國立臺灣大學95學年度碩士班招生考試試題

※ 注意：請於試卷上「選擇題作答區」依序作答。

Each problem may have **multiple** answers. To get credits you must correctly select all answers. For example, if answers are ABC, you get zero point no matter you choose A, BC, DE, ABCDE, or whatever else different from ABC.

Note that we consider only real-valued matrices.

1. (3%) Which of the followings are correct?

   (A) The determinant of
   $$\begin{bmatrix} 2 & 1 & 3 & 4 & 0 \\ 1 & 5 & 2 & 0 & 0 \\ 1 & 0 & 0 & 3 & 0 \\ 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$
   is 130.

   (B) The determinant of
   $$\begin{bmatrix} 2 & 1 & 3 & 4 & 0 \\ 1 & 5 & 2 & 0 & 0 \\ 1 & 0 & 0 & 3 & 0 \\ 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$
   is 140

   (C) The determinant of
   $$\begin{bmatrix} 2 & 1 & 3 & 4 \\ 1 & 5 & 2 & 0 \\ 1 & 0 & 0 & 3 \\ 2 & 0 & 0 & 1 \end{bmatrix}$$
   is 65.

   (D) The determinant of
   $$\begin{bmatrix} 2 & 1 & 3 & 4 \\ 1 & 5 & 2 & 0 \\ 1 & 0 & 0 & 3 \\ 2 & 0 & 0 & 1 \end{bmatrix}$$
   is 70.

接背面

(E) The determinant of

$$\begin{bmatrix} 2 & 1 & 3 & 4 \\ 1 & 5 & 2 & 0 \\ 1 & 0 & 0 & 3 \\ 2 & 0 & 0 & 1 \end{bmatrix}$$

is 80.

2. (3%) Let

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 5 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \qquad (1)$$

Which of the followings are correct?

(A) $u_{11} = 1$

(B) $u_{12} = 2$

(C) $l_{21} = 2$

(D) It is impossible to have (1), so no way to have u_{11}, u_{12}, etc.

(E) $u_{33} = 0$

3. (3%) Let $\det(A)$ be the determinant of the matrix $A \in R^{n \times n}$. Which of the followings are correct?

(A) $\det(A) \neq 0 \Leftrightarrow A$ invertible

(B) $\det(A^T) = \det(A)$

(C) $\det(AB) = \det(A)\det(B)$

(D) If A is invertible, then $\det(A^{-1}) = \det(A)^{-1}$

(E) $\det(cA) = c^{n-1}\det(A)$

4. (3%) What is the solution of

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 2 & 1 & 2 \\ 3 & 1 & 1 & 3 \\ 1 & 2 & 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 0 & 0 & 0 & 0 \\ 9 & 10 & 11 & 12 \end{bmatrix} + \begin{bmatrix} 1 & -2 & -3 & -1 \\ -2 & 2 & 1 & -2 \\ -3 & 1 & 1 & 3 \\ -1 & -2 & 3 & -2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 0 & 0 & 0 & 0 \\ 9 & 10 & 11 & 12 \end{bmatrix} ?$$

接次頁

(A) $\begin{bmatrix} 2 & 4 & 6 & 8 \\ 20 & 24 & 28 & 32 \\ 64 & 72 & 80 & 88 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ (B) $\begin{bmatrix} 2 & 4 & 6 & 8 \\ 20 & 24 & 28 & 32 \\ 64 & 72 & 88 & 80 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ (C) $\begin{bmatrix} 2 & 4 & 6 & 8 \\ 20 & 24 & 28 & 32 \\ 64 & 72 & 88 & 80 \\ 0 & 0 & 4 & 0 \end{bmatrix}$

(D) $\begin{bmatrix} 2 & 4 & 6 & 8 \\ 20 & 24 & 28 & 32 \\ 64 & 76 & 88 & 80 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ (E) $\begin{bmatrix} 2 & 4 & 6 & 8 \\ 20 & 20 & 28 & 32 \\ 60 & 76 & 88 & 88 \\ 0 & 0 & 4 & 0 \end{bmatrix}$

5. (3%) A $m \times n$ matrix A is full rank if $\text{rank}(A) = \min(m, n)$. Which of the followings are correct?

