0.1 Q1 – matrix representation for $|\phi_k\rangle\langle\phi_j|$, in the orthonormal $|\nu_i\rangle$ basis

Perform change of basis.

0.2 Q2 – positive operator is Hermitian

We first show that a positive operator is normal, and this automatically implies that it is Hermitian.

To show that a positive operator is normal, we consider $A^\dagger A$

Now that we know that it is normal, by spectral decomposition, it posesses an eigenbasis. We now show that all of its eigenvalues are real. This is now a matrix with real entries on the diagonal, which is hermitian.

To show that the eigenvalues are real, let $|\lambda\rangle$ be an eigenvector with magnitude 1 and eigenvalue λ.

$$\langle\lambda| A |\lambda\rangle \geqslant 0 \qquad \lambda \langle\lambda|\lambda\rangle = \lambda \geqslant 0$$

Hence, the eigenvalues are real and positive, and therefore it is Hermitian.

0.3 Q3 – $A^\dagger A$ is positive

$$\forall \nu \in V, \;\; \langle\nu| A^\dagger A |\nu\rangle = \langle A\nu| |A\nu\rangle = \|A\nu\|^2 \geqslant 0$$

Hence, $A^\dagger A$ is positive.

0.4 Q4. Eigenvalues of a projector P are either 0 or 1

Let $|\lambda\rangle$ be an eigenvector of P with associated eigenvalue λ.

$$P^2(|\lambda\rangle) = \lambda(P|\lambda\rangle) = \lambda^2 |\lambda\rangle \qquad P(|\lambda\rangle) = \lambda|\lambda\rangle$$

However, since P is a projector, $P^2 = P$, and therefore, $\lambda^2 = \lambda$. The roots of this equation are $0, 1$. Hence, $\lambda \in \{0, 1\}$.

0.5 Q5. Tensor product of two unitary operators is unitary

Let U, V be unitary operators.

$$
\begin{aligned}
&\langle Uu \otimes Vv| \, |Uu \otimes Vv\rangle = \\
&\langle u \otimes v| \, (U^\dagger \otimes V^\dagger)(U \otimes V) \, |u \otimes v\rangle = \\
&\langle u \otimes v| \, (U^\dagger U \otimes V^\dagger V) \, |u \otimes v\rangle = \\
&\langle u \otimes v| \, I \otimes I \, |u \otimes v\rangle = \\
&\langle u \otimes v| \, |u \otimes v\rangle =
\end{aligned}
$$

Hence, $U \otimes V$ is unitary since it preserves inner products.

0.6 Q6. Tensor product of projectors is a projector

0.7 Q7. Find $\log$ and square root of matrix

0.8 Q8. Trace properties

0.8.1 $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$

$$\mathrm{Tr}(AB) = \sum_z (AB)_{zz} = \sum_z \sum_k A_{zk} B_{kz} = \sum_z \sum_k B_{kz} A_{kz} = \sum_z (BA)_{zz} = \mathrm{Tr}(BA)$$

0.8.2 $\mathrm{Tr}(A + B) = \mathrm{Tr}(A) + \mathrm{Tr}(B)$

$$\mathrm{Tr}(A + B) = \sum_z (A + B)_{zz} = \sum_z A_{zz} + B_{zz} = \mathrm{Tr}(A) + \mathrm{Tr}(B)$$

0.8.3 $\mathrm{Tr}(2A) = 2\mathrm{Tr}(A)$

$$\mathrm{Tr}(2A) = \sum_z (2A)_{zz} = \sum_z 2A_{zz} 2 \sum_z A_{zz} = 2\,\mathrm{Tr}(A)$$

0.9 Commutator properties

0.9.1 $[A, B] = -[B, A]$

$$[A, B] = AB - BA = -(BA - AB) = -[B, A]$$

0.9.2 $\frac{[A,B]+\{A,B\}}{2} = AB$

$$\frac{[A, B] + \{A, B\}}{2} = \frac{(AB - BA) + (AB + BA)}{2} = AB$$

0.10 Express polar decomposition as outer product

0.11 Find left and right polar decomposition