0.1 Q1 – matrix representation for $|\phi_k\rangle\langle\phi_j|$, in the orthonormal $|\nu_i\rangle$ basis

$$\begin{aligned} O &= |\phi_k\rangle\langle\phi_j| = I\,|\phi_k\rangle\langle\phi_j|\,I \\ &= \left(\sum_l |\nu_l\rangle\langle\nu_l|\right)|\phi_k\rangle\langle\phi_j|\left(\sum_m |\nu_m\rangle\langle\nu_m|\right) \end{aligned}$$

Element at αth row, βth column of an operator O in the $\{\nu_i\}$ basis is $\langle\nu_\alpha| O |\nu_\beta\rangle$. So, in this case, it is:

$$\begin{aligned} &\langle\nu_\alpha|\left(\sum_l |\nu_l\rangle\langle\nu_l|\right)|\phi_k\rangle\langle\phi_j|\left(\sum_m |\nu_m\rangle\langle\nu_m|\right)|\nu_\beta\rangle \\ &= \left(\sum_l \langle\nu_\alpha|\nu_l\rangle\langle\nu_l|\right)|\phi_k\rangle\langle\phi_j|\left(\sum_m |\nu_m\rangle\langle\nu_m|\nu_\beta\rangle\right) \end{aligned}$$

$\langle\nu_\alpha|\nu_l\rangle = 1$ if $\alpha = l$, and 0 otherwise since $\{\nu_i\}$ are orthonormal. Similarly for β. Hence:
$= \langle\nu_\alpha|\phi_k\rangle\langle\phi_j|\nu_\beta\rangle$

0.2 Q2 – positive operator is Hermitian

We first show that a positive operator is normal, and this automatically implies that it is Hermitian.

To show that a positive operator is normal, we consider $A^\dagger A$

Now that we know that it is normal, by spectral decomposition, it posesses an eigenbasis. We now show that all of its eigenvalues are real. This is now a matrix with real entries on the diagonal, which is hermitian.

To show that the eigenvalues are real, let $|\lambda\rangle$ be an eigenvector with magnitude 1 and eigenvalue λ.

$$\langle\lambda| A |\lambda\rangle \geqslant 0 \qquad \lambda\langle\lambda|\lambda\rangle = \lambda \geqslant 0$$

Hence, the eigenvalues are real and positive, and therefore it is Hermitian.

0.3 Q3 – $A^\dagger A$ is positive

$$\forall v \in V, \ \langle v| A^\dagger A |v\rangle = \langle Av| |Av\rangle = \|Av\|^2 \geqslant 0$$

Hence, $A^\dagger A$ is positive.

0.4 Q4. Eigenvalues of a projector P are either 0 or 1

Let $|\lambda\rangle$ be an eigenvector of P with associated eigenvalue λ.

$$P^2(|\lambda\rangle) = \lambda(P|\lambda\rangle) = \lambda^2 |\lambda\rangle \qquad P(|\lambda\rangle) = \lambda |\lambda\rangle$$

However, since P is a projector, $P^2 = P$, and therefore, $\lambda^2 = \lambda$. The roots of this equation are $0, 1$. Hence, $\lambda \in \{0, 1\}$.

0.5 Q5. Tensor product of two unitary operators is unitary

Let U, V be unitary operators.

$$\begin{aligned}
&\langle Uu \otimes Vv| |Uu \otimes Vv\rangle = \\
&\langle u \otimes v| (U^\dagger \otimes V^\dagger)(U \otimes V) |u \otimes v\rangle = \\
&\langle u \otimes v| (U^\dagger U \otimes V^\dagger V) |u \otimes v\rangle = \\
&\langle u \otimes v| I \otimes I |u \otimes v\rangle = \\
&\langle u \otimes v| |u \otimes v\rangle =
\end{aligned}$$

Hence, $U \otimes V$ is unitary since it preserves inner products.

0.6 Q6. Tensor product of projectors is a projector

Let P, Q be projectors. $P \equiv \sum_{i=1}^{l} |i\rangle \langle i|$. $Q \equiv \sum_{j=1}^{k} |j\rangle \langle j|$.

$$P \otimes Q \equiv (\sum_{i=1}^{l} |i\rangle \langle i|) \otimes (\sum_{j=1}^{k} |j\rangle \langle j|)$$
$$\equiv \sum_{i=1}^{l} \sum_{j=1}^{k} |ij\rangle \langle ij|$$

Which is in the form of a projector, in that it leaves $|ij\rangle$ unchanged, and sends every other vector to 0. So, it projects vectors onto the subspace spanned by $|ij\rangle$.

0.7 Q7. Find log and square root of matrix

$$A = \begin{bmatrix} 4 & 3 \\ 3 & 4 \end{bmatrix}$$

Finding eigenvalues,

$$|A - \lambda I| = 0 \quad (4-\lambda)^2 - 9 = 0 \quad \lambda = 7, 1$$

Finding eigenvectors,

$$v = (1/\sqrt{2}, -1/\sqrt{2}) \quad w = (1/\sqrt{2}, 1/\sqrt{2})$$

hence, we can now write $A = U^{-1}DU$, where U transforms from the original basis to the eigenbasis, as:

$$D \equiv \begin{bmatrix} 7 & 0 \\ 0 & 1 \end{bmatrix} \qquad U \equiv \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} U^{-1} \equiv \begin{bmatrix} 1 \end{bmatrix}$$

0.7.1 Computing square root

$$S = U^{-1}\sqrt{D}U \qquad \sqrt{D} = \begin{bmatrix} \sqrt{7} & 0 \\ 0 & 1 \end{bmatrix} \qquad S = \begin{bmatrix} 1/2 + \sqrt{7}/2 & -1/2 + \sqrt{7}/2 \\ -1/2 + \sqrt{7}/2 & 1/2 + \sqrt{7}/2 \end{bmatrix}$$

