# VECTORS

Imagine a raft drifting down a river, carried by the current. The speed and direction of the raft at a point may be represented by an arrow (Figure 11.1). The length of the arrow represents the speed of the raft at that point; longer arrows correspond to greater speeds. The orientation of the arrow gives the direction in which the raft is headed at that point. The arrows at points $A$ and $C$ in Figure 11.1 have the same length and direction indicating that the raft has the same speed and heading at these locations. The arrow at $B$ is shorter and points to the left, indicating that the raft slows down as it nears the rock.

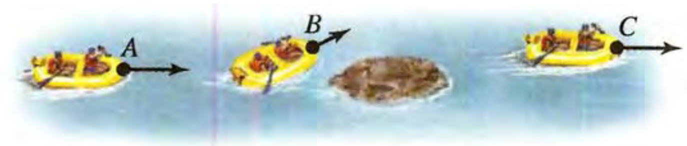
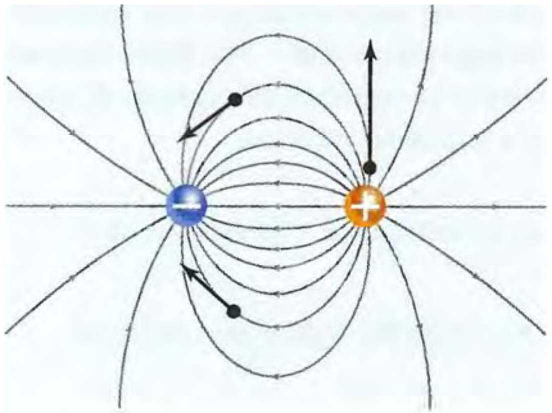


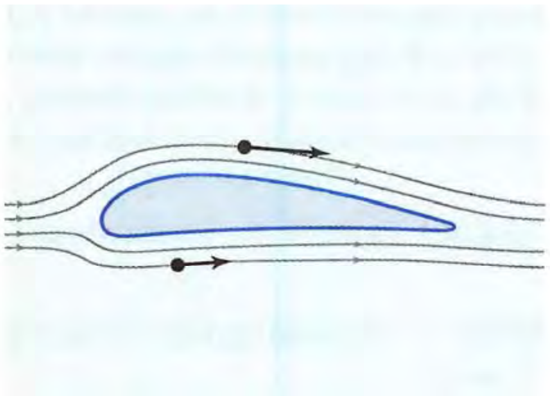

FIGURE 11.1

## Basic Vector Operations

The arrows that describe the raft's motion are examples of **vectors**—quantities that have both **length** (or **magnitude**) and **direction**. Vectors arise naturally in many situations. For example, electric and magnetic fields, the flow of air over an airplane wing, and the velocity and acceleration of elementary particles are described by vectors (Figure 11.2). In this section we examine vectors in the $xy$-plane and then extend the concept to three dimensions in Section 11.2.



Electric field vectors due to two charges



Velocity vectors of air flowing over an airplane wing

FIGURE 11.2

The vector whose *tail* is at the point $P$ and whose *head* is at the point $Q$ is denoted $\overrightarrow{PQ}$ (Figure 11.3). The vector $\overrightarrow{QP}$ has its tail at $Q$ and its head at $P$. We also label vectors with single, boldfaced characters such as $\mathbf{u}$ and $\mathbf{v}$.

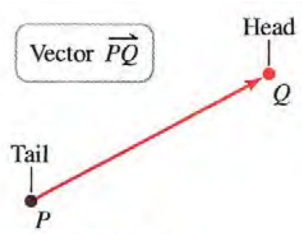


FIGURE 11.3

Two vectors $\mathbf{u}$ and $\mathbf{v}$ are **equal**, written $\mathbf{u} = \mathbf{v}$, if they have equal length and point in the same direction (Figure 11.4). An important fact is that equal vectors do not necessarily have the same location. *Any* two vectors with the same length and direction are equal.

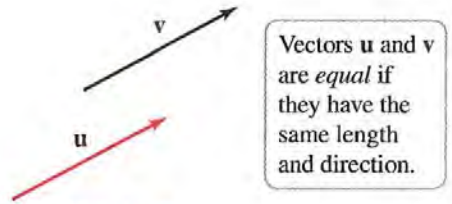


FIGURE 11.4

Not all quantities are represented by vectors. For example, mass, temperature, and price have magnitude, but no direction. Such quantities are described by real numbers and are called *scalars*.

### Vectors, Equal Vectors, Scalars, Zero Vector

**Vectors** are quantities that have both length (or magnitude) and direction. Two vectors are **equal** if they have the same magnitude and direction. Quantities having magnitude but no direction are called **scalars**. One exception is the **zero vector**, denoted $\mathbf{0}$: it has length 0 and no direction.

## Scalar Multiplication

A scalar $c$ and a vector $\mathbf{v}$ can be combined using **scalar-vector multiplication**, or simply **scalar multiplication**. The resulting vector, denoted $c\mathbf{v}$, is called a **scalar multiple** of $\mathbf{v}$. The magnitude of $c\mathbf{v}$ is $|c|$ multiplied by the magnitude of $\mathbf{v}$. The vector $c\mathbf{v}$ has the same direction as $\mathbf{v}$ if $c > 0$. If $c < 0$, then $c\mathbf{v}$ and $\mathbf{v}$ point in opposite directions. If $c = 0$, then $0 \cdot \mathbf{v} = \mathbf{0}$ (the zero vector).

For example, the vector $3\mathbf{v}$ is three times as long as $\mathbf{v}$ and has the same direction as $\mathbf{v}$. The vector $-2\mathbf{v}$ is twice as long as $\mathbf{v}$, but they point in opposite directions. The vector $\frac{1}{2}\mathbf{v}$ points in the same direction as $\mathbf{v}$ and has half the length of $\mathbf{v}$ (Figure 11.5). The vectors $\mathbf{v}$, $3\mathbf{v}$, $-2\mathbf{v}$, and $\mathbf{v}/2$ (that is, $\frac{1}{2}\mathbf{v}$) are examples of *parallel vectors*: each one is a scalar multiple of the others.

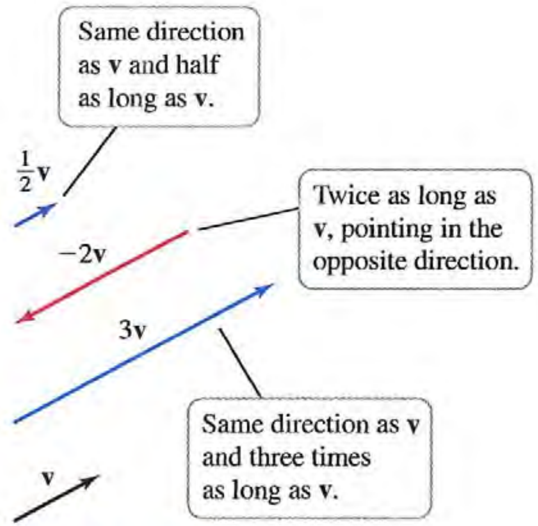


FIGURE 11.5

**DEFINITION Scalar Multiples and Parallel Vectors**

Given a scalar $c$ and a vector $\mathbf{v}$, the **scalar multiple** $c\mathbf{v}$ is a vector whose magnitude is $|c|$ multiplied by the magnitude of $\mathbf{v}$. If $c > 0$, then $c\mathbf{v}$ has the same direction as $\mathbf{v}$. If $c < 0$, then $c\mathbf{v}$ and $\mathbf{v}$ point in opposite directions. Two vectors are **parallel** if they are scalar multiples of each other.

Notice that two vectors are parallel if they point in the same direction (for example, $\mathbf{v}$ and $12\mathbf{v}$) *or* if they point in opposite directions (for example, $\mathbf{v}$ and $-2\mathbf{v}$). Also, because $0\mathbf{v} = \mathbf{0}$ for all vectors $\mathbf{v}$, it follows that *the zero vector is parallel to all vectors*. While it may seem counterintuitive, this result turns out to be a useful convention.

## Vector Addition and Subtraction

Figure 11.8 illustrates two ways to form the vector sum of two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ geometrically. The first method, called the **Triangle Rule**, places the tail of $\mathbf{v}$ at the head of $\mathbf{u}$. The sum $\mathbf{u} + \mathbf{v}$ is the vector that extends from the tail of $\mathbf{u}$ to the head of $\mathbf{v}$ (Figure 11.8b).

When $\mathbf{u}$ and $\mathbf{v}$ are not parallel, another way to form $\mathbf{u} + \mathbf{v}$ is to use the **Parallelogram Rule**. The *tails* of $\mathbf{u}$ and $\mathbf{v}$ are connected to form adjacent sides of a parallelogram; then, the remaining two sides of the parallelogram are sketched. The sum $\mathbf{u} + \mathbf{v}$ is the vector that coincides with the diagonal of the parallelogram, beginning at the tails of $\mathbf{u}$ and $\mathbf{v}$ (Figure 11.8c). The Triangle Rule and Parallelogram Rule each produce the same vector sum $\mathbf{u} + \mathbf{v}$.

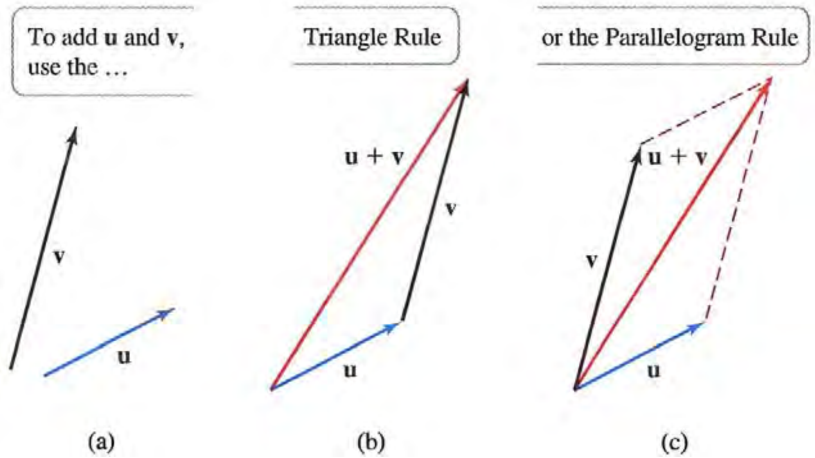



**FIGURE 11.8**

The difference $\mathbf{u} - \mathbf{v}$ is defined to be the sum $\mathbf{u} + (-\mathbf{v})$. By the Triangle Rule, the tail of $-\mathbf{v}$ is placed at the head of $\mathbf{u}$; then, $\mathbf{u} - \mathbf{v}$ extends from the tail of $\mathbf{u}$ to the head of $-\mathbf{v}$ (Figure 11.9a). Equivalently, when the tails of $\mathbf{u}$ and $\mathbf{v}$ coincide, $\mathbf{u} - \mathbf{v}$ has its tail at the head of $\mathbf{v}$ and its head at the head of $\mathbf{u}$ (Figure 11.9b).

