Topics in Physics - C. Mukku

Siddharth Bhat

Contents

Chapter 1

Lagrangian, Hamiltonian mechanics

Mechanics in terms of generalized coords.

1.1 Lagrangian

Define a functional. L over the config. space of partibles q^i, $qdot^i$. $L = L(q^i, qdot^i)$. We have an explicit dependence on t.

$L = KE - PE$

Assuming a 1-particle system of unit mass,

$$L = \frac{1}{2}\dot{q}^2 - V(q)$$

Assuming an n-particle system of unit mass,

$$L = \sum_i \frac{1}{2} qdot^{i2} - V(q^i)$$

1.2 Variational principle

Take a minimum path from A to B. Now notice that the path that is slightly different from this path will have some delta from the minimum.

Action

$$S(t0, t1) = \int L \mathrm{d}t = \int_{t0}^{t1} L(q^i, qdot^i) \mathrm{d}t$$

. Least action: $\delta S = 0$

Chapter 2

Functional calculus

this chapter develops a completely handwavy physics version of functional analysis.

Definition 1 *A* ***functional*** F *is a function:* $F : (\mathbb{R} \to \mathbb{R}) \to \mathbb{R}$

Notation 1 *Evaluation of a functional* F *with respect to* f *is denoted by* $F[f]$*.*

2.1 Functional Derivative - take 1

Consider a functional $F : (\mathbb{R} \to \mathbb{R}) \to \mathbb{R}$, a function $f : \mathbb{R} \to \mathbb{R}$, and a "test function" $\phi : \mathbb{R} \to \mathbb{R}$.

Consider a functional F. We only define the derivative of a functional F with respect to a function f by what happens under an integral sign as follows:

$$\int \frac{\delta F}{\delta f}(x)\phi(x)\mathrm{d}x = \lim_{\epsilon \to 0} \frac{F[f + \epsilon\phi] - F[f]}{\epsilon}$$

Now, we can define a small variation in F as:

$$\delta F : (\mathbb{R} \to \mathbb{R}) \times (\mathbb{R} \to \mathbb{R}) \to \mathbb{R}$$
$$\delta F(f, \phi) \equiv \int \frac{\delta F}{\delta f}(x)\phi(x)\mathrm{d}x$$

Intuitively, δF tells us the variation of the function f along a test function ϕ. So, it encapsulates some kind of "directional derivative".

So, we can look at $\frac{\delta F}{\delta f}$ as a functional as follows:

$$\frac{\delta F}{\delta f} : (\mathbb{R} \to \mathbb{R}) \to \mathbb{R}$$
$$\frac{\delta F}{\delta f}(\phi) = \delta F(f, \phi)$$

Wehre $\frac{\delta F}{\delta f}$ allows us to "test" the change of F with respect to f along a given "direction" ϕ.

2.2 Functional Derivative as taught in class

Substitute $\phi = \delta(x-p)$. Now, the quantity:

$$\frac{\delta F}{\delta f}\phi(x) = \delta F(f, \delta(x-p))$$

Rewriting δF by sticking it under an integral:

$$\int \frac{\delta F}{\delta f}(x)\delta(x-p)\mathrm{d}x = \lim_{\epsilon\to 0}\frac{F[f+\epsilon\delta(x-p)]-F[f]}{\epsilon}$$

$$\left.\frac{\delta F}{\delta f}\right|_p = \lim_{\epsilon\to 0}\frac{F[f+\epsilon\delta(x-p)]-F[f]}{\epsilon}$$

That is, we can start talking about "derivative of the functional F with respect to a function f at a point p" as long as we only test the functional F against δ-functions.

So, we can alternatively define this quantity as:

$$\left.\frac{\delta F}{\delta f}\right|_p \equiv \lim_{\epsilon\to 0}\frac{F[f+\epsilon\delta(x-p)]-F[f]}{\epsilon}$$

While this does not "look like a functional", it actually is, if we mentally replace:

$$p \to \int \_\ \delta(x-p)\mathrm{d}x$$

This is how mukku got that expression.

