

# General Relativity and Differential Geometry

Siddharth Bhat

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# Chapter 1

## Introduction

I am following the following sources:

- Susskind's General Relativity lectures as part of the Theoretical minimum: (The link is here). Note that the videos do not load. However, one can view the source to access the link to the iframe.
- Susskind's other General Relativity lectures, as part of his modern physics course: (link to playlist here). These seem to be taught at a much gentler pace.
- The book Gravitation by Misner, Thorne and wheeler.

The notes as scribed here are a mix from all of these sources, as well as tangential points I find interesting.

### 1.1 The equivalence principle

Gravity is in some sense the same thing as acceleration. First, an elementary derivation which formalizes the intuition of the equivalence principle.

We consider an elevator moving upward. Let its distance from the bottom be  $L(t)$ .

$$z' = z - L(t) \quad t' = t \quad x' = x$$

if  $\frac{d^2 L(t)}{dt^2} = 0$ , then the force is the same in the new coordinate system as that of the old coordinate system

## 1.2 Galileo's theory of flat space and gravitation

Newton's laws:

$$\vec{F} = m\vec{a}$$

$$\vec{F} = m \frac{d^2x}{dt^2}$$

Galileo's gravitation, under the approximation that the earth is flat: If we pick downwards to be negative direction along the 2 dimension, then his equation can be written as  $F_2 = -mg$  where  $g$  is a constant.

This is special, because the force is proportional to the mass, which is not the case of things like electromagnetism.

Combining the two equations, we get  $m \frac{d^2x}{dt^2} = -mg$ , or  $\frac{d^2x}{dt^2} = -g$ .

That the acceleration induced by the gravitational force is independent of the mass of the object is known as the *equivalence principle*. At this stage, we can say that gravity is equivalent across all objects independent of their mass.

Let's now consider a collection of point masses — A diffuse cloud of particles, and have it fall. Different particles maybe heavy, light, large, small. However, since all of them have the same acceleration, the point cloud looks unchanged as it falls. That is, the object will have no stresses or strains as it falls. We can't tell by looking at our neighbours that there is a force being exerted on us, since all our neighbours are moving along with us! We cannot tell the difference between being in free space versus being in a gravitational field.

## 1.3 Newton's theory of gravity

all bodies exert equal and opposite forces on each other. Given two bodies  $a$  and  $b$  of masses  $m$  and  $M$  with distance  $R$ , the force on  $a$  is  $F_a \equiv \frac{GmM}{R^2} \hat{r}$ , where  $\hat{r}$  is the direction from  $a$  to  $b$ .

Again, we can prove that the acceleration of an object  $a$  does not depend on its own mass.

Now that gravitation depends on distance, we can actually feel something if we are in a gravitational field, since different parts of a given object will have a different force on it, due to the varying distance from (say) the earth.

Gravitational field is defined as the force exerted on a test mass at every point in space.

### Gauss' theorem

$\int \nabla \cdot A dx dy dz = \int A_{\perp} d\sigma$  where  $\sigma$  is the differential unit of surface area of the surface.

Show that the divergence of a field in 3 dimensions will lead to an inverse square law.

## 1.4 Geometry and curvature

To describe a geometry, all we need is the distance between neighboring points on a black-board. In general, given a parametrization, we can draw a possibly distorted grid of lines

of constant coordinate. The distance between two points  $(x, y)$  and  $(x + dx, y + dy)$  will be  $ds^2 = g_{11}dx^2 + 2g_{12}dx dy + g_{22}dy^2$ .





## Chapter 2

# Opimisation on Steifel Manifolds

We have the space  $O(n) = \{X \in \mathbb{R}^{n \times n} | X^T X = I\}$ . To find an element of the tangent space at the point  $P \in O(n)$ , we parametrize a curve  $C(t) : \mathbb{R} \rightarrow O(n)$ , such that  $c(0) = P$ . Then, we differentiate the curve and evaluate it at 0. That is,  $\frac{dc}{dt}|_{t=0} \in T_P O(n)$ .

We consider  $C(t) : \mathbb{R} \rightarrow O(n)$ , such that  $C(0) = P$ . Since  $C(t) \in O(n)$ , we can write  $C(t)^T C(t) = I$ . This in index notation, is  $c^{ik}(t)c^{jk}(t) = \delta^{ij}$ . Differentiating with respect to  $t$ , we get:

$$\begin{aligned} C^T(t)C(t) &= I \\ c^{ik}(t)c^{jk}(t) &= \delta^{ij} \\ \frac{d(c^{ik}(t)c^{jk}(t))}{dt} &= \frac{d(\delta^{ij})}{dt} \\ \dot{c}^{ik}(t)c^{jk}(t) + c^{ik}(t)\dot{c}^{jk}(t) &= 0 \quad (\text{chain rule}) \\ \dot{C}^T(t)C(t) + C(t)^T \dot{C}(t) &= 0 \end{aligned}$$

Now, we know that  $C(0) = P$ , and hence  $\dot{C}(0) \in T_P O(n)$ . By evaluating the above equation at  $t = 0$ , we obtain the relation:

$$\begin{aligned} \dot{C}^T(0)C(0) + C(0)^T \dot{C}(0) &= 0 \\ \dot{C}^T(0)P + P^T \dot{C}(0) &= 0 \end{aligned}$$

Hence, we conclude that for all  $Z \in T_P O(n)$ ,  $Z^T P + P^T Z = 0$ . Indeed, we can characterize  $T_P O(n)$  this way and prove the reverse inclusion (how?), to show that:

$$T_P O(n) \equiv \{Z \mid Z^T P + P^T Z = 0\}$$

However, this equation for the  $Z$  is "ineffective", in that it does not tell us how to *compute* the set of  $Z$ s. We can only *check* if a particular  $Z_0 \in T_P O(n)$ .

We will solve the characterization of  $Z$  by first solving a slightly easier problem:  $T_I O(n)$ , where  $I$  is the identity matrix.

$$T_I O(n) \equiv \{Z \mid Z^T I + I^T Z = 0\} = \{Z \mid Z^T = -Z\}$$

We now have a complete enumeration of the *tangent space at the identity*: We know that this consists of all skew-symmetric matrices!

We now wish to transport this structure of  $T_I O(n)$  to an arbitrary  $T_P O(n)$ . Here, I will let you in on a secret about the structure of Lie groups, which we will later prove: The vector space at  $T_P O(n)$  is obtained by multiplying by  $P$  to  $T_I O(n)$ .

In this case, it tries to inform us that:

$$T_P O(n) = \{PZ \mid Z^T = -Z\}$$

We distrust this assertion at first, of course, so we can plug this equation back in the characterization of  $T_P O(n)$  that we had developed and see what pops out:

$$T_P O(n) \equiv \{X \mid X^T P + P^T X = 0\}$$

Let  $X = PZ$  where  $Z^T = -Z$ , and check that they satisfy the condition  $X^T P + P^T X = 0$

$$(PZ)^T P + P^T PZ = Z^T P^T P + P^T PZ = Z^T I + I^T Z = -Z + Z = 0$$

Hence, they do satisfy the condition, and we can assert that at least:

$$\{PZ \mid Z^T = -Z\} \subseteq \{X \mid X^T P + P^T X = 0\}$$

## Chapter 3

# Matrix Lie groups

We study the structure of Lie groups, and we prove theorems about the structure of the tangent spaces at the identity, and how this structure governs the behavior of the tangent spaces everywhere on the group.

