iterations is

$$\begin{aligned} \sum_{i=1}^{n-1} (n - i) &= \sum_{i=1}^{n-1} i \\ &= \frac{n(n-1)}{2}, \end{aligned}$$

so the running time of BUBBLESORT is  $\Theta(n^2)$ . It is the same as the worst-case running time of INSERTION-SORT, but worse than its best-case running time.

### 2-3

#### Correctness of Horner's rule

- a. The **for** loop of lines 2–3 makes  $n + 1$  iterations, each taking a constant time, so the running time of HORNER is  $\Theta(n)$ .
- b. The following procedure takes the same set of parameters as HORNER and evaluates  $P(x)$  using the naive method.

POLYNOMIAL-EVALUATE( $A, n, x$ )

```

1   $p = 0$ 
2  for  $i = 0$  to  $n$ 
3       $s = A[i]$ 
4      for  $j = 1$  to  $i$ 
5           $s = s \cdot x$ 
6       $p = p + s$ 
7  return  $p$ 
```

The **for** loop of lines ??–?? iterates  $i$  times for each  $i = 0, 1, \dots, n$ , for a total of  $\sum_{i=0}^n i = n(n+1)/2$  iterations. The running time of POLYNOMIAL-EVALUATE is therefore  $\Theta(n^2)$ , so the algorithm is less efficient as compared to HORNER.

c. We prove the three properties of the loop invariant:

**Initialization:** Prior to the first iteration,  $i = n$ , so

$$\begin{aligned} p &= \sum_{k=0}^{n-(i+1)} A[k+i+1] \cdot x^k \\ &= \sum_{k=0}^{-1} A[k+n+1] \cdot x^k \\ &= 0, \end{aligned}$$

that matches the initial value of  $p$ .

**Maintenance:** During each iteration  $p$  gets the value of  $A[i] + x \cdot p$ . Assuming that the invariant holds before the current iteration, we have

$$\begin{aligned} p &= A[i] + \sum_{k=0}^{n-(i+1)} A[k+i+1] \cdot x^{k+1} \\ &= A[i] \cdot x^0 + \sum_{k=1}^{n-i} A[k+i] \cdot x^k \\ &= \sum_{k=0}^{n-i} A[k+i] \cdot x^k. \end{aligned}$$

After decrementing  $i$  for the next iteration, the invariant is preserved.

**Termination:** The loop terminates, since it performs exactly  $n+1$  iterations. At termination,  $i = -1$ , so

$$\begin{aligned} p &= \sum_{k=0}^{n-(i+1)} A[k+i+1] \cdot x^k \\ &= \sum_{k=0}^n A[k] \cdot x^k. \end{aligned}$$

Therefore, the algorithm correctly evaluates the polynomial  $P(x)$  for a given  $x$ .

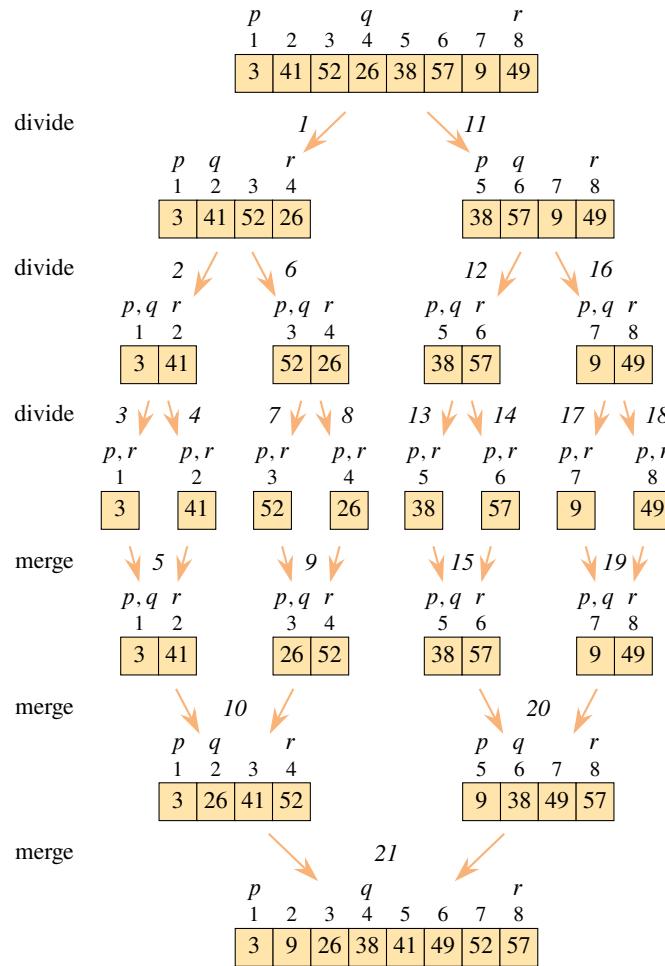
**2-4****Inversions**

- a.**  $(1, 5), (2, 5), (3, 4), (3, 5), (4, 5)$
- b.** An array of length  $n$  with the most inversions is the one sorted in descending order. In such an array, every pair  $(i, j)$ , where  $1 \leq i < j \leq n$ , is an inversion, and so the array has  $\binom{n}{2} = n(n - 1)/2$  inversions.
- c.** For an array  $A[1 : n]$ , by  $\text{inv}(i)$  let us denote the number of inversions  $(j, i)$  of  $A$ . In the INSERTION-SORT procedure, the **while** loop of lines 5–7 moves the element initially at position  $i$  by  $\text{inv}(i)$  positions to the left, thus eliminating exactly  $\text{inv}(i)$  inversions. The **while** loop runs for each  $i = 2, 3, \dots, n$ , for a total of  $I = \sum_{i=2}^n \text{inv}(i)$  iterations, which is exactly the number of inversions there were in the input array. We conclude that the running time of insertion sort is  $\Theta(n + I)$ .
- d.** Let  $A[1 : n]$  be an array representing the input permutation of  $n$  elements, and let  $a$  and  $b$  be two different elements. Suppose that during merge sort invoked for the array  $A$ , at some point in a call to MERGE,  $L[i] = a$  and  $R[j] = b$ . If the test in line 13 of MERGE holds, then the elements  $a$  and  $b$  don't make an inversion. Otherwise,  $a > b$ , and since the arrays  $L$  and  $R$  are sorted,  $b$  is less than any so far unprocessed element in  $L$  (i.e., not yet placed in a target position of  $A$  in line 14). These unprocessed elements occupy the subarray  $L[i : n_L - 1]$ , so  $b$  makes exactly  $n_L - i$  inversions with them. Once  $b$  is copied back to  $A$  in line 16, it will share the same merged subarray with the elements in  $L[i : n_L - 1]$ , so we won't count any inversion twice in subsequent recursive calls to MERGE.

In this way, by modifying merge sort we can determine the number of inversions of an array  $A[1 : n]$  of  $n$  distinct elements in  $\Theta(n \lg n)$  worst-case time. The side effect of running the modified procedure is having the input array sorted. The only modification to the original MERGE procedure is adding a new variable for counting inversions in a given recursive call. We will initialize the variable to 0 before line 12, and will be incrementing it by  $n_L - i$  in the **else** branch of lines 16–17, before returning it as a result of the modified procedure. We will use a similar counter in the modified MERGE-SORT procedure to sum up the number of inversions from the recursive calls, as well as from the call to the modified MERGE (which we call MERGE-INVERSIONS). The following pseudocode implements the modified MERGE-SORT.

```
COUNT-INVERSIONS( $A, p, r$ )
1   inversions = 0
2   if  $p < r$ 
3        $q = \lfloor (p + r)/2 \rfloor$ 
4       inversions = inversions + COUNT-INVERSIONS( $A, p, q$ )
5       inversions = inversions + COUNT-INVERSIONS( $A, q + 1, r$ )
6       inversions = inversions + MERGE-INVERSIONS( $A, p, q, r$ )
7   return inversions
```

The initial call  $\text{COUNT-INVERSIONS}(A, 1, n)$  returns the number of inversions of  $A[1 : n]$ .



**Figure 2.3-1** The operation of merge sort on the array  $A$  initially containing the sequence  $\{3, 41, 52, 26, 38, 57, 9, 49\}$ .

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# 3

## Characterizing Running Times

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### 3.1 $O$ -notation, $\Omega$ -notation, and $\Theta$ -notation

**3.1-1**

Suppose that the  $\lfloor n/3 \rfloor$  largest values of the input array  $A$  occupy the first  $\lfloor n/3 \rfloor$  positions  $A[1 : \lfloor n/3 \rfloor]$ . When the array is sorted, each of these values must pass through the middle  $n - 2\lfloor n/3 \rfloor$  positions  $A[\lfloor n/3 \rfloor + 1 : 2\lfloor n/3 \rfloor]$ , one position at a time, by at least  $n - 2\lfloor n/3 \rfloor$  executions of line 6. So the time taken by `INSERTION-SORT` in the worst case is at least proportional to

$$\begin{aligned} \lfloor n/3 \rfloor(n - 2\lfloor n/3 \rfloor) &\geq \lfloor n/3 \rfloor(n - 2n/3) && \text{(by inequality } \lfloor x \rfloor \leq x \text{ (3.2))} \\ &\geq ((n - 2)/3)(n/3) && \text{(by inequality (3.8))} \\ &\geq (n/6)(n/3) && \text{(as long as } n \geq 4) \\ &= n^2/18, \end{aligned}$$

which is  $\Omega(n^2)$ .

**3.1-2**

In `SELECTION-SORT` the outer loop runs  $n - 1$  times, and the inner loop runs at most  $n - 1$  times, spending a constant time per iteration. This situation is analogous to that we've seen in `INSERTION-SORT`, so the upper bound on the running times of both algorithms must be the same:  $O(n^2)$ .

Now observe that during those iterations of the outer loop, in which  $j = 1, 2, \dots, \lfloor n/2 \rfloor$ ,  $A[min]$  is compared with all elements of  $A[\lfloor n/2 \rfloor + 1 : n]$  in line 4. So in each of these  $\lfloor n/2 \rfloor$  iterations of the outer loop, the inner loop iterates at least  $n - \lfloor n/2 \rfloor$  times. The body of the inner loop takes a constant time per iteration, so the total running time of `SELECTION-SORT` in the worst case is proportional to the total number of iterations

of the inner loop, which is at least

$$\begin{aligned} \lfloor n/2 \rfloor(n - \lfloor n/2 \rfloor) &\geq ((n-1)/2)(n/2) && \text{(by inequalities (3.2) and (3.8))} \\ &\geq (n/4)(n/2) && \text{(as long as } n \geq 2\text{)} \\ &= n^2/8, \end{aligned}$$

or  $\Omega(n^2)$ .

Because we've shown that in all cases SELECTION-SORT runs in  $O(n^2)$  time and in  $\Omega(n^2)$  time, we can conclude that the running time of SELECTION-SORT is  $\Theta(n^2)$ .

### 3.1-3

If we assume that the  $\alpha n$  largest values of the input array occupy the first  $\alpha n$  positions, those values will need to pass through the  $(1 - 2\alpha)n$  middle positions, before the array is sorted. Those middle array positions must exist, so we'll require that  $1 - 2\alpha > 0$ , or  $\alpha < 1/2$ . The number of such moves is  $\alpha n(1 - 2\alpha)n$ . We can find the value of  $\alpha$  that maximizes this number for a given  $n$ , by analysing the quadratic function  $f(\alpha) = \alpha(1 - 2\alpha)$ . Its vertex form is  $f(\alpha) = -2(\alpha - 1/4) + 1/8$ , so the vertex of the parabola is  $(1/4, 1/8)$ , and so the function is maximized at  $\alpha = 1/4$ .

## 3.2 Asymptotic notation: formal definitions

### 3.2-1

Let  $n_0$  be the smallest value of  $n$  such that  $f(n) \geq 0$  and  $g(n) \geq 0$ . For all  $n \geq n_0$ ,

$$0 \leq \frac{f(n) + g(n)}{2} \leq \max \{f(n), g(n)\} \leq f(n) + g(n),$$

so it suffices to let  $c_1 = 1/2$  and  $c_2 = 1$  in the definition of  $\Theta$ -notation to obtain the desired result.

### 3.2-2

Let  $T(n)$  be the running time of algorithm  $A$ . The statement " $T(n)$  is at least  $O(n^2)$ " means that  $T(n) \geq f(n)$ , where  $f(n) = O(n^2)$ . In particular, we can choose  $f(n) = 0$ , and the statement will remain true for any  $T(n)$ . Thus, the statement does not characterize  $T(n)$  in any meaningful way.

### 3.2-3

Let's determine positive constants  $c$  and  $n_0$  such that  $0 \leq 2^{n+1} \leq c2^n$  for all  $n \geq n_0$ . Since  $2^{n+1} = 2 \cdot 2^n$ , we can pick  $c = 2$  and  $n_0 = 1$ . So  $2^{n+1} = O(2^n)$ .

Now let's assume that  $c$  and  $n_0$  are positive constants satisfying  $0 \leq 2^{2n} \leq c2^n$  for all  $n \geq n_0$ . Then  $2^{2n} = 2^n \cdot 2^n \leq c2^n$ , which implies that  $c \geq 2^n$ . The last inequality, however, does not hold regardless of a value we would choose for  $c$ , because  $2^n$  becomes arbitrarily large as  $n$  gets large. Hence,  $2^{2n} \neq O(2^n)$ .

**3.2-4**

By the definition of  $\Theta$ -notation,  $f(n) = \Theta(g(n))$  whenever there exist positive constants  $c_1, c_2$ , and  $n_0$  such that

$$0 \leq c_1g(n) \leq f(n) \leq c_2g(n)$$

for all  $n \geq n_0$ . This condition can be decomposed into the combination of inequalities  $0 \leq c_1g(n) \leq f(n)$  and  $0 \leq f(n) \leq c_2g(n)$ , both holding for all  $n \geq n_0$ . From the former follows  $f(n) = \Omega(g(n))$  and from the latter follows  $f(n) = O(g(n))$ .

For the proof of the opposite direction, suppose that  $f(n) = \Omega(g(n))$  and  $f(n) = O(g(n))$ . The former means that there exist positive constants  $c_1$  and  $n_1$  such that  $0 \leq c_1g(n) \leq f(n)$  for all  $n \geq n_1$ , and the latter means that there exist positive constants  $c_2$  and  $n_2$  such that  $0 \leq f(n) \leq c_2g(n)$  for all  $n \geq n_2$ . Then, by letting  $n_0 = \max\{n_1, n_2\}$  and merging both conditions, we get that

$$0 \leq c_1g(n) \leq f(n) \leq c_2g(n)$$

for all  $n \geq n_0$ . Thus,  $f(n) = \Theta(g(n))$ .

**3.2-5**

For the input size  $n$  let  $T(n)$  be the running time of an algorithm, and let  $T_{\text{worst}}(n)$  and  $T_{\text{best}}(n)$  be its worst-case running time and its best-case running time, respectively.

Suppose that  $T(n) = \Theta(g(n))$ . Among all inputs of size  $n$ , there is one that is the worst case and one that is the best case for the algorithm in terms of time required to run on these inputs. Since  $T(n)$  describes the running time for *all* cases, it also describes the running time for the worst case, and the running time for the best case. Thus,  $T_{\text{worst}}(n) = \Theta(g(n))$  and  $T_{\text{best}}(n) = \Theta(g(n))$ , and by Theorem 3.1 we have that  $T_{\text{worst}}(n) = O(g(n))$  and  $T_{\text{best}}(n) = \Omega(g(n))$ .

Now suppose that  $T_{\text{best}}(n) = \Omega(g(n))$  and  $T_{\text{worst}}(n) = O(g(n))$ . Let  $c_1$  and  $n_1$  be positive constants such that  $0 \leq c_1g(n) \leq T_{\text{best}}(n)$  for all  $n \geq n_1$ , and let  $c_2$  and  $n_2$  be positive constants such that  $0 \leq T_{\text{worst}}(n) \leq c_2g(n)$  for all  $n \geq n_2$ . Of course, for any input size  $n$ ,  $T_{\text{best}}(n) \leq T(n) \leq T_{\text{worst}}(n)$ . Then for all  $n \geq \max\{n_1, n_2\}$ ,

$$0 \leq c_1g(n) \leq T_{\text{best}}(n) \leq T(n) \leq T_{\text{worst}}(n) \leq c_2g(n),$$

so  $T(n) = \Theta(g(n))$ .

**3.2-6**

Suppose the set is nonempty and let  $f(n) \in o(g(n)) \cap \omega(g(n))$ . Then both relations  $f(n) = o(g(n))$  and  $f(n) = \omega(g(n))$  hold, which means that for any positive constants  $c_1$  and  $c_2$  there exists a positive constant  $n_0$  such that  $c_1g(n) < f(n) < c_2g(n)$  for all  $n \geq n_0$ . This statement is a contradiction, because it is not true that  $c_1 < c_2$  for every two numbers  $c_1$  and  $c_2$ . Hence,  $o(g(n)) \cap \omega(g(n)) = \emptyset$ .

3.2-7

$\Omega(g(n, m)) = \{f(n, m) : \text{there exist positive constants } c, n_0, \text{ and } m_0 \text{ such that } 0 \leq cg(n, m) \leq f(n, m) \text{ for all } n \geq n_0 \text{ or } m \geq m_0\},$

$\Theta(g(n, m)) = \{f(n, m) : \text{there exist positive constants } c_1, c_2, n_0, \text{ and } m_0 \text{ such that } c_1g(n, m) \leq f(n, m) \leq c_2g(n, m) \text{ for all } n \geq n_0 \text{ or } m \geq m_0\}.$

### 3.3 Standard notations and common functions

3.3-1

Let  $n \leq m$ . Then,  $f(n) \leq f(m)$  and  $g(n) \leq g(m)$ . If we combine these inequalities by adding one to the other, we obtain

$$f(n) + g(n) \leq f(m) + g(m),$$

which means that the function  $f(n) + g(n)$  is monotonically increasing. By plugging the values of  $g(n)$  into the function  $f(n)$ , we have that  $g(n) \leq g(m)$  implies  $f(g(n)) \leq f(g(m))$ , and so  $f(g(n))$  is monotonically increasing. If we additionally assume that  $f(n)$  and  $g(n)$  are nonnegative, we can combine the inequalities  $f(n) \leq f(m)$  and  $g(n) \leq g(m)$  by multiplying one with the other:

$$f(n) \cdot g(n) \leq f(m) \cdot g(m).$$

Thus,  $f(n) \cdot g(n)$  is monotonically increasing.

3.3-2

$$\begin{aligned} \lfloor \alpha n \rfloor + \lceil (1 - \alpha)n \rceil &= \lfloor \alpha n \rfloor + \lceil n + (-\alpha n) \rceil \\ &= \lfloor \alpha n \rfloor + n + \lceil -\alpha n \rceil && \text{(by equation (3.10))} \\ &= \lfloor \alpha n \rfloor + n - \lfloor \alpha n \rfloor && \text{(by equation (3.3))} \\ &= n \end{aligned}$$

3.3-3

Let  $f(n) = o(n)$ . Then, in particular, there exists a positive constant  $n_0$  such that  $0 \leq f(n) < n/2$  for all  $n \geq n_0$ . If  $k \geq 0$ , the function  $n^k$  is monotonically increasing, so we have

$$(n + f(n))^k \geq n^k$$

and

$$\begin{aligned} (n + f(n))^k &\leq (n + n/2)^k \\ &= (3/2)^k n^k. \end{aligned}$$

We can satisfy the condition in the definition of  $\Theta$ -notation for all  $n \geq n_0$ , by letting  $c_1 = 1$  and  $c_2 = (3/2)^k$ . In case where  $k < 0$ , the function  $n^k$  is monotonically decreasing, and the above inequalities need to be reversed:

$$(3/2)^k n^k \leq (n + f(n)) \leq n^k.$$

Then it suffices to plug  $c_1 = (3/2)^k$  and  $c_2 = 1$  to make the condition true. Hence,  $(n + o(n))^k = \Theta(n^k)$ .

Similarly, we can show that  $(n - o(n))^k = \Theta(n^k)$ . Suppose that  $k \geq 0$ . For any  $f(n) = o(n)$  satisfying  $0 \leq f(n) < n/2$  for all  $n \geq n_0$ , where  $n_0$  is a positive constant, we have

$$\begin{aligned} (n - f(n))^k &\geq (n - n/2)^k \\ &= (1/2)^k n^k \end{aligned}$$

and

$$(n - f(n))^k \leq n^k.$$

In the definition of  $\Theta$ -notation we let  $c_1 = (1/2)^k$  and  $c_2 = 1$ , so that the condition is met for all  $n \geq n_0$ . The case where  $k < 0$  is symmetric.

From inequalities (3.2), if  $k \geq 0$ , then

$$\begin{aligned} (n-1)^k &< \lceil n \rceil^k < (n+1)^k, \\ (n-1)^k &< \lfloor n \rfloor^k < (n+1)^k, \end{aligned}$$

and symmetrically, if  $k < 0$ . Then, by the fact shown above, we have  $\lceil n \rceil^k = \Theta(n^k)$  and  $\lfloor n \rfloor^k = \Theta(n^k)$ .

### 3.3-4

a. By applying equation (3.17) twice, we get

$$\begin{aligned} a^{\log_b c} &= (b^{\log_b a})^{\log_b c} \\ &= (b^{\log_b c})^{\log_b a} \\ &= c^{\log_b a}. \end{aligned}$$

b. We'll prove that

$$\lim_{n \rightarrow \infty} \frac{n!}{2^n} = \infty.$$

Note that in the product

$$\frac{n!}{2^n} = \left(\frac{1}{2}\right)\left(\frac{2}{2}\right)\cdots\left(\frac{n}{2}\right)$$

all the factors on the right-hand side are positive and, as  $n$  increases, the final factors can grow arbitrarily large, so the product diverges to  $\infty$ .

Similarly, we'll prove that

$$\lim_{n \rightarrow \infty} \frac{n!}{n^n} = 0.$$

We have

$$\frac{n!}{n^n} = \left(\frac{1}{n}\right) \left(\frac{2}{n}\right) \cdots \left(\frac{n}{n}\right).$$

Each factor on the right-hand side is positive and does not exceed 1. For  $n$  large enough the initial factors can be arbitrarily close to 0, so the product approaches its limit of 0.

For the proof of equation (3.28), observe that since  $n! \leq n^n$ , the upper bound on  $\lg(n!)$  is  $\lg n^n = n \lg n$ . We'll find an asymptotic lower bound by using Stirling's approximation. Let  $f(n) = 1 + \Theta(1/n) = \Theta(1)$  be the last factor on the right-hand side of formula (3.25). Because the function  $\sqrt{2\pi n}$  is strictly increasing, there exists a positive constant  $n_0$  such that  $\sqrt{2\pi n} f(n) \geq 1$  for all  $n \geq n_0$ . Also, we'll use the fact that  $e < 3$ , so  $e < \sqrt{n}$  when  $n \geq 9$ . For all  $n \geq \max\{n_0, 9\}$  we have

$$\begin{aligned} \lg(n!) &= \lg \left( \sqrt{2\pi n} \left( \frac{n}{e} \right)^n f(n) \right) \\ &\geq \lg \left( \frac{n}{e} \right)^n \\ &> \lg (\sqrt{n})^n \\ &= \frac{n \lg n}{2}. \end{aligned}$$

Combining both upper and lower bounds, we obtain  $\lg(n!) = \Theta(n \lg n)$ .

**c.** Let  $f(n) = \Theta(n)$ . By the definition of  $\Theta$ -notation, there exist positive constants  $c_1, c_2$ , and  $n_0$  such that

$$c_1 n \leq f(n) \leq c_2 n$$

for all  $n \geq n_0$ . Taking logarithms of each side of the above inequalities yields

$$\begin{aligned} \lg f(n) &\geq \lg(c_1 n) \\ &= \lg c_1 + \lg n \\ &\geq \lg \left( \frac{1}{\sqrt{n}} \right) + \lg n && \text{(as long as } n \geq 1/c_1^2) \\ &= \lg \sqrt{n} \\ &= \frac{\lg n}{2} \end{aligned}$$

and

$$\begin{aligned}\lg f(n) &\leq \lg(c_2 n) \\ &= \lg c_2 + \lg n \\ &\leq 2 \lg n \quad (\text{as long as } n \geq c_2).\end{aligned}$$

So,  $\lg f(n) = \Theta(\lg n)$ .

**3.3-5**

- ★ Let a function  $f(n)$  be polynomially bounded. Then  $f(n) = O(n^k)$  for some constant  $k$ . As we show in Problem ??(c), this fact implies that  $\lg f(n) = O(\lg n^k) = O(\lg n)$ . Now suppose that  $\lg f(n) = O(\lg n)$ . Then there exist positive constants  $c$  and  $n_0$  such that  $0 \leq \lg f(n) \leq c \lg n$  for all  $n \geq n_0$ . Then, we have

$$\begin{aligned}0 &\leq f(n) \\ &= 2^{\lg f(n)} \\ &\leq 2^{c \lg n} \\ &= (2^{\lg n})^c \\ &= n^c\end{aligned}$$

for all  $n \geq n_0$ , so that  $f(n)$  is polynomially bounded.

In the following proofs we use equation (3.28) proved in Exercise ?? and equation  $\lceil n \rceil = \Theta(n)$  proved in Exercise ??, after substituting different functions for  $n$  in those equations. We also rely on the observations and proofs from the solution to Problem ??.

We have

$$\begin{aligned}\lg(\lceil \lg n \rceil !) &= \Theta(\lceil \lg n \rceil \lg \lceil \lg n \rceil) \\ &= \Theta(\lg n \lg \lg n) \\ &= \omega(\lg n).\end{aligned}$$

We've shown that  $\lg(\lceil \lg n \rceil !) \neq O(\lg n)$ , which means that the function  $\lceil \lg n \rceil !$  is not polynomially bounded.

On the other hand,

$$\begin{aligned}\lg(\lceil \lg \lg n \rceil !) &= \Theta(\lceil \lg \lg n \rceil \lg \lceil \lg \lg n \rceil) \\ &= \Theta(\lg \lg n \lg \lg \lg n) \\ &= o((\lg \lg n)^2) \\ &= o(\lg n),\end{aligned}$$

where the last step results from equation (3.24) for  $a = 1$  and  $b = 2$  and after substituting  $\lg n$  for  $n$ . The obtained result implies in particular that  $\lg(\lceil \lg \lg n \rceil !) = O(\lg n)$ , so the function  $\lceil \lg \lg n \rceil !$  is polynomially bounded.

**3.3-6**

- ★ As stated in the text on page 67, any positive polynomial function grows faster than any polylogarithmic function, in particular  $m - 1 = \omega(\lg m)$ . So for any positive constant  $c$  there is a positive constant  $m_0$  such that  $0 \leq c \lg m < m - 1$  for all  $m \geq m_0$ . Let

$$n_0 = \underbrace{2^{2^{\dots^2}}}_{m_0}.$$

Notice that  $\lg^* n_0 = m_0$  and since the function  $\lg^* n$  is monotonically increasing, for  $n \geq n_0$  it takes all integer values greater than or equal to  $m_0$ . We can therefore substitute  $\lg^* n$  for  $m$  in the previous inequalities. More precisely, it is true that for any positive constant  $c$  there is a positive constant  $n_0$  such that

$$\begin{aligned} 0 &\leq c \lg(\lg^* n) \\ &< \lg^* n - 1 \\ &= \lg^*(\lg n) \end{aligned}$$

for all  $n \geq n_0$ . Therefore,  $\lg^*(\lg n) = \omega(\lg(\lg^* n))$ .