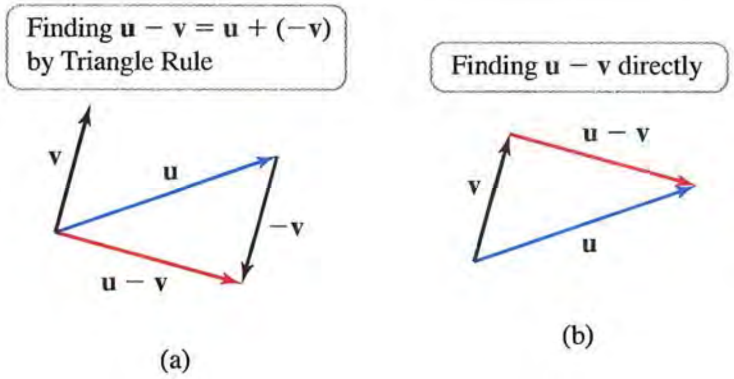



**FIGURE 11.9**

## Vector Components

So far, vectors have been examined from a geometric point of view. To do calculations with vectors, it is necessary to introduce a coordinate system. We begin by considering a vector $\mathbf{v}$ whose tail is at the origin in the Cartesian plane and whose head is at the point $(v_1, v_2)$ (Figure 11.12a).

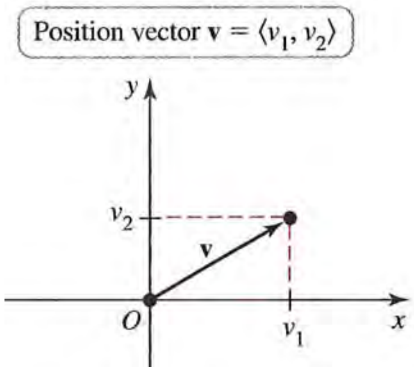



**FIGURE 11.12** (a)

**DEFINITION Position Vectors and Vector Components**

A vector **v** with its tail at the origin and head at $(v_1, v_2)$ is called a **position vector** (or is said to be in **standard position**) and is written $\langle v_1, v_2\rangle$. The real numbers $v_1$ and $v_2$ are the $x$- and $y$-**components** of **v**, respectively. The position vectors $\mathbf{u} = \langle u_1, u_2\rangle$ and $\mathbf{v} = \langle v_1, v_2\rangle$ are **equal** if and only if $u_1 = v_1$ and $u_2 = v_2$.

There are infinitely many vectors equal to the position vector **v**, all with the same length and direction (Figure 11.12b). It is important to abide by the convention that $\mathbf{v} = \langle v_1, v_2\rangle$ refers to the position vector **v** *or to any other vector equal to* **v**.

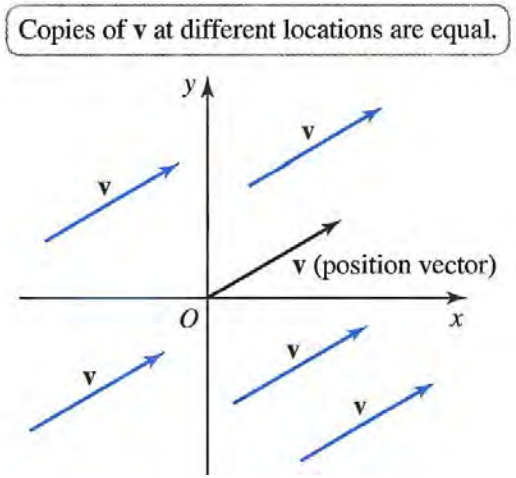



FIGURE 11.12 (b)

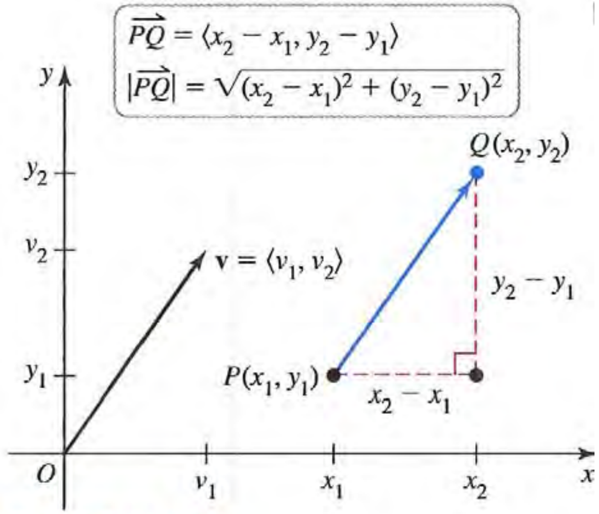



FIGURE 11.13

Now consider the vector $\overrightarrow{PQ}$, not in standard position, with its tail at the point $P(x_1, y_1)$ and its head at the point $Q(x_2, y_2)$. The $x$-component of $\overrightarrow{PQ}$ is the difference in the $x$-coordinates of $Q$ and $P$, or $x_2 - x_1$. The $y$-component of $\overrightarrow{PQ}$ is the difference in the $y$-coordinates, $y_2 - y_1$ (Figure 11.13). Therefore, $\overrightarrow{PQ}$ has the same length and direction as the position vector $\langle v_1, v_2\rangle = \langle x_2 - x_1, y_2 - y_1\rangle$, and we write $\overrightarrow{PQ} = \langle x_2 - x_1, y_2 - y_1\rangle$.

As already noted, there are infinitely many vectors equal to a given position vector. All these vectors have the same length and direction; therefore, they are all equal. In other words, two arbitrary vectors are **equal** if they are equal to the same position vector. For example, the vector $\overrightarrow{PQ}$ from $P(2,5)$ to $Q(6,3)$ and the vector $\overrightarrow{AB}$ from $A(7,12)$ to $B(11,10)$ are equal because they are both equal to the position vector $\langle 4, -2\rangle$.

**DEFINITION Magnitude of a Vector**

Given the points $P(x_1, y_1)$ and $Q(x_2, y_2)$, the **magnitude**, or **length**, of $\overrightarrow{PQ} = \langle x_2 - x_1, y_2 - y_1\rangle$, denoted $|\overrightarrow{PQ}|$, is the distance between $P$ and $Q$:

$$|\overrightarrow{PQ}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

The magnitude of the position vector $\mathbf{v} = \langle v_1, v_2\rangle$ is $|\mathbf{v}| = \sqrt{v_1^2 + v_2^2}$.

## EXAMPLE-4

**Calculating components and magnitude** Given the points $O(0,0)$, $P(-3,4)$, and $Q(6,5)$, find the components and magnitudes of the following vectors.

**a.** $\overrightarrow{OP}$ **b.** $\overrightarrow{PQ}$

**SOLUTION**

**a.** The vector $\overrightarrow{OP}$ is the position vector whose head is located at $P(-3,4)$. Therefore, $\overrightarrow{OP} = \langle -3, 4\rangle$ and $|\overrightarrow{OP}| = \sqrt{(-3)^2 + 4^2} = 5$.

**b.** $\overrightarrow{PQ} = \langle 6 - (-3), 5 - 4\rangle = \langle 9, 1\rangle$ and $|\overrightarrow{PQ}| = \sqrt{9^2 + 1^2} = \sqrt{82}$.

## Vector Operations in Terms of Components

We now show how vector addition, vector subtraction, and scalar multiplication are performed using components. Suppose $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$. The vector sum of $\mathbf{u}$ and $\mathbf{v}$ is $\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2 \rangle$. This definition of a vector sum is consistent with the Parallelogram Rule given earlier (Figure 11.14).

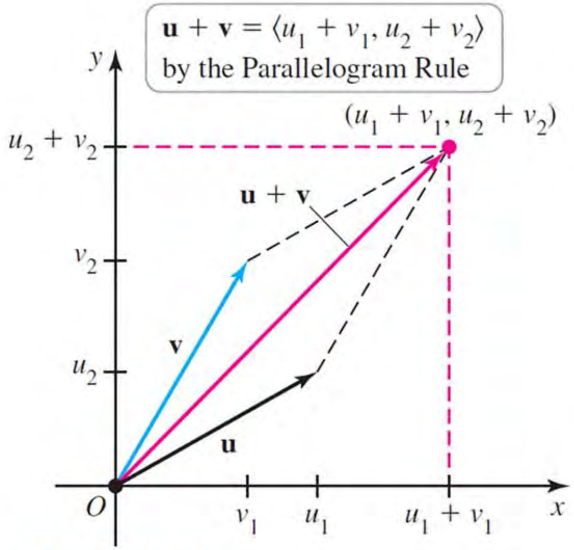



**FIGURE 11.14**

For a scalar $c$ and a vector $\mathbf{u}$, the scalar multiple $c\mathbf{u}$ is $c\mathbf{u} = \langle cu_1, cu_2 \rangle$; that is, the scalar $c$ multiplies each component of $\mathbf{u}$. If $c > 0$, $\mathbf{u}$ and $c\mathbf{u}$ have the same direction (Figure 11.15a). If $c < 0$, $\mathbf{u}$ and $c\mathbf{u}$ have opposite directions (Figure 11.15b). In either case, $|c\mathbf{u}| = |c||\mathbf{u}|$ (Exercise 81).

Notice that $\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v})$, where $-\mathbf{v} = \langle -v_1, -v_2 \rangle$. Therefore, the vector difference of $\mathbf{u}$ and $\mathbf{v}$ is $\mathbf{u} - \mathbf{v} = \langle u_1 - v_1, u_2 - v_2 \rangle$.