Consider a group of matrices  $(G, * : G \times G \rightarrow G, I : G)$ , which is equipped with a manifold structure, where  $I$  is the identity matrix, and  $*$  is matrix multiplication.

Now, we consider the tangent space at the identity element  $T_e G$ . What we want to do is to transport the structure of the tangent space at the identity to an arbitrary point  $p \in G$ . For this, we consider a manifold mapping  $f : G \rightarrow G$ , such that its pushforward  $df : T_X G \rightarrow T_{f(X)} G$  maps  $I$  to  $f(I)$ . That is, we would need a map such that  $f(I) = p$ , such that the pushforward pushes forward  $T_I G$  to  $T_{f(I)} G = T_p G$ .

We consider the map  $(f : G \rightarrow G; f(X) \equiv P * X)$ .  $f(I) = P * I = P$ , so  $f$  maps  $I$  to  $P$ .

Next, let's consider a curve  $c \in T_e G$ . That is,  $c : (-1, 1) \rightarrow G$ , such that  $c(0) = I$ . Recall that all the information about the curve  $c$  that defines its directional derivative is stored in  $\frac{d \text{ch} \circ c(t)}{dt}|_{t=0}$ .

Recall that the pushforward is defined as

$$\begin{aligned} df : T_X G &\rightarrow T_{f(X)} G \\ df : ((-1, 1) \rightarrow G) &\rightarrow ((-1, 1) \rightarrow G) \\ df(c)(t) &= \frac{d f(c(t))}{dt}|_{t=0} \quad (\text{where } c(0) = X) \\ \frac{d f(c(t))}{dt}|_{t=0} &= f'(c(t))c'(t)|_{t=0} = f'(c(0))c'(0) = f'(X)c'(0) \end{aligned}$$

Hence, we would like to pick a  $c$  such that  $c(0) = e$ , and  $\frac{d \text{ch} \circ c(t)}{dt}|_{t=0}$  is easy to compute, where  $\text{ch} : G \rightarrow \mathbb{R}^n$  is a chart for  $G$  around the point  $c(0)$ . In our case, the  $\text{ch}$  is just the identity function, since matrices can naturally be considered as elements of  $\mathbb{R}^{m \times n}$  if the matrices of  $G$  are  $m \times n$  matrices. So  $(\text{ch} = \text{id})$ , and the tangents reduce to  $\frac{dc(t)}{dt}|_{t=0}$ . to construct such a curve  $c$ , we first define the matrix exponential and use it to define the curve  $c$ :

$$\exp : \mathbf{G} \rightarrow \mathbf{G} \quad \exp(Y) \equiv I + Y + \frac{Y^2}{2!} + \frac{Y^3}{3!} + \cdots + \frac{Y^n}{n!} + \cdots$$

$$c : (-1, 1) \rightarrow \mathbf{G} \quad c(t) \equiv \exp(tX)$$

$$c(0) = \exp(0 \cdot X) = \exp(0) = I$$

$$\frac{dc}{dt}\bigg|_{t=0} = \frac{d\exp(tX)}{dt}\bigg|_{t=0} = X\exp(tX)\bigg|_{t=0} = X\exp(0X)$$

## Chapter 4

# Symplectic vector spaces

Let  $V$  be a  $m$ -dimensional real vector space. Let  $\Omega : V \times V \rightarrow \mathbb{R}$  be a bilinear form on  $V$  that is skew-symmetric:  $\forall a, b \in V, \Omega(a, b) = -\Omega(b, a)$ .

**Theorem 1** *Let  $\Omega$  be a bilinear skew-symmetric map on  $V$ . Then there is a basis  $u_1, u_2, \dots, u_k, e_1, e_2, \dots, e_n, f_1, f_2, \dots, f_n$  such that:*

- $\Omega(u_i, v) = 0 \quad \forall v \in V$
- $\Omega(e_i, e_j) = \Omega(f_i, f_j) = 0$
- $\Omega(e_i, f_j) = \delta_{ij}$

**Proof 2** *Generalize Gram-Schmidt process. Let  $U = \{u \in V | \Omega(u, v) = 0, \forall v \in V\}$   $U$  is a subspace of  $V$ . Let  $u_1, u_2, \dots, u_k$  be a basis of  $U$ .*

*Establish  $e_j, f_j$  by induction. Let  $V = U \oplus W$ . Take  $e_1 \in W$ . Then, there must exist an  $f_1 \in W$  such that  $\Omega(e_1, f_1) \neq 0$ . Otherwise, we could expand the size of  $U$  by adding  $e_1$  to  $U$ . But we assumed that  $U$  contains all such vectors.  $U$  and  $W$  share no non-zero vectors since  $V = U \oplus W$ . Define  $W_1 \equiv \text{span}(e_1, f_1)$ . Now build  $W_1^\Omega \equiv \{w \in W | \Omega(w, W_1) = 0\}$ .*

*We need the following facts:*

- $W_1^\Omega \cap W_1 = \{0\}$
- $W = W_1 \oplus W_1^\Omega$

*So, recurse on  $W_1^\Omega$ .*

*Finally,  $V = U \oplus W_1 \oplus W_1^\Omega \oplus \dots \oplus W_n$ .*

**Remark 3**  $\dim(U) = k$  is invariant for  $(V, \Omega)$  since  $U$  was defined in a coordinate free way. But, remember that  $k + 2n = m$ , and hence  $n$  is an invariant of  $(V, \Omega)$ .  $n$  is called as the rank of  $\Omega$ .

$$\Omega \text{ written in the basis of } \{u_i, e_i, f_i\} \text{ gives } \Omega = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & I \\ 0 & -I & 0 \end{bmatrix}$$

### 4.0.1 Symplectic maps

Let  $V$  be an  $m$  dimensional real vector space over  $\mathbb{R}$ . Let  $\Omega : V \times V \rightarrow \mathbb{R}$  be a skew-symmetric bilinear map. We define  $\tilde{\Omega} : V \rightarrow V^*$ ,  $\tilde{\Omega}(v)(u) \equiv \Omega(v, u)$ . The problem is that this map has a non-trivial kernel  $U = \{u_i\}$  in general, so we cannot use it like a metric to identify the two spaces.

**Definition 4** A skew-symmetric bilinear map  $\Omega : V \times V \rightarrow \mathbb{R}$  is Symplectic iff  $\tilde{\Omega}$  is bijective. ie,  $U = \{0\}$ .  $(V, \Omega)$  is then called a Symplectic vector space.

## 4.1 Properties of a Symplectic map

- $\tilde{\Omega}$  is an identification between  $V$  and  $V^*$ .
- Since each  $(e_i, f_i)$  come in pairs, the dimension of the vector space  $V$  is divisible by 2. ie,  $m = 2n$ .
- We have a basis  $(e_i, f_i)$  called the Symplectic basis for  $V$ .
- $\Omega$  in matrix form with respect to the Symplectic basis is  $\Omega = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$

### 4.1.1 Subspaces of a Symplectic vector space

**Definition 5** A subspace  $W$  of  $V$  is a Symplectic subspace if  $\Omega|_W$  is non-degenerate. For example, take the subspace  $\text{span}(e_1, f_1)$ .

**Definition 6** A subspace  $W$  of  $V$  is an Isotropic subspace if  $\Omega|_W = 0$ . For example, take the subspace  $\text{span}(e_1, e_2)$ .

