Topics in Physics - C. Mukku

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Chapter 1

Lagrangian, Hamiltonian mechanics

Mechanics in terms of generalized coords.

1.1 Lagrangian

Define a functional. L over the config. space of partibles q^i, $qdot^i$. $L = L(q^i, qdot^i)$. We have an explicit dependence on t.

$L = KE - PE$

Assuming a 1-particle system of unit mass,

$$L = \frac{1}{2}\dot{q}^2 - V(q)$$

Assuming an n-particle system of unit mass,

$$L = \sum_i \frac{1}{2} qdot^{i2} - V(q^i)$$

1.2 Variational principle

Take a minimum path from A to B. Now notice that the path that is slightly different from this path will have some delta from the minimum.

Action

$$S(t0, t1) = \int L \mathrm{d}t = \int_{t0}^{t1} L(q^i, qdot^i) \mathrm{d}t$$

. Least action: $\delta S = 0$

Chapter 2

Functional calculus

this chapter develops a completely handwavy physics version of functional analysis.

Definition 1 *A **functional** F is a function: $F : (\mathbb{R} \to \mathbb{R}) \to \mathbb{R}$*

Notation 1 *Evaluation of a functional F with respect to f is denoted by $F[f]$.*

2.1 Functional Derivative - take 1

Consider a functional $F : (\mathbb{R} \to \mathbb{R}) \to \mathbb{R}$, a function $f : \mathbb{R} \to \mathbb{R}$, and a "test function" $\phi : \mathbb{R} \to \mathbb{R}$.

Consider a functional F. We only define the derivative of a functional F with respect to a function f by what happens under an integral sign as follows:

$$\int \frac{\delta F}{\delta f}(x)\phi(x)\mathrm{d}x = \lim_{\epsilon \to 0} \frac{F[f + \epsilon\phi] - F[f]}{\epsilon}$$

Now, we can define a small variation in F as:

$$\begin{gathered}\delta F : (\mathbb{R} \to \mathbb{R}) \times (\mathbb{R} \to \mathbb{R}) \to \mathbb{R} \\ \delta F(f, \phi) \equiv \int \frac{\delta F}{\delta f}(x)\phi(x)\mathrm{d}x\end{gathered}$$

Intuitively, δF tells us the variation of the function f along a test function ϕ. So, it encapsulates some kind of "directional derivative".

So, we can look at $\frac{\delta F}{\delta f}$ as a functional as follows:

$$\begin{gathered}\frac{\delta F}{\delta f} : (\mathbb{R} \to \mathbb{R}) \to \mathbb{R} \\ \frac{\delta F}{\delta f}(\phi) = \delta F(f, \phi)\end{gathered}$$

Wehre $\frac{\delta F}{\delta f}$ allows us to "test" the change of F with respect to f along a given "direction" ϕ.

2.2 Functional Derivative as taught in class

Substitute $\phi = \delta(x-p)$. Now, the quantity:

$$\frac{\delta F}{\delta f}\phi(x) = \delta F(f, \delta(x-p))$$

Rewriting δF by sticking it under an integral:

$$\int \frac{\delta F}{\delta f}(x)\delta(x-p)\mathrm{d}x = \lim_{\epsilon \to 0} \frac{F[f+\epsilon\delta(x-p)] - F[f]}{\epsilon}$$

$$\left.\frac{\delta F}{\delta f}\right|_p = \lim_{\epsilon \to 0} \frac{F[f+\epsilon\delta(x-p)] - F[f]}{\epsilon}$$

That is, we can start talking about "derivative of the functional F with respect to a function f at a point p" as long as we only test the functional F against δ-functions.

So, we can alternatively define this quantity as:

$$\left.\frac{\delta F}{\delta f}\right|_p \equiv \lim_{\epsilon \to 0} \frac{F[f+\epsilon\delta(x-p)] - F[f]}{\epsilon}$$

While this does not "look like a functional", it actually is, if we mentally replace:

$$p \to \int \_\ \delta(x-p)\mathrm{d}x$$

This is how mukku got that expression.