**3.3-7**

- $\phi$  and  $\hat{\phi}$  should be given by equations (3.32) and (3.33), respectively. Corrected in fourth printing.

$$\begin{aligned} \phi^2 &= \left( \frac{1 + \sqrt{5}}{2} \right)^2 \\ &= \frac{1 + 2\sqrt{5} + 5}{4} \\ &= \frac{3 + \sqrt{5}}{2} \\ &= \frac{1 + \sqrt{5}}{2} + 1 \\ &= \phi + 1 \end{aligned}$$

and

$$\begin{aligned}\hat{\phi}^2 &= \left(\frac{1-\sqrt{5}}{2}\right)^2 \\ &= \frac{1-2\sqrt{5}+5}{4} \\ &= \frac{3-\sqrt{5}}{2} \\ &= \frac{1-\sqrt{5}}{2} + 1 \\ &= \hat{\phi} + 1,\end{aligned}$$

so both  $\phi$  and  $\hat{\phi}$  are the roots of the equation  $x^2 = x + 1$ , and therefore are the golden ratio and its conjugate, respectively.

**3.3-8**

$\phi$  and  $\hat{\phi}$  should be given by equations (3.32) and (3.33), respectively. Corrected in fourth printing.

We have

$$\begin{aligned}\frac{\phi^0 - \hat{\phi}^0}{\sqrt{5}} &= 0 \\ &= F_0\end{aligned}$$

and

$$\begin{aligned}\frac{\phi^1 - \hat{\phi}^1}{\sqrt{5}} &= \frac{1+\sqrt{5} - (1-\sqrt{5})}{2\sqrt{5}} \\ &= 1 \\ &= F_1,\end{aligned}$$

so the base case holds. Now suppose that  $i \geq 2$ . The inductive hypothesis is that

$$F_{i-2} = \frac{\phi^{i-2} - \hat{\phi}^{i-2}}{\sqrt{5}}$$

and

$$F_{i-1} = \frac{\phi^{i-1} - \hat{\phi}^{i-1}}{\sqrt{5}}.$$

Then, we have

$$\begin{aligned}
 F_i &= F_{i-2} + F_{i-1} \\
 &= \frac{\phi^{i-2} - \hat{\phi}^{i-2}}{\sqrt{5}} + \frac{\phi^{i-1} - \hat{\phi}^{i-1}}{\sqrt{5}} \\
 &= \frac{\phi^{i-2}(1 + \phi) - \hat{\phi}^{i-2}(1 + \hat{\phi})}{\sqrt{5}} \\
 &= \frac{\phi^i - \hat{\phi}^i}{\sqrt{5}}
 \end{aligned}
 \tag{by Exercise ??},$$

so the formula for  $F_i$  holds for all  $i \geq 0$ .

### 3.3-9

Suppose that  $k \lg k = \Theta(n)$ . By the definition of  $\Theta$ -notation for functions with two parameters (see Exercise ??), there exist positive constants  $c_1, c_2, k_0$ , and  $n_0$  such that

$$c_1 n \leq k \lg k \leq c_2 n$$

for all  $k \geq k_0$  or  $n \geq n_0$ . By placing the additional restriction of  $k \geq 2$  and taking logarithms on all parts of the above compound inequality, we get

$$\lg c_1 + \lg n \leq \lg k + \lg \lg k \leq \lg c_2 + \lg n.$$

Then, as long as  $n \geq c_2$ ,  $\lg c_2 \leq \lg n$ , and so

$$\begin{aligned}
 \lg k &\leq \lg c_2 + \lg n - \lg \lg k \\
 &\leq \lg n + \lg n \\
 &= 2 \lg n,
 \end{aligned}$$

or  $1 \geq \lg k / 2 \lg n$ . Therefore,

$$\begin{aligned}
 k &\geq \frac{k \lg k}{2 \lg n} \\
 &\geq \frac{c_1}{2} \cdot \frac{n}{\lg n}
 \end{aligned}$$

for all  $k \geq \max\{k_0, 2\}$  or  $n \geq \max\{n_0, c_2\}$ . Hence,  $k = \Omega(n / \lg n)$ .

Now, let's also assume that  $k \geq 16$ , which implies  $\lg k \leq \sqrt{k}$ , or  $\lg \lg k \leq (\lg k)/2$ . As long as  $n \geq 1/c_1^2$ ,  $\lg c_1 \geq -(\lg n)/2$ . Then,

$$\begin{aligned}
 \lg k &\geq \lg c_1 + \lg n - \lg \lg k \\
 &\geq -(\lg n)/2 + \lg n - (\lg k)/2 \\
 &= (\lg n)/2 - (\lg k)/2
 \end{aligned}$$

so  $\lg k \geq (\lg n)/3$  or, equivalently,  $1/3 \leq \lg k / \lg n$ . Then,

$$\begin{aligned} k &\leq \frac{3k \lg k}{\lg n} \\ &\leq 3c_2 \cdot \frac{n}{\lg n} \end{aligned}$$

for all  $k \geq \max\{k_0, 16\}$  or  $n \geq \max\{n_0, 1/c_1^2\}$ . Hence,  $k = O(n/\lg n)$ , and by Theorem 3.1, we obtain  $k = \Theta(n/\lg n)$ .

## Problems

### 3-1 Asymptotic behavior of polynomials

In this problem we rely on the fact noted in the text on page 65 — for an asymptotically positive polynomial  $p(n)$  of degree  $d$ ,  $p(n) = \Theta(n^d)$ . The proofs refer to the positive constants  $c_1, c_2$ , and  $n_0$  such that  $0 \leq c_1 n^d \leq p(n) \leq c_2 n^d$  for all  $n \geq n_0$ .

**a.**  $k \geq d$  implies that  $n^k \geq n^d$  for all  $n \geq 1$ , so

$$0 \leq p(n) \leq c_2 n^d \leq c_2 n^k$$

for all  $n \geq \max\{n_0, 1\}$ . Hence,  $p(n) = O(n^k)$ .

**b.** Symmetrically,  $k \leq d$  implies that  $n^k \leq n^d$  for all  $n \geq 1$ , so

$$0 \leq c_1 n^k \leq c_1 n^d \leq p(n)$$

for all  $n \geq \max\{n_0, 1\}$ . Hence,  $p(n) = \Omega(n^k)$ .

**c.** For  $k = d$  the equality  $p(n) = \Theta(n^d) = \Theta(n^k)$  holds trivially.

**d.** Let  $b_2$  be a positive constant. The inequality  $c_2 n^d < b_2 n^k$  can be rewritten as  $c_2/b_2 < n^{k-d}$ . If we let  $m = (c_2/b_2)^{1/(k-d)}$  we have that, since  $k > d$ , the inequality holds for all  $n > m$ . Thus, for any  $b_2 > 0$ ,

$$0 \leq p(n) \leq c_2 n^d < b_2 n^k$$

for all  $n \geq \max\{n_0, m + 1\}$ . Hence,  $p(n) = o(n^k)$ .

e. The proof is symmetric to the one from part (d). Let  $b_1$  be a positive constant and let  $m = (b_1/c_1)^{1/(d-k)}$ . Then, since  $k < d$ , the inequality  $b_1 n^k < c_1 n^d$  holds for all  $n > m$ . Thus, for any  $b_1 > 0$ ,

$$0 \leq b_1 n^k < c_1 n^d \leq p(n)$$

for all  $n \geq \max\{n_0, m + 1\}$ . Hence,  $p(n) = \omega(n^k)$ .

### 3-2

#### Relative asymptotic growths

a. By equation (3.24),  $\lg^k n = o(n^\epsilon)$ , which also implies  $\lg^k n = O(n^\epsilon)$ .

b. By equation (3.13),  $n^k = o(c^n)$ , which also implies  $n^k = O(c^n)$ .

c. The functions  $\sqrt{n}$  and  $n^{\sin n}$  cannot be compared using asymptotic notation, because the value of the exponent in  $n^{\sin n}$  takes all values between 0 and 1, while  $\sqrt{n} = n^{1/2}$ .

d. We have

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{2^n}{2^{n/2}} &= \lim_{n \rightarrow \infty} 2^{n/2} \\ &= \infty, \end{aligned}$$

so  $2^n = \omega(2^{n/2})$ , which also implies  $2^n = \Omega(2^{n/2})$ .

e. By equation (3.21), the functions  $n^{\lg c}$  and  $c^{\lg n}$  are equivalent.

f. By equation (3.28),  $\lg(n!) = \Theta(n \lg n)$ . Because  $\lg(n^n) = n \lg n = \Theta(n \lg n)$ , we have  $\lg(n!) = \Theta(\lg(n^n))$ .

The table in Figure ?? collects the obtained results.

### 3-3

#### Ordering by asymptotic growth rates

a. In our justifications we will make use of the following lemmas:

##### **Lemma 1**

If a function  $f(n)$  is asymptotically positive and if  $g(n) = \omega(h(n))$ , then  $f(n)g(n) = \omega(f(n)h(n))$ .

$A$	$B$	$O$	$o$	$\Omega$	$\omega$	$\Theta$
$\lg^k n$	$n^\epsilon$	yes	yes	no	no	no
$n^k$	$c^n$	yes	yes	no	no	no
$\sqrt{n}$	$n^{\sin n}$	no	no	no	no	no
$2^n$	$2^{n/2}$	no	no	yes	yes	no
$n^{\lg c}$	$c^{\lg n}$	yes	no	yes	no	yes
$\lg(n!)$	$\lg(n^n)$	yes	no	yes	no	yes

**Figure 3-2** Comparison of asymptotic growths of pairs of functions.

**Proof** Pick  $n_1$ , so that  $f(n) > 0$  for all  $n \geq n_1$ . The relation  $g(n) = \omega(h(n))$  implies that for any constant  $c > 0$  there exists a constant  $n_2 > 0$  such that

$$0 \leq ch(n) < g(n)$$

for all  $n \geq n_2$ . If we let  $n \geq \max\{n_1, n_2\}$  and multiply all parts of the above compound inequality by  $f(n)$ , we obtain the result. ■

**Lemma 2**

If  $f(n) = \omega(g(n))$  and  $\lim_{n \rightarrow \infty} g(n) = \infty$ , then  $2^{f(n)} = \omega(2^{g(n)})$ .

**Proof** By the definition of  $\omega$ -notation, for any constant  $c > 0$  there is a constant  $n_0 > 0$  such that  $f(n) > cg(n)$  for all  $n \geq n_0$ . For fixed  $c$ , let  $d$  be a number satisfying the inequality  $cg(n) \geq d + g(n)$  for all  $n \geq n_0$ . Then  $d \leq (c-1)g(n)$ , and because  $g(n)$  grows arbitrarily large as  $n$  grows,  $d$  can take any real value, and so  $2^d$  can take any positive value. Thus,

$$\begin{aligned} 2^{f(n)} &> 2^{cg(n)} \\ &\geq 2^{d+g(n)} \\ &= 2^d \cdot 2^{g(n)} \\ &> 0, \end{aligned}$$

so  $2^{f(n)} = \omega(2^{g(n)})$ . ■

**Lemma 3**

For  $f(n) = \Theta(g(n))$ ,  $h(n) = \omega(f(n))$  if and only if  $h(n) = \omega(g(n))$ .

**Proof** Let  $c$  and  $n_0$  be positive constants such that for all  $n \geq n_0$ ,

$$0 \leq cg(n) \leq f(n).$$

If  $h(n) = \omega(f(n))$ , then for any constant  $a > 0$  there exists a constant  $n_a > 0$  such that  $0 \leq af(n) < h(n)$  for all  $n \geq n_a$ . For any  $a$  and all  $n \geq \max\{n_0, n_a\}$ ,

$$\begin{aligned} h(n) &> af(n) \\ &\geq acg(n) \\ &\geq 0, \end{aligned}$$

or  $h(n) = \omega(g(n))$ , because the factor  $ac$  can take any positive value.

By symmetry of  $\Theta$ -notation,  $g(n) = \Theta(f(n))$ , so the proof in the backward direction is symmetric, with  $f(n)$  and  $g(n)$  swapped. ■

The following justifications show that either  $f(n) = \omega(g(n))$ —which implies that  $f(n) = \Omega(g(n))$ —or  $f(n) = \Theta(g(n))$ .

$$2^{2^n+1} = \omega(2^{2^n})$$

By Lemma ?? for  $f(n) = g(n) = 2^{2^n}$  and  $h(n) = 1$ .

$$2^{2^n} = \omega((n+1)!)$$

By equation (3.13),  $2^n = \omega(n^2)$ , and by Lemma ??,  $n^2 = \omega(n \lg n)$ , because  $n = \omega(\lg n)$ . Then, by transitivity of  $\omega$ -notation,  $2^n = \omega(n \lg n)$ . By equation (3.28) and the fact that  $0 < \lg(n+1) < \lg(n!)$  for  $n \geq 3$ ,

$$\begin{aligned} \lg((n+1)!) &= \lg(n+1) + \lg(n!) \\ &= \Theta(n \lg n), \end{aligned}$$

so thanks to Lemma ?? we have  $2^n = \omega(\lg(n+1)!)$  and the result follows from Lemma ??.

$$(n+1)! = \omega(n!)$$

Immediate from Lemma ?? for  $f(n) = n!$ ,  $g(n) = n+1$ , and  $h(n) = 1$ .

$$n! = \omega(e^n)$$

By equation (3.28) and Lemma ??,  $\lg(n!) = \Theta(n \lg n) = \omega(n)$ , while  $\lg e^n = \Theta(n)$ . Thanks to Lemma ?? we have  $\lg(n!) = \omega(\lg e^n)$  and the relation follows from Lemma ??.

$$e^n = \omega(n \cdot 2^n)$$

By equation (3.13),  $(e/2)^n = \omega(n)$ . Then, the relation follows from Lemma ?? for  $f(n) = 2^n$ ,  $g(n) = (e/2)^n$ , and  $h(n) = n$ .

$$n \cdot 2^n = \omega(2^n)$$

Immediate from Lemma ?? for  $f(n) = 2^n$ ,  $g(n) = n$ , and  $h(n) = 1$ .

$$2^n = \omega((3/2)^n)$$

Immediate from Lemma ?? for  $f(n) = (3/2)^n$ ,  $g(n) = (4/3)^n$ , and  $h(n) = 1$ .

$$\begin{aligned}
(3/2)^n &= \omega(n^{\lg \lg n}) \\
\lg(3/2)^n &= n \lg(3/2) \text{ and} \\
\lg(n^{\lg \lg n}) &= \lg n \lg \lg n \\
&< \lg n \lg n \\
&= \lg^2 n.
\end{aligned}$$

By equality (3.24),  $n \lg(3/2) = \omega(\lg^2 n)$ , and so  $\lg(3/2)^n = \omega(\lg(n^{\lg \lg n}))$ . Then, the result follows from Lemma ??.

$$\begin{aligned}
n^{\lg \lg n} &= \Theta((\lg n)^{\lg n}) \\
&\text{Immediate from equation (3.21).}
\end{aligned}$$

$$\begin{aligned}
(\lg n)^{\lg n} &= \omega((\lg n)!) \\
&\text{Using Stirling's approximation, we get}
\end{aligned}$$

$$\begin{aligned}
(\lg n)! &= \Theta\left(\sqrt{2\pi \lg n} \left(\frac{\lg n}{e}\right)^{\lg n}\right) \\
&= \Theta\left((\lg n)^{\lg n} \frac{\sqrt{\lg n}}{n^{\lg e}}\right).
\end{aligned}$$

Let

$$f(n) = (\lg n)^{\lg n} \frac{\sqrt{\lg n}}{n^{\lg e}}$$

and let

$$g(n) = \frac{n^{\lg e}}{\sqrt{\lg n}}.$$

Then,  $g(n) = \omega(1)$ , so by Lemma ??,  $f(n)g(n) = (\lg n)^{\lg n} = \omega(f(n))$ . Finally,  $(\lg n)^{\lg n} = \omega((\lg n)!)$  follows from Lemma ??.

$$(\lg n)! = \omega(n^3)$$

By equation (3.28) and Lemma ??, we get  $\lg((\lg n)!) = \Theta(\lg n \lg \lg n) = \omega(\lg n)$  and  $\lg n^3 = 3 \lg n$ , so  $\lg((\lg n)!) = \omega(\lg n^3)$  and now it suffices to apply Lemma ??.

$$n^3 = \omega(n^2)$$

Immediate from Problem ??(e).

$$n^2 = \Theta(4^{\lg n})$$

Immediate from equation (3.21).

$$4^{\lg n} = \omega(n \lg n)$$

By equation (3.21),  $4^{\lg n} = n^2$ , so we apply Lemma ?? for  $f(n) = g(n) = n$  and  $h(n) = \lg n$ .

$$n \lg n = \Theta(\lg(n!))$$

Immediate from equation (3.28).

$$\lg(n!) = \omega(n)$$

By equation (3.28) and Lemma ??,  $\lg(n!) = \Theta(n \lg n) = \omega(n)$ .

$$n = \Theta(2^{\lg n})$$

Immediate from equation (3.21).

$$2^{\lg n} = \omega((\sqrt{2})^{\lg n})$$

By equation (3.21),  $2^{\lg n} = n$ , as well as  $(\sqrt{2})^{\lg n} = n^{\lg \sqrt{2}} = \sqrt{n}$ . Thus, the relation follows from Problem ??(e).

$$(\sqrt{2})^{\lg n} = \omega(2^{\sqrt{2 \lg n}})$$

It holds that

$$\begin{aligned} \lg(\sqrt{2})^{\lg n} &= \lg n \lg \sqrt{2} \\ &= \frac{\lg n}{2} \\ &= \sqrt{2 \lg n} \cdot \frac{\sqrt{\lg n}}{2\sqrt{2}} \\ &= \sqrt{2 \lg n} \cdot \omega(1), \end{aligned}$$

so Lemma ?? implies  $\lg(\sqrt{2})^{\lg n} = \omega(\lg 2^{\sqrt{2 \lg n}})$ . Now it suffices to use Lemma ??.

$$2^{\sqrt{2 \lg n}} = \omega(\lg^2 n)$$

Consider  $\lg \lg 2^{\sqrt{2 \lg n}} = \Theta(\lg \lg n)$  and  $\lg \lg \lg^2 n = \Theta(\lg \lg \lg n)$ . We can show that  $\lg \lg n = \omega(\lg \lg \lg n)$  by substituting  $\lg \lg n$  for  $n$  in equation (3.24). Then we have  $\lg \lg 2^{\sqrt{2 \lg n}} = \omega(\lg \lg \lg^2 n)$ , and we apply Lemma ?? twice.

$$\lg^2 n = \omega(\ln n)$$

By Lemma ??,  $\lg^2 n = \omega(\lg n)$ . Also,  $\ln n = \Theta(\lg n)$ , so the relation follows from Lemma ??.

$$\ln n = \omega(\sqrt{\lg n})$$

We have

$$\begin{aligned} \ln n &= \frac{\lg n}{\lg e} \\ &= \sqrt{\lg n} \cdot \frac{\sqrt{\lg n}}{\lg e} \\ &= \sqrt{\lg n} \cdot \omega(1), \end{aligned}$$

so the relation follows from Lemma ??.

$$\sqrt{\lg n} = \omega(\ln \ln n)$$

We have  $\lg \sqrt{\lg n} = \Theta(\lg \lg n)$  and  $\lg \ln \ln n = \Theta(\lg \lg \lg n)$ . By equation (3.24), after substituting  $\lg \lg n$  for  $n$ ,  $\lg \lg n = \omega(\lg \lg \lg n)$ , so  $\lg \sqrt{\lg n} = \omega(\lg \ln \ln n)$ , and the relation follows from Lemma ??.

$$\ln \ln n = \omega(2^{\lg^* n})$$

$\lg \ln \ln n = \Theta(\lg \lg \lg n) = \omega(\lg^* n)$ , so we apply Lemma ??.

$$2^{\lg^* n} = \omega(\lg^* n)$$

$\lg 2^{\lg^* n} = \lg^* n$  and it holds that  $\lg^* n = \omega(\lg(\lg^* n))$ , which we prove in the following justifications. Then the relation follows from Lemma ??.

$$\lg^* n = \Theta(\lg^*(\lg n))$$

$\lg^*(\lg n) = \lg^* n - 1$  holds for all  $n \geq 2$ .

$$\lg^*(\lg n) = \omega(\lg(\lg^* n))$$

Immediate from Exercise ??.

$$\lg(\lg^* n) = \omega(n^{1/\lg n})$$

By equation (3.19),  $1/\lg n = \log_n 2$ , and by (3.21),

$$\begin{aligned} n^{1/\lg n} &= n^{\log_n 2} \\ &= 2^{\log_n n} \\ &= 2 \\ &= \Theta(1). \end{aligned}$$

On the other hand,  $\lg(\lg^* n) = \omega(1)$ , and so the relation follows from Lemma ??.

$$n^{1/\lg n} = \Theta(1)$$

Immediate from the above justification.

Below is the list of all examined functions  $g_1, g_2, \dots, g_{30}$  ordered according to relations between them in terms of  $\Omega$ -notation. Different lines on the list represent different equivalence classes of  $\Theta$ -notation.

$$g_1(n) = 2^{2^n+1}$$

$$g_2(n) = 2^{2^n}$$

$$g_3(n) = (n + 1)!$$

$$g_4(n) = n!$$

$$g_5(n) = e^n$$

$$g_6(n) = n \cdot 2^n$$

$$g_7(n) = 2^n$$

$$g_8(n) = (3/2)^n$$

$$g_9(n) = n^{\lg \lg n}, g_{10}(n) = (\lg n)^{\lg n}$$

$$g_{11}(n) = (\lg n)!$$

$$g_{12}(n) = n^3$$

$$g_{13}(n) = n^2, g_{14}(n) = 4^{\lg n}$$

$$g_{15}(n) = n \lg n$$

$$g_{16}(n) = \lg(n!)$$

$$g_{17}(n) = n, g_{18}(n) = 2^{\lg n}$$

$$g_{19}(n) = (\sqrt{2})^{\lg n}$$

$$g_{20}(n) = 2^{\sqrt{2 \lg n}}$$

$$g_{21}(n) = \lg^2 n$$

$$g_{22}(n) = \ln n$$

$$g_{23}(n) = \sqrt{\lg n}$$

$$g_{24}(n) = \ln \ln n$$

$$g_{25}(n) = 2^{\lg^* n}$$

$$g_{26}(n) = \lg^* n, g_{27}(n) = \lg^*(\lg n)$$

$$g_{28}(n) = \lg(\lg^* n)$$

$$g_{29}(n) = n^{1/\lg n}, g_{30}(n) = 1$$

**b.** Here is an example of a function that has the desired property:

$$f(n) = \left(2^{2^{n+1}}\right)^{n \bmod 2}.$$

Observe that  $f(n) = g_1(n)$  when  $n$  is odd, and  $f(n) = g_{30}(n)$  when  $n$  is even.

### 3-4

#### Asymptotic notation properties

**a.** False. Let  $f(n) = n$  and  $g(n) = n^2$ . Then  $f(n) = O(g(n))$  but  $g(n) \neq O(f(n))$ .

**b.** False. Let  $f(n) = n$  and  $g(n) = n^2$ . For all  $n \geq 1$ ,  $\min\{f(n), g(n)\} = f(n)$ , so  $f(n) + g(n) = O(g(n)) \neq O(f(n))$ .

**c.** True. Suppose that  $f(n) = O(g(n))$ . Let  $c$  and  $n_0$  be positive constants such that  $1 \leq f(n) \leq cg(n)$  and  $\lg g(n) \geq 1$  for all  $n \geq n_0$ . Then,

$$\begin{aligned}\lg f(n) &\leq \lg c + \lg g(n) \\ &\leq \lg c \cdot \lg g(n) + \lg g(n) \\ &= (\lg c + 1) \lg g(n) \\ &= O(\lg g(n)).\end{aligned}$$

**d.** False. Let  $f(n) = 2n$  and  $g(n) = n$ . It holds  $f(n) = O(g(n))$ , but  $2^{f(n)} \neq O(2^{g(n)})$  (see Exercise ??).

**e.** False. If  $f(n) = 1/n$ , then  $f^2(n) = 1/n^2$ . Since there doesn't exist any positive constant  $c$  such that  $1/n \leq c/n^2$  for arbitrarily large  $n$ , then  $f(n) \neq O(f^2(n))$ .

**f.** True. Suppose that  $f(n) = O(g(n))$ . Let  $c$  and  $n_0$  be positive constants such that  $0 \leq f(n) \leq cg(n)$  for all  $n \geq n_0$ . Dividing all parts of the inequality by  $c$  yields  $0 \leq f(n)/c \leq g(n)$ , and since  $1/c > 0$ , then  $g(n) = \Omega(f(n))$ .

**g.** False. Let  $f(n) = 2^n$ , so  $f(n/2) = 2^{n/2} = \sqrt{2^n}$ . Suppose that  $f(n) = O(f(n/2))$ . Then for a positive constant  $c$  and for sufficiently large  $n$ , it holds  $2^n \leq c\sqrt{2^n}$ . But then  $c \geq \sqrt{2^n}$  and  $c$  cannot be a constant. Therefore,  $f(n) \neq O(f(n/2))$ , which implies  $f(n) \neq \Theta(f(n/2))$ .

**h.** True. Let  $h(n) = o(f(n))$ . Then, for any positive constant  $c$  there exists a positive constant  $n_0$  such that  $0 \leq h(n) < cf(n)$  for all  $n \geq n_0$ . This implies that

$$\begin{aligned}f(n) &\leq f(n) + o(f(n)) \\ &= f(n) + h(n) \\ &< (c+1)f(n) \\ &< 2f(n),\end{aligned}$$

so  $f(n) + o(f(n)) = \Theta(f(n))$ .

3-5

***Manipulating asymptotic notation***

**a.** Let  $p(n) = \Theta(f(n))$  be the function on the left-hand side of the equation under the outer  $\Theta$ , and let  $c_1, c_2$ , and  $n_p$  be positive constants such that

$$0 \leq c_1 f(n) \leq p(n) \leq c_2 f(n)$$

for all  $n \geq n_p$ . Also, let  $q(n) = \Theta(p(n))$  and let  $d_1, d_2$ , and  $n_q$  be positive constants such that

$$0 \leq d_1 p(n) \leq q(n) \leq d_2 p(n)$$

for all  $n \geq n_q$ . Then, for all  $n \geq \max\{n_p, n_q\}$ ,

$$\begin{aligned} 0 &\leq c_1 d_1 f(n) \\ &\leq d_1 p(n) \\ &\leq q(n) \\ &\leq d_2 p(n) \\ &\leq c_2 d_2 f(n), \end{aligned}$$

which implies that  $q(n) = \Theta(f(n))$ .

**b.** Let  $p(n) = \Theta(f(n))$  and  $q(n) = O(f(n))$  be the two functions on the left-hand side of the equation. There exist positive constants  $c_1, c_2, d, n_p$ , and  $n_q$  such that

$$0 \leq c_1 f(n) \leq p(n) \leq c_2 f(n)$$

for all  $n \geq n_p$ , and

$$0 \leq q(n) \leq df(n)$$

for all  $n \geq n_q$ . Then, for all  $n \geq \max\{n_p, n_q\}$ ,

$$\begin{aligned} 0 &\leq c_1 f(n) \\ &\leq p(n) \\ &\leq p(n) + q(n) \\ &\leq c_2 f(n) + df(n) \\ &= (c_2 + d)f(n), \end{aligned}$$

which implies that  $p(n) + q(n) = \Theta(f(n))$ .

**c.** Let  $p(n) = \Theta(f(n))$  and  $q(n) = \Theta(g(n))$  be the two functions on the left-hand side of the equation. There exist positive constants  $c_1, c_2, d_1, d_2, n_p$ , and  $n_q$  such that

$$0 \leq c_1 f(n) \leq p(n) \leq c_2 f(n)$$

for all  $n \geq n_p$ , and

$$0 \leq d_1 g(n) \leq q(n) \leq d_2 g(n)$$

for all  $n \geq n_q$ . Then, for all  $n \geq \max\{n_p, n_q\}$ ,

$$\begin{aligned} 0 &\leq \min\{c_1, d_1\}(f(n) + g(n)) \\ &\leq c_1 f(n) + d_1 g(n) \\ &\leq p(n) + q(n) \\ &\leq c_2 f(n) + d_2 g(n) \\ &\leq \max\{c_2, d_2\}(f(n) + g(n)), \end{aligned}$$

which implies that  $p(n) + q(n) = \Theta(f(n) + g(n))$ .