   (A) A full rank $\Rightarrow AA^T$ invertible; AA^T invertible $\not\Rightarrow A$ full rank

   (B) A full rank $\not\Rightarrow AA^T$ invertible; AA^T invertible $\Rightarrow A$ full rank

   (C) A full rank $\not\Rightarrow AA^T$ invertible; AA^T invertible $\not\Rightarrow A$ full rank

   (D) A full rank $\Rightarrow AA^T$ invertible; AA^T invertible $\Rightarrow A$ full rank

   (E) All AA^T's eigenvalues are strictly positive

6. (3%) Which of the followings are correct?

   (A) If A and B are invertible, then so is AB

   (B) If A is invertible, then so is A^T

   (C) Eigenvalues of a triangular matrix are its diagonal elements

   (D) It is impossible to have real-valued matrix A such that $A^2 = -I$

   (E) It is possible that $AB \neq BA$

7. (3%) We decide to use a matrix storing all web connections. If web i has n out-links and j is one site that it connects to, then we put the ij element to be $1/n$. Otherwise the ij element is zero. If $n = 0$, then the ij element is zero. Which of the followings are incorrect?

   (A) Zero rows are possible as some pages have no out-link

   (B) Zero columns are possible as some pages are never linked

   (C) The rank of this matrix > (# total web pages -1)

(D) Sum of each row is 0 or 1

(E) Sum of each column is 1

8. (3%) Which of the followings are correct?

   (A) If $A, B \in R^{m\times k}$, then calculating $A+B$ costs $O(mk)$.

   (B) If $A \in R^{m\times k}, B \in R^{k\times n}$, then calculating AB costs $O(nmk)$.

   (C) If $A \in R^{m\times m}$ is invertible, using Gaussian elimination to find A^{-1} costs $O(m^3)$.

   (D) If $A \in R^{m\times k}, B \in R^{k\times m}, \mathbf{u} \in R^{m\times 1}$ and $k \ll m$, then calculating $(AB)\mathbf{u}$ costs less than $A(B\mathbf{u})$

   (E) If $A \in R^{m\times k}, B \in R^{k\times m}, \mathbf{u} \in R^{m\times 1}$ and $k \ll m$, then calculating $(AB)\mathbf{u}$ needs more storage than $A(B\mathbf{u})$

9. (3%) Given

$$A = \begin{bmatrix} 0.6 & 0.4 & 0.3 \\ 0.4 & 0.9 & 0.2 \\ 0.3 & 0.2 & 0.8 \end{bmatrix}.$$

   What is the sum of all A's eigenvalues? (A) 2.7 (B) 2.1 (C) 2.3 (D) 2.4 (E) 2.5

10. (3%) Let $A \in R^{n\times n}, U, V \in R^{n\times k}$. What is

$$(A+UV^T)^{-1}?$$

   Assume A and $(I+V^TA^{-1}U)$ are invertible.

   (A) $A^{-1}U(I+V^TA^{-1}U)^{-1}V^TA^{-1}$

   (B) $A^{-1}-A^{-1}U(I+V^TA^{-1}U)^{-1}V^TA^{-1}$

   (C) $A^{-1}-A^{-1}(I+V^TA^{-1}U)^{-1}V^TA^{-1}$

   (D) $A^{-1}-A^{-1}U(I+V^TA^{-1}U)^{-1}UV^TA^{-1}$

   (E) $A^{-1}-A^{-1}UV^T(I+V^TA^{-1}U)^{-1}V^TA^{-1}$

11. (3%) A palindrome is a sequence of symbols that reads the same left to right as right to left (e.g., ABCCBA and ABCBA). The number of length-5 palindromic strings, where each character is from { A, B, . . . , E }, is

接次頁

(A) 15
(B) 120
(C) 125
(D) 100
(E) 240

12. (3%) How many permutations of AOABOBEB are there?

(A) 1680
(B) 210
(C) 2520
(D) 20160
(E) 5040

13. (3%) Which of the following is $(x+x^{-1})^6$ equal to?

(A) $x^6+6x^4+15x^2+30+15x^{-2}+6x^{-4}+x^{-6}$
(B) $x^6+6x^4+15x^2+28+15x^{-2}+6x^{-4}+x^{-6}$
(C) $x^6+6x^4+15x^2+25+15x^{-2}+6x^{-4}+x^{-6}$
(D) $x^6+6x^4+15x^2+24+15x^{-2}+6x^{-4}+x^{-6}$
(E) $x^6+6x^4+15x^2+20+15x^{-2}+6x^{-4}+x^{-6}$

14. (3%) The number of nonnegative *integer* solutions of $x_1+x_2+x_3+x_4=3$ is

(A) 20
(B) 10
(C) 15
(D) 24
(E) 6

15. (3%) How many functions are there from A to B, where $|A|=m$ and $|B|=n$?

