EMU - Mathematics Department - Fall 2021-22
MATH152: FINAL Examination, January 9, 2023, 8:30

Group No: ☐ **Student No:** ☐ **Name:** ☐

Q	01	02	03	04	05	06	Total
M							

Important Notes: The exam consists of **6** classic questions with a total of **120** points. You have provide your solution and answer for each question. Time limit is **90** minutes. You are **not** allowed to use calculators. GOOD LUCK.

Q.1 (20 pts) Find the area of the plane region G in the first quadrant located between the graphs of $y = x^2$, $x + y = 2$, and x-axis.

Solution. We have

$$A = \int_0^1 \int_{\sqrt{y}}^{2-y} dxdy = \int_0^1 (2 - y - \sqrt{y})\, dy = 2 - \frac{1}{2} - \frac{2}{3} = \boxed{\frac{5}{6}}$$

Q.2 (20 pts) Use Green's theorem to evaluate

$$\oint_C (ye^{2xy} - 5y)\, dx + (xe^{2xy} - 2x)\, dy,$$

where C is the counterclockwise oriented boundary curve of the square with vertices at $(0,0)$, $(0,1)$, $(1,0)$, and (1.1).

Solution. We have

$$\oint_C (ye^{2xy} - 5y)\, dx + (xe^{2xy} - 2x)\, dy = \int_0^1 \int_0^1 (-2+5)\, dxdy = \boxed{3}$$

Q.3 (20 pts) Find the volume of the solid located between xy-plane, the paraboloid $z = x^2 + y^2$, and the circular cylinder $x^2 + y^2 = 1$.

Solution. We have

$$\text{Volume} = \int_0^{2\pi} \int_0^1 \int_0^{r^2} r\, dzdrd\theta = \int_0^{2\pi} \int_0^1 r^3\, drd\theta = \int_0^{2\pi} \frac{d\theta}{4} = \boxed{\frac{\pi}{2}}$$

Q.4 (20 pts) Use Divergence Theorem to evaluate

$$\iint_S \mathbf{F} \cdot \mathbf{N}\, dS$$

if $\mathbf{F} = x^3\mathbf{i} + y^3\mathbf{j} + 3z\mathbf{k}$, S is the surface of the solid between the cone $z = \sqrt{x^2 + y^2}$ and the paraboloid $z = 2 - x^2 - y^2$ and $\mathbf{N}$ over vthe xy-plane is an outward unit normal vector to S.

Solution. We have

$$\begin{aligned}\iint_S \mathbf{F}\cdot\mathbf{N}\,dS &= \iiint_Q \nabla\cdot\mathbf{F}\,dV = \iiint_Q 3(x^2+y^2+1)\,dV\\ &= 3\int_0^{2\pi}\int_0^1\int_r^{2-r^2} r(r^2+1)\,dzdrd\theta = 3\int_0^{2\pi}\int_0^1 (2-r^2-r)(r^3+r)drd\theta\\ &= 3\int_0^{2\pi}\int_0^1 (2r+r^3-r^2-r^4-r^5)drd\theta = 6\pi\frac{11}{20} = \boxed{\frac{33\pi}{10}}\end{aligned}$$

Q.5 (20 pts) Evaluate

$$\iint_S (\nabla\times\mathbf{F})\cdot\mathbf{N}\,dS$$

if $\mathbf{F} = -y\mathbf{i}+x\mathbf{j}+z\mathbf{k}$, S is the surface of the paraboloid $z = 1-x^2-y^2$ over the xy-plane and $\mathbf{N}$ is the upward unit normal vector to S.

Solution. We have

$$\iint_S (\nabla\times\mathbf{F})\cdot\mathbf{N}\,dS = \oint_C -y\,dx + x\,dy + z\,dz = \int_0^{2\pi}(\sin^2 t+\cos^2 t)\,dt = \boxed{2\pi}$$

Q.6 (20 pts) Given the vector field $\mathbf{F} = (6x^2y^2+3)\mathbf{i} + (4x^3y-1)\mathbf{j}$,

(a) Show that $\mathbf{F}$ is conservative.

(b) Find its potential function f.

(c) Evaluate the line integral

$$\int_{(0,1)}^{(1,2)} \mathbf{F}\cdot d\mathbf{r}.$$

Solution. We have

$$\begin{cases} P = 6x^2y^2+3 \Rightarrow P'_y = 12x^2y\\ Q = 4x^3y-1 \Rightarrow Q'_x = 12x^2y\end{cases} \quad \Rightarrow \boxed{\mathbf{F}\text{ is conservative}}$$

For potential function, solve

$$\begin{cases} f'_x = 6x^2y^2+3,\\ f'_y = 4x^3y-1.\end{cases}$$

Then

$$f(x,y) = \int (6x^2y^2+3)\,dx = 2x^3y^2+3x+c(y).$$

Use the second equation

$$f'_y = 4x^3y + c'(y) = 4x^3y-1 \Rightarrow c'(y) = -1 \Rightarrow c(y) = -y+k.$$

Thus, $f(x,y) = \boxed{2x^3y^2+3x-y+k}$. Then

$$\int_{(0,1)}^{(1,2)} \mathbf{F}\cdot d\mathbf{r} = f(1,2)-f(0,1) = (8+3-2+k)-(-1+k) = \boxed{10}$$