2.3 Common functional derivatives

2.3.1 Derivative of $F[\phi] \equiv \int (\frac{\partial \phi}{\partial y})^2$

$\left.\frac{\delta F}{\delta f}\right|_p = \int (\frac{\partial \phi}{\partial y})^2$

2.4 Deriving E-L from functional magic

Chapter 3

Maxwell's equations in Minkowski space

Let us first review Maxwell's equations:

$$\begin{aligned}
&\boldsymbol{\nabla} \cdot E = \frac{\rho}{\epsilon_0} \text{ (Electric charges produce fields)} \\
&\boldsymbol{\nabla} \cdot B = 0 \text{ (Only magnetic dipoles exist)} \\
&\boldsymbol{\nabla} \times E = -\frac{\partial B}{\partial t} \text{ (Lenz Law - time varying magnetic field induces current that opposes it)} \\
&\boldsymbol{\nabla} \times B = \mu_0 \left(J + \epsilon_0 \frac{\partial E}{\partial t} \right) \text{ (Ampere's law + fudge factor)}
\end{aligned}$$

3.1 Constructing F, or Tensorifying Maxwell's equations

Begin with the equation that $\boldsymbol{\nabla} \cdot B = 0$. This tells that B can be written as the curl of some other field:

$$\boxed{B \equiv \boldsymbol{\nabla} \times A} \tag{3.1}$$

Expanding this equation of B in tensorial form:

$$\boxed{B^i = \mathcal{E}^{ijk} \partial_j A^k} \tag{3.2}$$

Next, take $\boldsymbol{\nabla} \times E = -\frac{\partial B}{\partial t}$.

$$\nabla \times E = -\frac{\partial B}{\partial t} = \frac{\partial(\nabla \times A)}{\partial t} = \nabla \times \frac{\partial A}{\partial t}$$
$$\nabla \times \left(E + \frac{\partial A}{\partial t}\right) = 0$$
writing this as the divergence of some field ϕ scaled by $\alpha : \mathbb{R}$
$$E + \frac{\partial A}{\partial t} = \alpha(\nabla \cdot \phi)$$
$$E = \alpha \nabla \cdot \phi - \frac{\partial A}{\partial t}$$

Since electrostatics is time-independent, we choose to think of $\alpha = -1$, so we can interpret ϕ as the potential.

$$E^i = -\frac{\partial \phi}{\partial x^k} g^{ik} - \frac{\partial A^i}{\partial t} \tag{3.3}$$

A slight reformulation (since we know that in Minkowski space, $\partial_t = \partial_0$) we get the equation:

$$\boxed{E^i = -g^{ik}\partial_k \phi - \partial_0 A^i} \tag{3.4}$$

We get the metric $g^i k$ involved to raise the covariant $\frac{\partial \phi}{\partial x^k}$ into the contravariant E^i.
(**Sid question:** how does one justify switching $\nabla \times$ and ∂? It feels like some algebra)
Here be magic! We define A new rank-2 tensor in Minkowski space-time, called F (for Faraday),

$$\boxed{F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu} \tag{3.5}$$

(**Sid question:** why is this object $F_{\mu\nu}$ covariant? What does this *mean*?)

Lemma 1 *$F_{\mu\nu}$ is antisymmetric.*

Lemma 2 *$F_{\mu\nu}$ has 6 degrees of freedom*

Proof. Number of degrees of freedom of F:

$$\frac{4^2 \text{ (total)} - 4 \text{ (diagonal)}}{2 \text{ (anti-symmetry)}} = 6$$

Notice that F is a 1-form!