2.3 Common functional derivatives

2.3.1 Derivative of $F[\phi] \equiv \int(\frac{\partial\phi}{\partial y})^2$

$\left.\frac{\delta F}{\delta f}\right|_p = \int(\frac{\partial\phi}{\partial y})^2$

2.4 Deriving E-L from functional magic

Chapter 3

Maxwell's equations in Minkowski space

Let us first review Maxwell's equations:

$$\begin{aligned}
&\boldsymbol{\nabla} \cdot E = \frac{\rho}{\epsilon_0} \text{ (Electric charges produce fields)} \\
&\boldsymbol{\nabla} \cdot B = 0 \text{ (Only magnetic dipoles exist)} \\
&\boldsymbol{\nabla} \times E = -\frac{\partial B}{\partial t} \text{ (Lenz Law - time varying magnetic field induces current that opposes it)} \\
&\boldsymbol{\nabla} \times B = \mu_0 \left(J + \epsilon_0 \frac{\partial E}{\partial t} \right) \text{ (Ampere's law + fudge factor)}
\end{aligned}$$

Begin with the equation that $\boldsymbol{\nabla} \cdot B = 0$. This tells that B can be written as the curl of some other field — $B = \boldsymbol{\nabla} \times A$. Hence

$$\boxed{B^i = \mathcal{E}^{ijk} \partial_j A^k} \tag{3.1}$$

Next, take $\boldsymbol{\nabla} \times E = -\frac{\partial B}{\partial t}$.

$$\begin{aligned}
&\boldsymbol{\nabla} \times E = -\frac{\partial B}{\partial t} = \frac{\partial(\boldsymbol{\nabla} \times A)}{\partial t} = \boldsymbol{\nabla} \times \frac{\partial A}{\partial t} \\
&\boldsymbol{\nabla} \times \left(E + \frac{\partial A}{\partial t} \right) = 0 \\
&\text{writing this as the divergence of some field } \phi \text{ scaled by } \alpha : \mathbb{R} \\
&E + \frac{\partial A}{\partial t} = \alpha \big(\boldsymbol{\nabla} \cdot \phi \big) \\
&E = \alpha \boldsymbol{\nabla} \cdot \phi - \frac{\partial A}{\partial t}
\end{aligned}$$

Since electrostatics is time-independent, we choose to think of $\alpha = -1$, so we can interpret ϕ as the potential.

$$E^i = -\frac{\partial \phi}{\partial x^k} g^{ik} - \frac{\partial A^i}{\partial t} \tag{3.2}$$

A slight reformulation (since we know that in Minkowski space, $\partial_t = \partial_0$) we get the equation:

$$\boxed{E^i = -g^{ik}\partial_k \phi - \partial_0 A^i} \tag{3.3}$$

We get the metric $g^i k$ involved to raise the covariant $\frac{\partial \phi}{\partial x^k}$ into the contravariant E^i.
(**Sid question:** how does one justify switching $\boldsymbol{\nabla}\times$ and ∂? It feels like some algebra)

Here be magic! We define A new rank-2 tensor in Minkowski space-time, called F (for Faraday),

$$\boxed{F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu} \tag{3.4}$$

(**Sid question:** why is this object $F_{\mu\nu}$ covariant? What does this *mean*?)

Lemma 1 *$F_{\mu\nu}$ is antisymmetric.*

Lemma 2 *$F_{\mu\nu}$ has 6 degrees of freedom*

Proof. Number of degrees of freedom of F:

$$\frac{4^2 \text{ (total)} - 4 \text{ (diagonal)}}{2 \text{ (anti-symmetry)}} = 6$$

Notice that F is a 1-form!

We now wish to re-expresss B^{ij} and E^{ij} in terms of F, so that this F captures all of maxwell's equations.

$$B^i = \mathcal{E}^{ijk}\partial_j A^k = \mathcal{E}^{ikj}\partial_k A^j \qquad \text{by } k,\ j \text{ being free variables}$$

$$B^i = \frac{1}{2}\left(\mathcal{E}^{ijk}\partial_j A^k + \mathcal{E}^{ikj}\partial_k A^j\right)$$

Substituting $\partial_j A_k - \partial_k A_j = F_{jk}$,

$$B^i = \frac{1}{2}\mathcal{E}^{ijk}F_{jk}$$

So, B in terms of F is:

$$\boxed{B^i = \frac{1}{2}\mathcal{E}^{ijk}F_{jk}} \tag{3.5}$$

Similarly, we wish to write E in terms of F. The algebra is as follows:

$$E^i = -g^{ik}\partial_k \phi - \partial_0 A^i$$