**d.** For the same functions and constants as in the previous part, it is true that for all  $n \geq \max\{n_p, n_q\}$ ,

$$\begin{aligned} 0 &\leq \min\{c_1, d_1\}^2(f(n) \cdot g(n)) \\ &\leq c_1 f(n) \cdot d_1 g(n) \\ &\leq p(n) \cdot q(n) \\ &\leq c_2 f(n) \cdot d_2 g(n) \\ &\leq \max\{c_2, d_2\}^2(f(n) \cdot g(n)), \end{aligned}$$

which implies that  $p(n) \cdot q(n) = \Theta(f(n) \cdot g(n))$ .

**e.** Instead of  $a_1, b_1 > 0$  there should be  $a_1, a_2 > 0$ . Corrected in fourth printing.

We show the asymptotic upper bound and the asymptotic lower bound separately. For all  $n \geq a_2$ ,  $\lg a_2 \leq \lg n$ , so

$$\begin{aligned} (a_1 n)^{k_1} \lg^{k_2} (a_2 n) &= a_1^{k_1} n^{k_1} (\lg a_2 + \lg n)^{k_2} \\ &\leq a_1^{k_1} n^{k_1} (2 \lg n)^{k_2} \\ &= a_1^{k_1} 2^{k_2} \cdot n^{k_1} \lg^{k_2} n \\ &= O(n^{k_1} \lg^{k_2} n), \end{aligned}$$

because the factor  $a_1^{k_1} 2^{k_2}$  certainly is a positive constant.

Now, if we let  $n \geq 1/a_2^2$ , then  $\lg n \geq -2 \lg a_2$ , or  $\lg a_2 \geq -(\lg n)/2$ . Then,

$$\begin{aligned} (a_1 n)^{k_1} \lg^{k_2} (a_2 n) &= a_1^{k_1} n^{k_1} (\lg a_2 + \lg n)^{k_2} \\ &\geq a_1^{k_1} n^{k_1} ((\lg n)/2)^{k_2} \\ &= (a_1^{k_1}/2^{k_2}) \cdot n^{k_1} \lg^{k_2} n \\ &= \Omega(n^{k_1} \lg^{k_2} n), \end{aligned}$$

since  $a_1^{k_1}/2^{k_2}$  is a positive constant.

- ★ f. The statement in this part is flawed and cannot be solved. First, it lacks the assumption that both sums converge to positive numbers. Second, it fails to clarify the variable that the right-hand side  $\Theta$ -notation applies to, making the equation either trivial or unambiguous. Removed in fourth printing.
- ★ g. Similar as above.

### 3-6

#### Variations on $O$ and $\Omega$

- a. Let  $c$  be a positive constant. Suppose that  $f(n) \geq cg(n)$  for  $n$  in some set  $S$ . If  $S$  is infinite, then since  $g(n)$  is asymptotically nonnegative,  $f(n) = \overset{\infty}{\Omega}(g(n))$ . Now suppose  $S$  is finite and let  $n_0 \in S$  be the largest value for which the above inequality holds. Then,  $cg(n) > f(n)$  for all  $n \geq n_0 + 1$ , and since  $f(n)$  is asymptotically nonnegative,  $f(n) = O(g(n))$ .

Theorem 3.1 remains true if we substitute  $\overset{\infty}{\Omega}$  for  $\Omega$ .

- b. See the example given in Problem ??(b).

- c. The advantage of using  $\overset{\infty}{\Omega}$ -notation instead of  $\Omega$ -notation is the fact shown in part (a), that all asymptotically nonnegative functions become asymptotically comparable. In practice, if we can prove that a function  $f(n)$  is not  $O(g(n))$ , then we automatically get that  $f(n) = \overset{\infty}{\Omega}(g(n))$ .

On the other hand, the relation  $f(n) = \overset{\infty}{\Omega}(g(n))$  doesn't necessarily say that the function  $f(n)$  „dominates” over the function  $g(n)$  for sufficiently large  $n$ , because for any positive constant  $c$  there can still be  $f(n) < cg(n)$  for infinitely many integers  $n$ . For example,  $n^{1+\sin n} = \overset{\infty}{\Omega}(n)$ , since for infinitely many  $n$  the value of the exponent in  $n^{1+\sin n}$  is greater than 1, but at the same time for infinitely many  $n$  that exponent can take values less than 1.

Besides, from the practical point of view, the infinitely many integers referred to in the definition of  $\widetilde{\Omega}$ -notation can be far beyond the maximum input size of the algorithm whose running time we're trying to describe using this notation.

**d.** The relation  $f(n) = \Theta(g(n))$  implies that the function  $f(n)$  is asymptotically nonnegative, or  $f(n) = |f(n)|$  for sufficiently large  $n$ , so  $f(n) = O'(g(n))$  follows from  $f(n) = O(g(n))$ , and the „only if” direction remains true.

To examine the „if” direction, suppose that  $f(n) = O'(g(n))$  and  $f(n) = \Omega(g(n))$ . By definition, if  $f(n) = O'(g(n))$  then there exist positive constants  $c$  and  $n_0$  such that for all  $n \geq n_0$ ,

$$\begin{cases} 0 \leq f(n) \leq cg(n), & \text{if } f(n) \geq 0, \\ 0 \leq -f(n) \leq cg(n), & \text{if } f(n) < 0. \end{cases}$$

Suppose there is no  $n_1 \geq n_0$  such that  $f(n) \geq 0$  for all  $n \geq n_1$ . But then, for any positive constant  $d$  the inequalities  $0 \leq dg(n) \leq f(n)$  cannot hold for all sufficiently large  $n$ , which means that  $f(n) \neq \Omega(g(n))$  leading to a contradiction. Hence, the function  $f(n)$  is asymptotically nonnegative, as so  $f(n) = O(g(n))$ .

As we can see, by substituting  $O'$  for  $O$ , we aren't invalidating Theorem 3.1.

**e.** We define  $\widetilde{\Omega}$  and  $\widetilde{\Theta}$  as follows:

$$\widetilde{\Omega}(g(n)) = \{f(n) : \text{there exist positive constants } c, k, \text{ and } n_0 \text{ such that}$$

$$0 \leq cg(n) \lg^k n \leq f(n) \text{ for all } n \geq n_0\},$$

$$\widetilde{\Theta}(g(n)) = \{f(n) : \text{there exist positive constants } c_1, c_2, k_1, k_2, \text{ and } n_0 \text{ such that}$$

$$0 \leq c_1 g(n) \lg^{k_1} n \leq f(n) \leq c_2 g(n) \lg^{k_2} n \text{ for all } n \geq n_0\}.$$

The proof of Theorem 3.1 for  $\widetilde{O}$ -,  $\widetilde{\Omega}$ -, and  $\widetilde{\Theta}$ -notations is analogous to the proof in Exercise ?? to the original theorem.

By definition,  $f(n) = \widetilde{\Theta}(g(n))$  whenever there exist positive constants  $c_1, c_2, k_1, k_2$ , and  $n_0$  such that

$$0 \leq c_1 g(n) \lg^{k_1} n \leq f(n) \leq c_2 g(n) \lg^{k_2} n$$

for all  $n \geq n_0$ . This condition can be decomposed into the combination of inequalities  $0 \leq c_1 g(n) \lg^{k_1} n \leq f(n)$  and  $0 \leq f(n) \leq c_2 g(n) \lg^{k_2} n$ , both true for all  $n \geq n_0$ . The former implies  $f(n) = \widetilde{\Omega}(g(n))$  and the latter implies  $f(n) = \widetilde{O}(g(n))$ .

For the proof of the opposite direction, suppose that  $f(n) = \widetilde{\Omega}(g(n))$  and  $f(n) = \widetilde{O}(g(n))$ . Then, there exist positive constants  $c_1, k_1$ , and  $n_1$  such that

$$0 \leq c_1 g(n) \lg^{k_1} n \leq f(n)$$

for all  $n \geq n_1$ , and there exist positive constants  $c_2, k_2$ , and  $n_2$  such that

$$0 \leq f(n) \leq c_2 g(n) \lg^{k_2} n$$

for all  $n \geq n_2$ . Merging both inequalities yields

$$0 \leq c_1 g(n) \lg^{k_1} n \leq f(n) \leq c_2 g(n) \lg^{k_2} n$$

for all  $n \geq \max\{n_1, n_2\}$ . Hence,  $f(n) = \widetilde{\Theta}(g(n))$ .

### 3-7

#### *Iterated functions*

**a.** Since  $(n - 1)^{(i)} = n - i$ , we have  $f_0^*(n) = \max\{0, \lceil n \rceil\}$ , so  $f_0^*(n) = \Theta(n)$ .

**b.** By the definition of the iterated logarithm,  $f_1^*(n) = \lg^* n = \Theta(\lg^* n)$ .

**c.**  $(n/2)^{(i)} = n/2^i$ , so

$$f_1^*(n) = \begin{cases} 0 & \text{if } n \leq 1, \\ \lceil \lg n \rceil & \text{if } n > 1. \end{cases}$$

Hence,  $f_1^*(n) = \Theta(\lg n)$ .

**d.** As in part (c),

$$f_2^*(n) = \begin{cases} 0 & \text{if } n \leq 2, \\ \lceil \lg n \rceil - 1 & \text{if } n > 2. \end{cases}$$

Hence,  $f_2^*(n) = \Theta(\lg n)$ .

**e.**  $\sqrt{n} = n^{1/2}$ , so  $(\sqrt{n})^{(i)} = n^{1/2^i}$ . For  $n > 2$ , the solution to inequality  $n^{1/2^i} \leq 2$  with respect to  $i$  is  $i \geq \lg \lg n$ , so

$$f_2^*(n) = \begin{cases} 0 & \text{if } 0 \leq n \leq 2, \\ \lceil \lg \lg n \rceil & \text{if } n > 2. \end{cases}$$

Hence,  $f_2^*(n) = \Theta(\lg \lg n)$ .

**f.** If  $0 \leq n \leq 1$ ,  $f_1^*(n) = 0$ , and if  $n > 1$ ,  $f_1^*(n)$  is undefined, because for all integers  $i \geq 0$ ,  $(\sqrt{n})^{(i)} > 1$ .

**g.** This case is analogous to the one from part (e). Since  $(n^{1/3})^{(i)} = n^{1/3^i}$ ,

$$f_2^*(n) = \begin{cases} 0 & \text{if } n \leq 2, \\ \lceil \log_3 \lg n \rceil & \text{if } n > 2. \end{cases}$$

Hence,  $f_2^*(n) = \Theta(\lg \lg n)$ .

The table in Figure ?? collects the obtained results.

$f(n)$	$c$	$f_c^*(n)$
$n - 1$	0	$\Theta(n)$
$\lg n$	1	$\Theta(\lg^* n)$
$n/2$	1	$\Theta(\lg n)$
$n/2$	2	$\Theta(\lg n)$
$\sqrt{n}$	2	$\Theta(\lg \lg n)$
$\sqrt{n}$	1	undefined
$n^{1/3}$	2	$\Theta(\lg \lg n)$

**Figure 3-7** Asymptotic bounds of iterated functions.

## 4.1 Multiplying square matrices

**4.1-1** Observe, that for  $n \times n$  matrices  $A$  and  $B$ ,

$$\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} AB & 0 \\ 0 & 0 \end{pmatrix},$$

where 0 denotes zero matrices. In other words, when multiplying the square matrices  $A$  and  $B$ , we can pad them with zeros to obtain  $m \times m$  matrices  $A' = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$  and  $B' = \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix}$ , and multiply  $A'$  by  $B'$  instead. The submatrix formed by taking the intersection of the first  $n$  rows and the first  $n$  columns of  $A'B'$  is  $AB$ .

We implement this trick in the generalized version of the divide-and-conquer algorithm for multiplying  $n \times n$  matrices. First we find  $m$  — the least power of 2 greater than or equal to  $n$ . Next we create the  $m \times m$  matrices  $A'$  and  $B'$ , by copying the corresponding entries of the matrices  $A$  and  $B$  to the upper left  $n \times n$  submatrices of  $A'$  and  $B'$ , while setting the remaining entries to 0. For a given  $n \times n$  matrix  $C$ , where the matrix product  $AB$  will be added, we similarly pad it with zeros to create the  $m \times m$  matrix  $C'$ . Then we can call `MATRIX-MULTIPLY-RECURSIVE( $A'$ ,  $B'$ ,  $C'$ ,  $m$ )` and copy the relevant entries of the matrix  $C'$  to  $C$ .

Creating each matrix  $A'$ ,  $B'$ , and  $C'$  takes  $\Theta(m^2)$  time, and copying the result to the matrix  $C$  takes  $\Theta(n^2)$  time. Let  $T(n)$  and  $T'(n)$  be the worst-case time to multiply two  $n \times n$  matrices using the standard `MATRIX-MULTIPLY-RECURSIVE` procedure, and its generalized version, respectively. Then, omitting the base case, we get

$$T'(n) = T(m) + \Theta(m^2) + \Theta(n^2).$$

Note that  $n \leq m < 2n$ , so

$$\begin{aligned} T'(n) &\geq T(n) + \Theta(n^2) + \Theta(n^2) \\ &= \Omega(n^3) \end{aligned}$$

and

$$\begin{aligned} T'(n) &\leq T(2n) + \Theta((2n)^2) + \Theta(n^2) \\ &= O((2n)^3) \\ &= O(n^3). \end{aligned}$$

Hence,  $T'(n) = \Theta(n^3)$ .

#### 4.I-2

Let  $A_1, A_2, \dots, A_k$  and  $B_1, B_2, \dots, B_k$  be  $n \times n$  matrices. Then

$$A = \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_k \end{pmatrix}$$

is a  $k n \times n$  matrix,

$$B = (B_1 \ B_2 \ \cdots \ B_k)$$

is an  $n \times k n$  matrix, and

$$AB = \begin{pmatrix} A_1 B_1 & A_1 B_2 & \cdots & A_1 B_k \\ A_2 B_1 & A_2 B_2 & \cdots & A_2 B_k \\ \vdots & \vdots & \ddots & \vdots \\ A_k B_1 & A_k B_2 & \cdots & A_k B_k \end{pmatrix}.$$

Each submatrix product  $A_i B_j$  in the right-hand side matrix can be computed in  $\Theta(n^3)$  time using MATRIX-MULTIPLY-RECURSIVE, so the overall time to multiply  $A$  by  $B$  is  $\Theta(k^2 n^3)$ .

On the other hand,

$$BA = \sum_{i=1}^k B_i A_i,$$

and we can compute it by computing the  $k$  products  $B_i A_i$  in total time of  $\Theta(k n^3)$ , then adding them together in  $\Theta(k n^2)$  time.

Therefore, multiplying an  $n \times k n$  matrix by a  $k n \times n$  matrix is asymptotically faster than multiplying these matrices in reverse order, by a factor of  $k$ .

**4.1-3**

At each recursive step the time spent on partitioning the matrices and copying the results is  $\Theta(n^2)$ , so the recurrence (4.9) changes to

$$T(n) = 8T(n/2) + \Theta(n^2).$$

To show its has the same solution as the original recurrence, let's use the master method from Section 4.5. There exists a constant  $\epsilon > 0$  such that  $n^2 = O(n^{\log_2 8 - \epsilon})$ , so by the master theorem,  $T(n) = \Theta(n^3)$ .

**4.1-4**

We'll assume that  $n$  is an exact power of 2. The procedure MATRIX-ADD-RECURSIVE takes three  $n \times n$  matrices  $A$ ,  $B$  and  $C$ , and computes  $C = C + (A + B)$ . Like MATRIX-MULTIPLY-RECURSIVE, in the divide step it partitions the matrices as in equation (4.2). But in the conquer step it adds each submatrix of  $A$  to the corresponding submatrix of  $B$  and adds the sum to the corresponding submatrix of  $C$ :

$$\begin{aligned} C_{11} &= C_{11} + A_{11} + B_{11}, \\ C_{12} &= C_{12} + A_{12} + B_{12}, \\ C_{21} &= C_{21} + A_{21} + B_{21}, \\ C_{22} &= C_{22} + A_{22} + B_{22}, \end{aligned}$$

which involves four recursive calls.

```

MATRIX-ADD-RECURSIVE( $A, B, C, n$ )
1  if  $n == 1$ 
2      // Base case.
3       $c_{11} = c_{11} + a_{11} + b_{11}$ 
4      return
5  // Divide.
6  partition  $A$ ,  $B$ , and  $C$  into  $n/2 \times n/2$  submatrices
     $A_{11}, A_{12}, A_{21}, A_{22}; B_{11}, B_{12}, B_{21}, B_{22};$ 
    and  $C_{11}, C_{12}, C_{21}, C_{22}$ ; respectively
7  // Conquer.
8  MATRIX-ADD-RECURSIVE( $A_{11}, B_{11}, C_{11}, n/2$ )
9  MATRIX-ADD-RECURSIVE( $A_{12}, B_{12}, C_{12}, n/2$ )
10 MATRIX-ADD-RECURSIVE( $A_{21}, B_{21}, C_{21}, n/2$ )
11 MATRIX-ADD-RECURSIVE( $A_{22}, B_{22}, C_{22}, n/2$ )

```

Let  $T(n)$  be the worst-case time required by this algorithm to add two  $n \times n$  matrices. Each recursive call in lines ??–?? adds two  $n/2 \times n/2$  matrices contributing  $T(n/2)$  to the overall running time, so all four recursive calls take  $4T(n/2)$  time. If we assume that

the algorithm uses index calculations to partition the matrices in line ??, the recurrence for its running time is

$$T(n) = 4T(n/2) + \Theta(1).$$

Applying case 1 of the master theorem, we obtain the solution  $T(n) = \Theta(n^2)$ . However, if the matrices are partitioned by copying, then the recurrence changes to

$$T(n) = 4T(n/2) + \Theta(n^2),$$

which, by case 2 of the master theorem, has the solution  $T(n) = \Theta(n^2 \lg n)$ .

## 4.2 Strassen's algorithm for matrix multiplication

### 4.2-1

To simplify the notation we'll omit the parentheses surrounding the single entry of  $1 \times 1$  matrices and effectively treat such matrices as scalars.

We run Strassen's algorithm on the given  $2 \times 2$  input matrices  $A$  and  $B$ , and the  $2 \times 2$  output matrix  $C$ , initialized to 0. In step 1 the algorithm partitions the input matrices into the submatrices:

$$\begin{aligned} A_{11} &= 1, \\ A_{12} &= 3, \\ A_{21} &= 7, \\ A_{22} &= 5, \\ B_{11} &= 6, \\ B_{12} &= 8, \\ B_{21} &= 4, \\ B_{22} &= 2. \end{aligned}$$

Then, in step 2 the following matrices are created:

$$\begin{aligned} S_1 &= 8 - 2 = 6, \\ S_2 &= 1 + 3 = 4, \\ S_3 &= 7 + 5 = 12, \\ S_4 &= 4 - 6 = -2, \\ S_5 &= 1 + 5 = 6, \\ S_6 &= 6 + 2 = 8, \\ S_7 &= 3 - 5 = -2, \\ S_8 &= 4 + 2 = 6, \\ S_9 &= 1 - 7 = -6, \\ S_{10} &= 6 + 8 = 14. \end{aligned}$$

In step 3 the algorithm calls itself recursively to compute the matrices  $P_1, P_2, \dots, P_7$ . They are all products of two  $1 \times 1$  matrices, so each recursive call will return right after step 1:

$$\begin{aligned} P_1 &= 1 \cdot 6 = 6, \\ P_2 &= 4 \cdot 2 = 8, \\ P_3 &= 12 \cdot 6 = 72, \\ P_4 &= 5 \cdot (-2) = -10, \\ P_5 &= 6 \cdot 8 = 48, \\ P_6 &= (-2) \cdot 6 = -12, \\ P_7 &= (-6) \cdot 14 = -84. \end{aligned}$$

Finally, in step 4 the algorithm updates the four submatrices of the matrix  $C$ :

$$\begin{aligned} C_{11} &= 48 + (-10) - 8 + (-12) = 18, \\ C_{12} &= 6 + 8 = 14, \\ C_{21} &= 72 + (-10) = 62, \\ C_{22} &= 48 + 6 - 72 - (-84) = 66. \end{aligned}$$

The result of calling Strassen's algorithm on the given input is

$$C = \begin{pmatrix} 18 & 14 \\ 62 & 66 \end{pmatrix}.$$

#### 4.2-2

The pseudocode below calls two auxiliary procedures  $\text{MATRIX-ADD}(A, B, C, n)$  and  $\text{MATRIX-SUBTRACT}(A, B, C, n)$ . The former implements matrix addition on two  $n \times n$

matrices  $A$  and  $B$ , and accumulates the result into the third matrix:  $C = C + (A + B)$ . The latter subtracts the second matrix from the first in a similar fashion to compute  $C = C + (A - B)$ .

```

STRASSEN( $A, B, C, n$ )
1  if  $n == 1$ 
2     $c_{11} = c_{11} + a_{11} \cdot b_{11}$ 
3    return
4  partition  $A, B$ , and  $C$  into  $n/2 \times n/2$  submatrices
    $A_{11}, A_{12}, A_{21}, A_{22}; B_{11}, B_{12}, B_{21}, B_{22};$ 
   and  $C_{11}, C_{12}, C_{21}, C_{22}$ ; respectively
5  create  $n/2 \times n/2$  matrices  $S_1, S_2, \dots, S_{10}$ 
6  create and zero the entries of  $n/2 \times n/2$  matrices  $P_1, P_2, \dots, P_7$ 
7  MATRIX-SUBTRACT( $B_{12}, B_{22}, S_1, n/2$ )
8  MATRIX-ADD( $A_{11}, A_{12}, S_2, n/2$ )
9  MATRIX-ADD( $A_{21}, A_{22}, S_3, n/2$ )
10 MATRIX-SUBTRACT( $B_{21}, B_{11}, S_4, n/2$ )
11 MATRIX-ADD( $A_{11}, A_{22}, S_5, n/2$ )
12 MATRIX-ADD( $B_{11}, B_{22}, S_6, n/2$ )
13 MATRIX-SUBTRACT( $A_{12}, A_{22}, S_7, n/2$ )
14 MATRIX-ADD( $B_{21}, B_{22}, S_8, n/2$ )
15 MATRIX-SUBTRACT( $A_{11}, A_{21}, S_9, n/2$ )
16 MATRIX-ADD( $B_{11}, B_{12}, S_{10}, n/2$ )
17 STRASSEN( $A_{11}, S_1, P_1, n/2$ )
18 STRASSEN( $S_2, B_{22}, P_2, n/2$ )
19 STRASSEN( $S_3, B_{11}, P_3, n/2$ )
20 STRASSEN( $A_{22}, S_4, P_4, n/2$ )
21 STRASSEN( $S_5, S_6, P_5, n/2$ )
22 STRASSEN( $S_7, S_8, P_6, n/2$ )
23 STRASSEN( $S_9, S_{10}, P_7, n/2$ )
24 MATRIX-ADD( $P_5, P_4, C_{11}, n/2$ )
25 MATRIX-SUBTRACT( $P_6, P_2, C_{11}, n/2$ )
26 MATRIX-ADD( $P_1, P_2, C_{12}, n/2$ )
27 MATRIX-ADD( $P_3, P_4, C_{21}, n/2$ )
28 MATRIX-SUBTRACT( $P_5, P_3, C_{22}, n/2$ )
29 MATRIX-SUBTRACT( $P_1, P_7, C_{22}, n/2$ )

```

**4.2-3**

For convenience, we'll assume that  $n$  is an exact power of 3. Similarly to equation (4.2), let's partition the matrices into nine  $n/3 \times n/3$  submatrices. We can then treat the input matrices as two  $3 \times 3$  matrices that we multiply using  $k$  multiplications. Each of those is in fact a multiplication of two  $n/3 \times n/3$  matrices and can be performed by running the algorithm recursively.

Let  $T(n)$  be the worst-case time to multiply two  $n \times n$  matrices by this algorithm. If we did the partitioning using index calculations, then we get the recurrence

$$T(n) = kT(n/3) + \Theta(1)$$

which, by the master method, has the solution  $T(n) = \Theta(n^{\log_3 k})$ . According to Problem ??(d),  $T(n) = o(n^{\lg 7})$  if  $\log_3 k < \lg 7$ , which gives us  $k < 3^{\lg 7} \approx 21.85$ . Hence, the largest integer value for  $k$  is 21 and then the running time of the algorithm is  $T(n) = \Theta(n^{\log_3 21}) = O(n^{2.78})$ .

**4.2-4**

Like in the previous exercise, we can partition the  $n \times n$  matrices into either  $n/68 \times n/68$ ,  $n/70 \times n/70$ , or  $n/72 \times n/72$  submatrices, and use a corresponding method to multiply the partitioned matrices as if they were either  $68 \times 68$ ,  $70 \times 70$ , or  $72 \times 72$  matrices. Each multiplication of two submatrices made while applying these methods can be performed by invoking the algorithm recursively.

Let  $T_{68}(n)$ ,  $T_{70}(n)$ , and  $T_{72}(n)$  be the worst-case times to multiply two  $n \times n$  matrices using this algorithm, when using each given method. Then, assuming partitioning is done using index calculations, we get the recurrences:

$$T_{68}(n) = 132,464 \cdot T_{68}(n/68) + \Theta(1),$$

$$T_{70}(n) = 143,640 \cdot T_{70}(n/70) + \Theta(1),$$

$$T_{72}(n) = 155,424 \cdot T_{72}(n/72) + \Theta(1).$$

By the master method, their solutions are:

$$T_{68}(n) = \Theta(n^{\log_{68} 132,464}) = O(n^{2.795128488}),$$

$$T_{70}(n) = \Theta(n^{\log_{70} 143,640}) = O(n^{2.795122690}),$$

$$T_{72}(n) = \Theta(n^{\log_{72} 155,424}) = O(n^{2.795147392}).$$

Of all those methods, the one involving multiplying  $70 \times 70$  matrices yields the best asymptotic running time, and with this choice the algorithm performs better than the  $O(n^{2.81})$  Strassen's algorithm.

**4.2-5**

The algorithm will compute the three values:

$$\begin{aligned}\alpha &= ac, \\ \beta &= bd, \\ \gamma &= (a + b)(c + d),\end{aligned}$$

each requiring one multiplication of real numbers. Then,

$$\begin{aligned}(a + bi)(c + di) &= (ac - bd) + (ad + bc)i \\ &= (ac - bd) + ((a + b)(c + d) - ac - bd)i \\ &= (\alpha - \beta) + (\gamma - \alpha - \beta)i,\end{aligned}$$

so the algorithm will produce  $\alpha - \beta$  as the real component, and  $\gamma - \alpha - \beta$  as the imaginary component.

**4.2-6**

Given two  $n \times n$  matrices,  $A$  and  $B$ , let's create a new  $2n \times 2n$  matrix

$$D = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}$$

in  $\Theta((2n)^2) = \Theta(n^2)$  time. Then we can use the  $\Theta(n^\alpha)$ -time algorithm for squaring  $n \times n$  matrices to compute  $D^2$  in  $\Theta((2n)^\alpha) = \Theta(n^\alpha)$  time. The product  $AB$  is the upper right  $n \times n$  submatrix of  $D^2$ :

$$\begin{aligned}D^2 &= \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix} \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} A^2 & AB \\ 0 & 0 \end{pmatrix}.\end{aligned}$$

Since  $\alpha \geq 2$ , the overall running time of the algorithm is  $\Theta(n^2 + n^\alpha) = \Theta(n^\alpha)$ .