3.2 Expressing B, E in terms of F

We now wish to re-expresss B^{ij} and E^{ij} in terms of F, so that this F captures all of maxwell's equations.

$$
\begin{aligned}
B^i &= \mathcal{E}^{ijk}\partial_j A^k = \mathcal{E}^{ikj}\partial_k A^j && \text{by } k,\ j \text{ being free variables} \\
B^i &= \frac{1}{2}\left(\mathcal{E}^{ijk}\partial_j A^k + \mathcal{E}^{ikj}\partial_k A^j\right) \\
&\text{Substituting } \partial_j A_k - \partial_k A_j = F_{jk}, \\
B^i &= \frac{1}{2}\mathcal{E}^{ijk}F_{jk}
\end{aligned}
$$

So, B in terms of F is:

$$\boxed{B^i = \frac{1}{2}\mathcal{E}^{ijk}F_{jk}} \tag{3.6}$$

Similarly, we wish to write E in terms of F. The algebra is as follows:

$$
\begin{aligned}
E^i &= -g^{ik}\partial_k\phi - \partial_0 A^i \\
E^i &= -g^{ik}\partial_k\phi - \partial_0 g^{ik}A_k && \text{Is this allowed? Am I always allowed to insert the } g_{ik}\text{?} \\
E^i &= -g^{ik}(\partial_k\phi + \partial_0 A_k)
\end{aligned}
$$

Since $k = \{1, 2, 3\}$ (k is spacelike coordinates), and we would like to relate ϕ with A (to unify E), we **set**:

$$\boxed{A_0 \equiv -\phi} \tag{3.7}$$

Continuing the derivation,

$$
\begin{aligned}
E^i &= -g^{ik}(\partial_k(-A_0) + \partial_0 A_k) \\
E^i &= -g^{ik}(\partial_0 A_k - \partial_k A_0) \\
E^i &= -g^{ik}F_{0k}
\end{aligned}
$$

So, finally, the relation is:

$$\boxed{E^i = -g^{ik}F_{0k}} \tag{3.8}$$

TODO: Find out how $E^i = cF^{i0}$

$$\boxed{E^i = cF^{i0}} \tag{3.9}$$

3.3 Other ramifications of Maxwell's equations on F

3.3.1 Ramification 1

First part, using 4th equation

We next consider the 4th Maxwell equation:

$$
\begin{aligned}
\boldsymbol{\nabla} \times B &= \mu_0 J + \epsilon_0 \mu_0 \frac{\partial E}{\partial t} \\
\boldsymbol{\nabla} \times B &= \mu_0 J + \frac{1}{c^2} \frac{\partial E}{\partial t} \\
&\text{Converting to indeces,} \\
(\boldsymbol{\nabla} \times B)^i &= \mu_0 J^i + \frac{1}{c} \frac{\partial E^i}{\partial ct} && \text{(From } \partial_{ct} = \tfrac{1}{c}\partial_t\text{)} \\
&= \mu_0 J^i + \frac{1}{c} \frac{\partial E^i}{\partial X^0} \\
&= \mu_0 J^i + \frac{\partial F^{i0}}{\partial X^0} && \text{(From } E^i = cF^{i0}\text{)} \\
&= \mu_0 J^i + \partial_0 F^{i0}
\end{aligned}
$$

Now, we start to simplify the LHS, $\boldsymbol{\nabla} \times B$:

$$
\begin{aligned}
&(\boldsymbol{\nabla} \times B)^i = \mathcal{E}^{ijk} \partial_j B_k \\
&\text{Since } B^k = \frac{1}{2} \mathcal{E}^{klm} F_{lm}, \\
&B_k = \frac{1}{2} \mathcal{E}_{klm} F^{lm}, && (\textbf{TODO:} \text{ this is scam}) \\
&(\boldsymbol{\nabla} \times B)^i = \mathcal{E}^{ijk} \partial_j (\frac{1}{2} \mathcal{E}_{klm} F^{lm}) = \frac{1}{2} \mathcal{E}^{ijk} \mathcal{E}_{klm} \partial_j F^{lm}
\end{aligned}
$$

Aside: We need to know how to evaluate $\mathcal{E}^{ijk} \mathcal{E}_{klm}$:

$$
\mathcal{E}_{i_1, i_2, \ldots, i_n} \mathcal{E}_{j_1, j_2, \ldots j_n} = \det \left\{ \begin{vmatrix} \delta_{i_1 j_1} & \delta_{i_1 j_2} & \ldots & \delta_{i_1 j_n} \\ \delta_{i_2 j_1} & \delta_{i_2 j_2} & \ldots & \delta_{i_2 j_n} \\ \vdots & \vdots & \ddots & \vdots \\ \delta_{i_n j_1} & \delta_{i_n j_2} & \ldots & \delta_{i_n j_n} \end{vmatrix} \right\}
$$