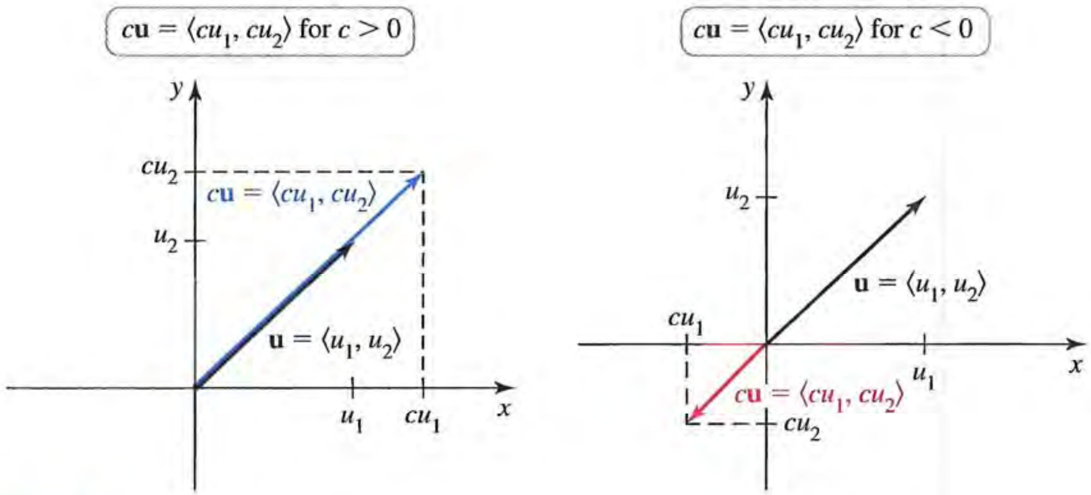



**FIGURE 11.15** (a) (b)

### Vector Operations

Suppose $c$ is a scalar, $\mathbf{u} = \langle u_1, u_2 \rangle$, and $\mathbf{v} = \langle v_1, v_2 \rangle$.

$$\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2 \rangle \quad \text{Vector addition}$$
$$\mathbf{u} - \mathbf{v} = \langle u_1 - v_1, u_2 - v_2 \rangle \quad \text{Vector subtraction}$$
$$c\mathbf{u} = \langle cu_1, cu_2 \rangle \quad \text{Scalar multiplication}$$

**EXAMPLE-5**

**Vector operations** Let $\mathbf{u} = \langle -1, 2 \rangle$ and $\mathbf{v} = \langle 2, 3 \rangle$.

**a.** Evaluate $|\mathbf{u} + \mathbf{v}|$. **b.** Simplify $2\mathbf{u} - 3\mathbf{v}$.

**c.** Find two vectors half as long as $\mathbf{u}$ and parallel to $\mathbf{u}$.

**SOLUTION**

**a.** Because $\mathbf{u} + \mathbf{v} = \langle -1, 2 \rangle + \langle 2, 3 \rangle = \langle 1, 5 \rangle$, we have $|\mathbf{u} + \mathbf{v}| = \sqrt{1^2 + 5^2} = \sqrt{26}$.

**b.** $2\mathbf{u} - 3\mathbf{v} = 2\langle -1, 2 \rangle - 3\langle 2, 3 \rangle = \langle -2, 4 \rangle - \langle 6, 9 \rangle = \langle -8, -5 \rangle$.

**c.** The vectors $\frac{1}{2}\mathbf{u} = \frac{1}{2}\langle -1, 2 \rangle = \left\langle -\frac{1}{2}, 1 \right\rangle$ and $-\frac{1}{2}\mathbf{u} = -\frac{1}{2}\langle -1, 2 \rangle = \left\langle \frac{1}{2}, -1 \right\rangle$ have half the length of $\mathbf{u}$ and are parallel to $\mathbf{u}$.

### Unit Vectors

A **unit vector** is any vector with length 1. Two useful unit vectors are the **coordinate unit vectors** $\mathbf{i} = \langle 1, 0 \rangle$ and $\mathbf{j} = \langle 0, 1 \rangle$ (Figure 11.16). These vectors are directed along the coordinate axes and allow us to express all vectors in an alternate form. For example, by the Triangle Rule (Figure 11.17a),

$$\langle 3, 4 \rangle = 3\langle 1, 0 \rangle + 4\langle 0, 1 \rangle = 3\mathbf{i} + 4\mathbf{j}.$$

In general, the vector $\mathbf{v} = \langle v_1, v_2 \rangle$ (Figure 11.17b) is also written

$$\mathbf{v} = v_1\langle 1, 0 \rangle + v_2\langle 0, 1 \rangle = v_1\mathbf{i} + v_2\mathbf{j}.$$

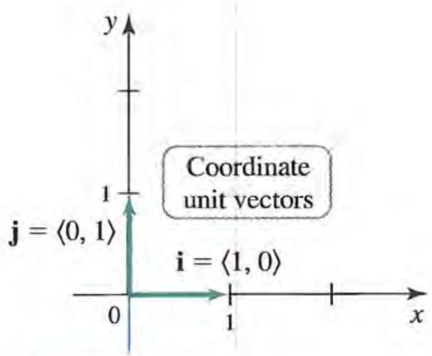



FIGURE 11.16

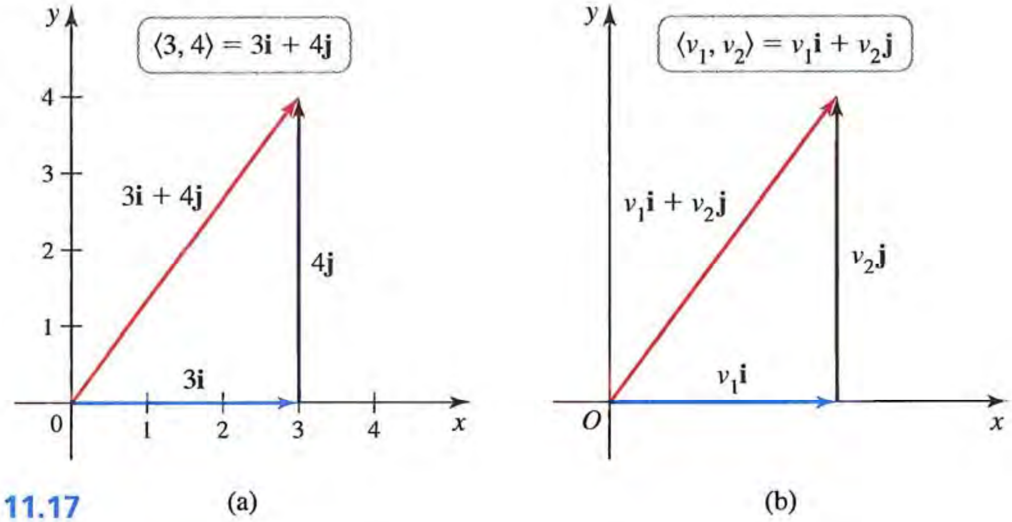



FIGURE 11.17 (a) (b)

Given a nonzero vector $\mathbf{v}$, we sometimes need to construct a new vector parallel to $\mathbf{v}$ of a specified length. Dividing $\mathbf{v}$ by its length, we obtain the vector $\mathbf{u} = \dfrac{\mathbf{v}}{|\mathbf{v}|}$. Because $\mathbf{u}$ is a positive scalar multiple of $\mathbf{v}$, it follows that $\mathbf{u}$ has the same direction as $\mathbf{v}$. Furthermore, $\mathbf{u}$ is a unit vector because $|\mathbf{u}| = \dfrac{|\mathbf{v}|}{|\mathbf{v}|} = 1$. The vector $-\mathbf{u} = -\dfrac{\mathbf{v}}{|\mathbf{v}|}$ is also a unit vector (Figure 11.18). Therefore, $\pm\dfrac{\mathbf{v}}{|\mathbf{v}|}$ are unit vectors parallel to $\mathbf{v}$ that point in opposite directions.

To construct a vector that points in the direction of $\mathbf{v}$ and has a specified length $c > 0$, we form the vector $\dfrac{c\mathbf{v}}{|\mathbf{v}|}$: It is a positive scalar multiple of $\mathbf{v}$, so it points in the direction of $\mathbf{v}$, and its length is $\left|\dfrac{c\mathbf{v}}{|\mathbf{v}|}\right| = |c|\dfrac{|\mathbf{v}|}{|\mathbf{v}|} = c$. The vector $-\dfrac{c\mathbf{v}}{|\mathbf{v}|}$ points in the opposite direction and also has length $c$.

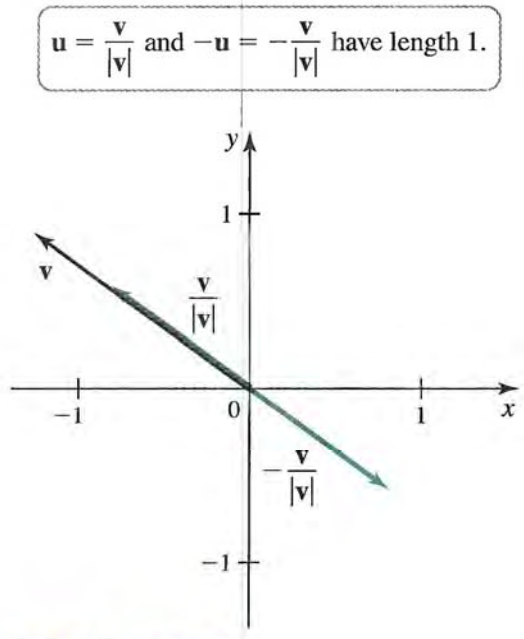



FIGURE 11.18

**DEFINITION Unit Vectors and Vectors of a Specified Length**

A **unit vector** is any vector with length 1. Given a nonzero vector **v**, $\pm\frac{\mathbf{v}}{|\mathbf{v}|}$ are unit vectors parallel to **v**. For a scalar $c > 0$, the vectors $\pm\frac{c\mathbf{v}}{|\mathbf{v}|}$ are vectors of length $c$ parallel to **v**.