### 4.3 The substitution method for solving recurrences

**4.3-1**

*Work in progress.*

**4.3-2**

*Work in progress.*

**4.3-3**

*Work in progress.*

---

## 4.4 The recursion-tree method for solving recurrences

**4.4-1**  Work in progress.

**4.4-2**  Work in progress.

**4.4-3**  Work in progress.

**4.4-4**  Work in progress.

---

## 4.5 The master method for solving recurrences

**4.5-1**  Work in progress.

**4.5-2**  Work in progress.

**4.5-3**  Work in progress.

**4.5-4**  Work in progress.

**4.5-5**  Work in progress.

---

## ★ 4.6 Proof of the continuous master theorem

**4.6-1**  Work in progress.

**4.6-2**  Work in progress.

**4.6-3**  Work in progress.

### ★ 4.7 Akra-Bazzi recurrences

**4.7-1**  Work in progress.

**4.7-2**  Work in progress.

**4.7-3**  Work in progress.

**4.7-4**  Work in progress.

**4.7-5**  Work in progress.

**4.7-6**  Work in progress.

## Problems

### 4-1 Recurrence examples

 a. Work in progress.

 b. Work in progress.

 c. Work in progress.

 d. Work in progress.

 e. Work in progress.

 f. Work in progress.

 g. Work in progress.

 h. Work in progress.

### 4-2 Parameter-passing costs

 a. Work in progress.



b. Work in progress.



c. Work in progress.

**4-3****Solving recurrences with a change of variables**

a. Work in progress.



b. Work in progress.



c. Work in progress.



d. Work in progress.



e. Work in progress.



f. Work in progress.

**4-4****More recurrence examples**

a. Work in progress.



b. Work in progress.



c. Work in progress.



d. Work in progress.



e. Work in progress.



f. Work in progress.



g. Work in progress.



h. Work in progress.



i. Work in progress.



j. Work in progress.

4-5

**Fibonacci numbers**



a. Work in progress.



b. Work in progress.



c. Work in progress.



d. Work in progress.



e. Work in progress.

4-6

**Chip testing**



a. Work in progress.



b. Work in progress.



c. Work in progress.



d. Work in progress.

4-7

**Monge arrays**



a. Work in progress.



b. Work in progress.



c. Work in progress.



d. Work in progress.



e. Work in progress.

---

## A

## Summations

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### A.1 Summation formulas and properties

**A.1-1**

Let  $g_1, g_2, \dots, g_n$  be functions, so that  $g_k(i) = O(f_k(i))$  for each  $k = 1, 2, \dots, n$ . From the definition of  $O$ -notation, there exist positive constants  $c_1, c_2, \dots, c_n$  and  $i_0$ , such that  $g_k(i) \leq c_k f_k(i)$  for all  $i \geq i_0$ . Let  $c = \max\{c_k : 1 \leq k \leq n\}$ . For  $i \geq i_0$  we have

$$\begin{aligned} \sum_{k=1}^n g_k(i) &\leq \sum_{k=1}^n c_k f_k(i) \\ &\leq c \sum_{k=1}^n f_k(i) \\ &= O\left(\sum_{k=1}^n f_k(i)\right). \end{aligned}$$

**A.1-2**

$$\begin{aligned} \sum_{k=1}^n (2k - 1) &= 2 \sum_{k=1}^n k - \sum_{k=1}^n 1 && \text{(by the linearity property)} \\ &= \frac{2n(n + 1)}{2} - n && \text{(by equation (A.1))} \\ &= n^2 \end{aligned}$$

**A.1-3** If we let  $x = 10$  in equation (A.6), we obtain:

$$\begin{aligned} 111,111,111 &= \sum_{k=0}^8 10^k \\ &= \frac{10^9 - 1}{10 - 1}. \end{aligned}$$

**A.1-4** Each subsequent term is the previous term multiplied by  $-1/2$ , so letting  $x = -1/2$  in equation (A.7) gives us the value of the series:

$$\begin{aligned} \sum_{k=0}^{\infty} (-1/2)^k &= \frac{1}{1 - (-1/2)} \\ &= 2/3. \end{aligned}$$

**A.1-5** For the upper bound, we bound  $k$  by  $n$ :

$$\begin{aligned} \sum_{k=1}^n k^c &\leq \sum_{k=1}^n n^c \\ &= n \cdot n^c \\ &= O(n^{c+1}). \end{aligned}$$

To get the lower bound, we drop around half of the smallest terms and bound the remaining ones:

$$\begin{aligned} \sum_{k=1}^n k^c &\geq \sum_{k=\lceil n/2 \rceil}^n k^c \\ &\geq \sum_{k=\lceil n/2 \rceil}^n \lceil n/2 \rceil^c \\ &= (n - \lceil n/2 \rceil + 1) \lceil n/2 \rceil^c \\ &= (\lfloor n/2 \rfloor + 1) \lceil n/2 \rceil^c \\ &\geq \lceil n/2 \rceil^{c+1} \\ &\geq n^{c+1}/2^{c+1} \\ &= \Omega(n^{c+1}) \quad (\text{since } 1/2^{c+1} \text{ is a constant}). \end{aligned}$$

By applying Theorem 3.1, we obtain the tight bound.

**A.I-6**

Let's differentiate both sides of equation (A.11):

$$\begin{aligned} \sum_{k=0}^{\infty} k \cdot kx^{k-1} &= \frac{(1-x)^2 + 2x(1-x)}{(1-x)^4} \\ &= \frac{(1-x) + 2x}{(1-x)^3} \\ &= \frac{1+x}{(1-x)^3}. \end{aligned}$$

To get the desired result, we multiply both sides of the equation by  $x$ .

**A.I-7**

The asymptotic upper bound:

$$\begin{aligned} \sum_{k=1}^n \sqrt{k \lg k} &\leq \sum_{k=1}^n \sqrt{n \lg n} \\ &= n \cdot \sqrt{n \lg n} \\ &= O(n^{3/2} \lg^{1/2} n). \end{aligned}$$

For the asymptotic lower bound, assume for convenience that  $n$  is even and at least 4. Then:

$$\begin{aligned} \sum_{k=1}^n \sqrt{k \lg k} &> \sum_{k=n/2+1}^n \sqrt{k \lg k} \\ &> \sum_{k=n/2+1}^n \sqrt{(n/2) \lg(n/2)} \\ &= (n/2) \sqrt{(n/2)(\lg n - 1)} \\ &\geq (n/2) \sqrt{(n/2)(\lg n)/2} \\ &= (n^{3/2} \lg^{1/2} n)/4 \\ &= \Omega(n^{3/2} \lg^{1/2} n). \end{aligned}$$

The asymptotically tight bound follows from Theorem 3.1.

**A.I-8**

$$\begin{aligned}
 \star \quad & \sum_{k=1}^n \frac{1}{2k-1} < 1 + \sum_{k=2}^{n+1} \frac{1}{2k-2} \\
 & = 1 + \sum_{k=1}^n \frac{1}{2k} \\
 & = 1 + \frac{H_n}{2} \\
 & = 1 + \frac{\ln n + O(1)}{2} \quad (\text{by equation (A.9)}) \\
 & = \ln \sqrt{n} + O(1)
 \end{aligned}$$

**A.I-9**

$$\begin{aligned}
 \star \quad & \sum_{k=0}^{\infty} \frac{k-1}{2^k} = \sum_{k=0}^{\infty} \frac{k}{2^k} - \sum_{k=0}^{\infty} \frac{1}{2^k} \\
 & = \sum_{k=0}^{\infty} k(1/2)^k - \sum_{k=0}^{\infty} (1/2)^k \\
 & = \frac{1/2}{(1-1/2)^2} - \frac{1}{1-1/2} \quad (\text{by equations (A.11) and (A.7)}) \\
 & = 0
 \end{aligned}$$

**A.I-10**

$$\begin{aligned}
 \star \quad & \sum_{k=1}^{\infty} (2k+1)x^{2k} = 2 \sum_{k=0}^{\infty} k(x^2)^k + \sum_{k=0}^{\infty} (x^2)^k - 1 \\
 & = \frac{2x^2}{(1-x^2)^2} + \frac{1}{1-x^2} - 1 \quad (\text{by equations (A.11) and (A.7)}) \\
 & = \frac{x^2(3-x^2)}{(1-x^2)^2}
 \end{aligned}$$

$$\begin{aligned}
\text{A.1-II} \quad \star \quad & \prod_{k=2}^n \left(1 - \frac{1}{k^2}\right) = \prod_{k=2}^n \frac{k^2 - 1}{k^2} \\
& = \frac{\prod_{k=2}^n (k^2 - 1)}{\prod_{k=2}^n k^2} \\
& = \frac{\prod_{k=2}^n (k-1) \cdot \prod_{k=2}^n (k+1)}{\left(\prod_{k=1}^n k\right)^2} \\
& = \frac{\prod_{k=1}^{n-1} k \cdot \prod_{k=3}^{n+1} k}{(n!)^2} \\
& = \frac{(n-1)! \cdot \frac{(n+1)!}{2}}{(n!)^2} \\
& = \frac{n+1}{2n}
\end{aligned}$$

## A.2 Bounding summations

**A.2-I** Using the properties of telescoping series, we get

$$\begin{aligned}
\sum_{k=1}^n \frac{1}{k^2} & \leq 1 + \sum_{k=2}^n \frac{1}{k(k-1)} \\
& = 1 + \sum_{k=2}^n \left( \frac{1}{k-1} - \frac{1}{k} \right) \\
& = 1 + 1 - \frac{1}{n} \\
& < 2.
\end{aligned}$$

$$\begin{aligned}
 \mathbf{A.2-2} \quad & \sum_{k=0}^{\lfloor \lg n \rfloor} \left\lceil \frac{n}{2^k} \right\rceil < \sum_{k=0}^{\lfloor \lg n \rfloor} \left( \frac{n}{2^k} + 1 \right) && \text{(by inequality (3.2))} \\
 & \leq \lg n + 1 + \sum_{k=0}^{\lfloor \lg n \rfloor} \frac{n}{2^k} \\
 & < \lg n + 1 + \sum_{k=0}^{\infty} \frac{n}{2^k} \\
 & = \lg n + 1 + \frac{n}{1 - 1/2} && \text{(by equation (A.7))} \\
 & = \lg n + 1 + 2n \\
 & = O(n)
 \end{aligned}$$

**A.2-3** Proceeding similarly as when searching for an upper bound on the  $n$ th harmonic number, we split the range 1 to  $n$  into  $\lfloor \lg n \rfloor$  pieces and lower-bound the contribution of each piece by  $1/2$ :

$$\begin{aligned}
 \sum_{k=1}^n \frac{1}{k} & \geq \sum_{i=0}^{\lfloor \lg n \rfloor - 1} \sum_{j=0}^{2^i - 1} \frac{1}{2^i + j} \\
 & \geq \sum_{i=0}^{\lfloor \lg n \rfloor - 1} \sum_{j=0}^{2^i - 1} \frac{1}{2^{i+1}} \\
 & = \sum_{i=0}^{\lfloor \lg n \rfloor - 1} (1/2) \\
 & = \lfloor \lg n \rfloor / 2 \\
 & = \Omega(\lg n).
 \end{aligned}$$

**A.2-4** The function  $f(k) = k^3$  is monotonically increasing, so we approximate the summation using inequality (A.18):

$$\int_0^n x^3 dx \leq \sum_{k=1}^n k^3 \leq \int_1^{n+1} x^3 dx.$$

The integral approximation gives the upper bound

$$\int_0^n x^3 dx = \frac{n^4}{4},$$

and the lower bound

$$\int_1^{n+1} x^3 dx = \frac{(n+1)^4 - 1}{4}.$$

**A.2-5** Applying inequality (A.19) to  $\sum_{k=1}^n 1/k$  leads to an improper integral:

$$\begin{aligned} \sum_{k=1}^n \frac{1}{k} &\leq \int_0^n \frac{dx}{x} \\ &= \lim_{a \rightarrow 0^+} \int_a^n \frac{dx}{x} \\ &= \infty. \end{aligned}$$

As a result, we don't get any useful information about the upper bound on the summation. By rewriting it as  $1 + \sum_{k=2}^n 1/k$ , we can apply formula (A.19) to the second term and upper-bound the initial summation by  $\ln n + 1$ .

## Problems

### A-1

#### *Bounding summations*

In order to determine the asymptotic tight bounds on each summation, we will find their asymptotic upper bound and asymptotic lower bound by substituting each term in the summations by appropriate values. Since the bounds on the summations don't depend on the parity of  $n$ , for simplicity we will assume that  $n$  is even.

**a.** An asymptotic upper bound:

$$\begin{aligned} \sum_{k=1}^n k^r &\leq \sum_{k=1}^n n^r \\ &= n \cdot n^r \\ &= O(n^{r+1}). \end{aligned}$$

An asymptotic lower bound:

$$\begin{aligned}\sum_{k=1}^n k^r &\geq \sum_{k=n/2+1}^n k^r \\ &\geq \sum_{k=n/2+1}^n (n/2)^r \\ &= (n/2) \cdot (n/2)^r \\ &= \Omega(n^{r+1}).\end{aligned}$$

Based on the results, we conclude that the asymptotic tight bound on the summation is

$$\sum_{k=1}^n k^r = \Theta(n^{r+1}).$$

**b.** An asymptotic upper bound:

$$\begin{aligned}\sum_{k=1}^n \lg^s k &\leq \sum_{k=1}^n \lg^s n \\ &= n \cdot \lg^s n \\ &= O(n \lg^s n).\end{aligned}$$

An asymptotic lower bound:

$$\begin{aligned}\sum_{k=1}^n \lg^s k &\geq \sum_{k=n/2+1}^n \lg^s k \\ &\geq \sum_{k=n/2+1}^n \lg^s(n/2) \\ &= (n/2) \cdot \lg^s(n/2) \\ &= \Omega(n \lg^s n).\end{aligned}$$

The asymptotic tight bound:

$$\sum_{k=1}^n \lg^s k = \Theta(n \lg^s n).$$

c. An asymptotic upper bound:

$$\begin{aligned} \sum_{k=1}^n k^r \lg^s k &\leq \sum_{k=1}^n n^r \lg^s n \\ &= n \cdot n^r \lg^s n \\ &= O(n^{r+1} \lg^s n). \end{aligned}$$

An asymptotic lower bound:

$$\begin{aligned} \sum_{k=1}^n k^r \lg^s k &\geq \sum_{k=n/2+1}^n k^r \lg^s k \\ &\geq \sum_{k=n/2+1}^n (n/2)^r \lg^s (n/2) \\ &= (n/2) \cdot (n/2)^r \lg^s (n/2) \\ &= \Omega(n^{r+1} \lg^s n). \end{aligned}$$

The asymptotic tight bound:

$$\sum_{k=1}^n k^r \lg^s k = \Theta(n^{r+1} \lg^s n).$$

Assuming  $s = 0$  and  $r = 0$  in the bound above, respectively, we get the summations and their asymptotic tight bounds from parts (a) and (b).

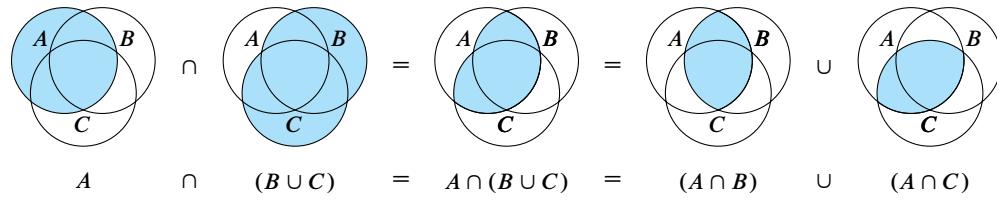
## B

## Sets, Etc.

### B.1 Sets

**B.1-1**

See Figure ??.



**Figure B.1-1** A Venn diagram illustrating the first of the distributive laws (B.1).

**B.1-2**

We prove the first formula by induction on  $n$ . For  $n = 1$  the proof is trivial, and for  $n = 2$  the formula is the first of DeMorgan's laws. Let's assume that  $n > 2$ , and that the formula holds for a family of  $n - 1$  sets. We have:

$$\begin{aligned} \overline{A_1 \cap A_2 \cap \cdots \cap A_{n-1} \cap A_n} &= \overline{(A_1 \cap A_2 \cap \cdots \cap A_{n-1}) \cap A_n} \\ &= \overline{A_1} \cup \overline{A_2} \cup \cdots \cup \overline{A_{n-1}} \cup \overline{A_n} \\ &= \overline{A_1} \cup \overline{A_2} \cup \cdots \cup \overline{A_{n-1}} \cup \overline{A_n}. \end{aligned}$$

The second equality follows from the first of DeMorgan's laws, and the third equality follows from the inductive hypothesis.

The proof of the second formula is similar. If  $n = 2$ , then the formula is DeMorgan's second law, and if  $n > 2$ , then it is enough to apply the above reasoning with the symbols  $\cup$  and  $\cap$  swapped, and use DeMorgan's second law.

**B.1-3**

- ★ We prove the principle of inclusion and exclusion by induction on  $n$ . For  $n = 1$  the proof is trivial, and for  $n = 2$  the formula is identical to (B.3). If  $n > 2$ , then by (B.3)

and the generalized first distributive law, we have:

$$\begin{aligned}
 |A_1 \cup A_2 \cup \cdots \cup A_n| &= |(A_1 \cup A_2 \cup \cdots \cup A_{n-1}) \cup A_n| \\
 &= |A_1 \cup A_2 \cup \cdots \cup A_{n-1}| + |A_n| \\
 &\quad - |(A_1 \cup A_2 \cup \cdots \cup A_{n-1}) \cap A_n| \\
 &= |A_1 \cup A_2 \cup \cdots \cup A_{n-1}| + |A_n| \\
 &\quad - |(A_1 \cap A_n) \cup \cdots \cup (A_{n-1} \cap A_n)|.
 \end{aligned}$$

We now apply the induction hypothesis to the first and the last summand:

$$\begin{aligned}
 |A_1 \cup A_2 \cup \cdots \cup A_{n-1}| &= \sum_{1 \leq i_1 < n} |A_{i_1}| \\
 &\quad - \sum_{1 \leq i_1 < i_2 < n} |A_{i_1} \cap A_{i_2}| \\
 &\quad + \sum_{1 \leq i_1 < i_2 < i_3 < n} |A_{i_1} \cap A_{i_2} \cap A_{i_3}| \\
 &\quad \vdots \\
 &\quad + (-1)^{n-2} |A_1 \cap A_2 \cap \cdots \cap A_{n-1}|,
 \end{aligned}$$

$$\begin{aligned}
 |(A_1 \cap A_n) \cup \cdots \cup (A_{n-1} \cap A_n)| &= \sum_{1 \leq i_1 < n} |A_{i_1} \cap A_n| \\
 &\quad - \sum_{1 \leq i_1 < i_2 < n} |A_{i_1} \cap A_{i_2} \cap A_n| \\
 &\quad \vdots \\
 &\quad + (-1)^{n-2} |A_1 \cap A_2 \cap \cdots \cap A_{n-1} \cap A_n|.
 \end{aligned}$$

We finish the proof by inserting the resulting expressions into the initial formula:

$$\begin{aligned}
 |A_1 \cup A_2 \cup \cdots \cup A_n| &= \sum_{1 \leq i_1 \leq n} |A_{i_1}| \\
 &\quad - \sum_{1 \leq i_1 < i_2 \leq n} |A_{i_1} \cap A_{i_2}| \\
 &\quad + \sum_{1 \leq i_1 < i_2 < i_3 \leq n} |A_{i_1} \cap A_{i_2} \cap A_{i_3}| \\
 &\quad \vdots \\
 &\quad + (-1)^{n-1} |A_1 \cap A_2 \cap \cdots \cap A_n|.
 \end{aligned}$$

#### B.1-4

An example of a one-to-one correspondence between  $\mathbb{N}$  and the set of odd natural

numbers is:  $n \rightarrow 2n + 1$ . Thus, the set of odd natural numbers is countable.

**B.1-5**

We prove by induction on the cardinality of the set  $S$ .

When  $|S| = 0$ , it holds that  $|2^S| = 2^{|S|} = 1$ , because  $S$  has exactly one subset—the empty set. Now let  $|S| > 0$  and assume that  $|2^S| = 2^{|S|}$ . Choose  $p \notin S$  and consider the set  $S' = S \cup \{p\}$ . We can divide the subsets of  $S'$  into those that contain  $p$  and those that do not contain  $p$ . By the inductive hypothesis there are  $|2^S| = |2^{S'-\{p\}}| = 2^{|S'-\{p\}|}$  of the latter. In fact there are as many subsets containing  $p$ , because each is formed as a union of the singleton  $\{p\}$  with some subset not containing  $p$ . Thus we have:

$$\begin{aligned} |2^{S'}| &= 2 \cdot 2^{|S'-\{p\}|} \\ &= 2^{|S'-\{p\}|+1} \\ &= 2^{|S'|}. \end{aligned}$$

**B.1-6**

An  $n$ -tuple of elements  $a_1, a_2, \dots, a_n$  is defined as:

$$(a_1, a_2, \dots, a_n) = \begin{cases} \emptyset & \text{if } n = 0, \\ \{a_1\} & \text{if } n = 1, \\ \{a_1, \{a_1, a_2\}\} & \text{if } n = 2, \\ (a_1, (a_2, \dots, a_n)) & \text{if } n \geq 3. \end{cases}$$

**B.2 Relations****B.2-1**

To show that the relation “ $\subseteq$ ” on the power set  $2^{\mathbb{Z}}$  is a partial order, we will show that “ $\subseteq$ ” is reflexive, antisymmetric and transitive. Let  $A, B, C \in 2^{\mathbb{Z}}$ . Of course,  $x \in A$  implies  $x \in A$ , so reflexivity trivially holds. If  $A \subseteq B$  and  $B \subseteq A$ , then  $x \in A$  implies  $x \in B$ , and  $x \in B$  implies  $x \in A$ . By the law of transitivity of implication,  $x \in A$  if and only if  $x \in B$ , and therefore  $A = B$ . The combination of  $A \subseteq B$  and  $B \subseteq C$  means that  $x \in A$  implies  $x \in B$ , and  $x \in B$  implies  $x \in C$ . Referring again to the law of transitivity of implication, we have that  $x \in A$  implies  $x \in C$ , that is,  $A \subseteq C$ .

The relation “ $\subseteq$ ” on  $2^{\mathbb{Z}}$  is not a total order, because, for example,  $\{0, 1\} \not\subseteq \{1, 2\}$  and  $\{1, 2\} \not\subseteq \{0, 1\}$ .

**B.2-2**

Let us denote the relation “equivalent modulo  $n$ ”, for  $n \in \mathbb{N} - \{0\}$ , by

$$R_n = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} : a = b \pmod{n}\}.$$

For any  $a \in \mathbb{Z}$  we have  $a = a \pmod{n}$ , because  $a - a = 0 \cdot n$ , so the relation  $R_n$  is reflexive. For any  $a, b \in \mathbb{Z}$ , if there exists  $q \in \mathbb{Z}$  that  $a - b = qn$ , then  $b - a = -qn$ , and therefore the fact that  $a = b \pmod{n}$  implies  $b = a \pmod{n}$ . Therefore,  $R_n$  is symmetric. For transitivity let's choose any  $a, b, c \in \mathbb{Z}$  such that  $a = b \pmod{n}$  and  $b = c \pmod{n}$ . This means that there exist  $q, r \in \mathbb{Z}$  such that  $a - b = qn$  and  $b - c = rn$ . Hence  $a - c = a - b + b - c = qn + rn = (q + r)n$ , and therefore  $a = c \pmod{n}$ .

Based on the proof above  $R_n$  is an equivalence relation that partitions the set  $\mathbb{Z}$  into  $n$  abstraction classes; the  $i$ th class, where  $i = 1, 2, \dots, n$ , is the set of integers that leave remainder  $i - 1$  when divided by  $n$ .

- B.2-3**
- a.** The relation  $\{(a, a), (b, b), (c, c), (a, b), (b, a), (a, c), (c, a)\}$  on the set  $\{a, b, c\}$ .
  - b.** The relation  $\{(a, a), (a, b), (b, b)\}$  on the set  $\{a, b\}$ .
  - c.** The empty relation on any nonempty set.

- B.2-4**
- If  $R$  is an equivalence relation, then  $s \in [s]$  for all  $s \in S$ . By antisymmetry of  $R$ , if  $s' R s$  and  $s R s'$ , then  $s = s'$ , so there are no such  $s'$  that  $s' \in [s] - \{s\}$ . This means that for all  $s \in S$ ,  $[s] = \{s\}$ .

- B.2-5**
- Symmetry and transitivity are defined using implications. For an implication “ $p$  implies  $q$ ” to hold true, it isn't required for  $p$  to be true. A relation on a set remains symmetric and transitive, if there is an element in the set, that is not related to any element in that set. Reflexivity, on the other hand, requires that each element is related to itself. Thus, there exist relations that are symmetric and transitive relations but not reflexive (an example of one is given in Exercise ??(c)).

## B.3 Functions

- B.3-1**
- a.** For any function  $f : A \rightarrow B$ ,  $f(A) \subseteq B$ , which results directly from the definition of the range of  $f$ . Additionally, if  $f$  is injective, then each argument  $a \in A$  has a distinct image  $f(a) \in f(A)$ , so  $|A| = |f(A)|$ . Hence,  $|A| \leq |B|$ .
  - b.** For any function  $f : A \rightarrow B$ , where  $A$  is finite, it holds  $|A| \geq |f(A)|$ , because there may be arguments  $a_1, a_2 \in A$  such that  $f(a_1) = f(a_2)$ . Additionally, if  $f$  is surjective, then  $f(A) = B$ , and therefore  $|A| \geq |B|$ .

**B.3-2**

The function  $f(x) = x + 1$ , with  $\mathbb{N}$  as both the domain and the codomain, is not bijective because there is no  $x \in \mathbb{N}$  such that  $f(x) = 0$ . If we consider  $\mathbb{Z}$  as the domain and the codomain, then  $f$  becomes bijective, since every integer  $y$  in the codomain is an image under  $f$  of a uniquely determined integer in the domain:  $y = f(y - 1)$ .

**B.3-3**

The definition comes as a natural generalization of the functional inverse. Let  $R$  be a binary relation on sets  $A$  and  $B$ . The binary relation  $R^{-1}$  on sets  $B$  and  $A$ , such that

$b R^{-1} a$  if and only if  $a R b$ ,

is called the *inverse* of  $R$ .

**B.3-4**

- ★ Let  $h : \mathbb{Z} \rightarrow \mathbb{Z} \times \mathbb{Z}$  be the function we are looking for. Since the inverse of a bijection is also a bijection, we will focus on finding the inverse  $h^{-1} : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}$ , which is equivalent to finding a way to sequentially number each ordered pair with integer elements, so that each integer is used as the number of some pair. We will now describe the construction of one of such mappings.