### 4.1.2 Morphisms: Symplectomorphism

**Definition 7** A map between  $(V, \Omega)$  and  $(V', \Omega')$  is a linear isomorphism  $\phi : V \rightarrow V'$  such that  $\phi^* \Omega' = \Omega$ . That is,  $\Omega'(\phi(v), \phi(w)) = \Omega(v, w)$ . ( $\phi^*$  is the pullback). The map  $\phi$  is called a Symplectomorphism and the spaces  $(V, \Omega)$  and  $(V', \Omega')$  are said to be Symplectomorphic

(Question: why do we need to define this in terms of a pullback? Why not pushforward? ie,  $\Omega(v, w) = \Omega'(\phi(v), \phi(w))$ ?)

### 4.1.3 Prototypical example of Symplectic space

Let  $V = \mathbb{R}^{2n}$ . Let  $e_j = (0_1, \dots, 1_j, \dots, 0_n; 0_{n+1}, \dots, 0_{2n})$ . Let  $f_j = (0_1, \dots, 0_n; 0_{n+1}, \dots, 1_{n+j}, \dots, 0_{2n})$ . take  $\{e_j, f_j\}$  as a basis for  $V$ .  $\Omega_0 = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$

Let  $\phi : V \rightarrow V$  be a Symplectomorphism of  $V$ . Then, we can show that  $M_\phi^T \Omega_0 M_\phi = \Omega_0$ . ( $M_\phi$  is the matrix of  $\phi$  associated to the  $\{e_j, f_k\}$  basis).

#### 4.1.4 Symplectic Manifolds

We are generalizing our symplectic vector space. We are postulating a space that locally looks like a symplectic vector space.

Let  $\omega : T_p M \times T_p M \rightarrow \mathbb{R}$  be a 2-form on a manifold. This is bilinear and skew-symmetric (by definition of being a differential form).

We say that  $\omega$  is closed iff  $d\omega = 0$ .

**Definition 8** The two form  $\omega$  is a symplectic form if it is closed and  $\omega_p$  is symplectic for all  $p \in M$ . That is, we need  $\omega_p$ .

**Definition 9** A symplectic manifold is a pair  $(M, \omega)$  where  $\omega$  is a symplectic form on  $M$ .

#### 4.1.5 Prototypical example of Symplectic manifold

Let  $M = \mathbb{R}^{2n}$  with coordinates  $X_1, \dots, X_n, Y_1, \dots, Y_n$ .  $\omega = \sum_i dx_i \wedge dy_i$ .  $\omega$  is symplectic since it has constant coefficients for each  $dx_i \wedge dy_i$ .

Symplectic basis for the tangent space  $T_p \mathbb{R}^{2n} \equiv \{\partial_{X_i}, \partial_{Y_i}\}$ .

#### 4.1.6 2 sphere as a symplectic manifold

Let  $M = S^2$  as an embedded manifold in  $\mathbb{R}^3$ .  $S^2 \equiv \{v \in \mathbb{R}^3, |v| = 1\}$ .  $T_p S^2 \equiv \{w \in \mathbb{R}^3 | p \cdot w = 0\}$ . This works because  $p$  is normal to the plane spanned by  $T_p S^2$ . We define  $\omega_p(u, v) \equiv p \cdot (u \times v)$ . Clearly, this is a 2-form since it is bilinear and anti-symmetric.  $\omega$  is also closed, since there cannot be any degree 3 forms on a 2D manifold! Hence,  $d\omega = 0$ .

We need to check that it is non-degenerate. For any point  $p \in S^2$ ,  $v \in T_p S^2$ , we can take  $u = p \times v$ . Now,  $\omega_p(u, v) = p \cdot (u \times v)$  will be a non-degenerate area of a parallelepiped. The parallelepiped is non-degenerate as  $v, p$  are perpendicular by definition.  $u$  is perpendicular to both  $v$  and  $p$  by construction (cross-product). (TODO: draw picture).

#### 4.1.7 Mapping between Symplectic Manifolds

Let  $(M, \Omega)$  and  $(M', \Omega')$  are Symplectomorphic if  $\phi : M \rightarrow M'$  is diffeomorphic, and such that  $\phi^* \Omega' = \Omega$ .

#### 4.1.8 Symplectic manifolds are locally like $\mathbb{R}^{2n}$

**Theorem 10 (Darboux):** Let  $(M, \omega)$  be a  $2n$  dimensional symplectic manifold. Let  $p \in M$ . Then, there is a coordinate chart called as the Darboux chart  $U$  with basis  $X_1, \dots, X_n, Y_1, \dots, Y_n$  such that the two-form  $\omega = \sum_i dx_i \wedge dy_i$ .

### 4.2 The cotangent bundle and Symplectic forms

Let  $X$  be an  $n$ -dimensional manifold. Let  $M \equiv T^*X$  be its cotangent bundle.  $M$  is also a manifold.

Define coordinate charts on  $M$ ,  $(T^*U : X_1, X_2, \dots, X_n, \xi_1, \xi_2, \dots, \xi_n)$ , where  $U$  is a chart on  $X$ ,  $X_1, \dots, X_n$  are coordinate functions for  $U$ , and  $\xi_1, \dots, \xi_n$  are coordinates for  $T^*X_{x_0}$  where  $x_0 \in X$ , defined by  $\xi_{x_0} = \sum_i \xi_i dx_{i_{x_0}}$ .

We need to show that the transition functions are smooth.

Let  $p \in U \cap U'$ . We need to express coordinates in one chart as a smooth function of coordinates in another chart.  $(U, X_1, \dots, X_n)$ ,  $(U', X'_1, \dots, X'_n)$  are coordinates, and let  $\xi \in T_p^*M$ .  $(p, \xi) \in T^*X \equiv M$ .

$$\xi = \sum_i \xi_i dx_i = \sum_{i,k} \xi_i \frac{\partial x_i}{\partial x'_k} dx'_k = \sum_k \xi'_k dx'_k$$

where the  $\xi'_k = \xi_i \frac{\partial x_i}{\partial x'_k}$  are smooth since the  $\xi_i$  are smooth, and that the transition maps derivatives are smooth since the charts are smooth.

#### 4.2.1 Canonical Symplectic structure of cotangent bundle

The physicist intuition for the following is that when we consider a particle moving on a manifold  $X$ . To fully specify the state of the particle, we need both position of the particle  $x \in X$ , and also momentum  $p \in T^*X$  (why does it belong to cotangent bundle? Intuition: given a velocity, it returns the momentum along it???. We can also supposedly look at how momentum transforms). This data is called "phase space" by physicists. Mathematically, this is the cotangent bundle.

Now, we will see the canonical symplectic structure on the cotangent bundle.

#### 4.2.2 Tautological and Canonical forms

Let  $(U, x_1, \dots, x_n)$  be a coordinate chart of  $X$  and  $(T^*U, x_1, \dots, x_n, \xi_1, \dots, \xi_n)$  be the corresponding chart of  $M \equiv T^*X$ . We will now define a 2-form on  $M$  on the chart  $T^*U$  via:

$$\omega = \sum_j dx_j \wedge d\xi_j$$

**Theorem 11** *This definition is coordinate independent*

**Proof 12** *Consider a one-form on  $T^*U \subset M$ . Let  $\alpha = \sum_j \xi_j dx_j$ . This is called as the tautological form.  $\alpha$  is a one-form on  $M \equiv T^*X$ , when  $T^*X$  is treated as a manifold. This is also a point on  $T^*X$ . Note that  $d\alpha = -\omega$ . But  $\alpha$  is intrinsically defined.  $\alpha = \sum_j \xi_j dx_j = \sum_j \xi'_j dx'_j$ .*

*TODO: I don't understand why this gives us coordinate independence.*

#### Coordinate-free definition

Let  $M \equiv T^*X$ . There is a canonical map  $\pi : M \rightarrow X$ ,  $\pi(x, \xi) = x$ . We are going to pullback  $T^*X$  along  $T^*M = T^*(T^*X)$   $(d\pi)^* : T^*X \rightarrow T^*M$ . Let  $p \in M$ ;  $p = (x, \xi)$ .  $\xi \in T_x^*X$ . We define  $\alpha$  pointwise.  $\forall p \in M, \alpha_p \equiv d\pi_p^* \xi_{\pi(p)}$ . Equivalently, let  $v_p \in T_p M \equiv T_p(T^*X)$ . Now,  $\alpha_p(v_p) \equiv \xi(d\pi_p(v_p))$ .

