國立交通大學 101 學年度碩士班考試入學試題

科目：線性代數與機率(1003)　　考試日期:101 年 2 月 16 日 第 2 節

系所班別：資訊聯招　　第 1 頁,共 4 頁

【不可使用計算機】＊作答前請先核對試題、答案卷(試卷)與准考證之所組別與考科是否相符！！

複選題，每題五分。該題答對一個選項得 1 分，答錯一個選項倒扣 1 分，該題分數扣完為止，整題未作答不給分。 選擇題請使用答案卡作答

1. Consider a 3×3 matrix $\mathbf{A}$ with *non-zero* elements. Assume $\mathbf{A}^T\mathbf{A}\mathbf{x}=\mathbf{0}$ has *non-trivial* solutions with two of them being $\mathbf{x}=[1,1,0]^T$ and $[1,0,0]^T$. Which of the following statements are *true* for $\mathbf{b},\mathbf{y}\in R^3$?
   (A) $\mathbf{A}\mathbf{y}=\mathbf{b}$ must have no solution for some $\mathbf{b}$ and exactly one solution for other $\mathbf{b}$.
   (B) $\mathbf{A}^T\mathbf{y}=[0,0,5]^T$ must have at least one exact solution.
   (C) The least-squares approximation error for the system $\mathbf{A}^T\mathbf{y}=[0,2,2]^T$ is 4.
   (D) Rank of $\mathbf{A}$ =2.
   (E) $\mathbf{A}$ must have zero rows.

2. Suppose $\mathbf{A}=\begin{bmatrix}1&2&1&b\\1&a&0&5\\0&c&0&d\end{bmatrix}$ has a *reduced row echelon* form $\mathbf{U}=\begin{bmatrix}1&2&1&2\\0&0&1&3\\0&0&0&0\end{bmatrix}$. Which of the following statements are *true* for $\mathbf{x},\mathbf{b}\in R^3$?
   (A) $\mathbf{A}\mathbf{x}=[3,1,0]^T$ has infinitely many solutions.
   (B) $\mathbf{A}\mathbf{x}=[0,1,0]^T$ has no solution.
   (C) In this case, Column Space of $\mathbf{A}$ = Column Space of $\mathbf{U}$.
   (D) The matrix $\mathbf{A}^T\mathbf{A}$ is invertible.
   (E) The projection of $\mathbf{x}=[2,1,1]^T$ onto the Column Space of $\mathbf{A}$ is the vector $[2,1,0]^T$.

3. Suppose $\mathbf{D}=\mathbf{A}\mathbf{B}$ with $\mathbf{A}=\begin{bmatrix}1&1&3\\1&2&4\\1&3&a\end{bmatrix}$ and $\mathbf{B}=\begin{bmatrix}1&0&1\\0&1&1\\1&1&b\end{bmatrix}$. Which of the following statements are *true?*
   (A) Null Space of $\mathbf{D}$ = Null Space of $\mathbf{B}$ *if and only if* $a\neq5$.
   (B) Column Space of $\mathbf{D}$ = Column Space of $\mathbf{A}$ *if and only if* $b\neq2$.
   (C) There exists values of a, b such that Rank of $\mathbf{D}$ = 1.
   (D) Rank of $\mathbf{D}=2$ *only if* $a=5$ and $b=2$.
   (E) Rank of $\mathbf{D}=3$ *only if* $a\neq5$ and $b\neq2$.

4. Which set is *not* a vector space?
   (A) The set of polynomials of degree exactly 3.
   (B) The set of polynomials having degree 0, 1, 2, or 3, together with the 0-polynomial.
   (C) The set of all vectors (x_1, x_2, x_3, x_4) in R^4 such that $x_2 = 5x_3 - 7x_4$.
   (D) The set of all vectors of the form $[a+3b, a-3b, 2a]$ in R^3.
   (E) The set of all vectors $[a+b^3, a-b^3]$ in R^2.

5. In a vector space V, we have the two sets $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ and $Q = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$. Which of these statements can be true?
   (A) S is linearly independent, Q spans V, and $n > m$.
   (B) V has dimension n, Q spans V, and $m \geq n$.
   (C) S and Q are bases for V, and $n = m$.
   (D) S is linearly independent, Q is a subset of S *and* Q is linearly dependent.
   (E) S is linearly dependent, Q is a subset of S *and* Q is linearly independent.