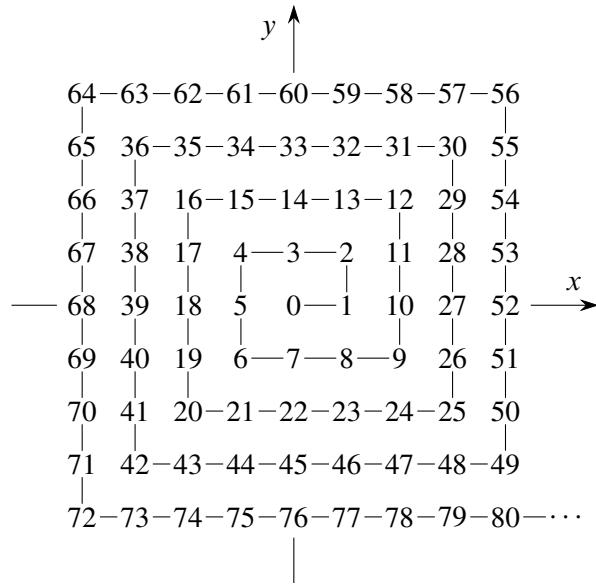
Let us make some simplification—instead of numbering pairs by integers, we will restrict ourselves to natural numbers. Let  $g : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{N}$  and  $f : \mathbb{N} \rightarrow \mathbb{Z}$  be such bijections that  $h^{-1} = f \circ g$ . It was already noted on page 1162 of the text that  $f(n) = (-1)^n \lceil n/2 \rceil$  is bijective, so all that remains is to find the function  $g$ .

Consider the numbering of ordered pairs with integer elements illustrated in Figure ?? in the form of a spiral. Since each such pair  $(x, y)$  is assigned a unique natural number, we can treat this spiral as a description of the function  $g$ . Let  $d = \max \{|x|, |y|\}$  and let  $D = (2d - 1)^2 - 1$ . Informally,  $d$  and  $D$  denote, respectively, the number of the “lap” around the origin covered by the spiral at the time of passing through  $(x, y)$ , and the largest number on the spiral during the previous “lap”. Then we can define the function  $g$  as follows:

$$g(x, y) = \begin{cases} 0 & \text{if } d = |x| = |y| = 0, \\ D + d + y & \text{if } d = x \neq |y|, \\ D + 3d - x & \text{if } d = y \neq 0, \\ D + 5d - y & \text{if } d = -x \neq |y|, \\ D + 7d + x & \text{if } d = -y \neq 0. \end{cases}$$

**B.4 Graphs****B.4-1**

We can represent the relation of “shaking hands” on the faculty party as an undirected graph  $G = (V, E)$ , where  $V$  is the set of professors attending the party and  $E$  is the



**Figure B.3-4** A bijection from  $\mathbb{Z} \times \mathbb{Z}$  to  $\mathbb{N}$ . Individual natural numbers denote the values of this bijection for points with integer coordinates in the Cartesian coordinate system.

set of unordered pairs of professors who shook hands. Summing up the degrees of all vertices of  $G$ , we get twice the number of its edges, because each edge is incident on exactly two vertices.

#### B.4-2

Let  $\langle v_0, v_1, \dots, v_n \rangle$  be a path from vertex  $u$  to vertex  $v$  in a directed or undirected graph. If the path is simple, the vertices in the path are distinct. Otherwise, the path contains a cycle  $\langle v_i, v_{i+1}, \dots, v_{i+k}, v_i \rangle$  for some  $i$  and  $k$ , and by eliminating the subpath  $\langle v_i, v_{i+1}, \dots, v_{i+k} \rangle$  from the path, we discard an extra repetition of  $v_i$ . By applying this modification until we remove all such repetitions, we obtain a simple path from  $u$  to  $v$ .

The proof for cycles is analogous with  $v_0 = v_n$  and we have to remember not to remove the very last repetition of  $v_0$ , required for the path to form a cycle.

#### B.4-3

We proceed by induction on the number of vertices in the graph  $G = (V, E)$ . When  $|V| = 1$ , the graph has no edges and the inequality trivially holds. For the inductive step, pick a vertex  $v \in V$ .  $G$  is connected so there must be at least one edge incident on  $v$  and another vertex from  $V$ . Let  $G' = (V', E')$  be the subgraph of  $G$  induced by  $V' = V - \{v\}$  (i.e., the graph  $G$  with vertex  $v$  and all edges incident on  $v$  removed).

We have  $|V'| = |V| - 1$  and  $|E'| \leq |E| - 1$ , so:

$$\begin{aligned} |E| &\geq |E'| + 1 \\ &\geq ((|V'| - 1) + 1) \quad (\text{by the inductive hypothesis applied for } G') \\ &= |V| - 1. \end{aligned}$$

**B.4-4**

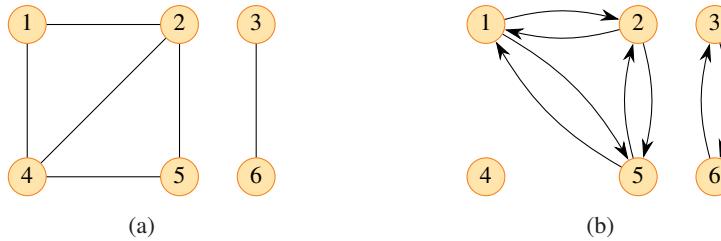
Every vertex of a directed or undirected graph is reachable from itself because there is a path of length 1 containing only that vertex, therefore the “is reachable from” relation is reflexive.

Let  $u, v, w$  be vertices of a directed or undirected graph such that  $u \xrightarrow{p} v$  and  $v \xrightarrow{q} w$ . Then there is a path from  $u$  to  $w$  which is a concatenation of the sequences  $p$  and  $q$  (with the extra  $v$  between them eliminated), which proves transitivity.

The relation is symmetric only in an undirected graph, because for its any vertices  $u, v$ , if  $u \xrightarrow{p} v$  then we can mirror the sequence  $p$  to obtain  $p'$  such that  $v \xrightarrow{p'} u$ . The sequence  $p'$  is a valid path, because if  $(x, y)$  is an edge, so is  $(y, x)$ . In a directed graph the edges are ordered pairs, so if  $(x, y)$  is an edge,  $(y, x)$  may not be. Thus mirroring may not produce a valid path and so symmetry does not hold in general for directed graphs.

**B.4-5**

See Figure ??.



**Figure B.4-5** (a) The undirected version of the directed graph in Figure B.2(a). (b) The directed version of the undirected graph in Figure B.2(b).

**B.4-6**

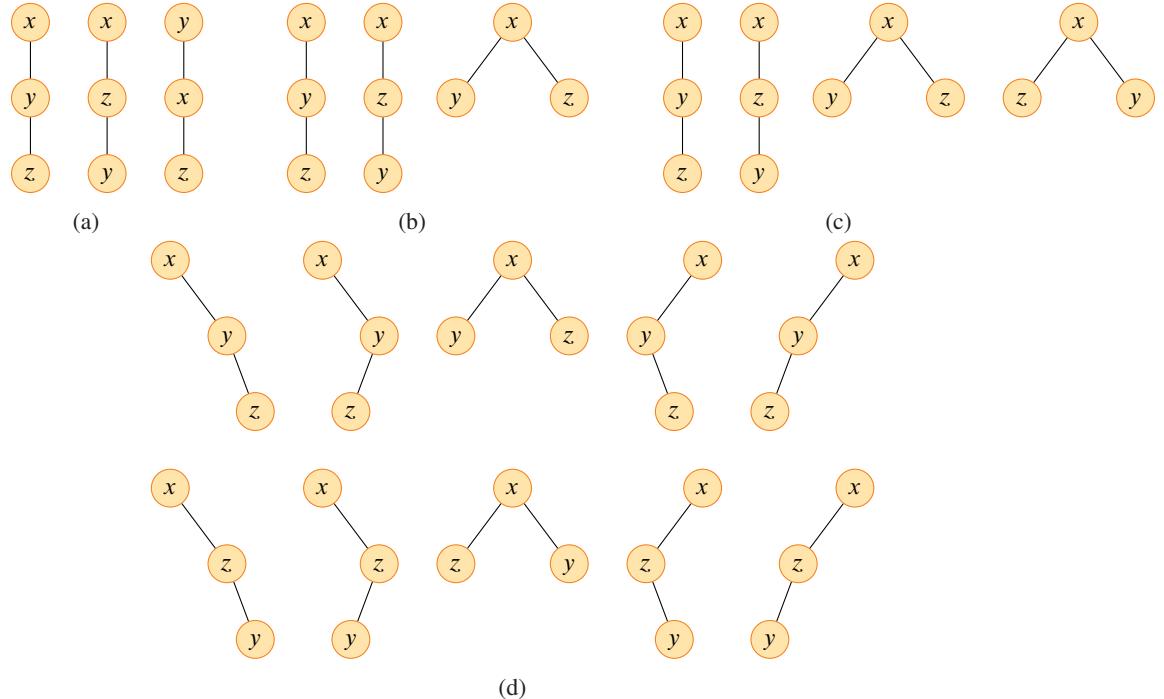
A hypergraph  $H = (V_H, E_H)$  can be represented as a bipartite graph  $G = (V_1 \cup V_2, E)$ , where  $V_1 = V_H$  and  $V_2 = E_H$ . A pair  $\{v_1, v_2\}$ , such that  $v_1 \in V_1$  and  $v_2 \in V_2$ , is an edge of the graph  $G$ , if and only if the hyperedge  $v_2$  is incident on vertex  $v_1$  in the hypergraph  $H$ . In the graph  $G$  there are no edges between elements of  $V_1$  or between elements of  $V_2$ , so  $G$  is indeed bipartite.

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## B.5 Trees

**B.5-1**

See Figure ??.



**Figure B.5-1** (a) All the free trees with vertices  $x$ ,  $y$ , and  $z$ . (b) All the rooted trees with nodes  $x$ ,  $y$ , and  $z$  with  $x$  as the root. (c) All the ordered trees with nodes  $x$ ,  $y$ , and  $z$  with  $x$  as the root. (d) All the binary trees with nodes  $x$ ,  $y$ , and  $z$  with  $x$  as the root.

**B.5-2**

Let  $G' = (V, E')$  be the undirected version of the graph  $G$ . Note that  $G'$  is connected, since every vertex  $v \in V$  is reachable from vertex  $v_0$  in  $G'$ .

Now suppose that  $G'$  does not form a tree. This means it has a cycle of length  $k \geq 3$ , in particular a simple cycle of such a length (after Exercise ??), say  $\langle v_1, v_2, \dots, v_{k+1} \rangle$ , where  $v_{k+1} = v_1$ . For simplicity, let  $v_{k+2} = v_2$ . The graph  $G$  is acyclic, so for some integer  $1 \leq l \leq k$  there must be edges  $(v_l, v_{l+1}), (v_{l+2}, v_{l+1}) \in E$ . By assumption, both  $v_l$  and  $v_{l+2}$  are reachable from  $v_0$ , so there are two different paths from  $v_0$  to  $v_{l+1}$ :  $v_0 \rightsquigarrow v_l \rightarrow v_{l+1}$  and  $v_0 \rightsquigarrow v_{l+2} \rightarrow v_{l+1}$ . The obtained contradiction leads to the conclusion that  $G'$  is acyclic and so it is a tree.

**B.5-3**

The proof is by induction on the number  $n$  of nodes in the tree. The base case is when  $n = 1$ . A binary tree with only one node has one leaf and no degree-2 nodes, thus the statement holds.

For the inductive step, suppose that  $n \geq 1$  and assume that the statement is true for all binary trees with  $n$  nodes. Let  $T$  be a binary tree with  $n + 1$  nodes. We can remove the root node and obtain two subtrees, each of which has less than  $n + 1$  nodes. If any of the subtrees is empty, the statement follows immediately from the inductive hypothesis applied to the other (nonempty) subtree, since the root of  $T$  is neither a leaf nor a degree-2 node.

Now suppose that both the left subtree and the right subtree are nonempty. Let  $L$  and  $D$  be, respectively, the number of leaves and the number of degree-2 nodes in  $T$ . Similarly, let  $L_1$  and  $D_1$  be the analogous numbers for the left subtree, and let  $L_2$  and  $D_2$  be the analogous numbers for the right subtree. It holds that  $L = L_1 + L_2$  and  $D = D_1 + D_2 + 1$ . Then:

$$\begin{aligned} D &= D_1 + D_2 + 1 \\ &= L_1 - 1 + L_2 - 1 + 1 && \text{(by the inductive hypothesis)} \\ &= L_1 + L_2 - 1 \\ &= L - 1. \end{aligned}$$

In a full binary tree each internal node is a degree-2 node, so based on the above result, such a tree has one fewer internal nodes than leaves.

**B.5-4**

We construct a sequence of full binary trees  $T_1, T_2, \dots$ , as follows. Let  $T_1$  be a binary tree containing a single node, so it has one leaf. For any  $k \geq 2$ , let  $T_k$  be a binary tree whose left subtree is  $T_{k-1}$  and whose right subtree is a one-node binary tree. Given that  $T_{k-1}$  has  $k - 1$  leaves,  $T_k$  has  $(k - 1) + 1 = k$  leaves.

**B.5-5**

The proof is by induction on the height  $h$  of the binary tree.

The only nonempty binary tree with  $h = 0$  is the one with a single node, so  $n = 1$  and the statement  $h \geq \lfloor \lg n \rfloor$  holds. Now let  $h > 0$  and suppose that the statement holds for all binary trees with heights at most  $h - 1$ . Consider a binary tree  $T$  with height  $h$ . One of its subtrees must have height  $h' = h - 1$ , and the other subtree must have height  $h'' \leq h - 1$ . Let  $n'$  and  $n''$  be the numbers of nodes in these subtrees, respectively. By the inductive hypothesis,  $h' \geq \lfloor \lg n' \rfloor$  and  $h'' \geq \lfloor \lg n'' \rfloor$ . Observe that for any integer  $k \geq 0$  it is true that  $\lfloor \lg 2^{k+1} \rfloor = k + 1$  but  $\lfloor \lg(2^{k+1} - 1) \rfloor = k$ . Therefore we have

$$n' \leq 2^{h'+1} - 1$$

and

$$n'' \leq 2^{h''+1} - 1.$$

The tree  $T$  has  $n = n' + n'' + 1$  nodes, so

$$\begin{aligned} n &= n' + n'' + 1 \\ &\leq 2^{h'+1} - 1 + 2^{h''+1} - 1 + 1 \\ &\leq 2^h + 2^h - 1 \\ &= 2^{h+1} - 1. \end{aligned}$$

Using the above observation again, we conclude that  $\lfloor \lg n \rfloor \leq h$ .

**B.5-6**

- ★ We proceed by induction on  $n$ . The base case is when  $n = 0$ . The only full binary tree that has no internal nodes is a one-node tree, for which we have  $e = i = 0$  and the statement holds.

Now let  $n \geq 1$  and suppose that the statement holds for all full binary trees with less than  $n$  internal nodes. We can construct an arbitrary full binary tree  $T$  with  $n$  internal nodes by choosing any pair of full binary trees, whose numbers of internal nodes sum up to  $n - 1$ , for its left subtree  $T_L$  and its right subtree  $T_R$ . For  $k \geq 0$ , if  $T_L$  has  $k$  internal nodes,  $T_R$  has  $n - k - 1$  internal nodes. By the conclusion made in Exercise ??,  $T_L$  and  $T_R$  have  $k + 1$  and  $n - k$  leaves, respectively.

Let  $i_L$  and  $i_R$  be the internal path lengths of  $T_L$  and  $T_R$ , respectively, and similarly, let  $e_L$  and  $e_R$  be their external path lengths. The depth of each node in  $T$  — other than the root, for which the depth is 0 — is 1 more than the depth of this node in either  $T_L$  or  $T_R$ . Hence,

$$\begin{aligned} i &= 0 + (i_L + k) + (i_R + n - k - 1) \\ &= i_L + i_R + n - 1 \end{aligned}$$

and

$$\begin{aligned} e &= (e_L + k + 1) + (e_R + n - k) \\ &= e_L + e_R + n + 1. \end{aligned}$$

By the inductive hypothesis,  $e_L = i_L + 2k$  and  $e_R = i_R + 2(n - k - 1)$ , so

$$\begin{aligned} e - i &= e_L + e_R + n + 1 - i_L - i_R - n + 1 \\ &= i_L + 2k + i_R + 2(n - k - 1) - i_L - i_R + 2 \\ &= 2n. \end{aligned}$$

Therefore, the statement holds for all full binary trees.

**B.5-7**

- ★ The proof is by induction on the number  $n$  of nodes of the tree. In the empty tree (where  $n = 0$ ) the sum of the leaf weights is 0, which obviously satisfies the inequality. When

$n = 1$ , then the tree consists of a single leaf with a depth of 0, so the sum of the leaf weights is  $2^{-0} = 1$ .

Now suppose that  $n > 1$  and that the inequality holds for trees with less than  $n$  nodes. Let  $T$  be a binary tree with  $n$  nodes, and let  $T_L, T_R$  be its left and right subtrees, respectively. By the inductive hypothesis, we have that the inequality is satisfied separately for trees  $T_L$  and  $T_R$ . The depth of a leaf in tree  $T$  is 1 greater than the depth of the same leaf in either  $T_L$  or  $T_R$ , hence the weights of the leaves of  $T$  are half as compared to the weights of the same leaves in its subtrees. Thus, in tree  $T$  both the sum of the leaf weights from  $T_L$ , as well as the sum of the leaf weights from  $T_R$  do not exceed  $1/2$ , and so the total sum of the leaf weights in  $T$  does not exceed 1.

- B.5-8** ★ Let  $T$  be a binary tree with  $L \geq 2$  leaves. Suppose that  $T$  does not contain a subtree having between  $L/3$  and  $2L/3$  leaves, inclusive. Since  $L \geq 2$ , the root of  $T$  is not a leaf. We will follow a path  $p$  from the root to a leaf, at each level traversing from a node to its child whose subtree has more leaves. The numbers of leaves in subtrees rooted at the visited nodes decrease from  $L$  to 1 as we move on the path  $p$ . By assumption, there is a node  $x$  on path  $p$  such that the subtree rooted at  $x$  has more than  $2L/3$  leaves, and that the subtree  $T_y$  rooted at its child  $y$  has less than  $L/3$  leaves. The subtree rooted at the other child of  $x$  has fewer leaves than  $T_y$  (0 if  $x$  has only one child). This leads to a contradiction because the total number of leaves in both subtrees is less than  $2L/3$ .

## Problems

### B-1 *Graph coloring*

a. For a given tree let us distinguish any of its nodes and view the tree as a rooted tree in that node. Edges in such a tree are incident on nodes from adjacent levels, so nodes can be colored according to the parity of their depth in the tree (e.g., those on odd levels get color 1, and those on even levels get color 2).

b. (1)  $\Rightarrow$  (2): Since  $G = (V, E)$  is a bipartite graph, the vertex set  $V$  can be partitioned into two subsets  $V_1$  and  $V_2$ , such that vertices from one subset aren't adjacent to vertices from the other subset. Thanks to this property it is possible to assign one color to each vertex from  $V_1$  and a different color to each vertex from  $V_2$ , obtaining a 2-coloring of graph  $G$ .

(2)  $\Rightarrow$  (3): Suppose that graph  $G$  has a cycle  $p = \langle v_0, v_1, \dots, v_{2k+1} \rangle$  for some  $k \geq 1$ . Let  $c$  be a 2-coloring of  $G$  and without loss of generality suppose that  $c(v_0) = 1$ . Then it must be  $c(v_{2i}) = 1$  and  $c(v_{2i+1}) = 2$  for all  $i = 0, 1, \dots, k$ . But since  $p$

is a cycle,  $v_{2k+1} = v_0$ , which yields the contradiction. Therefore, graph  $G$  must not contain a cycle of odd length.

(3)  $\Rightarrow$  (1): Suppose that graph  $G$  is connected. Otherwise, apply the following proof separately to each connected component of  $G$ .

Choose any  $v_0 \in V$ . Let  $V_1$  be the set of all vertices  $v \in V$  for which there exists a path from  $v_0$  to  $v$  of even length, and let  $V_2 = V - V_1$ . If there were vertices  $u, w \in V_1$ , such that there is a path from  $u$  to  $w$  of odd length, then  $v_0 \rightsquigarrow u \rightsquigarrow w \rightsquigarrow v_0$  would be a cycle of an odd length. Therefore, no two vertices from  $V_1$  are adjacent, and we can prove a similar fact for vertices in  $V_2$ . This means that  $G = (V_1 \cup V_2, E)$  is bipartite.

**c.** We carry out the proof by induction on the number of vertices in graph  $G = (V, E)$ . If the graph has only one vertex (with degree 0), then of course one color is enough.

Now suppose that  $|V| > 1$ . Let us choose an arbitrary vertex  $v \in V$  and denote by  $G'$  the subgraph of  $G$  induced by the set  $V - \{v\}$ . By the inductive hypothesis,  $G'$  can be colored with  $d' + 1$  colors, where  $d'$  is the maximum degree of any vertex in  $G'$ . It is true that  $d' \leq d$ , so  $G'$  can also be colored with  $d + 1$  colors. Since vertex  $v$  has at most  $d$  neighbors in  $G$ , among the colors used in a  $(d + 1)$ -coloring of  $G'$  there is one that is not assigned to any neighbor of  $v$  in graph  $G$ . By assigning that color to  $v$ , we obtain a  $(d + 1)$ -coloring of  $G$ .

**d.** For any  $k \geq 2$ , if graph  $G$  is  $k$ -colorable, but is not  $(k - 1)$ -colorable, then for every two different colors used in a  $k$ -coloring of  $G$  there are two adjacent vertices, one with the first color and the other with the second color. If that weren't true, there would be two different colors that we could not distinguish, and so we could reduce the number of colors required, obtaining a  $(k - 1)$ -coloring of  $G$ .

Based on the above reasoning, we have  $\binom{k}{2} \leq |E|$ . We use the fact that for  $k \geq 2$  it holds that  $k/2 \leq k - 1$ , and so

$$\begin{aligned} k^2 &= 4(k/2)(k/2) \\ &\leq 4(k - 1)k/2 \\ &= 4 \binom{k}{2} \\ &\leq 4|E| \\ &= O(|V|). \end{aligned}$$

This fact implies that there exist positive constants  $c$  and  $k_0$  such that  $0 \leq k^2 \leq c|V|$  for all  $k \geq k_0$ . By applying square roots, we obtain that  $0 \leq k \leq \sqrt{c}\sqrt{|V|}$  for all

$k \geq k_0$ , and since  $\sqrt{c}$  is a positive constant, we get  $k = O(\sqrt{|V|})$ .

## B-2

### Friendly graphs

a.

#### Theorem

In any undirected graph  $G = (V, E)$ , where  $|V| \geq 2$ , there are two vertices with the same degree.

**Proof** The degrees of the vertices in graph  $G$  are numbers from 0 to  $|V| - 1$ , inclusive. Observe, however, that  $G$  has an isolated vertex if and only if it doesn't have a vertex adjacent to any other vertex of  $G$ . Therefore, the degrees of the vertices of  $G$  can in fact take  $|V| - 1$  possible values, so some two vertices must have equal degrees. ■

b.

#### Theorem

An undirected graph  $G = (V, E)$  of 6 vertices, or its complement<sup>1</sup>  $\bar{G}$ , contain a clique<sup>2</sup> of size 3.

**Proof** Let's pick any  $v \in V$ . Since  $|V| = 6$ , there is a set  $U \subseteq V - \{v\}$  of three vertices, for which exactly one of the following cases holds:

1. each vertex in  $U$  is adjacent to  $v$ ,
2. no vertex in  $U$  is adjacent to  $v$ .

Since case 1 in  $G$  is equivalent to case 2 in  $\bar{G}$ , and vice versa, let us focus on case 1 only. If there is a pair of adjacent vertices among those in  $U$ , then the vertices of this pair together with  $v$  form a clique in  $G$ . Now suppose there is no such pair. Then, however, the vertices in  $U$  form a clique in  $\bar{G}$ . ■

c.

#### Theorem

For any undirected graph  $G = (V, E)$  the set  $V$  can be partitioned into two subsets such that for any vertex  $v \in V$  at least half of its neighbors do not belong to the subset that  $v$  belongs to.

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<sup>1</sup>See page 1085 of the text.

<sup>2</sup>See page 1081 of the text.

**Proof** Let us divide the vertices of  $G$  arbitrarily into two disjoint subsets  $V_1$  and  $V_2$ . Let

$$S = \{(v_1, v_2) \in E : v_1 \in V_1, v_2 \in V_2\}.$$

If more than half of the neighbors of some vertex  $v \in V_1$  are also in  $V_1$ , then moving  $v$  to  $V_2$  increases the size of set  $S$ . Similarly, for vertices  $v \in V_2$  with this property—by moving them to set  $V_1$ , we increase set  $S$ . This process must terminate after a finite number of steps, because  $S$  cannot grow indefinitely. The obtained partition  $V = V_1 \cup V_2$  satisfies the desired property. ■

d. Before we formulate the main theorem, we will need the following lemma.

**Lemma**

Let  $G = (V, E)$  be an undirected graph, where  $|V| \geq 3$  and

$$\text{degree}(u) + \text{degree}(v) \geq |V| \quad (1)$$

for every pair of non-adjacent vertices  $u, v \in V$ . Then  $G$  is hamiltonian<sup>3</sup>.

**Proof** Suppose the lemma does not hold, that is there is a graph  $G = (V, E)$  with at least 3 vertices, that satisfies the property (??), but is not hamiltonian. Among all such graphs let us consider that of the greatest number of edges  $|E|$ . Let  $n = |V|$ . Then  $G$  must have a hamiltonian path<sup>4</sup>  $\langle v_1, v_2, \dots, v_n \rangle$ —otherwise we could add the missing edges without violating (??) and get a graph with more than  $|E|$  edges. Since  $G$  does not have a hamiltonian cycle,  $(v_1, v_n) \notin E$ . By assumption, we have that  $\text{degree}(v_1) + \text{degree}(v_n) \geq n$ .

Now let us define the following sets:

$$S_1 = \{i : 2 \leq i \leq n \text{ and } (v_1, v_i) \in E\},$$

$$S_n = \{i : 2 \leq i \leq n \text{ and } (v_{i-1}, v_n) \in E\}.$$

Then  $|S_1| = \text{degree}(v_1)$  and  $|S_n| = \text{degree}(v_n)$ . Since  $|S_1| + |S_n| \geq n$  and the set  $S_1 \cup S_n$  has at most  $n - 1$  elements, the set  $S_1 \cap S_n$  must be non-empty. So there exists  $i$  for which  $(v_1, v_i), (v_{i-1}, v_n) \in E$ . Hence, the path

$$v_1 \rightsquigarrow v_{i-1} \rightarrow v_n \rightarrow v_{n-1} \rightsquigarrow v_i \rightarrow v_1$$

is a hamiltonian cycle in graph  $G$ , which is a contradiction. ■

**Theorem**

Let  $G = (V, E)$  be an undirected graph, where  $|V| \geq 3$  and

$$\text{degree}(v) \geq |V|/2$$

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<sup>3</sup>See page 1056 of the text.

<sup>4</sup>See Exercise 34.2-6.

for every vertex  $v \in V$ . Then  $G$  is hamiltonian.

**Proof** If the property holds for every  $v \in V$ , then

$$\text{degree}(u) + \text{degree}(v) \geq |V|$$

for every  $u, v \in V$  whether they are adjacent or not. So  $G$  satisfies the assumptions of the lemma, and thus it is hamiltonian.  $\blacksquare$

### B-3

#### Bisecting trees

**a.** The fact trivially holds for binary trees with at most three nodes, so in the rest of the proof we will assume that  $n \geq 4$ . We define a **branch** of a binary tree as a path  $\langle x_0, x_1, \dots, x_k \rangle$ , where  $x_0$  is the root,  $x_k$  is a leaf, and for  $i = 1, 2, \dots, k$ ,  $x_i$  is the root of the subtree of  $x_{i-1}$  with more nodes. Note that a tree may have more than one branch. Let's denote by  $s(x)$  the number of nodes in the subtree rooted at  $x$ . For every  $i = 0, 1, \dots, k-1$ ,

$$s(x_i) \leq 2s(x_{i+1}) + 1. \quad (2)$$

Of course, the sequence  $\langle s(x_0), s(x_1), \dots, s(x_k) \rangle$  decreases from  $n$  to 1, so there must be  $j$ , such that  $s(x_j) > n/4$  and  $s(x_{j+1}) \leq n/4$ . Thus, by removing edge  $(x_{j-1}, x_j)$ , we partition the nodes of the tree into two sets of sizes

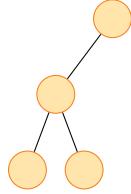
$$\begin{aligned} s(x_j) &\leq 2s(x_{j+1}) + 1 && \text{(by inequality (2))} \\ &= 2n/4 + 1 \\ &\leq 3n/4 \end{aligned}$$

and

$$\begin{aligned} n - s(x_j) &< n - n/4 \\ &= 3n/4. \end{aligned}$$

**b.** The constant  $3/4$  is sufficient to make balanced partitions, as we have shown in part (a). The example of the binary tree in Figure ?? shows that the constant cannot be lower.