*TODO: draw example (<https://www.youtube.com/watch?v=hAX7ZCMM2kQ>, 43:52)*



**Example 1**

Let  $X = \mathbb{R}$ . Now,  $M = \mathbb{R} \times \mathbb{R}$ .  $(x, y) \in M$  (position-momentum).  $\pi : M \rightarrow X; \pi((x, y)) = x$ .

$$\begin{aligned}\alpha_{(x,y)} &\equiv y dx & \omega &= -d\alpha = -(\partial_y y \cdot dy \wedge dx) = dx \wedge dy \\ \alpha_{(x,y)}(v_x, v_y) &= y dx(v_x \partial_x + v_y \partial_y) = y v_x\end{aligned}$$

**Example 1**

Let  $X = S^1$ . Now,  $M = S^1 \times \mathbb{R}$ .  $(x, y) \in M$  (position-momentum).  $\pi : M \rightarrow X; \pi((\theta, v)) = \theta$ .

$$\begin{aligned}\alpha_{(x,y)} &\equiv y d\theta \\ \omega_{(x,y)} &\equiv d\theta \wedge dy\end{aligned}$$

**4.2.3 Naturality of Tautological form**

Let  $X_1, X_2$  be  $n$ -dimensional manifolds with  $T^*X_1$  and  $T^*X_2$  as cotangent bundles. Let  $f : X_1 \rightarrow X_2$  is a diffeomorphism. We will show that the tautological manifolds also match.

We show that there is a diffeomorphism  $f_\# : T^*X_1 \rightarrow T^*X_2$  which lifts  $f$ . ie, if  $f_\#((x, \xi)) = (x', \xi')$ , where  $x' = f(x)$ ,  $\xi = df^* \xi'$ .

**Theorem 13** *The lift  $f_\#$  of  $f : X_1 \rightarrow X_2$  pulls the tautological form on  $M_2 \equiv T^*X_2$  onto the tautological form on  $M_1 \equiv T^*X_1$ .*

**Proof 14** *Pointwise, we wish to show that  $(df_\#^*)_{p_1}(\alpha_2)_{p_2} = (\alpha_1)_{p_1}$  where  $p_2 = f_\# p_1$ .*

$$p_2 = f_\# p_1 \implies p_2 = (x_2, \xi_2), x_2 = f(x_1) \wedge df^*(\xi_1) = \xi_2.$$

TODO!



## Chapter 5

# Vector fields and flows

TODO: integrate with notes on computer

**Theorem 15** For any point  $x \in M$ , there exists a differentiable function  $\sigma : \mathbb{R} \times M \rightarrow M$  such that  $\sigma(0, x) = x$ ,  $\sigma(t, \sigma(s, x)) = \sigma(t + s, x)$ , and the map  $t \mapsto \sigma(t, x)$  satisfies nice properties (which ones?)

**Proof 16**

**Definition 17** Let  $\sigma : \mathbb{R} \times M \rightarrow M$  be a flow. Write  $\sigma_t(p) \equiv \sigma(t, p)$ . The map  $\sigma_t$  is an **isotopy** if each  $\sigma_t : M \rightarrow M$  is a diffeomorphism and  $\sigma_0 \equiv \text{identity}$ .

Conversely, on a compact manifold  $M$ , there is a one-to-one correspondence between isotopies and time-dependent vector fields, given by the equation:

$$(\partial_x \sigma_t)(x_0) = v_t(\sigma_t(x_0))$$

Note that the situation can get complicated even on compact manifolds. Eg. a vector field on a torus with constant irrational slope — The space foliates with 1D subspaces. (TODO: add picture)

**Definition 18** When  $X = v_t$  is independent of  $t$ , the isotopy is said to be the **exponential map of the flow of  $X$** . It is denoted by  $\sigma^\mu(t, x) \equiv \exp(tX)x^\mu$ .  $\{\exp(tX) : M \rightarrow M \mid t \in \mathbb{R}\}$  is a unique, smooth family of diffeomorphisms, satisfying:

- $\exp(0X) = \text{id}$
- $\partial_t \exp(tX) = X \circ \exp(tX)$

Let us justify naming this object  $\exp$ :

$$\begin{aligned}\sigma^\mu(0 + t, x) &= \text{taylor series around } t = 0 \\ &= x^\mu + t(\partial_t \sigma^\mu)(0, x) + \frac{t^2}{2!}(\partial_t \partial_t \sigma^\mu)(0, x) + \dots \\ &= e^{t \partial_t} \sigma^\mu(t, x)|_{t=0} = e^{tX} \sigma^\mu(0, x)\end{aligned}$$

**Definition 19** The flow  $\sigma_t$  satisfies:

- $\sigma(0, x) = \exp(0X)$
- $\partial_t \sigma(t, x) = X(e^t X(x))$
- $\sigma(t, \sigma(s, x)) = \sigma(t, e^{sX}x) = e^{tX}(e^{sX}x) = e^{(t+s)X}x = \sigma(t+s, x)$

## Chapter 6

# Lie derivative: $\mathcal{L}_X Y$

### 6.1 The definition of the lie derivative

The problem with manifolds is that to compare values at  $x, y \in M$ , it's unclear how to compare objects. We simply cannot, since there is no structure available to do this. So, we construct the lie derivative. Given two vector fields  $X, Y$ . Let  $\sigma(s, x)$  and  $\tau(t, x)$  be the flows generated by  $X, Y$  respectively. Hence,  $(\partial_s \sigma^\mu)|_{(s,p)} = X(\sigma(s, p))$ ,  $(\partial_t \tau^\mu)|_{(t,p)} = Y(\tau(t, p))$ .

The derivative of  $Y$  along the integral curve  $\sigma$  generated by  $X$  is:

- map  $Y(\sigma_\epsilon(x)) : T_{\sigma_\epsilon(x)} M$  to  $T_x M$ , by  $\sigma_{-\epsilon*} : T_{\sigma_\epsilon(x)} M \rightarrow T_x M$ .
- Take the difference at  $x$ , between  $(\sigma_{-\epsilon*}(Y(\sigma_\epsilon(x))))$  and  $Y(x)$ .
- Let  $\epsilon \rightarrow 0$ :  $\mathcal{L}_X Y \equiv \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \sigma_{-\epsilon*}(Y(\sigma_\epsilon(x))) - Y(x)$ .

### 6.2 Coordinate definition of Lie derivative

Let  $(U, \phi)$  be a chart with coordinates  $X^\mu$ . We define  $e_\mu \equiv \partial_{X^\mu}$ . We write all our objects in terms of this chart.