國立臺灣大學95學年度碩士班招生考試試題

(A) m^n

(B) $\binom{n}{m}$

(C) $\binom{m}{n}$

(D) n^m

(E) 2^{nm}

16. (3%) Relation $\mathcal{R}$ is symmetric if $(x, y) \in \mathcal{R} \Rightarrow (y, x) \in \mathcal{R}$ for all $x, y \in A$. If $|A| = 4$, then how many symmetric relations on A are there?

(A) 2048

(B) 1024

(C) 512

(D) 256

(E) 128

17. (3%) Let $\phi(n)$ denote the number of positive integers $m \in \{1, 2, \ldots, n\}$ such that $\gcd(m, n) = 1$, where $n \geq 2$. Calculate $\phi(970)$.

(A) 547

(B) 450

(C) 96

(D) 480

(E) 384

18. (3%) Which of the following equals

$$\left(\sum_{k=0}^{\infty} \frac{1}{2^k}\right) \left(\sum_{k=0}^{\infty} \frac{1}{3^k}\right) \left(\sum_{k=0}^{\infty} \frac{1}{5^k}\right) \left(\sum_{k=0}^{\infty} \frac{1}{7^k}\right) \left(\sum_{k=0}^{\infty} \frac{1}{11^k}\right) \cdots ?$$

(A) $\sum_{n \geq 1} \ln n$

(B) $\sum_{n \geq 1} n^{-2}$

(C) $\sum_{n \geq 1} n^{-1}$

接次頁

(D) $\sum_{n\geq 1} n$

(E) $\sum_{n\geq 1} \frac{1}{\phi(n)}$

19. (3%) A tree can be colored in how many colors?

(A) 1

(B) 2

(C) 3

(D) 4

(E) 5

20. (3%) $\binom{6}{6}2^6 + \binom{6}{5}2^5 + \binom{6}{4}2^4 + \binom{6}{3}2^3 + \binom{6}{2}2^2 + \binom{6}{1}2 + 1$ equals

(A) 1000

(B) 64

(C) 256

(D) 729

(E) 625

21. (4%) Let k persons involve in m pairwise (i.e., two-person) games. We define an $m \times k$ game matrix G to show the setting: If the sth game involves persons i and j, then the sth row has the ith and jth entries to be one. Other entries are zeros. We assume that if two persons appear in one game, they do not appear together in another. Which of the followings are incorrect?

(A) Under this setting, there are at most $k(k-1)/2$ games

(B) Rank of $G \leq k-1$

(C) The ith diagonal element of $G'G$ = (#games the ith person involves -1)

(D) The number of off diagonal elements of $G'G$ is $k(k-1)$

(E) Any diagonal element of GG' is 1

接背面

22. (4%) Let $A \in R^{n\times n}$ be any symmetric positive definite matrix and we write it as

$$A = \begin{bmatrix} \alpha & \mathbf{v}^T \\ \mathbf{v} & B \end{bmatrix},$$

where α is a scalar, $\mathbf{v}$ is a vector, and $B \in R^{n-1\times n-1}$. Which of the followings are incorrect?

(A) A's principle sub-matrices are positive definite

(B) A's diagonal elements are all positive

(C) If $X \in R^{n\times k}$ has rank k, then $B = X^T A X \in R^{k\times k}$ may not be positive definite

(D) $B - \frac{\mathbf{v}\mathbf{v}^T}{\alpha}$ may not be positive definite

(E) We have

$$A = \begin{bmatrix} \beta & \mathbf{0}^T \\ \mathbf{v}/\beta & I_{n-1} \end{bmatrix} \begin{bmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & B - \frac{\mathbf{v}\mathbf{v}^T}{\alpha} \end{bmatrix} \begin{bmatrix} \beta & \mathbf{v}^T/\beta \\ \mathbf{0} & I_{n-1} \end{bmatrix},$$

where $\beta = \sqrt{\alpha}$, $\mathbf{0}$ is a zero vector and I_{n-1} is an $n-1\times n-1$ identity matrix.