**c.** Assume that  $n \geq 6$ . We'll remove one edge from the original tree to cut off a subtree of at most  $\lfloor n/2 \rfloor$  vertices. Consider any of the tree's branches  $\langle x_0, x_1, \dots, x_k \rangle$ , as well as the function  $s$ , both defined in the solution to part (a). There is an edge  $(x_j, x_{j+1})$ ,



**Figure B-3** The binary tree whose most evenly balanced partition upon removal of a single edge has a subset with 3 vertices.

such that  $s(x_j) > \lfloor n/2 \rfloor$  and  $s(x_{j+1}) \leq \lfloor n/2 \rfloor$ . From inequality (??) we have

$$\begin{aligned} s(x_{j+1}) &\geq (s(x_j) - 1)/2 \\ &> (\lfloor n/2 \rfloor - 1)/2 \\ &\geq \lfloor n/2 \rfloor / 3. \end{aligned}$$

We cut off the subtree rooted at  $x_{j+1}$  and continue partitioning the remaining tree, with at most

$$\lfloor n/2 \rfloor - s(x_{j+1}) < (2/3)\lfloor n/2 \rfloor$$

vertices still to remove from it. In each subsequent step the number of nodes remaining to be cut is reduced by a factor of  $2/3$ . Therefore, we'll need to remove  $O(\log_{3/2} n) = O(\lg n)$  edges.

## C

## Counting and Probability

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### C.1 Counting

#### C.1-1

We will only count non-empty  $k$ -substrings, so  $k \geq 1$ . The first  $k$ -substring occupies positions 1 through  $k$  in the  $n$ -string, the second one occupies positions 2 through  $k+1$ , and so on. The last  $k$ -substring ends at position  $n$ , so it must start at position  $n-k+1$ . So an  $n$ -string has  $n-k+1$   $k$ -substrings in total.

By the rule of sum, the total number of all substrings of an  $n$ -string is

$$\begin{aligned} \sum_{k=1}^n (n-k+1) &= \sum_{k=1}^n k \\ &= \frac{n(n+1)}{2} \quad (\text{by equation (A.1)}) \\ &= \binom{n+1}{2}. \end{aligned}$$

#### C.1-2

Consider the set of  $n$ -bit integers  $N = \{0, 1, \dots, 2^n - 1\}$  and the set of  $m$ -bit integers  $M = \{0, 1, \dots, 2^m - 1\}$ . We are counting functions  $f : N \rightarrow M$  or, equivalently, sequences  $\langle a_1, \dots, a_{2^n} \rangle$  with terms from  $M$ . We can choose each term in such a sequence in  $2^m$  ways, so following the rule of product gives us

$$(2^m)^{2^n} = 2^{m2^n}$$

possibilities in total for the number of sequences or the number of  $n$ -input,  $m$ -output boolean functions. Specifically, there is

$$2^{2^n}$$

$n$ -input, 1-output boolean functions.

**C.1-3**

Let  $S_n$  be the number of ways to arrange  $n$  professors around a circular table. A seating is indistinguishable from  $n - 1$  others that are formed from it by rotation. There are  $n!$  permutations of the  $n$  professors, so  $nS_n = n!$ . Hence

$$S_n = (n - 1)!.$$

**C.1-4**

Three integers add up to an even number only if all of them are even, or if exactly one of them is even. There are 49 even and 50 odd numbers in the set  $\{1, 2, \dots, 99\}$ , so in the first option we can make a choice in  $\binom{49}{3}$  ways, and in the second—in  $\binom{50}{2}\binom{49}{1}$  ways. Therefore, the total number of ways is

$$\binom{49}{3} + \binom{50}{2}\binom{49}{1} = 78,449.$$

**C.1-5**

$$\begin{aligned} \binom{n}{k} &= \frac{n!}{k!(n-k)!} \\ &= \frac{n}{k} \cdot \frac{(n-1)!}{(k-1)!(n-k)!} \\ &= \frac{n}{k} \binom{n-1}{k-1} \end{aligned}$$

**C.1-6**

$$\begin{aligned} \binom{n}{k} &= \frac{n!}{k!(n-k)!} \\ &= \frac{n}{n-k} \cdot \frac{(n-1)!}{k!(n-k-1)!} \\ &= \frac{n}{n-k} \binom{n-1}{k} \end{aligned}$$

**C.1-7**

Let  $S$  be an  $n$ -set and let  $s \in S$ . The set  $S$  has  $\binom{n}{k}$   $k$ -subsets, for  $0 \leq k \leq n$ . We can divide these  $k$ -subsets into those that do not contain  $s$  and those that do contain  $s$ . There are  $\binom{n-1}{k}$  of the former and  $\binom{n-1}{k-1}$  of the latter. Hence, we have

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

**C.1-8**

The first rows of Pascal's triangle:

$$\begin{array}{ccccccc}
 & & 1 & & & & \\
 & 1 & & 1 & & & \\
 & 1 & 2 & & 1 & & \\
 & 1 & 3 & 3 & & 1 & \\
 & 1 & 4 & 6 & 4 & & 1 \\
 & 1 & 5 & 10 & 10 & 5 & 1 \\
 & 1 & 6 & 15 & 20 & 15 & 6 & 1
 \end{array}$$

In the  $n$ th row the leftmost element is  $\binom{n}{0} = 1$ , and the rightmost element is  $\binom{n}{n} = 1$ . The remaining elements are determined based on the identity from Exercise ??.

**C.1-9**

$$\begin{aligned}
 \binom{n+1}{2} &= \frac{(n+1)!}{2!(n-1)!} \\
 &= \frac{n(n+1)}{2} \\
 &= \sum_{i=1}^n i \quad \text{(by equation (A.1))}
 \end{aligned}$$

**C.1-10**

Let's fix  $n$ . We'll view the expression  $\binom{n}{k}$  as a function  $b_n(k)$  for  $k = 0, 1, \dots, n$  and will find  $k$  for which the value  $b_n(k)$  is the greatest. Consider the ratio:

$$\begin{aligned}
 \frac{b_n(k+1)}{b_n(k)} &= \frac{\binom{n}{k+1}}{\binom{n}{k}} \\
 &= \frac{n!}{(k+1)!(n-k-1)!} \cdot \frac{k!(n-k)!}{n!} \\
 &= \frac{n-k}{k+1}.
 \end{aligned}$$

If  $n - k \geq k + 1$ , or  $k \leq (n - 1)/2$ , then function  $b_n$  is monotonically increasing. Conversely, when  $k \geq (n - 1)/2$ , then function  $b_n$  is monotonically decreasing. So

when  $n$  is odd,  $b_n$  reaches its maximum for  $k = (n - 1)/2 = \lfloor n/2 \rfloor$ . Moreover:

$$\begin{aligned} b_n((n - 1)/2) &= \binom{n}{(n - 1)/2} \\ &= \binom{n}{n - (n - 1)/2} \quad (\text{by equation (C.3)}) \\ &= \binom{n}{(n + 1)/2} \\ &= b_n((n + 1)/2), \end{aligned}$$

so the maximum value is also reached when  $k = (n + 1)/2 = \lceil n/2 \rceil$ .

In case  $n$  is an even number,  $b_n$  achieves the maximum value for either  $k = n/2$  or  $k = n/2 - 1$ . Let's see which value is bigger:

$$\begin{aligned} \frac{b_n(n/2)}{b_n(n/2 - 1)} &= \frac{\binom{n}{n/2}}{\binom{n}{n/2 - 1}} \\ &= \frac{n!}{(n/2)! (n/2)!} \cdot \frac{(n/2 - 1)! (n/2 + 1)!}{n!} \\ &= \frac{n/2 + 1}{n/2} \\ &> 1. \end{aligned}$$

So we have that  $b_n$  reaches its maximum for  $k = n/2 = \lfloor n/2 \rfloor = \lceil n/2 \rceil$ .

### C.1-II ★ First, let's observe that

$$\begin{aligned} (j + k)! &= j! (j + 1)(j + 2) \cdots k \\ &\geq j! 1 \cdot 2 \cdots k \\ &= j! k!, \end{aligned}$$

where equality holds only if  $j = 0$  or  $k = 0$ . Then

$$\begin{aligned} \binom{n}{j+k} &= \frac{n!}{(j+k)!(n-j-k)!} \\ &\leq \frac{n!}{j!k!(n-j-k)!} \\ &= \frac{n!}{j!(n-j)!} \cdot \frac{(n-j)!}{k!(n-j-k)!} \\ &= \binom{n}{j} \binom{n-j}{k}. \end{aligned}$$

We can interpret the expression  $\binom{n}{j+k}$  as the number of possible ways to choose  $j+k$  items out of  $n$ , and the expression  $\binom{n}{j} \binom{n-j}{k}$  as the number of ways to choose  $j$  items out of  $n$ , and then  $k$  items out of  $n-j$  left after the first choice. In each strategy we come up with a set of the  $j+k$  selected items. There is exactly one way to construct a given set of  $j+k$  items using the first strategy, and at least one if we follow the second strategy. It is so, because we can arbitrarily partition the set into the set of  $j$  items that will be selected in the first step and the set of  $k$  items that will be selected in the second step.

- ★ It's easy to check that the inequality holds for  $k = 0$ . Now let  $k \geq 1$  and let's assume that

$$\binom{n}{k-1} \leq \frac{n^n}{(k-1)^{k-1}(n-k+1)^{n-k+1}}.$$

Then

$$\begin{aligned} \binom{n}{k} &= \frac{n}{k} \binom{n-1}{k-1} && \text{(by Exercise ??)} \\ &= \frac{n}{k} \cdot \frac{n-k+1}{n} \binom{n}{k-1} && \text{(by Exercise ??)} \\ &\leq \frac{n^n}{k(k-1)^{k-1}(n-k+1)^{n-k}} && \text{(by the inductive hypothesis).} \end{aligned}$$

Now it suffices to show that

$$\frac{n^n}{k(k-1)^{k-1}(n-k+1)^{n-k}} \leq \frac{n^n}{k^k(n-k)^{n-k}}.$$

To prove this, we examine the ratio of both expressions:

$$\begin{aligned} \frac{\frac{n^n}{k(k-1)^{k-1}(n-k+1)^{n-k}}}{\frac{n^n}{k^k(n-k)^{n-k}}} &= \frac{k^{k-1}(n-k)^{n-k}}{(k-1)^{k-1}(n-k+1)^{n-k}} \\ &= \frac{\left(\frac{k}{k-1}\right)^{k-1}}{\left(\frac{n-k+1}{n-k}\right)^{n-k}} \\ &= \frac{\left(1 + \frac{1}{k-1}\right)^{k-1}}{\left(1 + \frac{1}{n-k}\right)^{n-k}}. \end{aligned}$$

The sequence  $e_n = (1 + 1/n)^n$  is increasing—therefore the above ratio does not exceed 1, as long as  $k-1 \leq n-k$  or, equivalently,  $k \leq (n+1)/2$ , and in particular when  $k \leq n/2$ . Thus, we have shown that inequality (C.7) holds for  $k \leq n/2$ .

When  $n/2 \leq k \leq n$ , then  $0 \leq n-k \leq n/2$ , so

$$\begin{aligned} \binom{n}{k} &= \binom{n}{n-k} && \text{(by equation (C.3))} \\ &\leq \frac{n^n}{(n-k)^{n-k} k^k}, \end{aligned}$$

where the last inequality holds after substituting  $k$  by  $n-k$  and applying the result from the first part of the exercise.

**C.1-13** ★ Using Stirling's approximation (3.25), we obtain

$$\begin{aligned} \binom{2n}{n} &= \frac{(2n)!}{(n!)^2} \\ &= \frac{\sqrt{4\pi n} (2n/e)^{2n} (1 + \Theta(1/n))}{2\pi n (n/e)^{2n} (1 + \Theta(1/n))^2} \\ &= \frac{2^{2n}}{\sqrt{\pi n}} \cdot \frac{1 + \Theta(1/n)}{(1 + \Theta(1/n))^2}. \end{aligned}$$

Now let's examine the last fraction, which we intentionally didn't reduce because the denominator may not be equal to the square of the numerator. Let  $c, d$  be constants such that  $c \geq d > 0$ , the expression  $1 + c/n$  bounds the numerator from above, and

the expression  $1 + d/n$  bounds the denominator from below. Then

$$\begin{aligned} \frac{1 + \Theta(1/n)}{(1 + \Theta(1/n))^2} &\leq \frac{1 + c/n}{(1 + d/n)^2} \\ &< \frac{1 + c/n}{1 + d/n} \\ &= \frac{n + c}{n + d} \\ &= 1 + \frac{c - d}{n + d} \\ &< 1 + \frac{c - d}{n} \\ &= 1 + O(1/n). \end{aligned}$$

Therefore,

$$\binom{2n}{n} = \frac{2^{2n}}{\sqrt{\pi n}} (1 + O(1/n)).$$

**C.1-14** ★ Let's differentiate the binary entropy function  $H$ :

$$\begin{aligned} H'(\lambda) &= -\left(\lg \lambda + \lambda \cdot \frac{1}{\lambda \ln 2}\right) - \left(-\lg(1-\lambda) + (1-\lambda) \cdot \frac{-1}{(1-\lambda) \ln 2}\right) \\ &= -\lg \lambda - \lg e + \lg(1-\lambda) + \lg e \\ &= \lg \frac{1-\lambda}{\lambda}, \\ H''(\lambda) &= \frac{\lambda}{(1-\lambda) \ln 2} \cdot \frac{-\lambda - (1-\lambda)}{\lambda^2} \\ &= -\frac{\lg e}{\lambda(1-\lambda)}. \end{aligned}$$

The first derivative reaches zero when  $\lambda = 1/2$ . For  $\lambda = 1/2$  the second derivative is negative, so the function  $H$  achieves its maximum of  $H(1/2) = 1$ .

- C.1-15** ★ For  $n = 0$  the equality holds trivially, so let  $n \geq 1$ . Then

$$\begin{aligned} \sum_{k=0}^n \binom{n}{k} k &= 0 + \sum_{k=1}^n \binom{n-1}{k-1} n && \text{(by identity (C.9))} \\ &= n \sum_{k=0}^{n-1} \binom{n-1}{k} \\ &= n 2^{n-1} && \text{(by identity (C.4) where } x = y = 1\text{).} \end{aligned}$$

- C.1-16** ★ We can enumerate the cases with  $n = 0, 1, 2, 3$  and  $k = 0, 1$ , and confirm that the identity holds for all of them.

Now let  $n \geq 4$ . We have

$$\begin{aligned} \binom{n}{k} &= \frac{n(n-1)\cdots(n-(k-1))}{k!} \\ &= \frac{n^k}{k!} \left(1 - \frac{1}{n}\right) \cdots \left(1 - \frac{k-1}{n}\right) \\ &\geq \frac{n^k}{k!} \left(1 - \frac{k-1}{n}\right)^{k-1}. \end{aligned}$$

Now consider the function

$$f(x, y) = \left(1 - \frac{x}{n}\right)^y,$$

where  $0 \leq x \leq n$  and  $y \geq 0$ . Observe that the function  $f$  monotonically decreases, when  $x$  increases for a fixed  $y$ , or when  $y$  increases for a fixed  $x$ . Therefore, since  $k-1 < \sqrt{n}$ ,

$$\begin{aligned} \left(1 - \frac{k-1}{n}\right)^{k-1} &\geq \left(1 - \frac{\sqrt{n}}{n}\right)^{k-1} \\ &\geq \left(1 - \frac{1}{\sqrt{n}}\right)^{\sqrt{n}}. \end{aligned}$$

It is true that

$$\left(1 - \frac{1}{t}\right)^t \geq \frac{1}{4}$$

for all  $t \geq 2$ . Therefore, when  $n \geq 4$ ,

$$\begin{aligned} \binom{n}{k} &\geq \frac{n^k}{k!} \left(1 - \frac{1}{\sqrt{n}}\right)^{\sqrt{n}} \\ &\geq \frac{n^k}{4k!}. \end{aligned}$$

## C.2 Probability

### C.2-1

Let's represent the outcome of the experiment as a 3-string over  $\{\text{H}, \text{T}\}$ , with the first two elements denoting the results of Professor Rosencrantz's both tosses, and the last element denoting the result of Professor Guildenstern's toss. The sample space consists of  $2^3 = 8$  elementary events. The probability that Professor Rosencrantz obtains strictly more heads than Professor Guildenstern is

$$\begin{aligned} \Pr \{\text{HHH}, \text{HHT}, \text{HTT}, \text{THT}\} &= \Pr \{\text{HHH}\} + \Pr \{\text{HHT}\} + \Pr \{\text{HTT}\} + \Pr \{\text{THT}\} \\ &= 1/2. \end{aligned}$$

### C.2-2

Let  $C_1, C_2, \dots$  be a finite or countably infinite sequence of events, where

$$C_i = A_i - \bigcup_{j=1}^{i-1} A_j$$

for every  $i = 1, 2, \dots$ . Because  $\bigcup_i A_i = \bigcup_i C_i$  and any two events  $C_j, C_k$ , where  $j \neq k$ , are mutually exclusive, we get

$$\begin{aligned} \Pr \left\{ \bigcup_i A_i \right\} &= \Pr \left\{ \bigcup_i C_i \right\} \\ &= \sum_i \Pr \{C_i\} \\ &\leq \sum_i \Pr \{A_i\}. \end{aligned}$$

The last inequality holds, since  $\Pr \{C_i\} \leq \Pr \{A_i\}$  for every  $i = 1, 2, \dots$ .

### C.2-3

Let  $n_1, n_2, n_3$  be the numbers on the selected cards, in order of removing the cards from the deck. Consider the events  $A = \{n_1 < n_2\}$  and  $B = \{n_2 < n_3\}$ . If  $A$  occurs, then  $n_1$  could have been chosen in  $n_2 - 1$  ways. Similarly, if  $B$  occurs, then  $n_3$  can be

chosen in  $10 - n_2$  ways. We need to calculate  $\Pr\{A \cap B\}$ . The number of outcomes in  $A \cap B$  is

$$\sum_{n_2=1}^{10} (n_2 - 1)(10 - n_2) = 120.$$

The sample space consists of all possible ordered triples of cards (3-permutations) selected from the deck. By (C.1), the sample space has the size of  $10!/7! = 720$ . Thus,

$$\begin{aligned}\Pr\{A \cap B\} &= 120/720 \\ &= 1/6.\end{aligned}$$

**C.2-4**

Since  $(A \cap B) \cup (\bar{A} \cap B) = B$ , and since events  $A \cap B$  and  $\bar{A} \cap B$  are mutually exclusive, we get

$$\begin{aligned}\Pr\{A \mid B\} + \Pr\{\bar{A} \mid B\} &= \frac{\Pr\{A \cap B\}}{\Pr\{B\}} + \frac{\Pr\{\bar{A} \cap B\}}{\Pr\{B\}} \\ &= \frac{\Pr\{B\}}{\Pr\{B\}} \\ &= 1.\end{aligned}$$

**C.2-5**

We prove by induction on the number of events  $n$ . For  $n = 1$  the equality trivially holds, so let's assume  $n \geq 2$ . We get

$$\begin{aligned}\Pr\left\{\bigcap_{i=1}^n A_i\right\} &= \Pr\left\{A_n \cap \bigcap_{i=1}^{n-1} A_i\right\} \\ &= \Pr\left\{\bigcap_{i=1}^{n-1} A_i\right\} \Pr\left\{A_n \mid \bigcap_{i=1}^{n-1} A_i\right\} \quad (\text{by (C.16)}) \\ &= \Pr\{A_1\} \Pr\{A_2 \mid A_1\} \Pr\{A_3 \mid A_1 \cap A_2\} \cdots \Pr\left\{A_n \mid \bigcap_{i=1}^{n-1} A_i\right\},\end{aligned}$$

where the last step follows from the inductive hypothesis.

**C.2-6**

- ★ Let  $n \geq 3$  and let  $S = \{s_{pq} : 1 \leq p, q \leq n\}$  be a sample space with the uniform probability distribution (i.e.,  $\Pr\{s_{pq}\} = 1/n^2$  for every  $p = 1, 2, \dots, n$  and  $q = 1, 2, \dots, n$ ). For each  $i = 1, 2, \dots, n$  consider the event

$$A_i = \{s_{pi} \in S : 1 \leq p \leq i\} \cup \{s_{iq} \in S : i < q \leq n\}.$$

Then  $|A_i| = n$ , so  $\Pr\{A_i\} = 1/n$ . Also,

$$|A_{i_1} \cap A_{i_2}| = 1$$

and

$$|A_{i_1} \cap A_{i_2} \cap A_{i_3}| = 0$$

for every distinct indices  $1 \leq i_1, i_2, i_3 \leq n$ . Therefore,

$$\begin{aligned}\Pr\{A_{i_1} \cap A_{i_2}\} &= 1/n^2 \\ &= \Pr\{A_{i_1}\} \Pr\{A_{i_2}\},\end{aligned}$$

and so the events  $A_1, A_2, \dots, A_n$  are pairwise independent. On the other hand, for each  $k > 2$  and distinct indices  $1 \leq i_1, \dots, i_k \leq n$ ,

$$\Pr\left\{\bigcap_{j=1}^k A_{i_j}\right\} = 0,$$

while

$$\prod_{j=1}^k \Pr\{A_{i_j}\} = 1/n^k,$$

so no  $k$ -subset of the events  $A_1, A_2, \dots, A_n$  is mutually independent.

**C.2-7**

- ★ Suppose we have two coins—a fair one and a biased one that always comes up heads. We will randomly pick one of the coins and flip it twice. Let  $A$  be the event, that the first flip resulted in heads, let  $B$  be the event that the second flip resulted in heads, and let  $C$  be the event that the fair coin was picked. Then,

$$\begin{aligned}\Pr\{A\} &= \Pr\{B\} \\ &= (1/2) \cdot (1/2) + (1/2) \cdot 1 \\ &= 3/4,\end{aligned}$$

$$\Pr\{C\} = 1/2.$$

The events  $A$  and  $B$  are not independent, since

$$\begin{aligned}\Pr\{A \cap B\} &= (1/2) \cdot (1/4) + (1/2) \cdot 1 \\ &= 5/8,\end{aligned}$$

while  $\Pr\{A\} \Pr\{B\} = 9/16$ . But  $A$  and  $B$  are conditionally independent, given  $C$ :

$$\begin{aligned}\Pr\{A \cap B \mid C\} &= 1/4 \\ &= (1/2) \cdot (1/2) \\ &= \Pr\{A \mid C\} \Pr\{B \mid C\}.\end{aligned}$$

**C.2-8**

- ★ Let  $J$ ,  $T$ , and  $C$  be the events that Jeff, Tim and Carmine will pass the course, respectively. Furthermore, let  $G$  be the event that Professor Gore pointed out that Jeff is the one who will fail. Since the professor can't reveal Carmine's outcome, there must be  $\Pr\{G \mid J\} = 0$ ,  $\Pr\{G \mid T\} = 1$ , and  $\Pr\{G \mid C\} = 1/2$ . Before talking to the professor, Carmine knows that his probability of passing is  $\Pr\{C\} = 1/3$ . His situation after the conversation is described by the event  $C \mid G$ .

Since  $G = (G \cap J) \cup (G \cap T) \cup (G \cap C)$  and since  $G \cap J$ ,  $G \cap T$  and  $G \cap C$  are mutually exclusive events,

$$\begin{aligned}\Pr\{G\} &= \Pr\{G \cap J\} + \Pr\{G \cap T\} + \Pr\{G \cap C\} \\ &= \Pr\{J\} \Pr\{G \mid J\} + \Pr\{T\} \Pr\{G \mid T\} + \Pr\{C\} \Pr\{G \mid C\}.\end{aligned}$$

Using Bayes's theorem, we obtain

$$\begin{aligned}\Pr\{C \mid G\} &= \frac{\Pr\{C\} \Pr\{G \mid C\}}{\Pr\{G\}} \\ &= \frac{\Pr\{C\} \Pr\{G \mid C\}}{\Pr\{J\} \Pr\{G \mid J\} + \Pr\{T\} \Pr\{G \mid T\} + \Pr\{C\} \Pr\{G \mid C\}} \\ &= \frac{(1/3) \cdot (1/2)}{(1/3) \cdot 0 + (1/3) \cdot 1 + (1/3) \cdot (1/2)} \\ &= 1/3.\end{aligned}$$

Thus, the information that Carmine has obtained does not change his chance of passing.

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### C.3 Discrete random variables

**C.3-1**

Define the random variable  $X$  to be the sum of the two values showing. Then

$$\begin{aligned} E[X] &= \sum_{x=2}^{12} x \Pr\{X = x\} \\ &= (1/36) \cdot (2 \cdot 1 + 3 \cdot 2 + 4 \cdot 3 + 5 \cdot 4 + 6 \cdot 5 + 7 \cdot 6 \\ &\quad + 8 \cdot 5 + 9 \cdot 4 + 10 \cdot 3 + 11 \cdot 2 + 12 \cdot 1) \\ &= 7. \end{aligned}$$

Define the random variable  $Y$  to be the maximum of the values showing. Then

$$\begin{aligned} E[Y] &= \sum_{y=1}^6 y \Pr\{Y = y\} \\ &= (1/36) \cdot (1 \cdot 1 + 2 \cdot 3 + 3 \cdot 5 + 4 \cdot 7 + 5 \cdot 9 + 6 \cdot 11) \\ &\approx 4.47. \end{aligned}$$

**C.3-2**

Let  $X$  be the random variable denoting the index of the maximum element in the array  $A[1:n]$ . For every  $i = 1, 2, \dots, n$ ,  $\Pr\{X = i\} = 1/n$ , and therefore

$$\begin{aligned} E[X] &= \sum_{i=1}^n i \Pr\{X = i\} \\ &= \frac{1}{n} \sum_{i=1}^n i \\ &= \frac{1}{n} \cdot \frac{n(n+1)}{2} \\ &= \frac{n+1}{2}. \end{aligned}$$

In fact, this result is identical for each element of the array, in particular for the maximum element and for the minimum element.

**C.3-3**

Let  $X$  be the random variable representing the player's gain (in dollars) from playing the carnival game once. For  $k = 0, 1, 2, 3$ , let  $A_k$  be the event that the player's number

appeared on exactly  $k$  dice. Then

$$\begin{aligned}\Pr\{A_k\} &= \binom{3}{k} \left(\frac{1}{6}\right)^k \left(\frac{5}{6}\right)^{3-k} \\ &= \binom{3}{k} \frac{5^{3-k}}{6^3}\end{aligned}$$

We have that

$$\begin{aligned}\mathbb{E}[X] &= (-1) \cdot \Pr\{A_0\} + 1 \cdot \Pr\{A_1\} + 2 \cdot \Pr\{A_2\} + 3 \cdot \Pr\{A_3\} \\ &= (1/6^3) \cdot ((-1) \cdot 1 \cdot 125 + 1 \cdot 3 \cdot 25 + 2 \cdot 3 \cdot 5 + 3 \cdot 1 \cdot 1) \\ &= -17/216 \\ &\approx -0.08,\end{aligned}$$

so the player will lose about 8 cents on average in this game.