We prove that S is the square root, since:

$$S^2 = (U^{-1}\sqrt{D}U)(U^{-1}\sqrt{D}U) = U^{-1}(\sqrt{D})^2 U = U^{-1}DU = A$$

0.7.2 Computing log

We can now show that if U is unitary and D is diagonal, then:

$$L \equiv \log(U^{-1}DU) = U^{-1}\log DU \qquad \log D = \begin{bmatrix} \log 7 & 0 \\ 0 & 0 \end{bmatrix}$$
$$L = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \log 7 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$
$$L = 1/2 \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \log 7 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$
$$L = 1/2 \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \log 7 & \log 7 \\ 0 & 0 \end{bmatrix}$$
$$L = 1/2 \begin{bmatrix} \log 7 & \log 7 \\ \log 7 & \log 7 \end{bmatrix}$$

0.8 Q8. Trace properties

0.8.1 $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$

$$\mathrm{Tr}(AB) = \sum_z (AB)_{zz} = \sum_z \sum_k A_{zk}B_{kz} = \sum_z \sum_k B_{kz}A_{kz} = \sum_z (BA)_{zz} = \mathrm{Tr}(BA)$$

0.8.2 $\mathrm{Tr}(A+B) = \mathrm{Tr}(A) + \mathrm{Tr}(B)$

$$\mathrm{Tr}(A+B) = \sum_z (A+B)_{zz} = \sum_z A_{zz} + B_{zz} = \mathrm{Tr}(A) + \mathrm{Tr}(B)$$

0.8.3 $\mathrm{Tr}(2A) = 2\mathrm{Tr}(A)$

$$\mathrm{Tr}(2A) = \sum_z (2A)_{zz} = \sum_z 2A_{zz} 2 \sum_z A_{zz} = 2\,\mathrm{Tr}(A)$$

0.9 Commutator properties

0.9.1 $[A, B] = -[B, A]$

$$[A, B] = AB - BA = -(BA - AB) = -[B, A]$$

0.9.2 $\frac{[A,B]+\{A,B\}}{2} = AB$

$$\frac{[A, B] + \{A, B\}}{2} = \frac{(AB - BA) + (AB + BA)}{2} = AB$$

0.10 Express polar decomposition of normal matrix as outer product

Since the matrix A is normal, it will posess an eigenbasis $|\lambda_i\rangle$ with eigenvalues λ_i. Hence,

$$A = \sum_i \lambda_i |\lambda_i\rangle \langle\lambda_i|$$

Let $A = UP$ where U is unitary and P is positive definite. From the definition, we can clearly pick P:

$$P = \sum_i |\lambda_i| |\lambda_i\rangle \langle\lambda_i|$$

such that P has positive eigenvalues.

We pick U as:

$$U = \sum_i \frac{\lambda_i}{|\lambda_i|} |\lambda_i\rangle \langle\lambda_i|$$

Clearly, U has orthogonal columns $\frac{\lambda_i}{|\lambda_i|} |\lambda_i\rangle$, which have length 1, hence the columns of U are orthonormal.

We can verify that $UP = A$ as follows:

$$
\begin{aligned}
UP &= \left(\sum_j \frac{\lambda_j}{|\lambda_j|} \left|\lambda_j\right\rangle \left\langle\lambda_j\right|\right)\left(\sum_i |\lambda_i| \left|\lambda_i\right\rangle \left\langle\lambda_i\right|\right) \\
&= \sum_j \sum_i \frac{\lambda_j}{|\lambda_j|}|\lambda_i| \left|\lambda_j\right\rangle \left\langle\lambda_j\right| \left|\lambda_i\right\rangle \left\langle\lambda_i\right| \\
&\text{Since } \left\langle\lambda_j\middle|\lambda_i\right\rangle = \delta_{ij},\ \frac{\lambda_j}{|\lambda_j|}|\lambda_i| = \lambda_i \text{ when } i = j\text{:} \\
&= \sum_i \lambda_i \left|\lambda_i\right\rangle \left\langle\lambda_i\right| \\
&= A
\end{aligned}
$$

0.11 Find left and right polar decomposition

matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

First, compute SVD, which gives $A = WD^{\frac{1}{2}}V^\dagger = WD^{\frac{1}{2}}$

So, the polar decompositions are

$$
\begin{aligned}
A &= WD^{\frac{1}{2}}V^\dagger = (WD^{\frac{1}{2}}W^\dagger)(WV^\dagger) \\
A &= WD^{\frac{1}{2}}V^\dagger = (WV^\dagger)(VD^{\frac{1}{2}}V^\dagger)
\end{aligned}
$$

where $WV^\dagger$ is unitary since W, V are unitary. $WD^{\frac{1}{2}}W^\dagger$ and $VD^{\frac{1}{2}}D^\dagger$ are positive definite since they are just similarity transforms of a positive definite matrix D.

Computing, this gives:

$$
\begin{aligned}
A = DU &= \begin{bmatrix} 0.89 & 0.45 \\ 0.45 & 1.34 \end{bmatrix} \begin{bmatrix} 0.89 & -0.45 \\ 0.45 & 0.89 \end{bmatrix} \\
A = VD' &= \begin{bmatrix} 0.89 & -0.45 \\ 0.45 & 0.89 \end{bmatrix} \begin{bmatrix} 1.34 & 0.45 \\ 0.45 & 0.89 \end{bmatrix}
\end{aligned}
$$