23. (4%) Let A be any $n \times n$ symmetric positive definite matrix and $\mathbf{u}$ be any non-zero vector. Consider the following $(n+1)\times(n+1)$ matrix:

$$\begin{bmatrix} A & \mathbf{u} \\ \mathbf{u}^T & 0 \end{bmatrix}. \tag{2}$$

Which of the following is incorrect:

(A) If we solve the following linear system

$$\begin{bmatrix} A & \mathbf{u} \\ \mathbf{u}^T & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ y \end{bmatrix} = \begin{bmatrix} \mathbf{b} \\ c \end{bmatrix}, \tag{3}$$

then

$$y = \frac{\mathbf{u}^T A^{-1}\mathbf{b}}{\mathbf{u}^T A^{-1}\mathbf{u}}$$

接次頁

(B) For (3), the solution satisfies

$$\mathbf{x} = A^{-1}(\mathbf{b} - y\mathbf{u}).$$

(C) The matrix (2) is positive semi-definite

(D) The matrix (2) is invertible

(E) The matrix (2) is symmetric

24. (4%) For $A \in R^{m\times n}$, define

$$\|A\|_F \equiv \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n} A_{ij}^2}.$$

Which of the followings are incorrect?

(A) $\|A\|_F = \sqrt{\text{trace}(AA^T)}$

(B) $\|A\|_F = \sqrt{\text{trace}(A^TA)}$

(C) For any scalar α, $\|\alpha A\|_F = |\alpha|\|A\|_F$

(D) $\|A+B\|_F = \|A\|_F + \|B\|_F$

(E) If $\mathbf{u} \in R^n, \mathbf{v} \in R^m$, and $A = \mathbf{u}\mathbf{v}^T$, then $\|A\|_F = 2\|\mathbf{u}\|_2\|\mathbf{v}\|_2$

25. (4%) Consider a function

$$f(W,H) = \|V - WH\|_F^2,$$

where $V \in R^{n\times m}$ is a constant matrix, $W \in R^{n\times r}, H \in R^{r\times m}$ are two matrix variables. $\|\cdot\|_F$ is defined in Problem 24. If we define a matrix $\nabla_W f(W,H)$ so that

$$\nabla_W f(W,H)_{ij} \equiv \frac{\partial f(W,H)}{\partial W_{ij}},$$

then what is $\nabla_W f(W,H)$:

(A) $(WH - V)V^TW$

(B) $(WH - V)H^T$

(C) $VW^T(WH - V)H^T$

接背面

(D) WHH^T

(E) $W^T(WH-V)$

26. (4%) Solve $a_{n+1} - 2 \times a_n = 3^n$ with $a_0 = 1$.

(A) $(-5 \times (-1)^n + 3^{2+n})/4$

(B) $(-7 + 3^{2+n})/2$

(C) $(-1 + 3^{1+n})/2$

(D) $(1 + 3^n)/2$

(E) 3^n

27. (4%) Which of the following defines an abelian group?

(A)

∘	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

(B)

∘	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	3	1
3	0	3	1	2

(C)

∘	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

(D)

∘	0	1	2	3
0	0	1	3	2
1	1	0	2	3
2	2	3	1	0
3	3	2	0	1

(E)

∘	0	1	2	3
0	1	1	2	3
1	1	1	3	2
2	2	3	1	1
3	3	2	1	1

28. (4%) Which of the following is a tautology?

(A) $(p \wedge \neg q) \vee (\neg p \wedge q)$

(B) $(p \to q) \wedge (\neg p \to \neg q)$

(C) $(p \vee q) \to (\neg p \vee \neg q)$

(D) $(p \wedge \neg q) \vee (\neg p \vee \neg q)$

(E) $[(p \to r) \wedge (\neg q \to p) \wedge \neg r] \to q$

接次頁

國立臺灣大學95學年度碩士班招生考試試題

29. (4%) Consider $2a_{n+3} = a_{n+2} + 2a_{n+1} - a_n$ with $a_0 = -1$, $a_1 = 0$, and $a_2 = 1$. Then

    (A) $a_n = (-1)^n(n-1)$

    (B) $a_n = 2^n(-13 - 3 \times (-1)^n + 10 \times n)/16$

    (C) $a_n = 2^{-n}(-4 + 4^n)/3$

    (D) $a_n = (9 + (-1)^n - 16 \times 2^{-n})/6$

    (E) $a_n = (-9 - (-1)^n + 4 \times 2^n)/6$

30. (4%) Let $p(m)$ denote the number of partitions of $m \in \mathbb{Z}^+$ into distinct positive integers where the order of summands is irrelevant. Calculate $p(8)$.

    (A) 5

    (B) 6

    (C) 7

    (D) 8

    (E) 9