### C.3-4

Consider the subset of the sample space, where  $X \geq Y$  holds. Because  $Y \geq 0$ , then  $\mathbb{E}[Y] \geq 0$ , and so

$$\begin{aligned}\mathbb{E}[\max\{X, Y\}] &= \mathbb{E}[X] \\ &\leq \mathbb{E}[X] + \mathbb{E}[Y].\end{aligned}$$

The case in which  $Y \geq X$  is similar.

### C.3-5

- ★ According to the definition of independent random variables  $X$  and  $Y$ , for all  $x$  and  $y$ , we have

$$\Pr\{X = x \text{ and } Y = y\} = \Pr\{X = x\} \Pr\{Y = y\}.$$

Let  $f$  and  $g$  be any functions over the reals. If  $X = x$ , then  $f(X) = f(x)$ , and similarly, if  $Y = y$ , then  $g(Y) = g(y)$ . Therefore, we can write the above formula as

$$\Pr\{f(X) = f(x) \text{ and } g(Y) = g(y)\} = \Pr\{f(X) = f(x)\} \Pr\{g(Y) = g(y)\},$$

from which we conclude that  $f(X)$  and  $g(Y)$  are independent random variables.

**C.3-6** ★ For any  $t > 0$ , we have

$$\begin{aligned}\mathbb{E}[X] &= \sum_{x \geq 0} x \Pr\{X = x\} \\ &= \sum_{0 \leq x < t} x \Pr\{X = x\} + \sum_{x \geq t} x \Pr\{X = x\} \\ &\geq \sum_{x \geq t} x \Pr\{X = x\} \\ &\geq t \sum_{x \geq t} \Pr\{X = x\} \\ &= t \Pr\{X \geq t\}.\end{aligned}$$

**C.3-7** ★ Let's pick any real  $t$ . Let  $A = \{s \in S : X(s) \geq t\}$  and  $A' = \{s \in S : X'(s) \geq t\}$ . It follows from the relationship between random variables  $X$  and  $X'$ , that if  $X'(s) \geq t$ , then  $X(s) \geq t$ , so  $A' \subseteq A$ . From the definition of a random variable, we have

$$\begin{aligned}\Pr\{X \geq t\} &= \sum_{s \in A} \Pr\{s\} \\ &= \sum_{s \in A'} \Pr\{s\} + \sum_{s \in A - A'} \Pr\{s\} \\ &\geq \sum_{s \in A'} \Pr\{s\} \\ &= \Pr\{X' \geq t\}.\end{aligned}$$

**C.3-8** Let  $f(x) = x^2$  and let  $0 \leq \lambda \leq 1$ . Then

$$\begin{aligned}f(\lambda x + (1 - \lambda)y) - (\lambda f(x) + (1 - \lambda)f(y)) &= \lambda^2 x^2 + (1 - \lambda)^2 y^2 + 2\lambda(1 - \lambda)xy - \lambda x^2 - (1 - \lambda)y^2 \\ &= -\lambda(1 - \lambda)x^2 - \lambda(1 - \lambda)y^2 + 2\lambda(1 - \lambda)xy \\ &= -\lambda(1 - \lambda)(x - y)^2 \\ &\leq 0,\end{aligned}$$

which means that the function  $f(x)$  is convex. By applying it to Jensen's inequality, we obtain

$$\mathbb{E}[X^2] \geq \mathbb{E}^2[X].$$

**C.3-9** Since  $X$  takes on only the values 0 and 1, we have  $X^2 = X$ . Then,

$$\begin{aligned}\text{Var}[X] &= \mathbb{E}[X^2] - \mathbb{E}^2[X] \\ &= \mathbb{E}[X] - \mathbb{E}^2[X] \\ &= \mathbb{E}[X](1 - \mathbb{E}[X]) \\ &= \mathbb{E}[X]\mathbb{E}[1 - X] \quad (\text{by linearity of expectation}).\end{aligned}$$

**C.3-10**

$$\begin{aligned}\text{Var}[aX] &= \mathbb{E}[a^2X^2] - \mathbb{E}^2[aX] \\ &= a^2\mathbb{E}[X^2] - a^2\mathbb{E}^2[X] \quad (\text{by equation (C.25)}) \\ &= a^2(\mathbb{E}[X^2] - \mathbb{E}^2[X]) \\ &= a^2\text{Var}[X]\end{aligned}$$

## C.4 The geometric and binomial distributions

**C.4-1** Let  $X$  be a random variable representing the number of trials before obtaining a success in a sequence of Bernoulli trials, where a success occurs with probability  $p > 0$  and a failure with probability  $q = 1 - p$ . Then,

$$\begin{aligned}\sum_{k=1}^{\infty} \Pr\{X = k\} &= \sum_{k=1}^{\infty} q^{k-1}p \\ &= p \sum_{k=0}^{\infty} q^k \\ &= \frac{p}{1-q} \quad (\text{by equation (A.7)}) \\ &= 1,\end{aligned}$$

so axiom 2 of the probability axioms holds.

**C.4-2** We have a success when of all six coins, any three came up heads and the other three came up tails—it is therefore  $\binom{6}{3} = 20$  ways to obtain a success. There are  $2^6 = 64$  possible outcomes, so the probability of success is  $p = 20/64 = 5/16$ . By (C.36), before obtaining one we need to make  $1/p \approx 3.2$  flips on average.

**C.4-3** Let  $X$  be a random variable having a geometric distribution with the probability  $p > 0$  of a success and the probability  $q = 1 - p$  of a failure. First, let's find the expectation

of  $X^2$ :

$$\begin{aligned}
 E[X^2] &= \sum_{k=1}^{\infty} k^2 \Pr\{X = k\} \\
 &= \sum_{k=1}^{\infty} k^2 q^{k-1} p \\
 &= \frac{p}{q} \sum_{k=0}^{\infty} k^2 q^k \\
 &= \frac{p}{q} \cdot \frac{q(1+q)}{(1-q)^3} \quad (\text{by Exercise ??}) \\
 &= (1+q)/p^2.
 \end{aligned}$$

From the definition (C.31) of variance and equation (C.36), we have

$$\begin{aligned}
 \text{Var}[X] &= E[X^2] - E^2[X] \\
 &= (1+q)/p^2 - 1/p^2 \\
 &= q/p^2.
 \end{aligned}$$

**C.4-4**

$$\begin{aligned}
 b(k; n, p) &= \binom{n}{k} p^k (1-p)^{n-k} \\
 &= \binom{n}{n-k} q^{n-k} (1-q)^k \quad (\text{by equation (C.3)}) \\
 &= b(n-k; n, q)
 \end{aligned}$$

**C.4-5**

The binomial distribution achieves a maximum at an integer  $k$ , such that  $np - q \leq k \leq (n + 1)p$ , so a good approximation of the maximum is the value for  $k = np$ . Since  $np$  may not be integer, we'll sacrifice mathematical rigor in order to simplify our calculations:

$$\begin{aligned}
 b(np; n, p) &= \binom{n}{np} p^{np} (1-p)^{n-np} \\
 &= \frac{n!}{(np)!(n-np)!} p^{np} (1-p)^{n-np} \\
 &= \frac{n!}{(np)!(nq)!} p^{np} q^{nq}.
 \end{aligned}$$

Using Stirling's approximation (3.25), we can simplify the first factor of the last expression above:

$$\begin{aligned}\frac{n!}{(np)!(nq)!} &\approx \frac{\sqrt{2\pi n} \left(\frac{n}{e}\right)^n}{\sqrt{2\pi np} \left(\frac{np}{e}\right)^{np} \sqrt{2\pi nq} \left(\frac{nq}{e}\right)^{nq}} \\ &= \frac{\left(\frac{n}{e}\right)^n \left(\frac{e}{np}\right)^{np} \left(\frac{e}{nq}\right)^{nq}}{\sqrt{2\pi npq}} \\ &= \frac{1}{p^{np} q^{nq} \sqrt{2\pi npq}}.\end{aligned}$$

Hence, the value of the maximum of  $b(k; n, p)$  is roughly equal to  $1/\sqrt{2\pi npq}$ .

- C.4-6** ★ We are interested in the value of  $b(0; n, 1/n)$  and we can approximate it by examining its limit:

$$\begin{aligned}\lim_{n \rightarrow \infty} b(0; n, 1/n) &= \lim_{n \rightarrow \infty} (1 - 1/n)^n \\ &= 1/e \quad (\text{by equation (3.16)}).\end{aligned}$$

In the second part we similarly approximate  $b(1; n, 1/n)$ :

$$\begin{aligned}\lim_{n \rightarrow \infty} b(1; n, 1/n) &= \lim_{n \rightarrow \infty} \frac{(1 - 1/n)^n}{1 - 1/n} \\ &= \frac{1/e}{1} \quad (\text{by equation (3.16)}) \\ &= 1/e.\end{aligned}$$

- C.4-7** ★ We'll calculate the probability that the professors get the same number of heads in two ways. In the first one, we'll treat the result of each experiment as a  $2n$ -element sequence of heads and tails, such that its initial  $n$  terms are the results obtained by Professor Rosencrantz, and the final  $n$  terms are the results obtained by Professor Guildenstern. There are  $2^{2n} = 4^n$  of all such sequences. Following the hint, we call a head a success for Professor Rosencrantz and we call a tail a success for Professor Guildenstern. Note that both professors obtain the same number of heads if and only if they achieve exactly  $n$  successes in total. The number of ways this can be done is the number of choices of  $n$  items responsible for successes among all  $2n$  items in the sequence. This number is equal to  $\binom{2n}{n}$ , so the probability we are looking for is  $\binom{2n}{n}/4^n$ .

Another approach to determine the desired probability, is to define random variables  $R$  and  $G$  denoting the number of heads obtained by Professor Rosencrantz and by Professor Guildenstern, respectively. Again, defining successes for each professor according to the hint, we get that  $R$  has the binomial distribution  $b(k; n, 1/2)$  and  $G$

has the binomial distribution  $b(n - k; n, 1/2)$ . The events  $R = k$  and  $G = k$  are independent, so the probability we are looking for, is

$$\begin{aligned} \sum_{k=0}^n \Pr\{R = k \text{ and } G = k\} &= \sum_{k=0}^n \Pr\{R = k\} \Pr\{G = k\} \\ &= \sum_{k=0}^n b(k; n, 1/2) b(n - k; n, 1/2) \\ &= \sum_{k=0}^n \binom{n}{k} \left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right)^{n-k} \binom{n}{n-k} \left(\frac{1}{2}\right)^{n-k} \left(\frac{1}{2}\right)^k \\ &= \frac{\sum_{k=0}^n \binom{n}{k}^2}{4^n}. \end{aligned}$$

Comparing the results obtained in both ways, we get the identity

$$\sum_{k=0}^n \binom{n}{k}^2 = \binom{2n}{n}.$$

- C.4-8** ★ Let  $0 \leq \lambda \leq 1$ . Using the inequality

$$\binom{n}{\lambda n} \leq 2^{nH(\lambda)},$$

that follows from (C.7), we obtain

$$\begin{aligned} b(k; n, 1/2) &= \binom{n}{k} \left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right)^{n-k} \\ &= \binom{n}{(k/n)n} \left(\frac{1}{2}\right)^n \\ &\leq \frac{2^{nH(k/n)}}{2^n} \\ &= 2^{nH(k/n)-n}. \end{aligned}$$

- C.4-9** ★ Let  $X'$  be the random variable denoting the number of successes in a series of Bernoulli trials, each with a probability  $p$  of success. Using the result from Exercise ??, where  $p'_i = p$  for every  $i = 1, 2, \dots, n$ , gives

$$\Pr\{X \geq k\} \leq \Pr\{X' \geq k\}.$$

The random variable  $X'$  has a binomial distribution, so the above inequality can be reformulated as

$$\Pr\{X \geq k\} \leq \sum_{i=k}^n b(i; n, p).$$

Then,

$$\begin{aligned}\Pr\{X < k\} &= 1 - \Pr\{X \geq k\} \\ &\geq \sum_{i=0}^n b(i; n, p) - \sum_{i=k}^n b(i; n, p) \quad (\text{by equation (C.40)}) \\ &= \sum_{i=0}^{k-1} b(i; n, p).\end{aligned}$$

- C.4-10** ★ Let  $S$  be the sample space containing all possible sequences of outcomes during  $n$  Bernoulli trials. A set  $A$  corresponds to a sequence  $s \in S$ , and  $X(s)$  is the number of successes in  $s$ .

We'll show how to obtain a set  $A'$  of Bernoulli trials by performing experiments on the trials of  $A$ . Let  $A_i$  denote the event that the  $i$ th trial of  $A$  is a success, and let  $A'_i$  be the event that the  $i$ th trial of  $A'$  is a success. If  $A_i$  occurs, we'll make  $A'_i$  to also occur. Otherwise, a success in the  $i$ th trial of  $A'$  will be generated with some probability  $r_i$ . Then,

$$\begin{aligned}p'_i &= \Pr\{A'_i\} \\ &= \Pr\{A'_i \cap A_i\} + \Pr\{A'_i \cap \overline{A_i}\} \\ &= p_i + (1 - p_i) \cdot r_i,\end{aligned}$$

and so,

$$r_i = \frac{p'_i - p_i}{1 - p_i}.$$

Reusing the sample space  $S$ , we'll denote by  $X'(s)$  the number of successes in a series of the  $n$  trials obtained by following the above procedure, based on the outcomes in  $s$ . For any  $s \in S$  and its corresponding set  $A$ , it's clear that after constructing the set  $A'$  this way, it's impossible that there are fewer successes in  $A'$  than in the original set  $A$ . In other words,  $X'(s) \geq X(s)$  holds. Using Exercise ??, we get the desired result.

## ★ C.5 The tails of the binomial distribution

C.5-1

- ★ Intuitively, the situation when each flip in a series results in a head is less likely when the series consists of twice more flips, while we expect the same number of heads as before. Let's prove our intuition. Let  $X_m$  for  $m \geq 0$  be the random variable denoting the number of heads obtained in  $m$  flips of a fair coin. When  $n \leq m$ ,  $\Pr\{X_m = n\} = b(n; m, 1/2)$ . Thus, the ratio of both studied probabilities is

$$\begin{aligned}\frac{\Pr\{X_{2n} = n\}}{\Pr\{X_n = n\}} &= \frac{\binom{2n}{n} \left(\frac{1}{2}\right)^{2n}}{\binom{n}{n} \left(\frac{1}{2}\right)^n} \\ &= \frac{\binom{2n}{n}}{2^n} \\ &= \frac{2^n}{\sqrt{\pi n}} (1 + O(1/n)) \quad (\text{by equation (C.11)}) \\ &> 1,\end{aligned}$$

which confirms our guess.

C.5-2

- ★ **Proof of Corollary C.6** Let  $Y$  be the random variable denoting the total number of failures in this experiment. Then,  $\Pr\{X > k\} = \Pr\{Y < n - k\}$ . Since  $np < k < n$ , or  $0 < n - k < nq$ , we can apply Theorem C.4 for  $Y$ , after inverting the roles of successes and failures, to obtain the desired inequality. ■

**Proof of Corollary C.7** Similarly as in the previous proof, let's treat the probability of getting more than  $k$  successes as the probability of getting fewer than  $n - k$  failures. Since  $(np + n)/2 < k < n$ , we have  $0 < n - k < nq/2$ , and after swapping the roles of successes and failures, the result follows from Corollary C.5. ■

C.5-3

- ★ Let  $p = a/(a + 1)$  and  $q = 1 - p = 1/(a + 1)$ , so that  $a = p/q$ . Then,

$$\begin{aligned}\sum_{i=0}^{k-1} \binom{n}{i} a^i &= \sum_{i=0}^{k-1} \binom{n}{i} \left(\frac{p}{q}\right)^i \\ &= \frac{\sum_{i=0}^{k-1} \binom{n}{i} p^i q^{n-i}}{q^n} \\ &= \frac{\sum_{i=0}^{k-1} b(i; n, p)}{q^n},\end{aligned}$$

so by applying Theorem C.4 to the last summation, we obtain

$$\begin{aligned} \sum_{i=0}^{k-1} \binom{n}{i} a^i &< \frac{kq}{q^n(np-k)} b(k;n,p) \\ &= \frac{k/(a+1)}{(1/(a+1))^n(na/(a+1)-k)} b(k;n,a/(a+1)) \\ &= (a+1)^n \frac{k}{na-k(a+1)} b(k;n,a/(a+1)). \end{aligned}$$

- C.5-4** ★ Using the fact that  $\binom{n}{i} \geq 1$  for all  $i = 0, 1, \dots, n$ , we have

$$\begin{aligned} \sum_{i=0}^{k-1} p^i q^{n-i} &\leq \sum_{i=0}^{k-1} \binom{n}{i} p^i q^{n-i} \\ &= \sum_{i=0}^{k-1} b(i;n,p) \\ &< \frac{kq}{np-k} b(k;n,p) && \text{(by Theorem C.4)} \\ &\leq \frac{kq}{np-k} \left(\frac{np}{k}\right)^k \left(\frac{nq}{n-k}\right)^{n-k} && \text{(by Lemma C.1).} \end{aligned}$$

- C.5-5** ★ Let  $Y = n - X$  and let  $v = E[Y]$ . By linearity of expectation,  $v = n - E[X] = n - \mu$ . Since  $r > v$ , we can use Theorem C.8 to obtain

$$\begin{aligned} \Pr\{\mu - X \geq r\} &= \Pr\{n - v - X \geq r\} \\ &= \Pr\{Y - v \geq r\} \\ &\leq \left(\frac{ve}{r}\right)^r \\ &= \left(\frac{(n-\mu)e}{r}\right)^r. \end{aligned}$$

Similarly, in the second part, if we let  $Y = n - X$ , then  $E[Y] = n - E[X] = n - np = nq$ . Since  $r > nq$ , we use Corollary C.9 to get

$$\begin{aligned} \Pr\{np - X \geq r\} &= \Pr\{n - nq - X \geq r\} \\ &= \Pr\{Y - nq \geq r\} \\ &\leq \left(\frac{nqe}{r}\right)^r. \end{aligned}$$

**C.5-6****★ Lemma 4**

Let  $\alpha$ ,  $p$ , and  $q$  be nonnegative reals, such that  $p + q = 1$ . Then,

$$pe^{\alpha q} + qe^{-\alpha p} \leq e^{\alpha^2/2}.$$

**Proof** Consider the function  $f(\alpha) = e^{\alpha^2/2} - (pe^{\alpha q} + qe^{-\alpha p})$  for  $\alpha \geq 0$ , and examine its derivatives:

$$\begin{aligned} f'(\alpha) &= \alpha e^{\alpha^2/2} - pq(e^{\alpha q} - e^{-\alpha p}), \\ f''(\alpha) &= \alpha^2 e^{\alpha^2/2} + e^{\alpha^2/2} - pq(qe^{\alpha q} + pe^{-\alpha p}). \end{aligned}$$

It's true that  $f''(0) = 1 - pq \geq 0$ , and we'll show that  $f''(\alpha) > 0$  for all  $\alpha > 0$ . Since

$$\begin{aligned} pq - 1/4 &= p(1-p) - 1/4 \\ &= -p^2 + p - 1/4 \\ &= -(p - 1/2)^2 \\ &\leq 0 \end{aligned}$$

or, equivalently, since  $pq \leq 1/4$ ,

$$\begin{aligned} pq(qe^{\alpha q} + pe^{-\alpha p}) &\leq (qe^{\alpha q} + pe^{-\alpha p})/4 \\ &\leq (e^{\alpha q} + e^{-\alpha p})/4 \\ &= e^{-\alpha p}(e^{\alpha q+\alpha p} + 1)/4 \\ &= e^{-\alpha p}(e^\alpha + 1)/4 \\ &\leq (e^\alpha + 1)/4. \end{aligned}$$

Combining this inequality with  $\alpha^2 e^{\alpha^2/2} > 0$ , that holds for all  $\alpha > 0$ , we get

$$f''(\alpha) > e^{\alpha^2/2} - (e^\alpha + 1)/4.$$

Now it suffices to show that  $4e^{\alpha^2/2} \geq e^\alpha + 1$  or, since  $e^{\alpha^2/2} > 1$ , that

$$3e^{\alpha^2/2} \geq e^\alpha.$$

This is equivalent to  $3e^{\alpha^2/2}/e^\alpha = 3e^{\alpha^2/2-\alpha} \geq 1$ , which holds when

$$\alpha^2/2 - \alpha + \ln 3 \geq 0.$$

The left-hand side expression attains the minimum at  $\alpha = 1$ , equal to  $\ln 3 - 1/2 > 0$ , which concludes the proof that  $f''(\alpha) \geq 0$  for  $\alpha \geq 0$ .

The fact we've just shown implies that the function  $f'(\alpha)$  is monotonically increasing for all  $\alpha \geq 0$ . Since  $f'(0) = 0$ , it must be  $f'(\alpha) \geq 0$  for all  $\alpha \geq 0$ , which in turn implies that the function  $f(\alpha)$  is monotonically increasing. And since  $f(0) = 0$ , it holds that  $f(\alpha) \geq 0$  for all  $\alpha \geq 0$ , which completes the proof. ■

Using the same notations as in the proof of Theorem C.8, we have

$$\begin{aligned}
 \mathbb{E} \left[ e^{\alpha(X-\mu)} \right] &= \prod_{i=1}^n \mathbb{E} \left[ e^{\alpha(X_i-p_i)} \right] && \text{(from the proof on page 1207)} \\
 &= \prod_{i=1}^n (e^{\alpha(1-p_i)} p_i + e^{\alpha(0-p_i)} q_i) \\
 &= \prod_{i=1}^n (p_i e^{\alpha q_i} + q_i e^{-\alpha p_i}) \\
 &\leq \prod_{i=1}^n e^{\alpha^2/2} && \text{(by Lemma ?? for } p = p_i \text{ and } q = q_i \text{)} \\
 &= \exp(\alpha^2 n/2).
 \end{aligned}$$

Then,

$$\begin{aligned}
 \Pr \{X - \mu \geq r\} &= \Pr \left\{ e^{\alpha(X-\mu)} \geq e^{\alpha r} \right\} && \text{(by equation (C.47))} \\
 &\leq \mathbb{E} \left[ e^{\alpha(X-\mu)} \right] e^{-\alpha r} && \text{(by inequality (C.48))} \\
 &\leq \exp(\alpha^2 n/2 - \alpha r).
 \end{aligned}$$

Now we need to choose the value of  $\alpha$  that minimizes the last expression. The argument of the exponential function in that expression is a quadratic function with respect to  $\alpha$ , so it's easy to show that it attains a minimum at  $\alpha = r/n$ . We finally get

$$\begin{aligned}
 \Pr \{X - \mu \geq r\} &\leq \exp((r/n)^2 n/2 - (r/n)r) \\
 &= e^{-r^2/2n}.
 \end{aligned}$$

- C.5-7** ★ The right-hand side of inequality (C.51) can be expressed as the function  $f(\alpha) = \exp(\mu e^\alpha - \alpha r)$ , where  $\alpha > 0$  and  $r > \mu > 0$ . We examine its derivatives:

$$\begin{aligned}
 f'(\alpha) &= (\mu e^\alpha - r) \exp(\mu e^\alpha - \alpha r), \\
 f''(\alpha) &= (\mu e^\alpha + (\mu e^\alpha - r)^2) \exp(\mu e^\alpha - \alpha r).
 \end{aligned}$$

$f'(\alpha) = 0$  for  $\alpha = \ln(r/\mu)$ , and  $f''(\alpha) > 0$  for all  $\alpha > 0$ . Therefore,  $f(\alpha)$  is minimized at  $\alpha = \ln(r/\mu)$ .

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**Problems**
**C-1      *The Monty Hall problem***

- a.** Let  $A$  be the event that the door we chose at first is the one with the automobile, and let  $W$  be the event of winning the automobile. Then  $\Pr\{A\} = 1/3$  and

$$\begin{aligned}\Pr\{W\} &= \Pr\{W \cap A\} + \Pr\{W \cap \bar{A}\} \\ &= \Pr\{A\} \Pr\{W | A\} + \Pr\{\bar{A}\} \Pr\{W | \bar{A}\} \quad (\text{by equation (C.16)}) \\ &= (1/3) \cdot \Pr\{W | A\} + (2/3) \cdot \Pr\{W | \bar{A}\}.\end{aligned}$$

Let's calculate the value of the above probability depending on the decision we made after Carol has revealed the first goat. If we decided to stick to our current choice,  $\Pr\{W | A\} = 1$  and  $\Pr\{W | \bar{A}\} = 0$ , so we win with probability  $\Pr\{W\} = 1/3$ . Otherwise, we switched, and then  $\Pr\{W | A\} = 0$  and  $\Pr\{W | \bar{A}\} = 1$ . In this case the probability of winning is  $\Pr\{W\} = 2/3$ . Thus, by choosing to switch, we double the chances of winning the prize.

- b.** In this game we make two decisions — first we choose one of the three doors, and then, after one of the doors have been opened, whether to stick or switch. There are therefore six outcomes, of which three correspond to winning the game. The results are detailed in Figure ??.

	stick	switch
the right door	$(1 - p_{\text{right}} p_{\text{switch}})/3$	$p_{\text{right}} p_{\text{switch}}/3$
the first wrong door	$(1 - p_{\text{wrong}} p_{\text{switch}})/3$	$p_{\text{wrong}} p_{\text{switch}}/3$
the second wrong door	$(1 - p_{\text{wrong}} p_{\text{switch}})/3$	$p_{\text{wrong}} p_{\text{switch}}/3$

**Figure C-1** The probability distribution in the Monty Hall's game. Rows represent our first choice and columns represent our second choice, and each cell contains the probability of the corresponding outcome. The cells shaded blue correspond to the winning outcomes.

- c.** By summing up the probabilities of the winning outcomes from the table in Figure ??, we get

$$\begin{aligned}p_{\text{win}} &= (1 - p_{\text{right}} p_{\text{switch}})/3 + p_{\text{wrong}} p_{\text{switch}}/3 + p_{\text{wrong}} p_{\text{switch}}/3 \\ &= \frac{1}{3}(2p_{\text{wrong}} p_{\text{switch}} - p_{\text{right}} p_{\text{switch}} + 1).\end{aligned}$$

**d.** In order to minimize our chances  $p_{\text{win}}$  of winning, given  $p_{\text{switch}} > 0$ , Monty needs to minimize the term  $2p_{\text{wrong}}p_{\text{switch}}$ , as well as maximize the term  $p_{\text{right}}p_{\text{switch}}$ . Therefore, his best strategy is  $p_{\text{right}} = 1$  and  $p_{\text{wrong}} = 0$ . With such values, our chances of winning are

$$\begin{aligned} p_{\text{win}} &= (1 - p_{\text{switch}})/3 \\ &< 1/3. \end{aligned}$$

**e.** If  $p_{\text{switch}} = 0$ , the winning probability reduces to  $p_{\text{win}} = 1/3$ , independently of any choice for  $p_{\text{right}}$  and  $p_{\text{wrong}}$ .

**f.** In order to maximize  $p_{\text{win}}$  for fixed  $p_{\text{right}}$  and  $p_{\text{wrong}}$ , we need to maximize the expression

$$2p_{\text{wrong}}p_{\text{switch}} - p_{\text{right}}p_{\text{switch}} = p_{\text{switch}}(2p_{\text{wrong}} - p_{\text{right}}). \quad (3)$$

If  $p_{\text{right}} \leq 2p_{\text{wrong}}$ , we just set  $p_{\text{switch}}$  to the maximum possible value of 1. Otherwise, the value of (??) is nonpositive, so by letting  $p_{\text{switch}} = 0$  we make it achieve 0.

