ECE368: Probabilistic Reasoning
Lab 2 – Part II: Hidden Markov Model

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You should hand in: 1) A scanned .pdf version of this sheet with your answers (file size should be under 2 MB); 2) one Python file inference.py that contains your code. The files should be uploaded to Quercus.

1. (a) Write down the formulas of the forward-backward algorithm to compute the marginal distribution $p(\mathbf{z}_i|(\hat{x}_0, \hat{y}_0), \ldots, (\hat{x}_{N-1}, \hat{y}_{N-1}))$ for $i = 0, 1, \ldots, N-1$. Your answer should contain the initializations of the forward and backward messages, the recursion relations of the messages, and the computation of the marginal distribution based on the messages. (1 **pt**)

   Forward Message

   Initialization: $\alpha(\boldsymbol{z_0}) = P(\boldsymbol{z_0}) * P\big((\hat{x}_0, \hat{y}_0)|\boldsymbol{z_0}\big)$

   Recursion: $\alpha(\boldsymbol{z_i}) = P\big((\hat{x}_i, \hat{y}_i)|\boldsymbol{z_i}\big) * \sum\limits_{\boldsymbol{z_{i-1}}} \Big(\alpha(\boldsymbol{z_{i-1}}) * P(\boldsymbol{z_i}|\boldsymbol{z_{i-1}})\Big)$

   Backward Message

   Initialization: $\beta(\boldsymbol{z_{N-1}}) = 1$

   Recursion: $\beta(\boldsymbol{z_i}) = \sum\limits_{\boldsymbol{z_{i+1}}} \Big(\beta(\boldsymbol{z_{i+1}}) * P(\boldsymbol{z_{i+1}}|\boldsymbol{z_i}) * P\big((\hat{x}_{i+1}, \hat{y}_{i+1})|\boldsymbol{z_{i+1}}\big)\Big)$

   Marginal Distribution

   $P\big(\boldsymbol{z_i}|(\hat{x}_0, \hat{y}_0),\ \ldots\ ,\ (\hat{x}_{N-1}, \hat{y}_{N-1})\big) = \frac{\alpha(\boldsymbol{z_i}) * \beta(\boldsymbol{z_i})}{\sum\limits_{\boldsymbol{z_i}} \big(\alpha(\boldsymbol{z_i}) * \beta(\boldsymbol{z_i})\big)}$

   (b) After you run the forward-backward algorithm on the data in test.txt, write down the obtained marginal distribution of the state at $i = 99$ (the last time step), i.e., $p(\mathbf{z}_{99}|(\hat{x}_0, \hat{y}_0), \ldots, (\hat{x}_{99}, \hat{y}_{99}))$. Only include states with non-zero probability in your answer. (2 **pt**)

$$P\big(\boldsymbol{z_{99}}|(\hat{x}_0, \hat{y}_0),\ \ldots\ ,\ (\hat{x}_{99}, \hat{y}_{99})\big) = \begin{cases} 0.01013 & \text{if } \boldsymbol{z_{99}} = (10, 1, \text{'down'}) \\ 0.17961 & \text{if } \boldsymbol{z_{99}} = (11, 0, \text{'right'}) \\ 0.81026 & \text{if } \boldsymbol{z_{99}} = (11, 0, \text{'stay'}) \end{cases}$$

2. Modify your forward-backward algorithm so that it can handle missing observations. After you run the modified forward-backward algorithm on the data in test_missing.txt, write down the obtained marginal distribution of the state at $i = 30$, i.e., $p(\mathbf{z}_{30}|(\hat{x}_0, \hat{y}_0), \ldots, (\hat{x}_{99}, \hat{y}_{99}))$. Only include states with non-zero probability in your answer. (1 **pt**)

$$P\big(\boldsymbol{z_{30}}|(\hat{x}_0, \hat{y}_0),\ \ldots\ ,\ (\hat{x}_{99}, \hat{y}_{99})\big) = \begin{cases} 0.04348 & \text{if } \boldsymbol{z_{30}} = (5, 7, \text{'right'}) \\ 0.04348 & \text{if } \boldsymbol{z_{30}} = (5, 7, \text{'stay'}) \\ 0.91304 & \text{if } \boldsymbol{z_{30}} = (6, 7, \text{'right'}) \end{cases}$$