## EXAMPLE-6

**Magnitude and unit vectors** Consider the points $P(1, -2)$ and $Q(6, 10)$.

**a.** Find $\overrightarrow{PQ}$ and two unit vectors parallel to $\overrightarrow{PQ}$.

**b.** Find two vectors of length 2 parallel to $\overrightarrow{PQ}$.

**SOLUTION**

**a.** $\overrightarrow{PQ} = \langle 6 - 1, 10 - (-2) \rangle = \langle 5, 12 \rangle$, or $5\mathbf{i} + 12\mathbf{j}$. Because $|\overrightarrow{PQ}| = \sqrt{5^2 + 12^2} = \sqrt{169} = 13$, a unit vector parallel to $\overrightarrow{PQ}$ is

$$\frac{\overrightarrow{PQ}}{|\overrightarrow{PQ}|} = \frac{\langle 5, 12 \rangle}{13} = \left\langle \frac{5}{13}, \frac{12}{13} \right\rangle = \frac{5}{13}\mathbf{i} + \frac{12}{13}\mathbf{j}.$$

Another unit vector parallel to $\overrightarrow{PQ}$ but having the opposite direction is $\left\langle -\frac{5}{13}, -\frac{12}{13} \right\rangle$.

**b.** To obtain two vectors of length 2 that are parallel to $\overrightarrow{PQ}$, we multiply the unit vector $\frac{5}{13}\mathbf{i} + \frac{12}{13}\mathbf{j}$ by $\pm 2$:

$$2\left(\frac{5}{13}\mathbf{i} + \frac{12}{13}\mathbf{j}\right) = \frac{10}{13}\mathbf{i} + \frac{24}{13}\mathbf{j} \quad \text{and} \quad -2\left(\frac{5}{13}\mathbf{i} + \frac{12}{13}\mathbf{j}\right) = -\frac{10}{13}\mathbf{i} - \frac{24}{13}\mathbf{j}$$

## Properties of Vector Operations

When we stand back and look at vector operations, ten general properties emerge. For example, the first property says that vector addition is commutative, which means $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$. This property is proved by letting $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$. By the commutative property of addition for real numbers,

$$\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2 \rangle = \langle v_1 + u_1, v_2 + u_2 \rangle = \mathbf{v} + \mathbf{u}.$$

### SUMMARY Properties of Vector Operations

Suppose $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors and $a$ and $c$ are scalars.

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
3. $\mathbf{v} + \mathbf{0} = \mathbf{v}$
4. $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$
5. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
6. $(a + c)\mathbf{v} = a\mathbf{v} + c\mathbf{v}$
7. $0\mathbf{v} = \mathbf{0}$
8. $c\mathbf{0} = \mathbf{0}$
9. $1\mathbf{v} = \mathbf{v}$
10. $a(c\mathbf{v}) = (ac)\mathbf{v}$

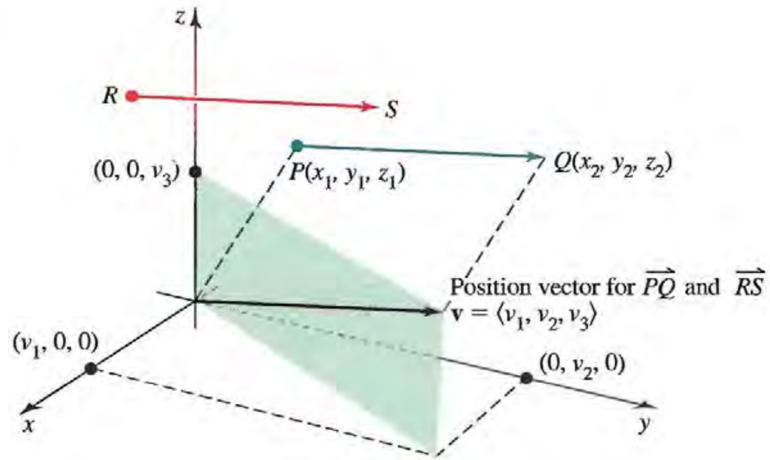



FIGURE 11.35

## Vectors in $\mathbf{R}^3$

Vectors in $\mathbf{R}^3$ are straightforward extensions of vectors in the $xy$-plane; we simply include a third component. The position vector $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ has its tail at the origin and its head at the point $(v_1, v_2, v_3)$. Vectors having the same magnitude and direction are equal. Therefore, the vector from $P(x_1, y_1, z_1)$ to $Q(x_2, y_2, z_2)$ is denoted $\overrightarrow{PQ}$ and is equal to the position vector $\langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$. It is also equal to all vectors such as $\overrightarrow{RS}$ that have the same length and direction as $\mathbf{v}$ (Figure 11.35).

The operations of vector addition and scalar multiplication in $\mathbf{R}^2$ generalize in a natural way to three dimensions. For example, the sum of two vectors is found geometrically using the Triangle Rule or the Parallelogram Rule (Section 11.1). The sum is found analytically by adding the respective components of the two vectors. As with two-dimensional vectors, scalar multiplication corresponds to stretching or compressing a vector, possibly with a reversal of direction. Two nonzero vectors are parallel if one is a scalar multiple of the other (Figure 11.36).

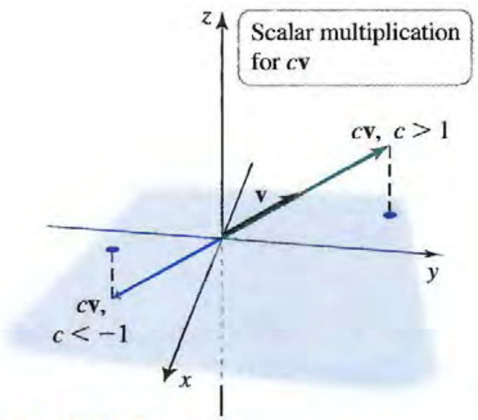


FIGURE 11.36

**DEFINITION** **Vector Operations in $\mathbf{R}^3$**

Let $c$ be a scalar, $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$, and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$.

$$\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle \quad \text{Vector addition}$$

$$\mathbf{u} - \mathbf{v} = \langle u_1 - v_1, u_2 - v_2, u_3 - v_3 \rangle \quad \text{Vector subtraction}$$

$$c\mathbf{u} = \langle cu_1, cu_2, cu_3 \rangle \quad \text{Scalar multiplication}$$

## EXAMPLE-7

**Vectors in $\mathbf{R}^3$** Let $\mathbf{u} = \langle 2, -4, 1 \rangle$ and $\mathbf{v} = \langle 3, 0, -1 \rangle$. Find the components of the following vectors and draw them in $\mathbf{R}^3$.

**a.** $2\mathbf{u}$ **b.** $-2\mathbf{v}$ **c.** $\mathbf{u} + 2\mathbf{v}$

**SOLUTION**

**a.** Using the definition of scalar multiplication, $2\mathbf{u} = 2\langle 2, -4, 1 \rangle = \langle 4, -8, 2 \rangle$. The vector $2\mathbf{u}$ has the same direction as $\mathbf{u}$ with twice the magnitude of $\mathbf{u}$ (Figure 11.37).

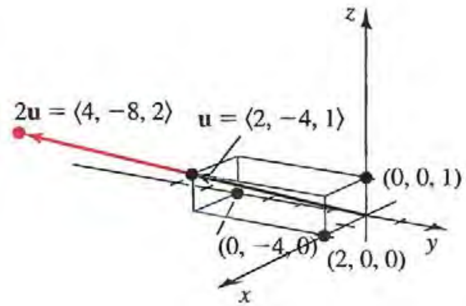


FIGURE 11.37

**b.** Using scalar multiplication, $-2\mathbf{v} = -2\langle 3, 0, -1 \rangle = \langle -6, 0, 2 \rangle$. The vector $-2\mathbf{v}$ has the opposite direction as $\mathbf{v}$ and twice the magnitude of $\mathbf{v}$ (Figure 11.38).

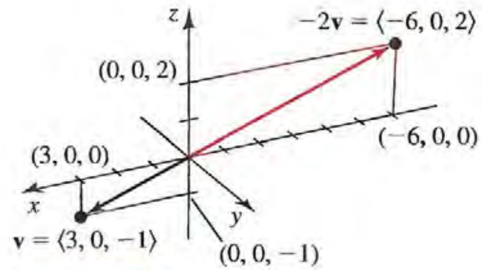


FIGURE 11.38

**c.** Using vector addition and scalar multiplication,

$$\mathbf{u} + 2\mathbf{v} = \langle 2, -4, 1 \rangle + 2\langle 3, 0, -1 \rangle = \langle 8, -4, -1 \rangle.$$

The vector $\mathbf{u} + 2\mathbf{v}$ is drawn by applying the Parallelogram Rule to $\mathbf{u}$ and $2\mathbf{v}$ (Figure 11.39).

**DEFINTION** **Magnitude of a Vector**

The **magnitude** (or **length**) of the vector $\overrightarrow{PQ} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$ is the distance from $P(x_1, y_1, z_1)$ to $Q(x_2, y_2, z_2)$:

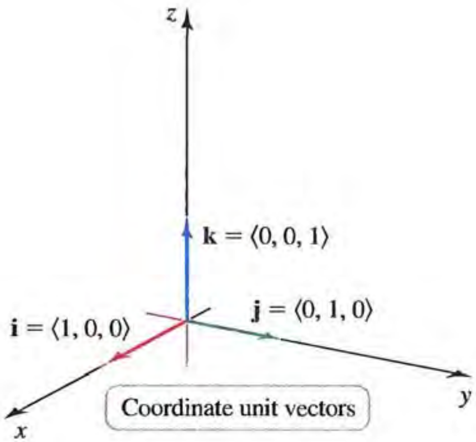
$$|\overrightarrow{PQ}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

The coordinate unit vectors introduced in Section 11.1 extend naturally to three dimensions. The three coordinate unit vectors in $\mathbf{R}^3$ are (Figure 11.41)

$$\mathbf{i} = \langle 1, 0, 0 \rangle, \quad \mathbf{j} = \langle 0, 1, 0 \rangle, \quad \text{and} \quad \mathbf{k} = \langle 0, 0, 1 \rangle.$$

These unit vectors give an alternative way of expressing position vectors. If $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$, then we have

$$\mathbf{v} = v_1\langle 1, 0, 0 \rangle + v_2\langle 0, 1, 0 \rangle + v_3\langle 0, 0, 1 \rangle = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}.$$



