Instructor: Subhadip Mitra (*Office:* A3-313C, *Phone:* 1587, *E-mail:* subhadip.mitra@iiit.ac.in) **Date:** JANUARY 10, 2018

Problem Set 1

Topic: Vector Analysis in Cartesian Coordinates

$$\text{Kronecker delta: } \delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{otherwise} \end{cases} \tag{1.1}$$

$$\text{Kronecker delta is symmetric: } \delta_{ij} = \delta_{ji} \tag{1.2}$$

$$\text{Levi-Civita in 3D: } \varepsilon_{ijk} = \begin{cases} 1 & \text{if } i,j,k \text{ is an even permutation of 123} \\ & (\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312}) \\ -1 & \text{if } i,j,k \text{ is an odd permutation of 123} \\ & (\varepsilon_{132} = \varepsilon_{321} = \varepsilon_{213}) \\ 0 & \text{otherwise} \end{cases} \tag{1.3}$$

$$\text{Levi-Civita is anti-symmetric in any pair of indices: } \varepsilon_{ijk} = \varepsilon_{jki} = \varepsilon_{kij} = -\varepsilon_{jik} = -\varepsilon_{ikj} = -\varepsilon_{kji} \tag{1.4}$$

Summation convention: repeated indices are summed over unless specified otherwise. Repeated indices are called *dummy*. We can use any symbol for them.

$$\vec{A}\cdot\vec{B} = A_iB_i = A_iB_j\delta_{ij} \tag{1.5}$$

$$\left(\vec{A}\times\vec{B}\right)_i = \varepsilon_{ijk}A_jB_k \tag{1.6}$$

$$\delta_{ij}\delta_{ik} = \delta_{ik}, \quad \delta_{ij}\delta_{ij} = \delta_{ii} = 3, \quad \delta_{ij}\varepsilon_{ijk} = 0 \tag{1.7}$$

$$\varepsilon_{ijk}\varepsilon_{abc} = \begin{vmatrix} \delta_{ia} & \delta_{ib} & \delta_{ic} \\ \delta_{ja} & \delta_{jb} & \delta_{jc} \\ \delta_{ka} & \delta_{kb} & \delta_{kc} \end{vmatrix}, \quad \varepsilon_{ijk}\varepsilon_{iab} = \delta_{ja}\delta_{kb} - \delta_{jb}\delta_{ka}, \quad \varepsilon_{ijk}\varepsilon_{ija} = 2!(3-2)!\delta_{ka}, \quad \varepsilon_{ijk}\varepsilon_{ijk} = 3! \tag{1.8}$$

Q 1. Consider a transformation from an unprimed (x) to a primed (x') coordinate system. Prove $\vec{A}\cdot\vec{B}$ is a scalar and $\vec{A}\times\vec{B}$ is a vector in 3D. Remember $x'_i = R_{ij}x_j$ where $R_{ij} = \partial x'_i/\partial x_j = \partial x_j/\partial x'_i = \cos(x'_i, x_j)$.

Q 2. Convince yourself $R_{ij} = \partial x'_i/\partial x_j = \partial x_j/\partial x'_i$.

Q 3. In 2D, we can define an anti-symmetric symbol, ε_{ij} – the 2D Levi-Civita.* Given

$$\varepsilon_{ij}\varepsilon_{ab} = \begin{vmatrix} \delta_{ia} & \delta_{ib} \\ \delta_{ja} & \delta_{jb} \end{vmatrix},$$

simplify $\varepsilon_{ij}\varepsilon_{ab}\delta_{ia}$ and $\varepsilon_{ij}\varepsilon_{ab}\delta_{ia}\delta_{jb}$.

Q 4. Let us define a quantity with six indices $X_{ijkabc} \stackrel{\text{def.}}{=} \varepsilon_{ijk}\varepsilon_{abc}$ (Levi-Civita in 3D).† (Don't worry about what a monster X_{ijkabc} is ! It does not matter for us.) Simplify

(a) $Y_{jkbc} \stackrel{\text{def.}}{=} \delta_{ia}X_{ijkabc}$,

(b) $Z_{jc} \stackrel{\text{def.}}{=} \delta_{kb}Y_{jkbc}$,

(c) $S \stackrel{\text{def.}}{=} \delta_{jc}Z_{jc}$.

This process of multiplying with Kronecker delta to reduce the number of *free* (not dummy) indices is called "contraction". (Going through the steps here may not be a bad idea.)

