

ECE368: Probabilistic Reasoning

Lab 3: Hidden Markov Model

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You should hand in: 1) A scanned .pdf version of this sheet with your answers (file size should be under 2 MB); 2) one Python file `inference.py` that contains your code. The files should be uploaded to Quercus.

1. (a) Write down the formulas of the forward-backward algorithm to compute the marginal distribution $p(\mathbf{z}_i | (\hat{x}_0, \hat{y}_0), \dots, (\hat{x}_{N-1}, \hat{y}_{N-1}))$ for $i = 0, 1, \dots, N-1$. Your answer should contain the initial-izations of the forward and backward messages, the recursion relations of the messages, and the computation of the marginal distribution based on the messages. (1 pt)

then normalise marginal distribution

$$p(\mathbf{z}_i | (\hat{x}_0, \hat{y}_0), \dots, (\hat{x}_{N-1}, \hat{y}_{N-1})) = \frac{\alpha(\mathbf{z}_i) \beta(\mathbf{z}_i)}{\sum_{\mathbf{z}_i} \alpha(\mathbf{z}_i) \beta(\mathbf{z}_i)}$$

Note: $p(\hat{x}_i, \hat{y}_i | \mathbf{z}_i)$ would be treated as 1 if (\hat{x}_i, \hat{y}_i) is not observed (missing)

$$p(\mathbf{z}_i | (\hat{x}_0, \hat{y}_0), \dots, (\hat{x}_{N-1}, \hat{y}_{N-1})) = \delta(\mathbf{z}_i) = \alpha(\mathbf{z}_i) \beta(\mathbf{z}_i)$$

where $\alpha(\cdot)$ is the forward message, $\beta(\cdot)$ is the backward message

initialization: $\alpha(\mathbf{z}_0) = p(\mathbf{z}_0) p(\hat{x}_0, \hat{y}_0 | \mathbf{z}_0)$, $\beta(\mathbf{z}_{N-1}) = 1$

recursion relations:

$$\alpha(\mathbf{z}_i) = p(\hat{x}_i, \hat{y}_i | \mathbf{z}_i) \cdot \sum_{\mathbf{z}_{i-1}} \alpha(\mathbf{z}_{i-1}) p(\mathbf{z}_i | \mathbf{z}_{i-1})$$

$$\beta(\mathbf{z}_i) = \sum_{\mathbf{z}_{i+1}} \beta(\mathbf{z}_{i+1}) p(\hat{x}_{i+1}, \hat{y}_{i+1} | \mathbf{z}_{i+1}) p(\mathbf{z}_{i+1} | \mathbf{z}_i)$$

Note: $\alpha(\mathbf{z}_i)$ and $\beta(\mathbf{z}_i)$ will both be normalized.

Normalise means compute $\alpha(\mathbf{z}_i)$ & $\beta(\mathbf{z}_i)$ for all possible \mathbf{z}_i values and for each divide by the sum

- (b) After you run the forward-backward algorithm on the data in `test.txt`, write down the obtained marginal distribution of the state at $i = 99$ (the last time step), i.e., $p(\mathbf{z}_{99} | (\hat{x}_0, \hat{y}_0), \dots, (\hat{x}_{99}, \hat{y}_{99}))$. Only include states with non-zero probability in your answer. (2 pt)

$$p(\mathbf{z}_{99} | (\hat{x}_0, \hat{y}_0), \dots, (\hat{x}_{99}, \hat{y}_{99})) = \begin{cases} 0.010128 & \mathbf{z}_{99} = (10, 1, \text{'down'}) \\ 0.179608 & \mathbf{z}_{99} = (11, 0, \text{'right'}) \\ 0.810263 & \mathbf{z}_{99} = (11, 0, \text{'stay'}) \\ 0 & \text{otherwise} \end{cases}$$

2. Modify your forward-backward algorithm so that it can handle missing observations. After you run the modified forward-backward algorithm on the data in `test_missing.txt`, write down the obtained marginal distribution of the state at $i = 30$, i.e., $p(\mathbf{z}_{30} | (\hat{x}_0, \hat{y}_0), \dots, (\hat{x}_{99}, \hat{y}_{99}))$. Only include states with non-zero probability in your answer. (1 pt)

$$p(\mathbf{z}_{30} | (\hat{x}_0, \hat{y}_0), \dots, (\hat{x}_{99}, \hat{y}_{99})) = \begin{cases} 0.043478 & \mathbf{z}_{30} = (5, 2, \text{'right'}) \\ 0.043478 & \mathbf{z}_{30} = (5, 2, \text{'stay'}) \\ 0.913043 & \mathbf{z}_{30} = (6, 2, \text{'right'}) \\ 0 & \text{otherwise} \end{cases}$$

3. (a) Write down the formulas of the Viterbi algorithm using \mathbf{z}_i and $(\hat{x}_i, \hat{y}_i), i = 0, 1, \dots, N - 1$. Your answer should contain the initialization of the messages and the recursion of the messages in the Viterbi algorithm. (1 pt)

initialization: $W_0(z_0) = \ln p(z_0) + \ln p(\hat{x}_0, \hat{y}_0 | z_0)$
 $i \in [1, N-1], W_i(z_i) = \ln p(\hat{x}_i, \hat{y}_i | z_i) + \max_{z_{i-1}} \{ \ln p(z_i | z_{i-1}) + W_{i-1}(z_{i-1}) \}$
 $z_i^* = \arg \max_{z_i} W_i(z_i)$
 then, back track to find z_i^* -- using the ϕ function,
 where $z_i^* = \phi_i(z_i^*) = \arg \max_{z_{i-1}} (W_i(z_i^*))$
 Remark: when (\hat{x}_i, \hat{y}_i) is not observed, $p(\hat{x}_i, \hat{y}_i | z_i) = 1$

- (b) After you run the Viterbi algorithm on the data in test_missing.txt, write down the last 10 hidden states of the most likely sequence (i.e., $i = 90, 91, 92, \dots, 99$) based on the MAP estimate. (3 pt)

$z_{90} = (11, 5, 'down')$
 $z_{91} = (11, 6, 'down')$
 $z_{92} = (11, 7, 'down')$
 $z_{93} = (11, 7, 'stay')$
 $z_{94} = (11, 7, 'stay')$
 $z_{95} = (10, 7, 'left')$
 $z_{96} = (9, 7, 'left')$
 $z_{97} = (8, 7, 'left')$
 $z_{98} = (7, 7, 'left')$
 $z_{99} = (6, 7, 'left')$

4. Compute and compare the error probabilities of $\{\tilde{z}_i\}$ and $\{\hat{z}_i\}$ using the data in test_missing.txt. The error probability of $\{\tilde{z}_i\}$ is 0.03. The error probability of $\{\hat{z}_i\}$ is 0.02. (1 pt)
5. Is sequence $\{\tilde{z}_i\}$ a valid sequence? If not, please find a small segment $\tilde{z}_i, \tilde{z}_{i+1}$ that violates the transition model for some time step i . Your answer should specify the value of i as well as the corresponding states $\tilde{z}_i, \tilde{z}_{i+1}$. (1 pt)

$z_{64} = (3, 7, 'stay')$
 $z_{65} = (2, 7, 'stay')$
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 wrong
 it's invalid because when the power
 drops at $i=64$, it should
 be in position of $(3, 7)$ at $i=65$