ECE368: Probabilistic Reasoning

Lab 3: Hidden Markov Model

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You should hand in: 1) A scanned .pdf version of this sheet with your answers (file size should be under 2 MB); 2) one Python file inference.py that contains your code. The files should be uploaded to Quercus.

1. (a) Write down the formulas of the forward-backward algorithm to compute the marginal distribution $p(\mathbf{z}_i|(\hat{x}_0,\hat{y}_0),\dots,(\hat{x}_{N-1},\hat{y}_{N-1}))$ for $i = 0, 1, \dots, N-1$. Your answer should contain the initializations of the forward and backward messages, the recursion relations of the messages, and the computation of the marginal distribution based on the messages. (1 **pt**)

then normalize marginal distribution

$$p(z_i|(\hat{x}_0,\hat{y}_0)\dots(\hat{x}_{N-1},\hat{y}_{N-1})) = \frac{\alpha(z_i)\beta(z_i)}{\sum_{z_i}(\alpha(z_i)\cdot\beta(z_i))}$$

Note: $p((\hat{x}_i,\hat{y}_i)|z_i)$ would be treat as 1 if $(\hat{x}_i,\hat{y}_i)$ is not observed (missing)

$p(z_i|(\hat{x}_0,\hat{y}_0)\dots(\hat{x}_{N-1},\hat{y}_{N-1})) = \gamma(z_i) = \alpha(z_i)\beta(z_i)$

where $\alpha(\cdot)$ is the forward message, $\beta(\cdot)$ is the backward message

initialization: $\alpha(z_0) = p(z_0)p((\hat{x}_0,\hat{y}_0)|z_0)$, $\beta(z_{N-1}) = 1$

$$\left.\begin{aligned}\alpha(z_i) &= p((\hat{x}_i,\hat{y}_i)|z_i)\cdot\sum_{z_{i-1}}\alpha(z_{i-1})p(z_i|z_{i-1})\\ \beta(z_i) &= \sum_{z_{i+1}}\beta(z_{i+1})p((\hat{x}_{i+1},\hat{y}_{i+1})|z_{i+1})p(z_{i+1}|z_i)\end{aligned}\right\}\text{recursion relations}$$

Note: $\alpha(z_i)$ and $\beta(z_i)$ will both be normalized.

Normalize means compute $\alpha(z_i)$ & $\beta(z_i)$ for all possible z_i values and for each divide by the sum

(b) After you run the forward-backward algorithm on the data in test.txt, write down the obtained marginal distribution of the state at $i = 99$ (the last time step), i.e., $p(\mathbf{z}_{99}|(\hat{x}_0,\hat{y}_0),\dots,(\hat{x}_{99},\hat{y}_{99}))$. Only include states with non-zero probability in your answer. (2 **pt**)

$$p(z_{99}|(\hat{x}_0,\hat{y}_0)\dots(\hat{x}_{99},\hat{y}_{99})) = \begin{cases} 0.010128 & z_{99} = (10, 1, \text{'down'}) \\ 0.179608 & z_{99} = (11, 0, \text{'right'}) \\ 0.810263 & z_{99} = (11, 0, \text{'stay'}) \\ 0 & \text{otherwise} \end{cases}$$

2. Modify your forward-backward algorithm so that it can handle missing observations. After you run the modified forward-backward algorithm on the data in test_missing.txt, write down the obtained marginal distribution of the state at $i = 30$, i.e., $p(\mathbf{z}_{30}|(\hat{x}_0,\hat{y}_0),\dots,(\hat{x}_{99},\hat{y}_{99}))$. Only include states with non-zero probability in your answer. (1 **pt**)

$$p(z_{30}|(\hat{x}_0,\hat{y}_0)\dots(\hat{x}_{99},\hat{y}_{99})) = \begin{cases} 0.043478 & z_{30} = (5, 7, \text{'right'}) \\ 0.043478 & z_{30} = (5, 7, \text{'stay'}) \\ 0.913043 & z_{30} = (6, 7, \text{'right'}) \\ 0 & \text{otherwise} \end{cases}$$

3. (a) Write down the formulas of the Viterbi algorithm using $\mathbf{z}_i$ and $(\hat{x}_i, \hat{y}_i), i = 0, 1, \ldots, N-1$. Your answer should contain the initialization of the messages and the recursion of the messages in the Viterbi algorithm. (1 **pt**)

initialization: $w_0(z_0) = \ln p(z_0) + \ln p((\hat{x}_0, \hat{y}_0) | z_0)$

$i \in [1, N-1]$, $w_i(z_i) = \ln p((\hat{x}_i, \hat{y}_i) | z_i) + \max_{z_{i-1}} \{\ln p(z_i | z_{i-1}) + w_{i-1}(z_{i-1})\}$

$z_i^* = \arg\max_{z_i} w_i(z_i)$

then, back track to find z_{i-1}^*, ... using the ϕ function,

where $z_{i-1}^* = \phi_i(z_i) = \arg\max_{z_{i-1}} (w_i(z_i^*))$

Remark: when $(\hat{x}_i, \hat{y}_i)$ is not observed, $p((\hat{x}_i, \hat{y}_i) | z_i) = 1$

(b) After you run the Viterbi algorithm on the data in test_missing.txt, write down the last 10 hidden states of the most likely sequence (i.e., $i = 90, 91, 92, \ldots, 99$) based on the MAP estimate. (3 **pt**)

z_{90} = (11, 5, 'down')

z_{91} = (11, 6, 'down')

z_{92} = (11, 7, 'down')

z_{93} = (11, 7, 'stay')

z_{94} = (11, 7, 'stay')

z_{95} = (10, 7, 'left')

z_{96} = (9, 7, 'left')

z_{97} = (8, 7, 'left')

z_{98} = (7, 7, 'left')

z_{99} = (6, 7, 'left')

4. Compute and compare the error probabilities of $\{\tilde{\mathbf{z}}_i\}$ and $\{\check{\mathbf{z}}_i\}$ using the data in test_missing.txt. The error probability of $\{\tilde{\mathbf{z}}_i\}$ is $\boxed{0.03}$. The error probability of $\{\check{\mathbf{z}}_i\}$ is $\boxed{0.02}$. (1 **pt**)

5. Is sequence $\{\check{\mathbf{z}}_i\}$ a valid sequence? If not, please find a small segment $\check{\mathbf{z}}_i, \check{\mathbf{z}}_{i+1}$ that violates the transition model for some time step i. You answer should specify the value of i as well as the corresponding states $\check{\mathbf{z}}_i, \check{\mathbf{z}}_{i+1}$. (1 **pt**)

z_{64} = (3, 7, 'stay')

z_{65} = (2, 7, 'stay')

wrong

It's invalid because when the rover stays at i=64, it should be in position of (3, 7) at i=65