6. Consider $\mathbf{A} = \begin{bmatrix} 1 & 3 & 2 & 12 & 4 \\ -3 & -9 & 6 & 24 & 7 \\ 2 & 6 & 3 & 19 & 1 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 4 & 2 & 12 & 3 & 1 \\ 7 & 6 & 24 & -9 & -3 \\ 1 & 3 & 19 & 6 & 2 \end{bmatrix}$. Each is obtained from the other by rearranging columns. Which statement is correct?
   (A) Ker(**A**) = Ker(**B**).
   (B) Range(**A**) = Range(**B**).
   (C) **A** and **B** have the same reduced row echelon form.
   (D) Column space of **A** = Column space of **B**.
   (E) Row space of **A** = Row space of **B**.

7. Let **A** be an $m \times n$ matrix whose null space has dimension k. Which conclusion is correct?
   (A) The dimension of Null($\mathbf{A}^T$) is k.
   (B) The dimension of row space of **A** is m-k.
   (C) The dimension of column space of **A** is m-k.
   (D) The dimension of row space of **A** is n-k.
   (E) The dimension of column space of **A** is n-k.

8. Let matrices **A**, **B**, and **C** be square matrices. Choose the incorrect arguments
   (A) The determinant det(**AB**) = det(**A**)det(**B**)
   (B) Let matrix **A** can be decomposed into **A=QR**, where **Q** is the orthogonal matrix and **R** is the upper triangular matrix. The determinant det(**A**) = det(**R**).
   (C) Let matrix **A** be diagonalizable; that is, $\mathbf{A}=\mathbf{X}^{-1}\mathbf{D}\mathbf{X}$. The determinant det(**A**)=det(**D**).
   (D) Let matrices **A** and **B** are similar. The determinant det(**A**)=det(**B**).
   (E) Let matrix **B** be the Hermitian transpose of matrix **A**; that is $\mathbf{B}=\mathbf{A}^{\mathbf{H}}$. The determinant det(**A**)=det(**B**).

9. Choose the incorrect arguments.
   (A) Let **A** be the Hermitian matrix. Then, matrix **A** is diagonalizable; that is, $\mathbf{A}=\mathbf{X}^{-1}\mathbf{D}\mathbf{X}$.
   (B) For a square matrix **A**, the eigenvectors correspond to different eigenvalues are linearly independent.
   (C) Two similar matrices have the same characteristic polynomial.
   (D) Let **A** be an *mxn* real matrix. Then $\mathbf{A}^{\mathbf{T}}\mathbf{A}$ is diagonalizable.
   (E) Let matrix **A** be diagonalizable. Then, matrix **A** is not singular.

10. Let **A** be an *nxn* matrix and **b** be an arbitrary *nx1* vector. Select the incorrect arguments.

(A) If determinant det(**A**) $\neq$ 0, then the system equation **Ax=b** has exactly one single solution.

(B) If matrix **A** is diagonalizable, then the system equation **Ax=b** has exactly one single solution.

(C) Let matrix **A** can be decomposed into **A=QR**, where **Q** is the orthogonal matrix and **R** is the upper triangular matrix. The system equation **Ax=b** has exactly one single solution.

(D) Let matrix **A** have *n* independent eigenvectors. The matrix A should have *n* distinct eigenvalues.

(E) The system equation **Ax=b** can be solved by Cramer's rule either the system is consistent or inconsistent.

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非選擇題請用答案卷作答

11. A stack of cards consists of seven red and five blue cards. A second stack of cards consists of nine red cards. A stack is selected at random and three of its cards are drawn. If all of them are red, what is the probability that the first stack was selected? (Show your calculation steps.) (10 points)

12. (a) If two fair dice are rolled 4 times, what is the probability of at least one 6 (on either die) in exactly two of these 4 rolls? (5 points)

(b) Suppose that, on average, in every two pages of a book there is one typographical error. What is the probability of at least one error on a specific page of the book? (5 points)

13. Let the density function of X be

$$f(x) = \begin{cases} e^{-x} & \text{if } x > 0 \\ 0 & \text{elsewhere.} \end{cases}$$

Find the density functions of $Y=X\sqrt{X}$ and $Z=e^{-2X}$. (10 points)

14. Let the joint probability density function of X and Y be given by

$$f(x,y) = \begin{cases} c/x & \text{if } 0 < y < x, 0 < x < 2 \\ 0 & \text{elsewhere.} \end{cases}$$

(a) Determine the value of c. (4 points)

(b) Find the marginal probability density functions of Y. (6 points)

15. (a) If X is a normal random variable with parameters $\mu = 3$ and $\sigma^2 = 16$, find P(1 < X < 11). (4 points)

(b) If 48 random numbers are selected independently from the interval (0, 1), what is the approximate probability that the sum of these numbers is at least 20? (6 points)

(Express your solutions in (a) and (b) by using the standard normal distribution function Φ.)