- $X \equiv X^\mu e_\mu$
- $Y \equiv Y^\mu e_\mu$
- $\sigma_\epsilon^i(x) = x^i + \epsilon X^i$
- $\sigma_{-\epsilon}^i(x) = x^i - \epsilon X^i$
- $Y^j(\sigma_\epsilon(x)) = Y^j(x + \epsilon X) = Y^j + \epsilon X^k \partial_k Y^j$
- $\sigma_{-\epsilon*}^i(v) = v^j \partial_j \sigma_{-\epsilon}^i = v^j \partial_j (x^i - \epsilon X^i) = v^j (\delta_j^i - \epsilon \partial_j X^i)$
- $\sigma_{-\epsilon*}^i(Y(\sigma_\epsilon(x))) = Y(\sigma_\epsilon(x))^j (\delta_j^i - \epsilon \partial_j X^i) = (Y^j + \epsilon X^k \partial_k Y^j)(\delta_j^i - \epsilon \partial_j X^i) = Y^j \delta_j^i + \epsilon (X^k \partial_k Y^j \delta_j^i - Y^j \partial_j X^i) + \epsilon^2(\dots) = Y^i + \epsilon (X^k \partial_k Y^i - Y^k \partial_k X^i)$
- $(\lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \sigma_{-\epsilon*}(Y(\sigma_\epsilon(x))) - Y(x))^i = \frac{1}{\epsilon} (Y^i + \epsilon (X^k \partial_k Y^i - Y^k \partial_k X^i) - Y^i) = X^k \partial_k Y^i - Y^k \partial_k X^i$

**Example 20** Manifold  $M$ , chart  $\phi$ , coordinates  $X^1, X^2$ .  $P \equiv -X^2 \partial_{X^1} + X^1 \partial_{X^2}$ ,  $Q \equiv (X^1)^2 \partial_{X^1} + X^2 \partial_{X^2}$ . The lie derivative is computed as:

$$\begin{aligned} \mathcal{L}_X Y &= \{(-X^2 \partial_{X^1} + X^1 \partial_{X^2})(X^1)^2 - ((X^1)^2 \partial_{X^1} + X^2 \partial_{X^2})(-X^2)\} e_1 + (\dots) e_2 \\ &= \{(-X^2(2X^1) + 0) - (0 + X^2(-1))\} e_1 + (\dots) e_2 \end{aligned}$$

### 6.2.1 The lie bracket

**Definition 21** Let  $X^\mu \partial_\mu, Y^\mu \partial_\mu \in \mathfrak{X}(M)$ . We define the lie derivative as  $[X, Y] \equiv [X, Y](f) = X(Y(f)) - Y(X(f))$ .

Now this might have second-order derivatives. However, for it to be a vector field, it is only allowed to have first order derivatives. Let's prove that the lie bracket only contains first-order derivatives.

$$\begin{aligned} [X, Y]f &= \sum_{\mu, \nu} X^\mu \partial_\mu (Y^\nu \partial_\nu f) - Y^\mu \partial_\mu (X^\nu \partial_\nu f) \\ &= \sum_{\mu, \nu} X^\mu (Y^\nu \partial_\mu \partial_\nu f + \partial_\nu f \partial_\mu Y^\nu) - Y^\mu (X^\nu \partial_\mu \partial_\nu f + \partial_\nu f \partial_\mu X^\nu) \\ &= (X^\mu Y^\nu \partial_\mu \partial_\nu f - Y^\mu X^\nu \partial_\mu \partial_\nu f + \partial_\nu f (X^\mu \partial_\mu Y^\nu - Y^\mu \partial_\mu X^\nu)) = 0 + \mathcal{L}_X Y f \end{aligned}$$

The terms are zero since  $X^\mu Y^\nu \partial_\mu \partial_\nu f - Y^\mu X^\nu \partial_\mu \partial_\nu f$  vanishes:  $\partial_\mu \partial_\nu f = \partial_\nu \partial_\mu f$  by smoothness. Then the sum  $X^\mu Y^\nu - Y^\mu X^\nu$  disappears due to the double summation.

### 6.2.2 Properties of the lie bracket

**Lemma 22** The lie bracket is bilinear:  $[X, cY + dY'] = c[X, Y] + d[X, Y']$ .

**Proof 23** TODO

**Lemma 24** The lie bracket is skew-symmetric:  $[X, Y] = -[Y, X]$ .

**Proof 25** TODO

**Lemma 26** The lie bracket satisfies the Jacobi identity:  $[[X, Y], Z] + [[Y, Z], X] + [[X, Z], Y] = 0$ .

**Proof 27** TODO

Let us define  $(fX)(p) \equiv f(p)X^\mu(p)e_\mu$ .

**Lemma 28**  $\mathcal{L}_{fX} Y = f[X, Y] - Y[f]X$

**Proof 29** TODO

**Lemma 30**  $\mathcal{L}_Y(fX) = f[X, Y] + X[f]Y$

**Proof 31** TODO

**Lemma 32**  $f_*[X, Y] = [f_*X, f_*Y]$

**Proof 33** TODO

**6.2.3 Lie bracket as failure of flows to commute**

(TODO: draw picture)

**Lemma 34**  $\tau(\delta, \sigma(\epsilon, x)) - \sigma(\epsilon, \tau(\delta, x)) = \epsilon[X, Y] + O(\epsilon^2)$ .

**Proof 35** *TODO*

**6.2.4 Lie derivatives for one forms**

$$\mathfrak{L}_X \omega \equiv \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} [\sigma_{-\epsilon}^*(\omega(\sigma_\epsilon(x))) - \omega(x)]$$

That is, replace pushforward with pullbacks everywhere.

In components, the formula will be:

$$\sigma_{-\epsilon}^*(\omega(\sigma_\epsilon(x))) = \omega_i(x) dx^i + (\epsilon X^k \partial_k \omega_j + \omega_k \partial_j X^k) dx^j$$

The derivation is: (TODO)

**6.2.5 Lie derivative for k-forms**

The formula is exactly the same as that of the one-form case:

$$\mathfrak{L}_X \omega \equiv \frac{d}{dt} (\exp(tX))^* \omega|_{t=0}$$

**6.2.6 An axiomatic characterization of the Lie derivative**

The lie derivative satisfies:

$$\mathfrak{L}_X(t_1 + t_2) = \mathfrak{L}_X t_1 + \mathfrak{L}_X t_2$$

where  $t_1, t_2$  are objects of the same type.

It also satisfies:

$$\mathfrak{L}_X(t_1 \otimes t_2) = (\mathfrak{L}_X t_1) \otimes t_2 + t_1 \otimes (\mathfrak{L}_X t_2)$$

for arbitrary  $t_1, t_2$ .

Since we know the definition for a vector and a 1-form, we can use this to work out the value of the lie derivative for k-forms.