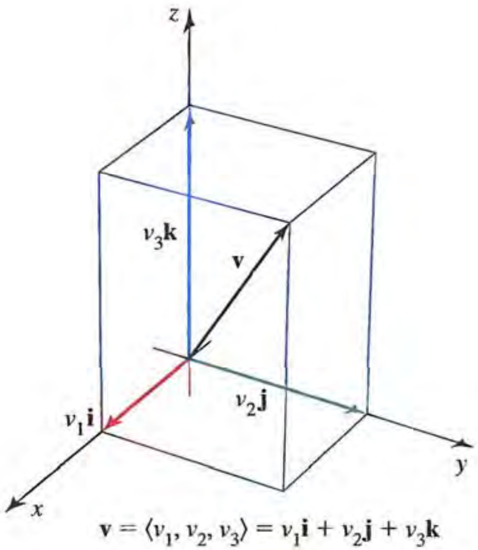




**FIGURE 11.41**

## EXAMPLE-8

**Magnitudes and unit vectors** Consider the points $P(5,3,1)$ and $Q(-7,8,1)$.

**a.** Express $\overrightarrow{PQ}$ in terms of the unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$.

**b.** Find the magnitude of $\overrightarrow{PQ}$.

**c.** Find the position vector of magnitude 10 in the direction of $\overrightarrow{PQ}$.

### SOLUTION

**a.** $\overrightarrow{PQ}$ is equal to the position vector $\langle -7-5, 8-3, 1-1\rangle = \langle -12, 5, 0\rangle$. Thus, $\overrightarrow{PQ} = -12\mathbf{i} + 5\mathbf{j}$.

**b.** $|\overrightarrow{PQ}| = |-12\mathbf{i} + 5\mathbf{j}| = \sqrt{12^2 + 5^2} = \sqrt{169} = 13$

**c.** The unit vector in the direction of $\overrightarrow{PQ}$ is $\mathbf{u} = \dfrac{\overrightarrow{PQ}}{|\overrightarrow{PQ}|} = \frac{1}{13}\langle -12, 5, 0\rangle$. Therefore, a vector in the direction of $\mathbf{u}$ with a magnitude of 10 is $10\mathbf{u} = \frac{10}{13}\langle -12, 5, 0\rangle$.

# DOT PRODUCT; PROJECTIONS

The *dot product* is used to determine the angle between two vectors. It is also a tool for calculating *projections*—the measure of how much of a given vector lies in the direction of another vector.

The dot product is also called the *scalar product*, a term we do not use in order to avoid confusion with *scalar multiplication*.

**DEFINITION** **Dot Product**

Given two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ in two or three dimensions, their **dot product** is

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos \theta,$$

where $\theta$ is the angle between $\mathbf{u}$ and $\mathbf{v}$ with $0 \leq \theta \leq \pi$ (Figure 11.44). If $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$, then $\mathbf{u} \cdot \mathbf{v} = 0$, and $\theta$ is undefined.

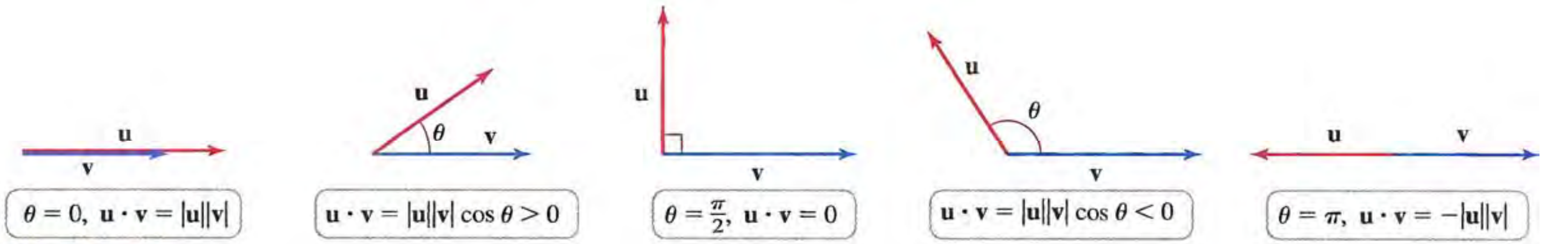



FIGURE 11.44

The dot product of two vectors is itself a scalar. Two special cases immediately arise:

- $\mathbf{u}$ and $\mathbf{v}$ are parallel ($\theta = 0$ or $\theta = \pi$) if and only if $\mathbf{u} \cdot \mathbf{v} = \pm|\mathbf{u}||\mathbf{v}|$.
- $\mathbf{u}$ and $\mathbf{v}$ are perpendicular ($\theta = \pi/2$) if and only if $\mathbf{u} \cdot \mathbf{v} = 0$.

The second case gives rise to the important property of **orthogonality**.

**DEFINITION** **Orthogonal Vectors**

Two vectors $\mathbf{u}$ and $\mathbf{v}$ are **orthogonal** if and only if $\mathbf{u} \cdot \mathbf{v} = 0$. The zero vector is orthogonal to all vectors. In two or three dimesions, two nonzero orthogonal vectors are perpendicular to each other.

## EXAMPLE-1

**Dot products** Compute the dot products of the following vectors.

**a.** $\mathbf{u} = 2\mathbf{i} - 6\mathbf{j}$ and $\mathbf{v} = 12\mathbf{k}$

**b.** $\mathbf{u} = \langle\sqrt{3}, 1\rangle$ and $\mathbf{v} = \langle 0, 1\rangle$

### SOLUTION

**a.** The vector $\mathbf{u}$ lies in the $xy$-plane and the vector $\mathbf{v}$ is perpendicular to the $xy$-plane. Therefore, $\theta = \frac{\pi}{2}$, $\mathbf{u}$ and $\mathbf{v}$ are orthogonal, and $\mathbf{u} \cdot \mathbf{v} = 0$ (Figure 11.45a).

**b.** As shown in Figure 11.45b, $\mathbf{u}$ and $\mathbf{v}$ form two sides of a 30–60–90 triangle in the $xy$-plane, with an angle of $\pi/3$ between them. Because $|\mathbf{u}| = 2$, $|\mathbf{v}| = 1$, and $\cos \pi/3 = 1/2$, the dot product is

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos\theta = 2 \cdot 1 \cdot \tfrac{1}{2} = 1.$$

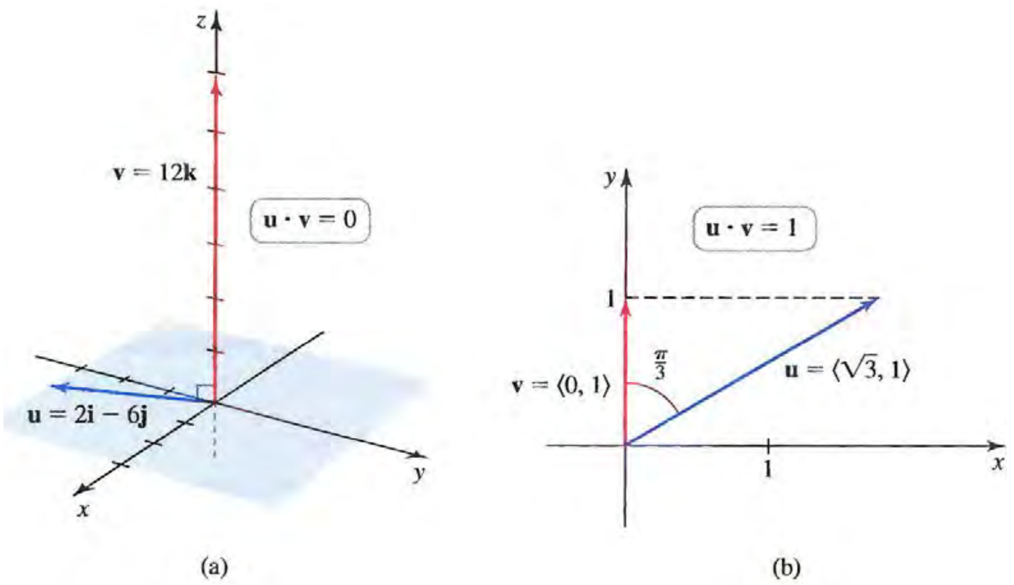



**FIGURE 11.45**

The definition of the dot product requires knowing the angle $\theta$ between the vectors. Often the angle is not known; in fact, it may be exactly what we seek. For this reason, we present another method for computing the dot product that does not require knowing $\theta$.

**THEOREM 11.1 Dot Product**
Given two vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$,

$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3.$$

Theorem 11.1 extends to vectors with any number of components. If $\mathbf{u} = \langle u_1, \ldots, u_n \rangle$ and $\mathbf{v} = \langle v_1, \ldots, v_n \rangle$, then

$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + \cdots + u_nv_n.$$

The properties in Theorem 11.2 also apply in two or more dimensions.

This new representation of $\mathbf{u} \cdot \mathbf{v}$ has two immediate consequences.

1. Combining it with the definition of dot product gives

$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3 = |\mathbf{u}||\mathbf{v}| \cos \theta.$$

If $\mathbf{u}$ and $\mathbf{v}$ are both nonzero, then

$$\cos \theta = \frac{u_1v_1 + u_2v_2 + u_3v_3}{|\mathbf{u}||\mathbf{v}|},$$

and we have a way to compute $\theta$.

2. Notice that $\mathbf{u} \cdot \mathbf{u} = u_1^2 + u_2^2 + u_3^2 = |\mathbf{u}|^2$. Therefore, we have a relationship between the dot product and the magnitude of a vector: $|\mathbf{u}| = \sqrt{\mathbf{u} \cdot \mathbf{u}}$ or $|\mathbf{u}|^2 = \mathbf{u} \cdot \mathbf{u}$.

## EXAMPLE-2

**Dot products and angles** Let $\mathbf{u} = \langle \sqrt{3}, 1, 0 \rangle$, $\mathbf{v} = \langle 1, \sqrt{3}, 0 \rangle$, and $\mathbf{w} = \langle 1, \sqrt{3}, 2\sqrt{3} \rangle$.

a. Compute $\mathbf{u} \cdot \mathbf{v}$.

b. Find the angle between $\mathbf{u}$ and $\mathbf{v}$.

c. Find the angle between $\mathbf{u}$ and $\mathbf{w}$.