**g.** As a function of  $p_{\text{right}}$  and  $p_{\text{wrong}}$ , the expression (??) is minimized, when  $p_{\text{right}} = 1$  and  $p_{\text{wrong}} = 0$ . Then,  $p_{\text{win}} = (1 - p_{\text{switch}})/3$ , so our best strategy to maximize  $p_{\text{win}}$  is to choose  $p_{\text{switch}} = 0$ .

**h.** Let  $M$  be the event that Monty gives us an opportunity to switch, and let  $A$  and  $W$  be the events of the same meaning as in the solution to part (a). Then,  $\Pr\{M | A\} = p_{\text{right}}$  and  $\Pr\{M | \bar{A}\} = p_{\text{wrong}}$ , and so

$$\begin{aligned} \Pr\{M\} &= \Pr\{A\} \Pr\{M | A\} + \Pr\{\bar{A}\} \Pr\{M | \bar{A}\} \\ &= (1/3)p_{\text{right}} + (2/3)p_{\text{wrong}}. \end{aligned}$$

Now let  $S$  be the event that we switched. Then,

$$\begin{aligned} W \cap M &= ((A \cap \bar{S}) \cup (\bar{A} \cap S)) \cap M \\ &= ((A \cap M) \cap \bar{S}) \cup ((\bar{A} \cap M) \cap S), \end{aligned}$$

and the events  $A \cap M$  and  $\bar{S}$  are independent, and so are the events  $\bar{A} \cap M$  and  $S$ .

Thus, by the definition (C.16) of conditional probability,

$$\begin{aligned}
 \Pr\{W | M\} &= \frac{\Pr\{W \cap M\}}{\Pr\{M\}} \\
 &= \frac{\Pr\{(A \cap M) \cap \bar{S}\} + \Pr\{(\bar{A} \cap M) \cap S\}}{\Pr\{M\}} \\
 &= \frac{\Pr\{A\} \Pr\{M | A\} \Pr\{\bar{S}\} + \Pr\{\bar{A}\} \Pr\{M | \bar{A}\} \Pr\{S\}}{\Pr\{M\}} \\
 &= \frac{(1/3)p_{\text{right}}(1 - p_{\text{switch}}) + (2/3)p_{\text{wrong}}p_{\text{switch}}}{(1/3)p_{\text{right}} + (2/3)p_{\text{wrong}}} \\
 &= \frac{p_{\text{right}} - p_{\text{right}}p_{\text{switch}} + 2p_{\text{wrong}}p_{\text{switch}}}{p_{\text{right}} + 2p_{\text{wrong}}}.
 \end{aligned}$$

The conditional probability  $\Pr\{W | M\}$  is defined whenever  $\Pr\{M\} \neq 0$ , which holds when  $p_{\text{right}} + 2p_{\text{wrong}} \neq 0$  or, equivalently, when  $p_{\text{right}} \neq 0$  or  $p_{\text{wrong}} \neq 0$ .

*i.* The value of the expression for  $p_{\text{switch}} = 1/2$  is

$$\begin{aligned}
 \Pr\{W | M\} &= \frac{p_{\text{right}}/2 + p_{\text{wrong}}}{p_{\text{right}} + 2p_{\text{wrong}}} \\
 &= 1/2.
 \end{aligned}$$

When  $p_{\text{switch}} < 1/2$ , Monty can select  $p_{\text{right}} = 0$  and any  $p_{\text{wrong}} > 0$ , so that

$$\begin{aligned}
 \Pr\{W | M\} &= \frac{2p_{\text{wrong}}p_{\text{switch}}}{2p_{\text{wrong}}} \\
 &= p_{\text{switch}} \\
 &< 1/2.
 \end{aligned}$$

Similarly, when  $p_{\text{switch}} > 1/2$ , Monty's best strategy is to choose any  $p_{\text{right}} > 0$  and  $p_{\text{wrong}} = 0$ , in which case

$$\begin{aligned}
 \Pr\{W | M\} &= \frac{p_{\text{right}} - p_{\text{right}}p_{\text{switch}}}{p_{\text{right}}} \\
 &= 1 - p_{\text{switch}} \\
 &< 1/2.
 \end{aligned}$$

*j.* In part (i) we showed that for any value of  $p_{\text{switch}}$  other than  $1/2$ , Monty can always find a strategy that will reduce our chances of winning, as compared to the situation when we choose  $p_{\text{switch}} = 1/2$ .

In this problem we studied different strategies for Monty and for the player. The analysis of the original problem showed that switching can actually double the player's chances of winning the game, which for most people may seem counterintuitive. We have seen that for any Monty's strategy, there is an optimal strategy that the player can follow in order to maximize their chances to win, given that they know the Monty's plan. And vice versa, knowing how the player will act, Monty can always reduce the player's chances.

**C-2*****Balls and bins***

**a.** We place each ball in one of the  $b$  bins. There are  $b$  ways to choose a bin for each ball. Thus, we can make  $n$  such choices in  $b^n$  ways.

**b.** We can view this problem as counting all possible arrangements of  $n$  distinguishable balls and  $b - 1$  indistinguishable sticks. Informally speaking, the sticks in such arrangements partition the balls into subsequences of balls that end up in different bins.

There are  $(b + n - 1)!$  of all such arrangements, but since we do not distinguish sticks, we divide this number by the number of permutations of the sticks,  $(b - 1)!$ , to obtain the number  $\frac{(b+n-1)!}{(b-1)!}$  of ways to place the balls in the bins in this variant.

**c.** Compared to the situation from part (b), here we are not distinguishing the balls, so every permutation of them—with the positions of the sticks unchanged—represents the same arrangement of the balls in the bins. Thus, we have  $\frac{(b+n-1)!}{n!(b-1)!} = \binom{b+n-1}{n}$  ways to place the balls.

**d.** Out of  $b$  bins we choose  $n$  that will contain a ball. There are  $\binom{b}{n}$  ways to do this.

**e.** First, we place a ball in each bin, so that none of the bins are empty. By part (c), we can place the remaining  $n - b$  balls in the  $b$  bins in  $\binom{b+(n-b)-1}{n-b} = \binom{n-1}{n-b} = \binom{n-1}{b-1}$  ways.

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## D Matrices

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### D.1 Matrices and matrix operations

**D.1-1** By the definition of a symmetric matrix,  $A = A^T$  and  $B = B^T$  or, equivalently, for  $i = 1, 2, \dots, n$  and  $j = 1, 2, \dots, n$ ,  $a_{ij} = a_{ji}$  and  $b_{ij} = b_{ji}$ . Let  $C = A + B$  and  $D = A - B$ . Then for all  $i = 1, 2, \dots, n$  and all  $j = 1, 2, \dots, n$ ,

$$\begin{aligned}c_{ij} &= a_{ij} + b_{ij} \\&= a_{ji} + b_{ji} \\&= c_{ji},\end{aligned}$$

and, similarly,

$$\begin{aligned}d_{ij} &= a_{ij} - b_{ij} \\&= a_{ji} - b_{ji} \\&= d_{ji},\end{aligned}$$

so  $C$  and  $D$  are symmetric.

**D.1-2** Suppose that  $A$  is a  $p \times q$  matrix and  $B$  is a  $q \times r$  matrix. We'll denote  $A^T = (a_{ij}^T) = (a_{ji})$  and  $B^T = (b_{ij}^T) = (b_{ji})$ . Let  $C = (AB)^T$ . Then,  $C$  is an  $r \times p$  matrix, such that for  $i = 1, 2, \dots, p$  and  $j = 1, 2, \dots, r$ ,

$$\begin{aligned}c_{ji} &= \sum_{k=1}^q a_{ik} b_{kj} \\&= \sum_{k=1}^q b_{jk}^T a_{ki}^T,\end{aligned}$$

so  $C = B^T A^T$ .

This fact implicates that

$$\begin{aligned}(A^T A)^T &= A^T (A^T)^T \\ &= A^T A,\end{aligned}$$

and so the matrix  $A^T A$  is symmetric.

**D.1-3** Let  $L' = (l'_{ij})$  and  $L'' = (l''_{ij})$  be  $n \times n$  lower-triangular matrices. Let  $L = L'L''$ . For  $1 \leq i < j \leq n$ ,

$$\begin{aligned}l_{ij} &= \sum_{k=1}^n l'_{ik} l''_{kj} \\ &= \sum_{k=1}^{j-1} l'_{ik} l''_{kj} + \sum_{k=j}^n l'_{ik} l''_{kj} \\ &= \sum_{k=1}^{j-1} l'_{ik} \cdot 0 + \sum_{k=j}^n 0 \cdot l''_{kj} \\ &= 0,\end{aligned}$$

so  $L$  is lower-triangular.

**D.1-4** Let  $PA = A' = (a'_{ij})$ . Suppose that for  $i = 1, 2, \dots, n$ , row  $i$  of the matrix  $P$  has 1 in column  $c_i$  (i.e.,  $p_{ic_i} = 1$ ). Then,

$$\begin{aligned}a'_{ij} &= \sum_{k=1}^n p_{ik} a_{kj} \\ &= p_{ic_i} a_{c_i j} \\ &= a_{c_i j}.\end{aligned}$$

Therefore, the element  $a'_{ij}$  in the matrix  $PA$  is an element from  $A$  from the same row but possibly from a different column. This means that  $PA$  is formed by permuting rows of  $A$ .

Similarly, let  $AP = A'' = (a''_{ij})$  and for  $j = 1, 2, \dots, n$  let  $r_j$  be the row number, such that  $p_{r_j j} = 1$ . Then,

$$\begin{aligned}a''_{ij} &= \sum_{k=1}^n a_{ik} p_{kj} \\ &= a_{ir_j} p_{r_j j} \\ &= a_{ir_j},\end{aligned}$$

which implies that  $AP$  is  $A$  with its columns permuted.

Now suppose that  $A$  is a permutation matrix. Permuting rows of  $A$  preserves the property of permutation matrices that each row and each column has exactly one 1, and 0s elsewhere. Therefore,  $PA$  is also a permutation matrix.

## D.2 Basic matrix properties

$$\begin{aligned} D.2-1 \quad B &= BI \\ &= B(AC) \\ &= (BA)C \\ &= IC \\ &= C \end{aligned}$$

- D.2-2** Let  $L$  be an  $n \times n$  lower-triangular matrix. We'll prove by induction on  $n$  that the determinant of  $L$  is the product of its diagonal elements. The base case is when  $n = 1$ , and from the definition of determinant we have  $\det(L) = l_{11}$ . Now suppose that  $n \geq 2$ . The minor  $L_{[11]}$  is a lower-triangular matrix, so let's assume as an inductive hypothesis, that  $\det(L_{[11]}) = \prod_{i=2}^n l_{ii}$ . Then we have

$$\begin{aligned} \det(L) &= \sum_{j=1}^n (-1)^{1+j} l_{1j} \det(L_{[1j]}) \\ &= (-1)^{1+1} l_{11} \det(L_{[11]}) \\ &= l_{11} \prod_{i=2}^n l_{ii} \\ &= \prod_{i=1}^n l_{ii}, \end{aligned} \tag{4}$$

which concludes the inductive proof.

Now let  $U$  be an  $n \times n$  upper-triangular matrix. There exists a lower-triangular matrix

$L$ , such that  $U = L^T$ . Then  $U$  and  $L$  have the same diagonal elements, and so

$$\begin{aligned}\det(U) &= \det(L^T) \\ &= \det(L) && \text{(by Theorem D.4)} \\ &= \prod_{i=1}^n l_{ii} && \text{(by equation (??))} \\ &= \prod_{i=1}^n u_{ii}.\end{aligned}$$

Now let  $L$  be a nonsingular  $n \times n$  lower-triangular matrix and let  $L' = (l'_{ij})$  be the inverse of  $L$ . We use induction to show that  $L'$  is a lower-triangular matrix. When  $n = 1$ ,  $L'$  is a  $1 \times 1$  matrix, which is trivially lower-triangular. For the inductive step, let  $n \geq 2$  and assume that the minor  $L'_{[11]}$  itself is lower-triangular. For  $j = 1, 2, \dots, n$ , the element in the first row and the  $j$ th column of  $I_n$  is equal to  $\sum_{k=1}^n l_{1k} l'_{kj} = l_{11} l'_{1j}$ . If  $j = 1$ , then  $l_{11} l'_{11} = 1$ , so  $l'_{11} = 1/l_{11} \neq 0$ . Note that the existence of the inverse of  $L$  implies  $l_{11} \neq 0$ . If  $j > 1$ ,  $l_{11} l'_{1j} = 0$ , so  $l'_{1j}$  has to be 0. Combining these results with the inductive hypothesis, we have that  $L'$  is lower-triangular.

### D.2-3

Suppose that  $P$  is an  $n \times n$  matrix. We'll prove that  $P$  is invertible by showing that  $P^T = (p_{ij}^T)$  is its inverse. Suppose that for  $i = 1, 2, \dots, n$ ,  $c_i$  is the number of column, such that  $p_{ic_i} = 1$ . Then  $p_{c_i i}^T = 1$  is the only nonzero element in column  $i$  of  $P^T$ .

If we let  $A = PP^T$ , then for every  $1 \leq i \leq n$  and  $1 \leq j \leq n$ , we have that

$$\begin{aligned}a_{ij} &= \sum_{k=1}^n p_{ik} p_{kj}^T \\ &= \sum_{k=1}^n p_{ik} p_{jk} \\ &= p_{ic_i} p_{jc_i} \\ &= p_{jc_i}.\end{aligned}$$

By the fact that  $P$  is a permutation matrix, we have that if  $i = j$ , then  $a_{ij} = 1$ , otherwise  $a_{ij} = 0$ . That is,  $A = I_n$ , and so  $P^T = P^{-1}$ .

In a permutation matrix  $P$  each row and each column has exactly one 1, and 0s elsewhere. The rows of  $P$  are the columns of  $P^T$ , and the columns of  $P$  are the rows of  $P^T$ , so the permutation matrix property is preserved in  $P^T$ .

### D.2-4

Let  $C$  be the  $n \times n$  matrix with  $c_{ij} = 1$ , and 0s elsewhere. Observe that  $CA$  is the matrix, in which the  $i$ th row is the  $j$ th row of  $A$ , while other rows consist of 0s. Similarly,  $BC$

is the matrix, in which the  $j$ th column is the  $i$ th column of  $B$ , while other columns consist of 0s. Let  $D = I + C$  and  $D' = I - C$ . Then,  $A' = DA$  and  $B' = BD'$ .

Observe, that since  $i \neq j$ , for any  $1 \leq p, q \leq n$  the terms  $c_{pk}$  and  $c_{kq}$  can't be both nonzero. Thus,  $\sum_{k=1}^n c_{pk}c_{kq} = 0$ , and so  $CC$  is a zero matrix. Then,

$$\begin{aligned} DD' &= (I + C)(I - C) \\ &= II + CI - IC - CC \\ &= I + C - C - CC \\ &= I, \end{aligned}$$

which means that  $D' = D^{-1}$ .

Returning to the matrices  $A'$  and  $B'$ , we get

$$\begin{aligned} A'B' &= DABD' \\ &= D(AB)D^{-1} \\ &= (DI)D^{-1} \\ &= DD^{-1} \\ &= I, \end{aligned}$$

hence  $B' = (A')^{-1}$ .

### D.2-5

We need only prove the forward direction, since the inverse of  $A^{-1}$  is  $(A^{-1})^{-1} = A$  and the backward direction is symmetric. Suppose that every entry of  $A^{-1} = (a_{ij}^{-1})$  is real. We'll use induction to show that every entry of  $A$  is real.

If  $n = 1$ ,  $A^{-1}$  has a single real entry  $a_{11}^{-1}$ , and the only entry of  $A$  is  $a_{11} = 1/a_{11}^{-1}$ , so it is real. Now let  $n \geq 2$  and assume that every entry of  $A_{[11]}$  is real. Suppose that there exists  $2 \leq i \leq n$ , such that  $a_{i1}$  is complex but not real. Since  $AA^{-1} = I_n$ , we have

$$\begin{aligned} 1 &= \sum_{k=1}^n a_{ik}a_{ki}^{-1} \\ &= a_{i1}a_{1i}^{-1} + \sum_{k=2}^n a_{ik}a_{ki}^{-1}. \end{aligned}$$

The term  $a_{i1}a_{1i}^{-1}$  is not real, and so can't be the last summation, in order to cancel out the imaginary components on the right-hand side and sum up to 1. This means that some other element from the  $i$ th row of  $A$  can't be real, which contradicts our assumption. Using the identity  $A^{-1}A = I_n$  and a similar reasoning we can show that all elements  $a_{1j}$ , where  $2 \leq j \leq n$ , are real. This fact implies, that in the following

equation

$$1 = a_{11}a_{11}^{-1} + \sum_{k=2}^n a_{1k}a_{k1}^{-1}$$

the last summation is a real number, so  $a_{11}$  has to be real as well.

**D.2-6**

*In the second part of the exercise we also need to assume that  $A$  is a symmetric  $n \times n$  matrix. Corrected in fourth printing.*

We have

$$\begin{aligned}(A^{-1})^T &= (A^T)^{-1} \\ &= A^{-1},\end{aligned}$$

so  $A^{-1}$  is symmetric.

In the second part, since  $A$  is symmetric,

$$\begin{aligned}(BAB^T)^T &= (B^T)^T A^T B^T \\ &= BAB^T,\end{aligned}$$

so the product  $BAB^T$  gives a symmetric matrix.

**D.2-7**

Let  $A$  be an  $m \times n$  matrix that has full column rank and let  $x$  be an  $n$ -vector such that  $Ax = 0$ . If we denote the columns of  $A$  by  $a_1, a_2, \dots, a_n$ , then the product  $Ax$  can be expressed as the linear combination  $\sum_{j=1}^n a_j x_j$ . Because the column rank of the matrix  $A$  is  $n$ , all the column vectors are linearly independent, so  $x_j$  — viewed as coefficients from the definition of linear dependency — are all 0. Therefore,  $x = 0$ .

For the other direction, assume that for an  $m \times n$  matrix  $A$ , the condition  $Ax = 0$  implies  $x = 0$ , and that the matrix  $A$  has the rank less than  $n$ . This means that there is some set of column vectors of  $A$  which are linearly dependent. Note that we can extend this set to include all column vectors of  $A$ ,  $a_1, a_2, \dots, a_n$ , and they all will remain linearly dependent. Let  $c_1, c_2, \dots, c_n$  be coefficients, not all of which are 0, such that  $\sum_{j=1}^n a_j c_j = 0$ . If we consider the  $n$ -vector  $c = (c_j)$ , then the equation can be viewed as  $Ac = 0$ . But  $c \neq 0$ , so we have a contradiction.

**D.2-8**

Let  $A$  be an  $m \times p$  matrix and let  $B$  be  $p \times n$  matrix. From the alternate definition of the rank,  $r = \text{rank}(AB)$  is the smallest number such that  $AB = CD$  for some matrices  $C$  and  $D$  of respective sizes  $m \times r$  and  $r \times n$ . Let  $r' = \text{rank}(A)$  and  $r'' = \text{rank}(B)$  and let  $C', D', C'', D''$  be matrices of respective sizes  $m \times r'$ ,  $r' \times p$ ,  $p \times r''$ ,  $r'' \times n$  such that  $A = C'D'$  and  $B = C''D''$ . Since  $AB = (C'D'C'')D''$  and  $C'D'C''$

is an  $m \times p$  matrix, we have that  $r \leq r'$ . Similarly, if we represent the product as  $AB = C'(D'C''D'')$ , we conclude that  $r \leq r''$ . Therefore,  $r \leq \min\{r', r''\}$ .

Now consider a special case, where  $p = m$  and  $A$  is nonsingular. Let  $r, r', r'', C$  and  $D$  have the same meaning as before. Of course,  $r'' \leq \min\{m, n\} \leq m$  and, by Theorem D.1,  $r' = m$ . Also,

$$\begin{aligned} B &= A^{-1}AB \\ &= (A^{-1}C)D, \end{aligned}$$

and since  $A^{-1}C$  is an  $m \times r$  matrix, we have that  $r'' \leq r$ . Combining this with the inequality  $r \leq r''$  shown in the proof for a general case, we obtain  $r = r'' = \min\{r', r''\}$ . The proof for the case in which  $p = n$  and  $B$  is nonsingular, is symmetric.

## Problems

### D-1

#### Vandermonde matrix

The proof is by induction on  $n$  and is based on the fact that if we add to a column of a matrix the product by a scalar of another column, then the determinant remains unchanged. This fact can be devised from Theorem D.4—first multiply a column by a scalar, then add this column to another column, then divide the original column by the scalar.

For the base case of the induction, let  $n = 1$ . Then the single entry of  $V(x_0)$  is 1. Clearly,  $\det(V(x_0)) = 1$ , that matches the empty product  $\prod_{0 \leq j < k \leq 0} (x_k - x_j)$ , by convention equal to 1.

For the inductive step, let  $n \geq 2$ . As the hint suggests, for  $i = n-1, n-2, \dots, 1$ , we will multiply column  $i$  by  $-x_0$  and add it to column  $i+1$ , to obtain the matrix  $W$  shown below:

$$\left( \begin{array}{cccccc} 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & x_1 - x_0 & x_1(x_1 - x_0) & x_1^2(x_1 - x_0) & \cdots & x_1^{n-2}(x_1 - x_0) \\ 1 & x_2 - x_0 & x_2(x_2 - x_0) & x_2^2(x_2 - x_0) & \cdots & x_2^{n-2}(x_2 - x_0) \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n-1} - x_0 & x_{n-1}(x_{n-1} - x_0) & x_{n-1}^2(x_{n-1} - x_0) & \cdots & x_{n-1}^{n-2}(x_{n-1} - x_0) \end{array} \right).$$

By the definition of determinant,  $\det(W) = \det(W_{[11]})$ . As all the entries in the  $i$ th row of  $W_{[11]}$  have a factor of  $x_i - x_0$ , using Theorem D.4 we can take these factors out

and obtain

$$\begin{aligned}
 \det(V(x_1, x_2, \dots, x_{n-1})) &= \det(W) \\
 &= \det(W_{[11]}) \\
 &= \prod_{i=1}^{n-1} (x_i - x_0) \cdot \det(V(x_1, x_2, \dots, x_{n-1})) \\
 &= \prod_{i=1}^{n-1} (x_i - x_0) \prod_{1 \leq j < k \leq n-1} (x_k - x_j) \\
 &= \prod_{0 \leq j < k \leq n-1} (x_k - x_j).
 \end{aligned}$$

## D-2

### *Permutations defined by matrix-vector multiplication over GF(2)*

**a.** Let  $a_0, a_1, \dots, a_{n-1}$  be the columns of  $A$  and let  $J \subseteq \{0, 1, \dots, n-1\}$  be the set of indices of all linearly independent columns of  $A$ . Consider vectors  $x, x' \in S_n$ , such that there exists  $j \in J$  for which  $x_j \neq x'_j$ , and that for all  $j \in \{0, 1, \dots, n-1\} - J$ ,  $x_j = x'_j = 0$ . Then,

$$\begin{aligned}
 Ax - Ax' &= A(x - x') \\
 &= \sum_{j=0}^{n-1} a_j (x_j - x'_j) \\
 &= \sum_{j \in J} a_j (x_j - x'_j) \\
 &\neq 0,
 \end{aligned}$$

so  $Ax \neq Ax'$ . Because  $|J| = r$ , there are  $2^r$  ways of picking the vector  $x$ , and each such  $x$  is mapped to a distinct value  $Ax$ . Therefore,  $|R(A)| \geq 2^r$ .

To get the upper bound, observe that any column  $a_j$ , where  $j \notin J$ , is in fact a linear combination of the  $r$  linearly independent columns of  $A$ . Thus, for any vector  $x$ ,

$$\begin{aligned}
 Ax &= \sum_{j=0}^{n-1} a_j x_j \\
 &= \sum_{j \in J} a_j x'_j
 \end{aligned}$$

for some coefficients  $x'_j$ . In other words, for any  $x$  there exists a vector  $x'$ , with zeros on positions  $j \notin J$ , mapped to the same value as  $x$ . There are  $2^r$  ways of choosing

such  $x'$ , so  $|R(A)| \leq 2^r$ .

If  $\text{rank}(A) < n$ , then  $|R(A)| < 2^n = |S_n|$ , so  $A$  can't define a permutation on  $S_n$ .

**b.** We showed in part (a), that the value produced by multiplying a vector  $x$  by  $A$  does not depend on the  $n - r$  entries of  $x$  occupying the positions corresponding to the linear dependent columns of  $A$ . Therefore, for any  $y \in R(A)$ ,  $|P(A, y)| \geq 2^{n-r}$ .

On the other hand, there are  $2^r$  elements in  $R(A)$ , each with the preimage consisting of at least  $2^{n-r}$  elements. This means, there are at least  $2^r \cdot 2^{n-r} = 2^n$  elements in all preimages in total. Since  $|S_n| = 2^n$ , each preimage must have exactly  $2^{n-r}$  elements.

**c.** First observe that  $B(S', m)$  consists of size- $2^m$  blocks which contain a value produced by any  $x \in S$ . Since the first  $m$  rows of  $A$  only affect the first  $m$  entries of  $Ax$ , they can affect the value of  $Ax$  by at most  $2^m - 1$ . The block where  $Ax$  ends up is therefore determined by the last  $n - m$  rows of  $A$ . What is more, the numbers in block  $i$  differ from the numbers in block 0 by  $i2^m$ , therefore the produced values are also shifted by a constant offset. Thus, without loss of generality, we may assume that  $S$  is block 0, so that only the first  $m$  columns of  $A$  are relevant during multiplication.

By similar reasoning as in part (a), only the linearly independent rows of the lower left  $(n-m) \times m$  submatrix of  $A$  can determine the block where  $Ax$  will end up, since the entries of  $Ax$  on the positions that correspond to the other rows are linear combinations of the other entries of  $Ax$ . Thus,  $Ax$  spans  $2^r$  blocks, so  $|B(S', m)| = 2^r$ .

Similarly, we could identify the  $r$  linearly independent columns of the submatrix that decide on the block for  $Ax$ . A given block is uniquely chosen by a combination of the  $r$  entries of  $x$  that are multiplied by the elements of these columns, while the remaining  $m - r$  entries (besides zeros at positions  $m, m + 1, \dots, n - 1$ ) can be arbitrary. Thus, each block must be hit by the same number of times, equal to  $2^{m-r}$ .

**d.** The number of linear permutations is bounded above by the number of pairs  $(A, c)$ , where  $A$  is an  $n \times n$  0-1 matrix and  $c$  is an  $n$ -bit vector. There are  $2^{n^2} \cdot 2^n = 2^{n(n+1)}$  of such pairs. On the other hand, there are  $(2^n)!$  permutations of  $S_n$ . For  $n \geq 3$ ,  $(2^n)^{n+1} \leq (2^n)!$ .

**e.** Observe that  $A \cdot 0 + c = c$ , and  $Ax = \sum_{j=0}^{n-1} a_j x_j$ , where  $a_0, a_1, \dots, a_{n-1}$  are the columns of  $A$ , so if we let  $x = 2^i$ , then  $Ax = a_i$ . Therefore, the linear permutation induced by the pair  $(A, 0)$  maps  $x = 0$  to 0, and maps  $x = 2^i$  to the number whose binary representation is the  $i$ th column of  $A$ .

Consider a permutation  $\pi_{A,c}$  of  $S_2$ , such that  $\pi_{A,c}(x) = x$ , for  $x \in \{0, 1, 2\}$ . Based on the above observation, this is enough to determine both the matrix  $A$  and the vector  $c$ :  $A = I$ ,  $c = 0$ . These in turn determine the value  $\pi_{A,c}(3) = 3$ , which makes  $\pi_{A,c}$

the identity permutation. Therefore, any permutation of  $S_2$  that maps each  $x \in \{0, 1, 2\}$  to  $x$  and 3 to anything else than 3, is impossible to achieve by any linear permutation.