1. **(8 points)** Let V be the vector space of 2×2 matrices with real entries, and P_3 the vector space of real polynomials of degree 3 or less. Define the linear transformation $T : V \to P_3$ by

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = 2a + (b - d)x - (a + c)x^2 + (a + b - c - d)x^3.$$

   Find the rank and nullity of T.

2. **(5 points)** Let A be an $n \times n$ matrix with real entries and n is odd. Show that it is not possible for $A^2 + I = O$, in which I is the identity matrix and O is the zero matrix.

3. Let $C[-1, 1]$ be the vector space over R of all continuous functions defined on the interval $[-1, 1]$. Let $V : \{f(x) \in C[-1,1] | f(x) = ae^x + be^{2x} + ce^{3x}, a, b, c \in R\}$.

   a. **(2 points)** Prove that V is a subspace of $C[-1, 1]$.

   b. **(5 points)** Prove that $B = \{e^x, e^{2x}, e^{3x}\}$ is a basis of V.

   c. **(5 points)** Prove that $B' = \{e^x - 2e^{3x}, e^x + e^{2x} + 2e^{3x}, 3e^{2x} + e^{3x}\}$ is a basis of V.

4. Given $A = \begin{bmatrix} 1 & -3 & -5 \\ 1 & 1 & -2 \\ 1 & -3 & 1 \\ 1 & 1 & 4 \end{bmatrix}$ and $b = \begin{bmatrix} -6 \\ 1 \\ 1 \\ 6 \end{bmatrix}$.

   If the Gram-Schmidt process is applied to determine an orthonormal basis for $R(A) = \{b \in R^m | b = A_{mn}x\}$ and QR factorization of A, then, after the first one orthonormal vector q_1 and r_{11} are computed, we have

   $$Q = [q_1 \quad q_2 \quad q_2] = \begin{bmatrix} 0.5 & - & - \\ 0.5 & - & - \\ 0.5 & - & - \\ 0.5 & - & - \end{bmatrix} \text{ and } R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & r_{33} \end{bmatrix} = \begin{bmatrix} 2 & - & - \\ 0 & - & - \\ 0 & - & - \end{bmatrix}.$$

   a. **(5 points)** Finish above process and determine q_2 and q_3, and fill in the columns of Q.

   b. **(5 points)** Finish above process and determine R.

   c. **(5 points)** Use the QR factorization to find the least squares solution to $Ax = b$.

5. Let $C = \begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & -2 \\ 2 & -1 & -1 \end{bmatrix}$ is a 3x3 matrix.

   a. **(5 points)** Find the value of $\lim_{n \to \infty} C^n$.

   b. **(5 points)** Compute the value of e^C.

6. Let $Q(x)$ be the statement "$x + 1 > 2x$." If the domain consists of all integers, what are these truth values?

   a. **(2 points)** $\exists x \, Q(x)$

   b. **(2 points)** $\forall x \, Q(x)$

   c. **(2 points)** $\exists x \, \neg Q(x)$

   d. **(2 points)** $\forall x \, \neg Q(x)$

7. Determine whether each of the following conditional statements is a tautology or not. If yes, provide a proof. If no, provide a counter example.

   a. **(3 points)** $p \to (\neg q \lor r)$

   b. **(3 points)** $\neg p \to (q \to r)$

   c. **(3 points)** $(p \to q) \lor (\neg p \to r)$

8. Determine whether each of these functions is a bijection from R to R.

   a. **(2 points)** $f(x) = -3x + 4$

   b. **(2 points)** $f(x) = -3x^2 + 7$

   c. **(2 points)** $f(x) = (x + 1)/(x + 2)$

   d. **(2 points)** $f(x) = x^5 + 1$

國立交通大學 108 學年度碩士班考試入學招生試題

科目：線性代數與離散數學(1102)　　　　考試日期：108 年 2 月 13 日　第 2 節
系所班別：資訊聯招　　　　　　　　　　第 2 頁,共 2 頁
【不可使用計算機】＊作答前請先核對試題、答案卷(試卷)與准考證之所組別與考科是否相符！！

9. Let R and B be sets of red and blue balls, respectively, with $m = |R|$ and $n = |B|$. Suppose that $m < n$. Let $G = (V, E)$ be a undirected graph with vertex set V and edge set E such that $V = R \cup B$ and $E = \{(u, v) \mid u \in R$ and $v \in B\}$. Let E' be an edge cut of G with the minimum number of edges.
   a. (4 points) What is the value of $|E'|$?
   b. (6 points) Let $G' = G - E'$. What is the value of $\sum_{v \in G'} \deg(v)$, where $\deg(v)$ is the node degree of v?

10. (5 points) Let S be the set of all bit strings of length n. Let $\angle$ be a binary relation defined on S such that $a \angle b$ if and only if a and b differ in exactly k bit positions for any $a, b \in S$. Let $R = \{(a, b) \mid a \angle b, a, b \in S\}$. If we represent R using a zero-one matrix, how many 1's are there in the matrix?

11. A DNA sequence of length n is a sequence of n molecules, where each molecule is represented by either 'C', 'G', 'A', or 'T'. Consider a sequence $a_1, a_2, \ldots, a_n$, where a_i, $1 \le i \le n$, denotes the number of DNA sequences of length i that contain two consecutive 'G's. We may express a_n as a recurrence relation
$$a_n = s \times a_{n-1} + t \times a_{n-2} + f(n),$$
where s and t are positive integers and $f(n)$ is a function of n.
   a. (3 points) $a_4 = ?$
   b. (3 points) What is the value of $s + t$?
   c. (4 points) What is the definition of $f(n)$?