**SOLUTION**

**a.** $\mathbf{u}\cdot\mathbf{v} = \langle\sqrt{3},1,0\rangle\cdot\langle 1,\sqrt{3},0\rangle = \sqrt{3}+\sqrt{3}+0 = 2\sqrt{3}$

**b.** Note that $|\mathbf{u}| = \sqrt{\mathbf{u}\cdot\mathbf{u}} = \sqrt{\langle\sqrt{3},1,0\rangle\cdot\langle\sqrt{3},1,0\rangle} = 2$ and similarly $|\mathbf{v}| = 2$. Therefore,

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|} = \frac{2\sqrt{3}}{2\cdot 2} = \frac{\sqrt{3}}{2}.$$

Because $0 \le \theta \le \pi$, it follows that $\theta = \pi/6$.

**c.** $$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{w}}{|\mathbf{u}||\mathbf{w}|} = \frac{\langle\sqrt{3},1,0\rangle\cdot\langle 1,\sqrt{3},2\sqrt{3}\rangle}{|\langle\sqrt{3},1,0\rangle||\langle 1,\sqrt{3},2\sqrt{3}\rangle|} = \frac{2\sqrt{3}}{2\cdot 4} = \frac{\sqrt{3}}{4}$$ It follows that

$$\theta = \cos^{-1}\left(\frac{\sqrt{3}}{4}\right) \approx 1.12\text{ rad} \approx 64.3^\circ.$$

## DIRECTION ANGLES

In an $xy$-coordinate system, the direction of a nonzero vector $\mathbf{v}$ is completely determined by the angles $\alpha$ and $\beta$ between $\mathbf{v}$ and the unit vectors $\mathbf{i}$ and $\mathbf{j}$ (Figure 11.3.4), and in an $xyz$-coordinate system the direction is completely determined by the angles $\alpha$, $\beta$, and $\gamma$ between $\mathbf{v}$ and the unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ (Figure 11.3.5). In both 2-space and 3-space the angles between a nonzero vector $\mathbf{v}$ and the vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are called the ***direction angles*** of $\mathbf{v}$, and the cosines of those angles are called the ***direction cosines*** of $\mathbf{v}$. Formulas for the direction cosines of a vector can be obtained from Formula (2). For example, if $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$, then

$$\cos\alpha = \frac{\mathbf{v}\cdot\mathbf{i}}{\|\mathbf{v}\|\|\mathbf{i}\|} = \frac{v_1}{\|\mathbf{v}\|},\quad \cos\beta = \frac{\mathbf{v}\cdot\mathbf{j}}{\|\mathbf{v}\|\|\mathbf{j}\|} = \frac{v_2}{\|\mathbf{v}\|},\quad \cos\gamma = \frac{\mathbf{v}\cdot\mathbf{k}}{\|\mathbf{v}\|\|\mathbf{k}\|} = \frac{v_3}{\|\mathbf{v}\|}$$

Thus, we have the following theorem.

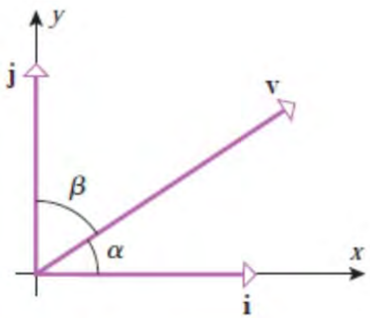
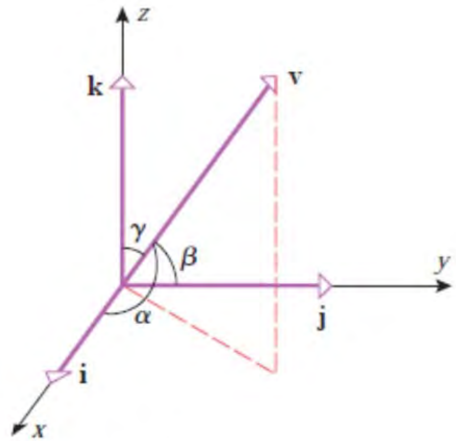



Figure 11.3.4




▲ Figure 11.3.5

**11.3.4 THEOREM** *The direction cosines of a nonzero vector* $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ *are*

$$\cos\alpha = \frac{v_1}{\|\mathbf{v}\|},\quad \cos\beta = \frac{v_2}{\|\mathbf{v}\|},\quad \cos\gamma = \frac{v_3}{\|\mathbf{v}\|}$$

The direction cosines of a vector $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ can be computed by normalizing $\mathbf{v}$ and reading off the components of $\mathbf{v}/\|\mathbf{v}\|$, since

$$\frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{v_1}{\|\mathbf{v}\|}\mathbf{i} + \frac{v_2}{\|\mathbf{v}\|}\mathbf{j} + \frac{v_3}{\|\mathbf{v}\|}\mathbf{k} = (\cos\alpha)\mathbf{i} + (\cos\beta)\mathbf{j} + (\cos\gamma)\mathbf{k}$$

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

**EXAMPLE-3**

Find the direction cosines of the vector $\mathbf{v} = 2\mathbf{i} - 4\mathbf{j} + 4\mathbf{k}$, and approximate the direction angles to the nearest degree.

***Solution.*** First we will normalize the vector $\mathbf{v}$ and then read off the components. We have $\|\mathbf{v}\| = \sqrt{4 + 16 + 16} = 6$, so that $\mathbf{v}/\|\mathbf{v}\| = \frac{1}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}$. Thus,

$$\cos\alpha = \tfrac{1}{3}, \quad \cos\beta = -\tfrac{2}{3}, \quad \cos\gamma = \tfrac{2}{3}$$

With the help of a calculating utility we obtain

$$\alpha = \cos^{-1}\left(\tfrac{1}{3}\right) \approx 71^\circ, \quad \beta = \cos^{-1}\left(-\tfrac{2}{3}\right) \approx 132^\circ, \quad \gamma = \cos^{-1}\left(\tfrac{2}{3}\right) \approx 48^\circ \quad ◄$$

## Orthogonal Projections

Given vectors **u** and **v**, how closely aligned are they? That is, how much of **u** points in the direction of **v**? This question is answered using *projections*. As shown in Figure 11.47a, the projection of the vector **u** onto a nonzero vector **v**, denoted $\text{proj}_{\mathbf{v}}\mathbf{u}$, is the "shadow" cast by **u** onto the line through **v**. The projection of **u** onto **v** is itself a vector; it points in the same direction as **v** if the angle between **u** and **v** lies in the interval $0 \le \theta < \pi/2$ (Figure 11.47b); it points in the direction opposite to that of **v** if the angle between **u** and **v** lies in the interval $\pi/2 < \theta \le \pi$ (Figure 11.47c).

To find the projection of **u** onto **v**, we proceed as follows: With the tails of **u** and **v** together, we drop a perpendicular line segment from the head of **u** to the point $P$ on the line through **v** (Figure 11.48). The vector $\overline{OP}$ is the *orthogonal projection of* **u** *onto* **v**.

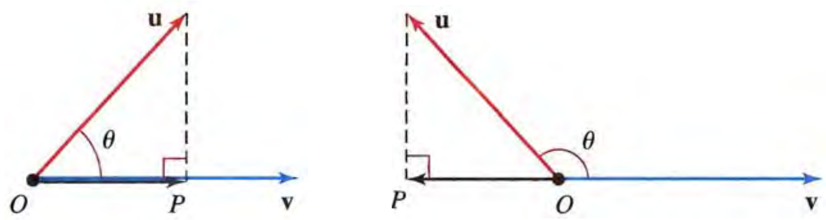



FIGURE 11.48

**DEFINITION (Orthogonal) Projection of u onto v**

The **orthogonal projection of u onto v**, denoted $\text{proj}_{\mathbf{v}}\mathbf{u}$, where $\mathbf{v} \neq \mathbf{0}$, is

$$\text{proj}_{\mathbf{v}}\mathbf{u} = |\mathbf{u}| \cos\theta \left(\frac{\mathbf{v}}{|\mathbf{v}|}\right).$$

The orthogonal projection may also be computed with the formulas

$$\text{proj}_{\mathbf{v}}\mathbf{u} = \text{scal}_{\mathbf{v}}\mathbf{u}\left(\frac{\mathbf{v}}{|\mathbf{v}|}\right) = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v},$$

where the **scalar component of u in the direction of v** is

$$\text{scal}_{\mathbf{v}}\mathbf{u} = |\mathbf{u}| \cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|}.$$

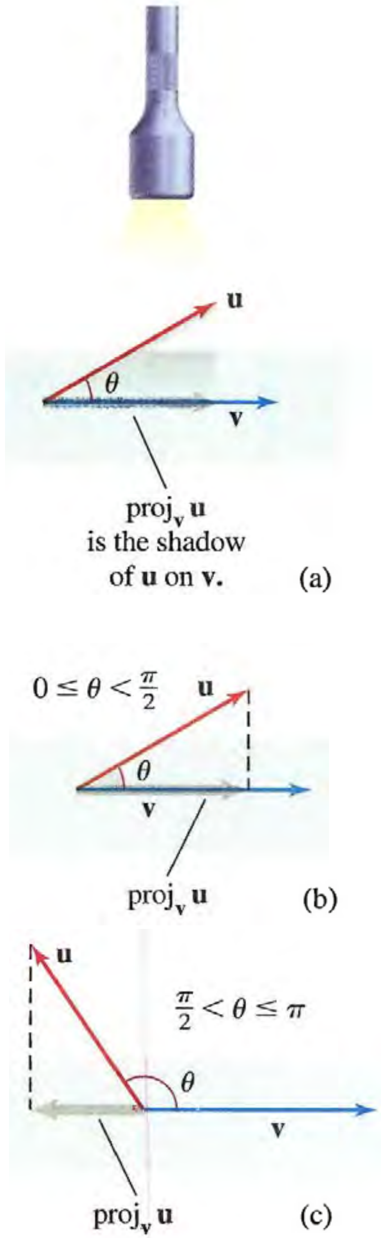



FIGURE 11.47

## EXAMPLE-4

**Orthogonal projections** Find $\text{proj}_{\mathbf{v}}\mathbf{u}$ and $\text{scal}_{\mathbf{v}}\mathbf{u}$ for the following vectors and illustrate each result.