## Chapter 7

# Interior products

The interior product is an operation that takes a  $k$ -form and a vector field, and produces a  $(k-1)$ -form. For programmers, this is literally just partial application of the  $k$ -form.

$$\forall X \in \mathfrak{X}(M) \quad i_X : \Omega^k(M) \rightarrow \Omega^{k-1}(M) \quad (i_X \omega)(X_1, \dots, X_{k-1}) \equiv \omega(X, X_1, \dots, X_{k-1})$$

Let  $\omega \in \Omega^k(M)$ . In components relative to a chart  $(U, \phi)$  with basis  $x^i$ ,  $\omega$  can be written as:

$$\omega = \frac{1}{k!} \omega_{i_1, i_2, \dots, i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k}$$

where the coefficients  $\omega_{i_1, \dots, i_k}$  are anti-symmetric on exchanging indices.

$$(i_X \omega) = \frac{1}{(k-1)!} X^j \omega_{j, i_2, \dots, i_k} dx^{i_2} \wedge \dots \wedge dx^{i_k}$$

$$(i_X \omega) = \frac{1}{(k-1)!} (-1)^{j-1} X^j \omega_{i_1, \dots, j, \dots, i_k} dx^{i_1} \wedge \widehat{dx^j} \wedge \dots \wedge dx^{i_k}$$

where  $\widehat{dx^j}$  means we do not include that  $dx^j$  in the wedge

### 7.0.1 Example of interior product computation

- Let  $M = \mathbb{R}^3$ . Let  $\omega = dx^1 \wedge dx^2$ . Let  $X = 1 \cdot \partial_{x^1}$ . Then,  $i_X \omega = 1 \cdot dx^2$
- Let  $M = \mathbb{R}^3$ . Let  $\omega = dx^1 \wedge dx^3$ . Let  $X = 1 \cdot \partial_{x^3}$ . Then,  $i_X \omega = i_X(dx^1 \wedge dx^3) = i_X(-dx^3 \wedge dx^1) = -dx^1$

### 7.0.2 Cartan's Magic formula

Let  $\omega \in \Omega^1(M)$ ,  $\omega \equiv \omega_i dx^i$ . Let  $X \in \mathfrak{X}(M)$ ,  $X = X^i \partial_i$ .

$$i_X \omega = \omega_i X^i$$

$$d(i_X \omega) = \partial_i (\omega_i X^i) dx^i$$

$$i_X(d\omega) = i_X(\partial_j \omega_i dx^j \wedge dx^i) = \text{TODO}$$

So, we get the formula:

$$(\mathrm{di}_X + \mathrm{i}_X \mathrm{d})\omega = \mathrm{L}_X \omega$$

### 7.0.3 Relationships between interior product and k-forms

**Lemma 36**  $\mathrm{i}_{[X,Y]}(\omega) = X(\mathrm{i}_Y(\omega)) - Y(\mathrm{i}_X(\omega))$

**Proof 37** *TODO*

**Lemma 38**  $\mathrm{i}_X(\omega \wedge \zeta) = \omega \mathrm{i}_X(\zeta) - 1^n$

**Proof 39** *TODO*

## Chapter 8

# Tensor network diagrams

Elements in  $v^j \partial_j \in TM$  is drawn as

$j \text{---} (v)$

. A leg is an index, a ball is a tensor. elements of  $w_j dx^j \in TM^*$  is drawn as

$(w) \text{---} j$

. Tensor contraction is by joining legs.  $v^j w_j$  is

$(w) \text{---} (v)$

.

A k form is drawn as (TODO)



## Chapter 9

# Symplectic Manifolds

Let  $M, \omega$  be a symplectic manifold. A vector field  $X \in \mathfrak{X}(M)$  that preserves  $\omega$  — that is,  $\frac{d}{dt}(\sigma_t^*(\omega)) = 0$  is called a symplectic vector field.

Notice that all hamilton vector fields are symplectic.

### 9.1 Hamiltonian vector fields for free

Suppose  $M$  is a manifold and let  $X \in \mathfrak{X}(M)$ . Then there is a unique  $X_\sharp \in \mathfrak{X}(T^*M)$  whose flow  $\sigma_t^\sharp$  is a lift of  $\sigma_t$ . Let  $\alpha$  be the tautological 1-form on  $T^*M$ . Let  $\omega = -d\alpha$ . Show that  $X_\sharp$  is a Hamiltonian vector field, with  $H = i_{X_\sharp}\alpha$ .

TODO: write down proof!

### 9.2 When is a Symplectic vector field Hamilton?

#### 9.2.1 Conditions for a vector field to be Symplectic

**Theorem 40**  $X$  is symplectic if and only if  $i_X\omega$  is closed.

**Proof 41**  $X$  is symplectic  $\implies i_X\omega$  is closed.

$$\begin{aligned} d(i_X\omega) &= d(i_X\omega) + i_X(d\omega) && \text{Since } d\omega = 0, \text{ we can add } i_X(d\omega) \\ &= \mathfrak{L}_X\omega && \text{By the identity } \mathfrak{L}_X = di_X + i_Xd \\ &= 0 && \text{By the definition of being Symplectic} \end{aligned}$$

**Proof 42**  $\alpha_X = i_X\omega$  is closed  $\implies X$  is symplectic. TODO

#### 9.2.2 Conditions for a vector field to be Hamilton

$X$  is Hamiltonian if  $\exists H \in M \rightarrow \mathbb{R}, i_X\omega = dH$ . That is, the vector field  $i_X\omega$  is exact.

The exactness is a *global* structure, since it's asking for the existence of a "potential function"  $H$ . This is equivalent to asking for information about the De Rham cohomology, which in turn relates to the homology.

Note that locally, on contractible open sets, closed and exact are the same thing, and therefore at least locally, all symplectic vector fields are hamilton.

### 9.2.3 Equivalent ways of stating that a vector field is Symplectic

In summary, the following are all equivalent:

- $X \in \mathfrak{X}(M)$  is symplectic
- The flow  $\sigma_X = \exp(tX)$  preserves  $\omega$
- $\mathcal{L}_X \omega = 0$
- $i_X \omega$  is **closed**

## 9.3 Classical Mechanics: Redux

Consider Euclidian space  $\mathbb{R}^{2n}$  with coordinates  $(q_1, q_2, \dots, q_n, p_1, p_2, \dots, p_n)$ , and with a two-form  $\omega \equiv \sum_i dq_i \wedge dp_i$ . We have a hamilton function on phase space  $H : \mathbb{R}^{2n} \rightarrow \mathbb{R}$ . Let  $X_H$  be the corresponding vector field, defined as

$$X_H = \sum_{j=1}^n \frac{\partial H}{\partial p_j} \frac{\partial}{\partial q_j} - \frac{\partial H}{\partial q_j} \frac{\partial}{\partial p_j}$$

We assert (from a computation we will perform) that:  $i_{X_H} \omega = dH$ .

To perform the above computation, we first need a lemma about computing interior product on two forms:

**Lemma 43**  $i_X(da \wedge db) = (i_X da) \wedge db - da \wedge (i_X db)$

**Proof 44** We let  $X \equiv X_a \partial_a + X_b \partial_b$ ,  $Y \equiv Y_a \partial_a + Y_b \partial_b$ .

Next, we write down the definition of the interior product when applied to a vector field  $Y$ :

$$\begin{aligned} i_X(da \wedge db)(Y) &= (da \wedge db)(X, Y) && \text{(by definition, } i_X \text{ is the partial application of } X) \\ &= (da \wedge db)(X_a \partial_a + X_b \partial_b, Y_a \partial_a + Y_b \partial_b) && \text{(Substituting } X, Y) \\ &= (da \wedge db)(X_a \partial_a, Y_a \partial_a) + (da \wedge db)(X_a \partial_a, Y_b \partial_b) + \\ &\quad (da \wedge db)(X_b \partial_b, Y_a \partial_a) + (da \wedge db)(X_b \partial_b, Y_b \partial_b) && \text{(Multi-linearity of } da \wedge db) \\ &= (da \wedge db)(X_a \partial_a, Y_b \partial_b) - (da \wedge db)(Y_a \partial_a, X_b \partial_b) + \\ &\quad \cancel{(da \wedge db)(X_b \partial_b, Y_b \partial_b)} + \cancel{(da \wedge db)(X_a \partial_a, Y_a \partial_a)} && \text{(Anti-symmetry)} \\ &= 1 \cdot (X_a Y_b) - 1 \cdot (X_b Y_a) \text{ TODO} \end{aligned}$$