3. (a) Write down the formulas of the Viterbi algorithm using $\mathbf{z}_i$ and $(\hat{x}_i, \hat{y}_i), i = 0, 1, \ldots, N-1$. Your answer should contain the initialization of the messages and the recursion of the messages in the Viterbi algorithm. (1 **pt**)

Initialization:

$$w_0(\boldsymbol{z_0}) = ln\Big(P(\boldsymbol{z_0}) * P\big((\hat{x}_0, \hat{y}_0)|\boldsymbol{z_0}\big)\Big)$$

Recursion:

$$w_i(\boldsymbol{z_i}) = ln\Big(P\big((\hat{x}_i, \hat{y}_i)|\boldsymbol{z_i}\big)\Big) + \max_{\boldsymbol{z_{i-1}}}\Big\{ln\Big(P(\boldsymbol{z_i}|\boldsymbol{z_{i-1}})\Big) + w_{i-1}(\boldsymbol{z_{i-1}})\Big\}$$

(b) After you run the Viterbi algorithm on the data in test_missing.txt, write down the last 10 hidden states of the most likely sequence (i.e., $i = 90, 91, 92, \ldots, 99$) based on the MAP estimate. (3 **pt**)

$$\tilde{\boldsymbol{z}}_{\boldsymbol{90}} = (11,\ 5,\ \text{'down'}) \qquad \tilde{\boldsymbol{z}}_{\boldsymbol{95}} = (10,\ 7,\ \text{'left'})$$
$$\tilde{\boldsymbol{z}}_{\boldsymbol{91}} = (11,\ 6,\ \text{'down'}) \qquad \tilde{\boldsymbol{z}}_{\boldsymbol{96}} = (9,\ 7,\ \text{'left'})$$
$$\tilde{\boldsymbol{z}}_{\boldsymbol{92}} = (11,\ 7,\ \text{'down'}) \qquad \tilde{\boldsymbol{z}}_{\boldsymbol{97}} = (8,\ 7,\ \text{'left'})$$
$$\tilde{\boldsymbol{z}}_{\boldsymbol{93}} = (11,\ 7,\ \text{'stay'}) \qquad \tilde{\boldsymbol{z}}_{\boldsymbol{98}} = (7,\ 7,\ \text{'left'})$$
$$\tilde{\boldsymbol{z}}_{\boldsymbol{94}} = (11,\ 7,\ \text{'stay'}) \qquad \tilde{\boldsymbol{z}}_{\boldsymbol{99}} = (6,\ 7,\ \text{'left'})$$

4. Compute and compare the error probabilities of $\{\tilde{\mathbf{z}}_i\}$ and $\{\check{\mathbf{z}}_i\}$ using the data in test_missing.txt. The error probability of $\{\tilde{\mathbf{z}}_i\}$ is $\boxed{0.03}$. The error probability of $\{\check{\mathbf{z}}_i\}$ is $\boxed{0.02}$. (1 **pt**)

5. Is sequence $\{\check{\mathbf{z}}_i\}$ a valid sequence? If not, please find a small segment $\check{\mathbf{z}}_i, \check{\mathbf{z}}_{i+1}$ that violates the transition model for some time step i. You answer should specify the value of i as well as the corresponding states $\check{\mathbf{z}}_i, \check{\mathbf{z}}_{i+1}$. (1 **pt**)

$\{\check{\boldsymbol{z}}_{\boldsymbol{i}}\}$ is an invalid sequence.

$$\check{\boldsymbol{z}}_{\boldsymbol{64}} = (3, 7, \text{'stay'})$$
$$\check{\boldsymbol{z}}_{\boldsymbol{65}} = (2, 7, \text{'stay'})$$