**a.** $\mathbf{u} = \langle 4, 1\rangle, \mathbf{v} = \langle 3, 4\rangle$ **b.** $\mathbf{u} = \langle -4, -3\rangle, \mathbf{v} = \langle 1, -1\rangle$

### SOLUTION

**a.** The scalar component of $\mathbf{u}$ in the direction of $\mathbf{v}$ (Figure 11.49) is

$$\text{scal}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|} = \frac{\langle 4,1\rangle\cdot\langle 3,4\rangle}{|\langle 3,4\rangle|} = \frac{16}{5}.$$

Because $\dfrac{\mathbf{v}}{|\mathbf{v}|} = \left\langle \frac{3}{5}, \frac{4}{5}\right\rangle$, we have

$$\text{proj}_{\mathbf{v}}\mathbf{u} = \text{scal}_{\mathbf{v}}\mathbf{u}\left(\frac{\mathbf{v}}{|\mathbf{v}|}\right) = \frac{16}{5}\left\langle \frac{3}{5}, \frac{4}{5}\right\rangle = \frac{16}{25}\langle 3, 4\rangle.$$

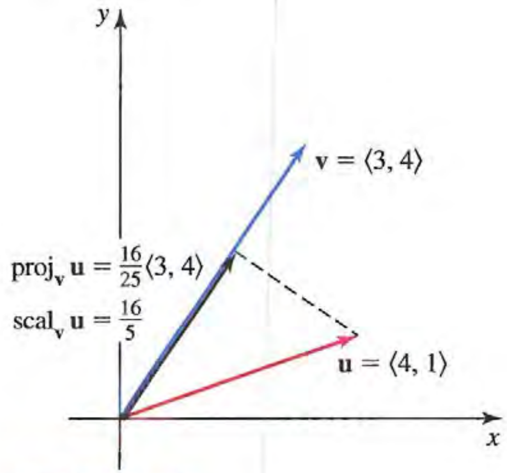



FIGURE 11.49

**b.** Using another formula for $\text{proj}_{\mathbf{v}}\mathbf{u}$, we have

$$\text{proj}_{\mathbf{v}}\mathbf{u} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v} = \left(\frac{\langle -4,-3\rangle\cdot\langle 1,-1\rangle}{\langle 1,-1\rangle\cdot\langle 1,-1\rangle}\right)\langle 1,-1\rangle = -\frac{1}{2}\langle 1,-1\rangle.$$

The vectors $\mathbf{v}$ and $\text{proj}_{\mathbf{v}}\mathbf{u}$ point in opposite directions because $\pi/2 < \theta \leq \pi$ (Figure 11.50). This fact is reflected in the scalar component of $\mathbf{u}$ in the direction of $\mathbf{v}$, which is negative:

$$\text{scal}_{\mathbf{v}}\mathbf{u} = \frac{\langle -4,-3\rangle\cdot\langle 1,-1\rangle}{|\langle 1,-1\rangle|} = -\frac{1}{\sqrt{2}}.$$

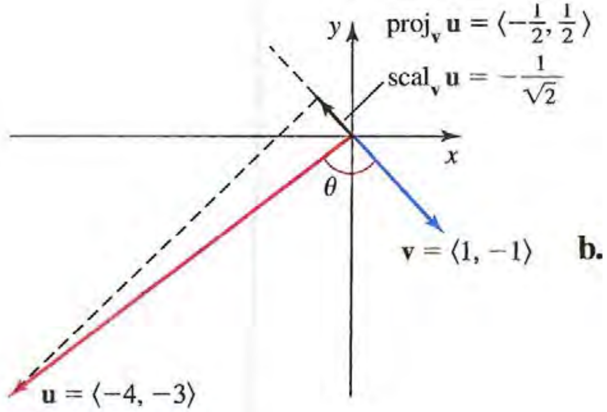



FIGURE 11.50

# CROSS PRODUCT

**DEFINITION Cross Product**

Given two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbf{R}^3$, the **cross product** $\mathbf{u} \times \mathbf{v}$ is a vector with magnitude

$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin\theta,$$

where $0 \leq \theta \leq \pi$ is the angle between $\mathbf{u}$ and $\mathbf{v}$. The direction of $\mathbf{u} \times \mathbf{v}$ is given by the **right-hand rule**: When you put the vectors tail to tail and let the fingers of your right hand curl from $\mathbf{u}$ to $\mathbf{v}$, the direction of $\mathbf{u} \times \mathbf{v}$ is the direction of your thumb, orthogonal to both $\mathbf{u}$ and $\mathbf{v}$ (Figure 11.56). When $\mathbf{u} \times \mathbf{v} = \mathbf{0}$, the direction of $\mathbf{u} \times \mathbf{v}$ is undefined.

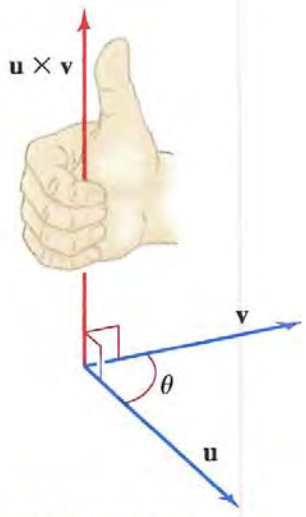


FIGURE 11.56

**THEOREM 11.3 Geometry of the Cross Product**

Let $\mathbf{u}$ and $\mathbf{v}$ be two nonzero vectors in $\mathbf{R}^3$.

1. The vectors $\mathbf{u}$ and $\mathbf{v}$ are parallel ($\theta = 0$ or $\theta = \pi$) if and only if $\mathbf{u} \times \mathbf{v} = \mathbf{0}$.
2. If $\mathbf{u}$ and $\mathbf{v}$ are two sides of a parallelogram (Figure 11.57), then the area of the parallelogram is
$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin\theta.$$

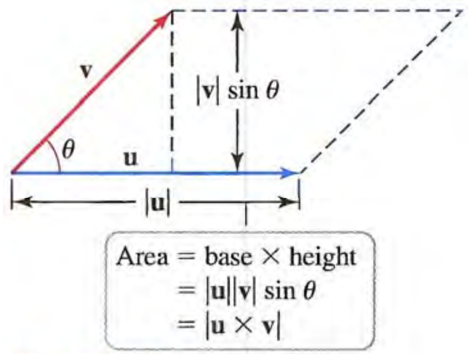


FIGURE 11.57

## EXAMPLE-5

**A cross product** Find the magnitude and direction of $\mathbf{u} \times \mathbf{v}$, where $\mathbf{u} = \langle 1, 1, 0 \rangle$ and $\mathbf{v} = \langle 1, 1, \sqrt{2} \rangle$.

**SOLUTION** Because $\mathbf{u}$ is one side of a 45–45–90 triangle and $\mathbf{v}$ is the hypotenuse (Figure 11.58), we have $\theta = \pi/4$ and $\sin\theta = \frac{1}{\sqrt{2}}$. Also, $|\mathbf{u}| = \sqrt{2}$ and $|\mathbf{v}| = 2$, so the magnitude of $\mathbf{u} \times \mathbf{v}$ is

$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin\theta = \sqrt{2} \cdot 2 \cdot \frac{1}{\sqrt{2}} = 2.$$

The direction of $\mathbf{u} \times \mathbf{v}$ is given by the right-hand rule: $\mathbf{u} \times \mathbf{v}$ is orthogonal to $\mathbf{u}$ and $\mathbf{v}$

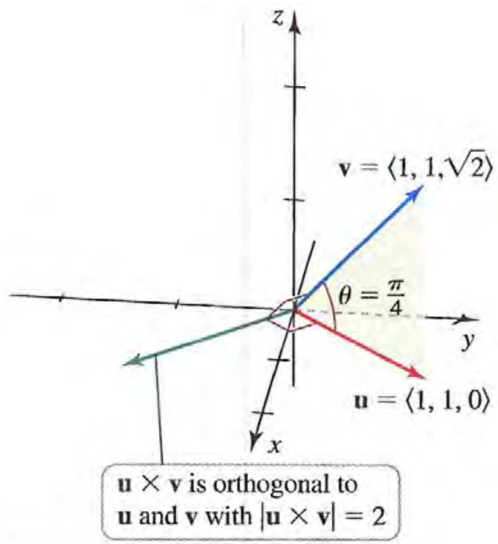



**FIGURE 11.58**

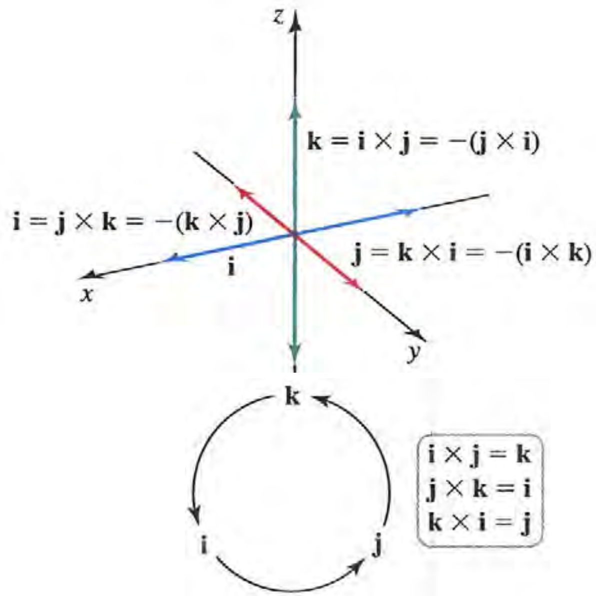
**THEOREM 11.4 Properties of the Cross Product**
Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be nonzero vectors in $\mathbf{R}^3$, and let $a$ and $b$ be scalars.

1. $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$ — Anticommutative property
2. $(a\mathbf{u}) \times (b\mathbf{v}) = ab(\mathbf{u} \times \mathbf{v})$ — Associative property
3. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$ — Distributive property
4. $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \times \mathbf{w}) + (\mathbf{v} \times \mathbf{w})$ — Distributive property

## EXAMPLE-6

**Cross products of unit vectors** Evaluate all the cross products among the coordinate unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$.