We now prove the main assertion, the structure of  $X_H$ :

**Theorem 45** *Given that  $X_H$  is defined as:*

$$X_H \equiv \sum_{j=1}^n \frac{\partial H}{\partial p_j} \frac{\partial}{\partial q_j} - \frac{\partial H}{\partial q_j} \frac{\partial}{\partial p_j}$$

*We assert that  $i_{X_H} \omega = dH$ .*

**Proof 46**

$$\begin{aligned} i_{X_H} \omega &= i_{X_H} \left( \sum_{j=1}^n dq_j \wedge dp_j \right) && \text{(Definition of } \omega) \\ &= \sum_{j=1}^n i_{X_H} (dq_j \wedge dp_j) && \text{(Linearity of } i) \\ &= \sum_{j=1}^n (i_{X_H} dq_j) \wedge dp_j - dq_j \wedge (i_{X_H} dp_j) && \text{(Lemma above 43)} \\ &= \sum_{j=1}^n \frac{\partial H}{\partial p_j} p_j - \left( -\frac{\partial H}{\partial q_j} dq_j \right) && \text{(Defn of } X_H) \\ &= dH && \left( df = \frac{\partial f}{\partial x_i} x_i \right) \end{aligned}$$





## Chapter 10

# Optimisation on manifolds

### 10.0.1 Sketch of optimisation on manifolds

We now consider manifold optimisation techniques on embedded riemannian manifolds  $M$ , equipped with the metric  $g : (p : M) \rightarrow T_p M \times T_p M \rightarrow \mathbb{R}$ . The metric at a point  $g(p)$  provides an inner product structure on the point  $T_p M$  for a  $p \in M$ .

where we are optimising a cost function  $c : M \rightarrow \mathbb{R}$ . We presume that we have a diffeomorphism  $E : M \rightarrow \mathbb{R}^n$  (Embedding) which preserves the metric structure. We will elucidate this notion of preserving the metric structure once we formally define the mapping between tangent spaces. This allows us to treat  $M$  as a subspace of  $\mathbb{R}^n$ .

For any object  $X$  defined with respect to the manifold, we define a new object  $\bar{X}$ , which is the embedded version of  $X$  in  $\mathbb{R}^n$ .

We define  $\bar{M} \subset \mathbb{R}^n; \bar{M} \equiv \text{image}(E)$ . We define  $\bar{c} : \bar{M} \subseteq \mathbb{R}^n \rightarrow \mathbb{R}; \bar{c} \equiv c \circ E^{-1}$

We then need two operators, that allows us to project onto the tangent space and the normal space. The tangent space at a point  $x_0 \in M$ ,  $T_{x_0} \bar{M} \equiv \text{span}(\partial_i E|_{E(x_0)})$ . We get an induced mapping of tangent spaces  $dE : T_{x_0} M$  and  $T_{x_0} \bar{M}$ .

we consider the gradient  $\nabla c : (p : \bar{M}) \rightarrow T_p \bar{M}; \nabla c \equiv dE dc$

The normal space,  $N_{x_0} \bar{M}$  is the orthogonal complement of the tangent space, defined as  $N_{x_0} \bar{M} \equiv \{v \in \mathbb{R}^n \mid \langle v | T_{x_0} \bar{M} \rangle = 0\}$ . It is often very easy to derive the projection onto the normal space, from whose orthogonal complement we derive the projection of the tangent space.

The final piece that we require is a retraction  $R : \mathbb{R}^n \rightarrow \bar{M} \subseteq \mathbb{R}^n$ . This allows us to project elements of the ambient space that are not on the manifold. The retraction must obey the property  $R(p \in \bar{M}) = p$ . (TODO: is this correct? Do we need  $R(\bar{M}) = \bar{M}$  or is this pointwise?) (what are the other conditions on the retraction? smoothness?)

Given all of this machinery, the algorithm is indeed quite simple.

- $x \in \bar{M} \subseteq \mathbb{R}^n$  is the current point on the manifold as an element of  $\mathbb{R}^n$
- Compute  $g = \nabla c(x) \in T_x \mathbb{R}^n$  is the gradient with respect to  $\mathbb{R}^n$ .
- $\bar{g} = P_{T_x} g \in T_x M$  is the projection of the gradient with respect to  $\mathbb{R}^n$  onto the tangent space of the manifold.
- $x_{\text{mid}} \in \mathbb{R}^n \equiv x + \eta \bar{g}$ , a motion along the tangent vector, giving a point in  $\mathbb{R}^n$ .

- $\bar{x}_{\text{next}} : \bar{M} \equiv R(x_{\text{mid}})$ , the retraction of the motion along the tangent vector, giving a point on the manifold  $\bar{M}$ .

## Chapter 11

# Exterior and Geometric algebras

While not strictly in the realm of differential geometry, geometric algebras are an interesting beast. Essentially, they answer the question:

What if we could study vectors on equal footing with arbitrary subspaces?

### 11.1 Exterior Algebras

We first define the *exterior algebra of degree 2* of a vector space  $V$ . Intuitively, these represent *oriented areas* of a space. Later, we generalize these to volumes and hypervolumes.

Given a vector space  $V$  of dimension  $n$ , we define a space  $(\Omega^2(V))$ , called as the exterior algebra of  $V$  of degree 2. We can construct elements of  $(\Omega^2(V))$  by using an operator  $(\wedge)$ , defined by the following axioms:

$$\wedge : V \times V \rightarrow \Omega^2(V)$$

$$\vec{v} \wedge \vec{w} = -\vec{w} \wedge \vec{v} \quad (\text{skew-symmetry})$$

$$\forall \alpha, \beta, \gamma, \delta \in \mathbb{R} \quad \vec{u}, \vec{v}, \vec{w}, \vec{x} \in V,$$

$$(\alpha \vec{u} + \beta \vec{v}) \wedge (\gamma \vec{w} + \delta \vec{x}) = \alpha \gamma (\vec{u} \wedge \vec{w}) + \alpha \delta (\vec{u} \wedge \vec{x}) + \beta \gamma (\vec{v} \wedge \vec{w}) + \beta \delta (\vec{v} \wedge \vec{x}) \quad (\text{bilinearity})$$

### 11.2 Intuition of the definition of wedge products on $\mathbb{R}^2$

From the above axioms, we can derive an intuition for the wedge product in  $\mathbb{R}^2$ . let us consider  $\vec{x}, \vec{y}$  as unit basis vectors of  $\mathbb{R}^2$ .

We first show:  $[\vec{x} \wedge \vec{x} = -(\vec{x} \wedge \vec{x}) \implies 2(\vec{x} \wedge \vec{x} = 0) \implies (\vec{x} \wedge \vec{x} = 0)]$ . Similarly,  $(\vec{y} \wedge \vec{y} = 0)$ . We also note that  $\vec{x} \wedge \vec{y} = -\vec{y} \wedge \vec{x}$ , by using the skew-symmetry rule.

