Homework - 10
COEN 240-Machine Learning

Pujitha Kallu
ID : W1653660
pkallu@scu.edu

Consider a dataset with 2 points in 1D: $(x_1 = 0, y_1 = -1)$ and $\left(x_2 = \sqrt{2}, y_2 = 1\right)$. Map each point to 3D using the feature vector $\phi(x) = \left[1, \sqrt{2}x, x^2\right]^T$. This is equivalent to using a second order polynomial kernel. The SVM classifier has the form

$$\min \|\boldsymbol{w}\|^2 \; s.t.$$

$$y_1(\boldsymbol{w}^T\phi(x_1) + w_0) \geq 1$$

$$y_2(\boldsymbol{w}^T\phi(x_2) + w_0) \geq 1$$

(a). Find the corresponding points in 3D. That is, $\phi(x_1)$ and $\phi(x_2)$.

Sol:

Solution:

a) Given, $x_1 = 0$, $y_1 = -1$

$x_2 = \sqrt{2}$, $y_2 = 1$

$$\phi(x) = [1, \sqrt{2}x, x^2]^T$$

$$\phi(x_1) = [1, \sqrt{2}(0), 0^2]^T$$

$$\phi(x_1) = [1, 0, 0]^T$$

$$\phi(x_2) = [1, \sqrt{2}\sqrt{2}, \sqrt{2}^2]^T$$

$$\phi(x_2) = [1, 2, 2]^T$$

(b). What is the value of the margin? Notice since there are only 2 points in the dataset, those points are the support vectors. Hence, the margin is the distance between each of them in 3D to the decision boundary, which lies in the middle.

Sol:

b) The decision boundary is in the mid way between 2 vectors. $\phi(x_1)$ & $\phi(x_2)$

$$\text{Midpoint} = \frac{(1,0,0)+(1,2,2)}{2} = \frac{(2,2,2)}{2}$$

$$= (1, 1,).$$

$$\text{distance} = \sqrt{(1-1)^2+(0-2)^2+(0-2)^2} = \sqrt{0+4+4}$$

$$\text{distance} = 2\sqrt{2}$$

The margin is the distance of each vector to the decision boundary which lies in middle

$$\text{Margin} = \frac{2\sqrt{2}}{2} = \sqrt{2}$$

(c). The margin obtained from part (b) is in fact equal to $1/\|\boldsymbol{w}\|$. Determine the vector $\boldsymbol{w}$. Recall this vector is the line through $\phi(x_1)$ and $\phi(x_2)$, which is perpendicular to the decision boundary.

Sol:

c) Given

$$\text{Margin Assumption} = \frac{1}{\|w\|} = \sqrt{2}$$

$$w = \sum_{i=1}^{n} \alpha_i y_i \phi(x_i)$$

$$= \alpha_1 y_1 \phi(x_1) + \alpha_2 y_2 \phi(x_2)$$

$$= \alpha_1(-1)\begin{bmatrix}1\\0\\0\end{bmatrix} + \alpha_2(1)\begin{bmatrix}1\\2\\2\end{bmatrix}$$

$$= \begin{bmatrix}-\alpha_1\\0\\0\end{bmatrix} + \begin{bmatrix}\alpha_2\\2\alpha_2\\2\alpha_2\end{bmatrix}$$

$$= \begin{bmatrix}-\alpha_1+\alpha_2\\2\alpha_2\\2\alpha_2\end{bmatrix}$$

we know that

$$\sum_{i=1}^{n} \alpha_i y_i = 0$$

$$\alpha_1 y_1 + \alpha_2 y_2 = 0$$

by substituting $w = \begin{bmatrix}0\\2\alpha_2\\2\alpha_2\end{bmatrix}$

from the given assumption,

$$\|w\| = \frac{1}{\sqrt{2}}$$

$$\sqrt{0+4\alpha_2^2+4\alpha_2^2} = \frac{1}{\sqrt{2}}$$

$$2\sqrt{2}\,\alpha_2 = \frac{1}{\sqrt{2}}$$

$$\alpha_2 = \frac{1}{4}$$

by substituting $w = \begin{bmatrix} 0 \\ 1/2 \\ 1/2 \end{bmatrix}$

(d). Solve for w_0 using your value for $\boldsymbol{w}$ and the above equations. Since the points are on the decision boundary, the inequalities will become equalities.

Sol:

$$\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$$

d) Using the SVM conditions $y_1(w^T\phi(x_1)+w_0)\geq 1$ & $y_2(w^T\phi(x_2)+w_0)\geq 1$ as equalities:

for $y_1(w^T\phi(x_1)+w_0)=1 \Rightarrow$

$$-1\left((0, 1/2, 1/2)\begin{pmatrix}1\\0\\0\end{pmatrix}+w_0\right)=1$$

$$w_0=-1$$

Same for $y_2(w^T\phi(x_2)+w_0)=1$

$$\Rightarrow 1\left((0, 1/2, 1/2)\begin{pmatrix}1\\2\\2\end{pmatrix}+w_0\right)=1$$

$$2+w_0=1$$

$$w_0=-1$$

(e). Write the function $f(x) = w_0 + \boldsymbol{w}^T\phi(x)$ as an explicit function of x.

Sol:

c) $f(x) = w_0 + w^T \phi(x)$

we have $\phi(x) = \begin{bmatrix} 1 \\ \sqrt{2}\,x \\ x^2 \end{bmatrix}$

$$w^T = \begin{bmatrix} 0 & 1/2 & 1/2 \end{bmatrix}$$

$$w_0 = -1$$

By substituting we get:

$$f(x) = -1 + \underset{1\times 3}{\begin{bmatrix} 0 & 1/2 & 1/2 \end{bmatrix}} \underset{3\times 1}{\begin{bmatrix} 1 \\ \sqrt{2}x \\ x^2 \end{bmatrix}}$$

$$= -1 + \left[0 + \frac{1}{2}(\sqrt{2}x) + \frac{1}{2}x^2 \right]$$

$$\boxed{f(x) = -1 + \frac{x}{\sqrt{2}} + \frac{x^2}{2}}$$

References:

1. Class notes:Support vector machines (SVMs)
2. https://scikit-learn.org/stable/modules/svm.html
3. https://www.geeksforgeeks.org/support-vector-machine-algorithm/