Homework -4
COEN 240-Machine Learning

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1.

Given the cost function $J(w_0, w_1) = \frac{1}{2m}\sum_{i=1}^{m}\left(w_0 + w_1\mathbf{x}^{(i)} - y_i\right)^2$, determine the definiteness of its Hessian matrix and the convexity of the function. Assume 1-D dataset and $m = 1$. Show your work.

Solution below →

Sol:

Given cost function:

$$J(w_0, w_1) = \frac{1}{2}(w_0 + w_1 x - y)^2$$

$$\frac{\partial J(w)}{\partial w_0} = w_0 + w_1 x - y$$

$$\frac{\partial^2 J(w)}{\partial w_0^2} = 1$$

$$\frac{\partial J(w)}{\partial w_0 \partial w_1} = x.$$

$$\frac{\partial J(w)}{\partial w_1} = (w_0 + w_1 x - y) \times x$$

$$\frac{\partial^2 J(w)}{\partial w_1^2} = x^2$$

$$\frac{\partial J(w)}{\partial w_1 \partial w_0} = x$$

$$H(w) = \begin{bmatrix} \frac{\partial^2 J(w)}{\partial w_0^2} & \frac{\partial J(w)}{\partial w_0 \partial w_1} \\ \frac{\partial J(w)}{\partial w_1 \partial w_0} & \frac{\partial^2 J(w)}{\partial w_1^2} \end{bmatrix}$$

$$= \begin{bmatrix} 1-\lambda & x \\ x & x^2-\lambda \end{bmatrix} = 0$$

$$\Rightarrow (1-\lambda)(x^2-\lambda) - x^2 = 0$$

$$\cancel{x^2} - \lambda - \lambda x^2 + \lambda^2 - \cancel{x^2} = 0$$

$$\lambda^2 - \lambda(x^2+1) = 0$$

$$\lambda^2 - \lambda(x^2+1) = 0$$

$$\lambda(\lambda - x^2 - 1) = 0$$

$$\Rightarrow \quad \lambda_1 = 0 \qquad \lambda_2 = 1 + x^2 > 0 \text{ (or:}$$

$$\lambda_2 = x^2 + 1 > 0$$

Hessian : Positive Semi definite

$\therefore$ The function is Convex.

Reference:

1. Class notes
2. https://people.eecs.berkeley.edu/~jordan/courses/294-fall09/lectures/optimization/slides.pdf
3. https://d2l.ai/chapter_optimization/convexity.html