Now, we observe the value of  $(\vec{v} \wedge \vec{w})$  for a general  $(\vec{v}, \vec{w} \in \mathbb{R}^2)$  and we provide a geometric interpretation. We first write  $\vec{v}, \vec{w}$  in terms of the basis vectors  $\vec{x}, \vec{y}$  as:  $(\vec{v} = \alpha \vec{x} + \beta \vec{y}), (\vec{w} = \gamma \vec{x} + \delta \vec{y})$ . Next, we expand  $(\vec{v} \wedge \vec{w})$  as:

$$\begin{aligned}
\vec{v} \wedge \vec{w} &= (\alpha\vec{x} + \beta\vec{y}) \wedge (\gamma\vec{x} + \delta\vec{y}) \\
&= \alpha\gamma(\vec{x} \wedge \vec{x}) + \alpha\delta(\vec{x} \wedge \vec{y}) + \beta\gamma(\vec{y} \wedge \vec{x}) + \beta\delta(\vec{y} \wedge \vec{y}) \\
&= \alpha\gamma(0) + \alpha\delta(\vec{x} \wedge \vec{y}) + \beta\gamma(-\vec{x} \wedge \vec{y}) + \beta\delta(0) \\
&= (\alpha\delta - \beta\gamma)(\vec{x} \wedge \vec{y})
\end{aligned}$$

We know from 2D geometry that  $(\alpha\delta - \beta\gamma)$  is the area of the parallelogram spanned by vectors  $(\vec{v}, \vec{w})$ . Now,  $\vec{x} \wedge \vec{x} = 0$  can be interpreted as "the parallelogram made up of two sides which are collinear has 0 area". The fact  $(\vec{x} \wedge \vec{y} = -\vec{y} \wedge \vec{x})$  can be interpreted as *orientation*. We view  $(\vec{x} \wedge \vec{y})$  as an "anti-clockwise" orientation going from the positive  $x$ -axis (3 o'clock) to the positive  $y$ -axis (12 o'clock).  $(\vec{y} \wedge \vec{x})$  is a clockwise orientation, going from the positive  $y$ -axis to the positive  $x$ -axis.

### 11.3 Generalizing exterior algebras to arbitrary dimensions

Given a vector space  $V$  of dimension  $k$ , we define the  $k$ -dimensional exterior algebra space inductively as follows:

$$\begin{aligned}
\Omega^k(n) &\equiv \{\vec{x}_1 \wedge \vec{x}_2 \wedge \dots \wedge \vec{x}_k \mid \vec{x}_1, \vec{x}_2, \dots, \vec{x}_k \in V\} \\
(\vec{x}_1 \wedge \dots \wedge \vec{x}_k) &= \text{sign}(P)(\vec{x}_{P(1)} \wedge \vec{x}_{P(2)} \wedge \dots \wedge \vec{x}_{P(k)}) \quad (\text{skew-symmetry})
\end{aligned}$$

where  $(P : \{1, 2, \dots, k\} \rightarrow \{1, 2, \dots, k\})$  is a permutation (bijection).  $\text{sign}(P)$  is  $+1$  if the permutation is an even permutation (contains an even number of swaps), and  $-1$  if it is an odd permutation (contains an odd number of swaps)

$$\begin{aligned}
(\vec{x}_1 \wedge \dots \wedge (\alpha\vec{y}_1 + \beta\vec{y}_2) \wedge \vec{x}_k) &= \\
&\alpha(\vec{x}_1 \wedge \dots \wedge \vec{y}_1 \wedge \dots \wedge \vec{x}_k) + \beta(\vec{x}_1 \wedge \dots \wedge \vec{y}_2 \wedge \dots \wedge \vec{x}_k) \quad (\text{multi-linearity})
\end{aligned}$$

Note that this clearly extends the situation as described in 2D to  $n$ D.

### 11.4 Geometric algebra

The geometric algebra of a vector space  $V$  over the reals  $\mathbb{R}$  of dimension  $n$  is called as  $\mathcal{G}(V)$ .  $\mathcal{G}(V)$  contains all *formal linear combinations* of elements from  $(\mathbb{R}, V, \Omega^2(V), \Omega^3(V), \dots, \Omega^n(V))$ . For example, if we consider the space  $\mathcal{G}(\mathbb{R}^2)$ , an element of this space is  $(2 - 3\vec{x} + 4\vec{y} + 5\vec{x} \wedge \vec{y})$ .

This is a unique space, because it allows us to combine objects such as scalars, vectors, areas, volumes, etc. This is the power that we shall exploit to model a wide variety of situations.

In general, if the vector space  $V$  is of dimension  $n$ , then the geometric algebra  $\mathcal{G}(V)$  will have dimension  $2^n$ , since it will contain as a basis all possible collections of subspaces.

## 11.5 The philosophy of geometric algebra

In general, within vector spaces, vectors are privileged. Subspaces on the other hand are defined with equations: for example, in 3D, the subspace spanned by the  $\vec{x}, \vec{y}$  axes would be  $\text{span}(\vec{x}, \vec{y}) = \{\lambda_x \vec{x} + \lambda_y \vec{y} \mid \lambda_x, \lambda_y \in \mathbb{R}\}$ . This is a *set of vectors*, and not an *element of the vector space*.

In a geometric algebra, we would represent the subspace (roughly) as  $(\vec{x} \wedge \vec{y})$ . This allows us to treat vectors, scalar, volumes, and hypervolumes on equal footing, and develop a theory that includes all of these objects.

It also provides a *geometric product*, that allows us to easily relate the regular *inner product* to the *exterior product*, thereby creating a unifying theory of vectors and all differential forms.

### 11.5.1 The geometric product

We first define the geometric product for vectors:

$$ab \equiv (a \cdot b) + (a \wedge b)$$

### 11.5.2 Construction: A non-commutative, bilinear structure for geometric algebra

We define a non-commutative, bilinear structure  $(\langle \cdot | \cdot \rangle : \mathcal{G}(V) \times \mathcal{G}(V) \rightarrow \mathbb{R})$ . We define the effect on the basis elements, and then extend it to the full space. Let the basis of  $V$  be  $\{b_1, b_2, \dots, b_k\}$ .

$$\langle b_{i_1} \wedge b_{i_2} \wedge \dots \wedge b_{i_n} | c_{j_1} \wedge c_{j_2} \wedge \dots \wedge c_{j_m} \rangle \equiv \begin{cases} 1 & \text{span}(b_{i_1}, b_{i_2}, \dots, b_{i_n}) \subseteq \text{span}(c_{j_1}, c_{j_2}, \dots, c_{j_m}) \\ 0 & \text{otherwise} \end{cases}$$

Then we extend this multi-linearly to the full space, since we have defined its action on the basis.

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$$\begin{aligned} & \langle 2 + 3\vec{x} + 4\vec{x} \wedge \vec{y} | 4\vec{y} \rangle \\ &= \langle 2 | 4\vec{y} \rangle + \langle 3\vec{x} | 4\vec{y} \rangle + \langle 4\vec{x} \wedge \vec{y} | 2\vec{x} \wedge \vec{y} \rangle \\ &= (2 \cdot 4) \langle 1 | \vec{y} \rangle + (3 \cdot 4) \langle \vec{x} | \vec{y} \rangle + (4 \cdot 4) \langle \vec{x} \wedge \vec{y} | \vec{y} \rangle \\ &= 8(1) + 12(0) + 16(0) = 8 \end{aligned}$$

Note that  $(\langle \vec{x} | \vec{y} \rangle = 0)$ , since the space spanned by  $\vec{x}$  is not contained in the space spanned by  $\vec{y}$ . Similarly,  $(\langle \vec{x} \wedge \vec{y} | \vec{y} \rangle = 0)$ , since the subspace spanned by  $(\vec{x}, \vec{y})$  is strictly bigger than the subspace spanned by  $(\vec{y})$ .

This dot product structure captures the asymmetric notion of "containment":  $\langle b | c \rangle \neq 0$  iff the span of  $b$  is contained in the span of  $c$ . This allows us to model situations where we wish to have a notion of containment *across subspaces* of a given space.