IIIT Hyderabad

# MATHEMATICAL METHODS

Spring 2018 – Mathematics elective – Credit 4

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## Problem Set 2

Topic: Vector Analysis in Cartesian Coordinates

Fundamental theorem of calculus: if $f(x)$ is a function of only one variable, $x$,

$$\int_a^b \frac{d}{dx}\left[f(x)\right]\, dx = f(b) - f(a). \tag{2.1}$$

Fundamental theorem with gradient: let $\mathscr{P}$ denote *any* path from a point $a$ to $b$,

$$\int_{\substack{a\\ \mathscr{P}}}^{b} \vec{\nabla}\phi \cdot d\vec{\lambda} = \int_{\substack{a\\ \mathscr{P}}}^{b} d\phi = \phi(b) - \phi(a). \tag{2.2}$$

Fundamental theorem with divergence (**Gauss's Theorem**): let $S$ be the surface enclosing a volume $V$,

$$\int_V \vec{\nabla}\cdot\vec{F}\, d\tau = \oint_S \vec{F}\cdot d\vec{\sigma}. \tag{2.3}$$

Fundamental theorem with curl (**Stokes' Theorem**): let $C$ be the boundary of a surface $S$,

$$\int_S \vec{\nabla}\times\vec{F}\cdot d\vec{\sigma} = \oint_C \vec{F}\cdot d\vec{\lambda}. \tag{2.4}$$

Dirac delta function (see e.g., en.wikipedia.org/wiki/Dirac_delta_function):

$$\delta(x-a) \overset{def.}{=} \left\{ \begin{array}{cl} +\infty & \text{if } x = a \\ 0 & \text{if } x \neq a \end{array} \right\}, \qquad \int_{-\infty}^{+\infty} \delta(x-a)dx \overset{def.}{=} 1, \tag{2.5}$$

$$\int_{-\infty}^{+\infty} \delta(x-a) f(x) dx = f(a), \tag{2.6}$$

$$\delta(x-a) = \delta(a-x), \qquad x\frac{d}{dx}\delta(x) = -\delta(x), \tag{2.7}$$

$$\delta(ax) = \frac{\delta(x)}{|a|}, \tag{2.8}$$

$$\text{more generally,} \quad \int_{-\infty}^{+\infty} \delta(g(x)) f(x) dx = \sum_i \left[\frac{f(x)}{\left|\frac{d}{dx}g(x)\right|}\right]_{x=x_i} \quad \text{where, } g(x_i) = 0. \tag{2.9}$$

**Q 19.** If a force field $\vec{F}$ can be expressed as $\vec{F}(x_i) = -\vec{\nabla}\phi(x_i)$, it is called a **conservative force**. Using Stokes' theorem [Eq. (2.4)], show that for such a force,

$$\oint_C \vec{F}\cdot d\vec{\lambda} = 0,$$

over any closed loop $C$. Confirm that this also follows from Eq. (2.2).

**Q 20.** Gravity is such a force. If we set our origin at the centre of the earth, its gravitational pull on an object of mass $m$ is,

$$\vec{F}_G = -\frac{G M_{\text{Earth}}\, m}{r^2}\hat{r},$$

where $\vec{r}$ is the position of the object, $G$ is the Newton's constant and $M_{\text{Earth}}$ is the mass of the earth. Generally we set our definitions such that the earth's gravitational potential, $\phi_G$ (rather, any potential) is zero at infinite distance away from the origin. Use Eq. (2.2) to show,

$$\phi_G = -\frac{G M_{\text{Earth}}}{r}.$$

**Q 21.** If $S$ is a surface enclosing a volume $V$ [as in Eq. (2.3)], evaluate $\oint_S \vec{\nabla}\times\vec{A}\cdot d\vec{\sigma}$.