Hence, **TODO: HOW?** $\mathcal{E}^{ijk} \mathcal{E}^{ilm} = \frac{-1}{2} (\delta_i^j \delta_m^k - \delta_m^j \delta_l^k)$

Plugging both equations together,

$$\frac{1}{2}\mathcal{E}^{ijk}\mathcal{E}_{klm}\partial_j F^{lm} = \mu_0 J^i + \partial_0 F^{i0}$$
$$\frac{1}{2}\Big[\frac{-1}{2}\big(\delta^i_l\delta^j_m - \delta^i_m\delta^j_l\big)\Big]\partial_j F^{lm} = \mu_0 J^i + \partial_0 F^{i0}$$

Something is fucked here with respect to $\partial_m F^{mi}$

$$\frac{1}{2}\Big[\frac{-1}{2}\big(\partial_m F^{im} - \partial_m F^{mi}\big)\Big] = \mu_0 J^i + \partial_0 F^{i0}$$

F is anti-symmetric, so rewriting $-\partial_m F^{mi} = \partial_m F^{im}$

$$-\frac{1}{2}\big[\partial_m F^{im}\big] = \mu_0 J^i + \partial_0 F^{i0}$$

In the notes, the $\frac{1}{2}$ does not exist

$$-\big[\partial_m F^{im}\big] = \mu_0 J^i + \partial_0 F^{i0}$$
$$\mu_0 J^i + \partial_0 F^{i0} + \partial_m F^{im} = 0$$
$$\mu_0 J^i + \partial_\mu F^{i\mu} = 0 \qquad (\mu = \{0,1,2,3\})$$

This gives us a continuity-style equation, linking the current density J to the rate of change of F.

$$\boxed{\mu_0 J^i + \partial_\mu F^{i\mu} = 0} \qquad (\mu = \{0,1,2,3\})$$

Second part, using 1st equation

$$\boldsymbol{\nabla} E = \frac{\rho}{\epsilon_0}$$
$$\partial_i E^i = \frac{\phi}{\epsilon_0}$$

Substituting $E^i = cF^{i0}$,

$$c\partial_i F^{i0} = \frac{\rho}{\epsilon_0} = \frac{\rho\mu_0}{\mu_0\epsilon_0} = \rho c^2$$
$$\partial_i F^{i0} = \mu_0 c\rho$$

Since F is anti-symmetric, $F^{00} = 0$, Hence: $\partial_0 F^{00} + \partial_i F^{i0} = \mu_0 c\rho$

$$\partial_\mu F^{\mu 0} = \mu_0 c\rho$$

$$\boxed{\partial_\mu F^{\mu 0} = \mu_0 c\rho} \tag{3.10}$$

Combining part 1 and part 2:

$$\mu_0 J^i + \partial_\mu F^{i\mu} = 0 \qquad \text{(From } B\text{)}$$
$$\partial_\mu F^{i\mu} = -\mu_0 J^i \partial_\mu F^{\mu 0} = \mu_0 c\rho$$
$$\partial_\mu F^{0\mu} = -\mu_0 c\rho$$

To combine these equations, **we set:**

$$\boxed{J^0 \equiv c\rho} \tag{3.11}$$

We arrive at the unified equation:

$$\partial_\mu F^{\nu\mu} = -\mu_0 J^\nu$$

Choose units such that $c = \frac{h}{2\pi} = G_n = 1$, which gives us:

$$\partial_\mu F^{\nu\mu} = -J^\nu$$

F **is antisymmetric, so flipping indeces**

$$\partial_\mu F^{\mu\nu} = J^\nu$$

$$\boxed{\partial_\mu F^{\mu\nu} = J^\nu} \tag{3.12}$$

Note that this is **Ampere's law!**