Q 5. Show $\vec{A}\cdot\left(\vec{B}\times\vec{C}\right) = \vec{B}\cdot\left(\vec{C}\times\vec{A}\right) = \vec{C}\cdot\left(\vec{A}\times\vec{B}\right)$. If A,B,C are non-zero, show $\vec{A}\cdot\left(\vec{B}\times\vec{C}\right) = 0 \Leftrightarrow A,B,C$ are coplanar.

Q 6. Show $\vec{A}\times\left(\vec{B}\times\vec{C}\right) = \vec{B}\left(\vec{C}\cdot\vec{A}\right) - \vec{C}\left(\vec{A}\cdot\vec{B}\right)$.

*Levi-Civita was an Italian mathematician (en.wikipedia.org/wiki/Tullio_Levi-Civita) who lived and died in 3D space (unless there are extra space dimensions that we don't know anything about yet) but the meaning of '2D Levi-Civita' should be clear to all who were in the class!

†We shall use the symbol $\stackrel{\text{def.}}{=}$ to mean'defined as'.

Q 7. Derive the identities for

(a) $\left(\vec{A}\times\vec{B}\right)\cdot\left(\vec{C}\times\vec{D}\right)$

(b) $\left(\vec{A}\times\vec{B}\right)\times\left(\vec{C}\times\vec{D}\right)$

Q 8. If $\phi = \phi(x_i)$ is a scalar function, prove $\vec{\nabla}\phi$ is a vector.

Q 9. If $f(r)$ is a scalar function of r, the magnitude of the position vector (i.e., the radial distance from origin) show

$$\vec{\nabla} f(r) = \hat{r}\frac{df}{dr}.$$

What is $\vec{\nabla} r$?

Q 10. Show $\vec{\nabla}(uv) = u\vec{\nabla}v + v\vec{\nabla}u$. If $\vec{\nabla}u\times\vec{\nabla}v = 0$, show that u, v are related by some function $f(u,v) = 0$ & *vice versa*.

Q 11. Show $\vec{\nabla}\cdot\left(f\vec{A}\right) = \left(\vec{\nabla}f\right)\cdot\vec{A} + f\left(\vec{\nabla}\cdot\vec{A}\right)$. Compute $\vec{\nabla}\cdot(r^n\vec{r})$.

Q 12. Show $\vec{\nabla}\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot\left(\vec{\nabla}\times A\right) - \vec{A}\cdot\left(\vec{\nabla}\times B\right)$.

Q 13. Show $\vec{\nabla}\times\left(f\vec{V}\right) = f\left(\vec{\nabla}\times\vec{V}\right) + \left(\vec{\nabla}f\right)\times\vec{V}$. Evaluate $\vec{\nabla}\times(\vec{r}f(r))$ (the answer should be obvious even before you start deriving).

Q 14. Derive the expression for $\vec{\nabla}\cdot\vec{\nabla}f(r) \stackrel{\text{def.}}{=} \nabla^2 f(r)$. What is $\nabla^2 r^n$? For what value of n, r^n is a solution for Laplace's equation $\left(\nabla^2\phi = 0\right)$?

Q 15. Is it true that since a magnetic field $\vec{B}$ can be written as the curl of the magnetic vector potential $\vec{A}$, it must be *solenoidal* (whose divergence vanishes)? Also, is it OK to call an electric field *irrotational* as it can be written as a gradient of a scalar function, $\vec{E} = -\vec{\nabla}\phi$?[‡] If $\vec{A}$ is irrotational, show that $\vec{A}\times\vec{r}$ is solenoidal.

Q 16. What is $\vec{\nabla}\times\left(S\vec{\nabla}S\right)$?

Q 17. Show that $\vec{\nabla}\times\left(\vec{\nabla}\times\vec{V}\right) = \vec{\nabla}\left(\vec{\nabla}\cdot\vec{V}\right) - \nabla^2\vec{V}$. This defines $\left(\vec{\nabla}\cdot\vec{\nabla}\right)\vec{V} \equiv \nabla^2\vec{V}$.

Q 18. Show $\vec{\nabla}\left(\vec{A}\cdot\vec{B}\right) = \left(\vec{A}\cdot\vec{\nabla}\right)\vec{B} + \left(\vec{B}\cdot\vec{\nabla}\right)\vec{A} + \vec{A}\times\left(\vec{\nabla}\times\vec{B}\right) + \vec{B}\times\left(\vec{\nabla}\times\vec{A}\right)$.

[‡] In physics, these arguments are generally used in reverse.