**SOLUTION** These vectors are mutually orthogonal, which means the angle between any two distinct vectors is $\theta = \pi/2$ and $\sin\theta = 1$. Furthermore, $|\mathbf{i}| = |\mathbf{j}| = |\mathbf{k}| = 1$. Therefore, the cross product of any two distinct vectors has magnitude 1. By the right-hand rule, when the fingers of the right hand curl from $\mathbf{i}$ to $\mathbf{j}$, the thumb points in the direction of the positive $z$-axis (Figure 11.59). The unit vector in the positive $z$-direction is $\mathbf{k}$, so $\mathbf{i} \times \mathbf{j} = \mathbf{k}$. Similar calculations show that $\mathbf{j} \times \mathbf{k} = \mathbf{i}$ and $\mathbf{k} \times \mathbf{i} = \mathbf{j}$.



By property 1 of Theorem 11.4, $\mathbf{j} \times \mathbf{i} = -(\mathbf{i} \times \mathbf{j}) = -\mathbf{k}$, so $\mathbf{j} \times \mathbf{i}$ and $\mathbf{i} \times \mathbf{j}$ point in opposite directions. Similarly, $\mathbf{k} \times \mathbf{j} = -\mathbf{i}$ and $\mathbf{i} \times \mathbf{k} = -\mathbf{j}$. These relationships are easily remembered with a circle diagram (Figure 11.59). Finally the angle between any unit vector and itself is $\theta = 0$. Therefore, $\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0}$.

**FIGURE 11.59**

What is missing so far is a method for finding the components of the cross product of two vectors in $\mathbf{R}^3$. Let $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$. Using the distributive properties of the cross product (Theorem 11.4) we have

$$\begin{aligned}
\mathbf{u} \times \mathbf{v} &= (u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}) \times (v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}) \\
&= u_1v_1\underbrace{(\mathbf{i} \times \mathbf{i})}_{\mathbf{0}} + u_1v_2\underbrace{(\mathbf{i} \times \mathbf{j})}_{\mathbf{k}} + u_1v_3\underbrace{(\mathbf{i} \times \mathbf{k})}_{-\mathbf{j}} \\
&\quad + u_2v_1\underbrace{(\mathbf{j} \times \mathbf{i})}_{-\mathbf{k}} + u_2v_2\underbrace{(\mathbf{j} \times \mathbf{j})}_{\mathbf{0}} + u_2v_3\underbrace{(\mathbf{j} \times \mathbf{k})}_{\mathbf{i}} \\
&\quad + u_3v_1\underbrace{(\mathbf{k} \times \mathbf{i})}_{\mathbf{j}} + u_3v_2\underbrace{(\mathbf{k} \times \mathbf{j})}_{-\mathbf{i}} + u_3v_3\underbrace{(\mathbf{k} \times \mathbf{k})}_{\mathbf{0}}.
\end{aligned}$$

**THEOREM 11.6 Evaluating the Cross Product**

Let $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$. Then,

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}\mathbf{k}.$$

## EXAMPLE-7

**Area of a triangle** Find the area of the triangle with vertices $O(0, 0, 0)$, $P(2, 3, 4)$, and $Q(3, 2, 0)$ (Figure 11.60).

**SOLUTION** First consider the parallelogram, two of whose sides are the vectors $\overrightarrow{OP}$ and $\overrightarrow{OQ}$. By Theorem 11.3, the area of this parallelogram is $|\overrightarrow{OP} \times \overrightarrow{OQ}|$. Computing the cross product, we find that

$$\overrightarrow{OP} \times \overrightarrow{OQ} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 4 \\ 3 & 2 & 0 \end{vmatrix} = \begin{vmatrix} 3 & 4 \\ 2 & 0 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & 4 \\ 3 & 0 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix}\mathbf{k}$$
$$= -8\mathbf{i} + 12\mathbf{j} - 5\mathbf{k}.$$

Therefore, the area of the parallelogram is

$$|\overrightarrow{OP} \times \overrightarrow{OQ}| = |-8\mathbf{i} + 12\mathbf{j} - 5\mathbf{k}| = \sqrt{233} \approx 15.26.$$

The triangle with vertices $O$, $P$, and $Q$ comprises half of the parallelogram, so its area is $\sqrt{233}/2 \approx 7.63$.

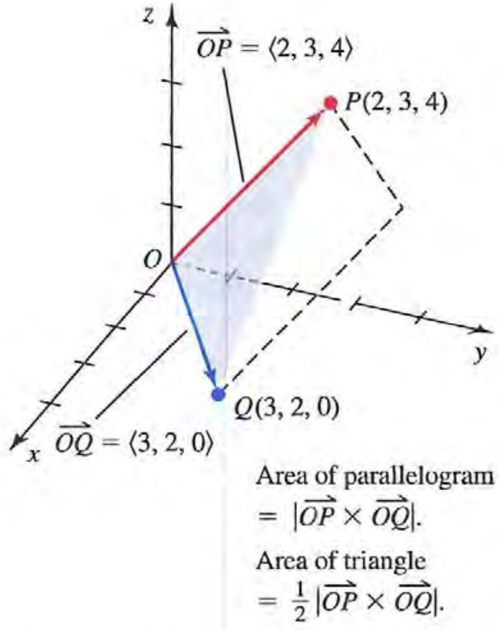



**FIGURE 11.60**

## EXAMPLE-8

**Vector normal to two vectors** Find a vector normal (or orthogonal) to the two vectors $\mathbf{u} = -\mathbf{i} + 6\mathbf{k}$ and $\mathbf{v} = 2\mathbf{i} - 5\mathbf{j} - 3\mathbf{k}$.

**SOLUTION** A vector normal to $\mathbf{u}$ and $\mathbf{v}$ is parallel to $\mathbf{u} \times \mathbf{v}$ (Figure 11.61). One normal vector is

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 0 & 6 \\ 2 & -5 & -3 \end{vmatrix}$$
$$= (0 + 30)\mathbf{i} - (3 - 12)\mathbf{j} + (5 - 0)\mathbf{k}$$
$$= 30\mathbf{i} + 9\mathbf{j} + 5\mathbf{k}.$$

Any scalar multiple of this vector is also orthogonal to $\mathbf{u}$ and $\mathbf{v}$.

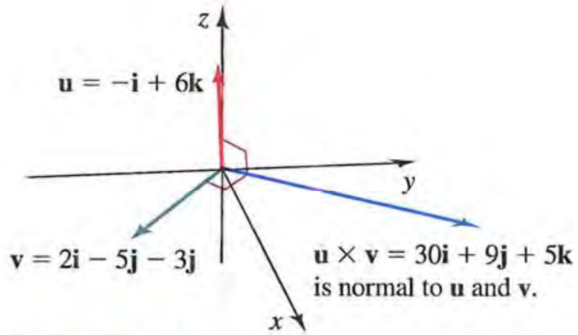



**FIGURE 11.61**

## SCALAR TRIPLE PRODUCTS

If $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$, $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$, and $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$ are vectors in 3-space, then the number

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$$

is called the ***scalar triple product*** of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$. It is not necessary to compute the dot product and cross product to evaluate a scalar triple product—the value can be obtained directly from the formula

$$\boxed{\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}}$$

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \mathbf{u} \cdot \left( \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} \mathbf{k} \right)$$

$$= u_1 \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} - u_2 \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} + u_3 \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

## EXAMPLE-9

Calculate the scalar triple product $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$ of the vectors

$$\mathbf{u} = 3\mathbf{i} - 2\mathbf{j} - 5\mathbf{k}, \quad \mathbf{v} = \mathbf{i} + 4\mathbf{j} - 4\mathbf{k}, \quad \mathbf{w} = 3\mathbf{j} + 2\mathbf{k}$$

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} 3 & -2 & -5 \\ 1 & 4 & -4 \\ 0 & 3 & 2 \end{vmatrix} = 49$$

## GEOMETRIC PROPERTIES OF THE SCALAR TRIPLE PRODUCT

If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are nonzero vectors in 3-space that are positioned so their initial points coincide, then these vectors form the adjacent sides of a parallelepiped (Figure 11.4.5). The following theorem establishes a relationship between the volume of this parallelepiped and the scalar triple product of the sides.

**11.4.6 THEOREM** *Let* $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *be nonzero vectors in 3-space.*

(a) *The volume $V$ of the parallelepiped that has* $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *as adjacent edges is*

$$\boxed{V = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|} \qquad (10)$$

(b) $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = 0$ *if and only if* $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *lie in the same plane.*

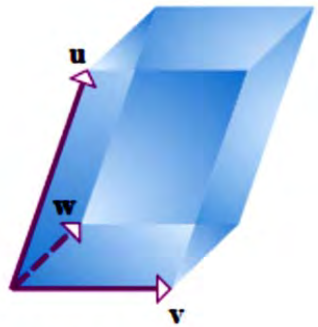


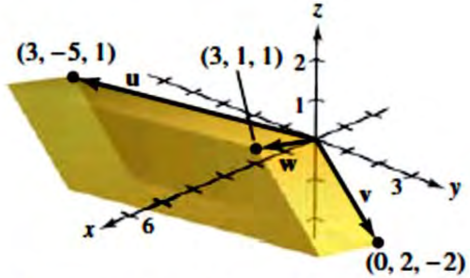

Figure 11.4.5

## EXAMPLE-10

**Find the volume of the parallelepiped shown in Figure 11.42 having $\mathbf{u} = 3\mathbf{i} - 5\mathbf{j} + \mathbf{k}$, $\mathbf{v} = 2\mathbf{j} - 2\mathbf{k}$, and $\mathbf{w} = 3\mathbf{i} + \mathbf{j} + \mathbf{k}$ as adjacent edges.**

**Solution** By Theorem 11.10, you have

$$\begin{aligned} V &= |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| \qquad \text{Triple scalar product} \\ &= \begin{vmatrix} 3 & -5 & 1 \\ 0 & 2 & -2 \\ 3 & 1 & 1 \end{vmatrix} \\ &= 3\begin{vmatrix} 2 & -2 \\ 1 & 1 \end{vmatrix} - (-5)\begin{vmatrix} 0 & -2 \\ 3 & 1 \end{vmatrix} + (1)\begin{vmatrix} 0 & 2 \\ 3 & 1 \end{vmatrix} \\ &= 3(4) + 5(6) + 1(-6) \\ &= 36. \end{aligned}$$




The parallelepiped has a volume of 36.
**Figure 11.42**