**Q 22.** Evaluate $\oint_C \vec{r}\cdot d\vec{\lambda}$.

**Q 23.** Just as we do 'integration by parts' for ordinary functions,

$$\int_a^b f(x)\frac{d}{dx}g(x)dx \;=\; [fg]_a^b - \int_a^b g(x)\frac{d}{dx}f(x)dx, \tag{2.10}$$

we can do it for vector integrations too. Show that (meanings of $C,\ S$ & $V$ should be clear from the context),

(a) $\displaystyle\int_V f\,\vec{\nabla}\cdot\vec{A}\,d\tau \;=\; \oint_S f\vec{A}\cdot d\vec{\sigma} - \int_V \vec{A}\cdot\vec{\nabla}f\,d\tau,$

(b) $\displaystyle\int_S f\,\vec{\nabla}\times\vec{A}\,\cdot d\vec{\sigma} \;=\; \oint_C f\vec{A}\cdot d\vec{\lambda} + \int_S \vec{A}\times\vec{\nabla}f\cdot d\vec{\sigma},$

(c) $\displaystyle\int_V \vec{B}\cdot\vec{\nabla}\times\vec{A}\,d\tau \;=\; \oint_S \vec{A}\times\vec{B}\cdot d\vec{\sigma} + \int_V \vec{A}\cdot\vec{\nabla}\times\vec{B}\,d\tau.$

**Q 24.** The force on a charge $q$ that is moving with velocity $\vec{v}$ in an electric field $\vec{E}$ and a magnetic field $\vec{B}$ is given by the **Lorentz force law**, $\vec{F}=q(\vec{E}+\vec{v}\times\vec{B})$. Do you realize that the magnetic field does no work to move the charge? *

**Q 25.** In general, the electric and magnetic fields are functions of space as well as time: $\vec{E}=\vec{E}(t,x,y,z)$, $\vec{B}=\vec{B}(t,x,y,z)$. The **homogeneous Maxwell's equations** tell you

$$\vec{\nabla}\cdot\vec{B} \;=\; 0, \tag{2.11}$$

$$\vec{\nabla}\times\vec{E}+\frac{\partial\vec{B}}{\partial t} \;=\; 0. \tag{2.12}$$

They imply that we can express these two fields in terms of two potential functions. The first one implies,

$$\vec{B}=\vec{\nabla}\times\vec{A}\ \text{(a vector potential)}, \tag{2.13}$$

and the second one implies,

$$\vec{\nabla}\times\left(\vec{E}+\frac{\partial\vec{A}}{\partial t}\right)=0\Rightarrow\left(\vec{E}+\frac{\partial\vec{A}}{\partial t}\right)=-\vec{\nabla}\phi\ \text{(a scalar potential)}. \tag{2.14}$$

Using these definitions, show that the Lorentz force can be written as (notice the total derivative of $\vec{A}$),

$$\vec{F} \;=\; q\left[-\vec{\nabla}\phi-\frac{d\vec{A}}{dt}+\vec{\nabla}\left(\vec{v}\cdot\vec{A}\right)\right].$$

**Q 26.** Show that

(a) if $\vec{B}$ is a constant then $\vec{A}=\frac{1}{2}\left(\vec{B}\times\vec{r}\right)$,

(b) the flux of the magnetic field through any surface enclosed by a loop $C$ is given by $\oint_C\vec{A}.d\vec{\lambda}$, and

(c) $\vec{A}'=\vec{A}+\vec{\nabla}\psi$ (where $\psi$ is any scalar function) also works as a vector potential. (This transformation $\vec{A}\to\vec{A}'=\vec{A}+\vec{\nabla}\psi$ is known as a **gauge transformation**.)

**Q 27.** Show that

$$\delta\left[(x-a)(x-b)\right]=\frac{1}{|a-b|}\left[\delta(x-a)+\delta(x-b)\right].$$

**Q 28.** Evaluate

(a) $\int_2^6\left(3x^2-2x-1\right)\delta(x-3)dx$,

(b) $\int_{-\infty}^{\infty}\ln(x+3)\delta(x+2)dx$,

(c) $\int_{-2}^{2}(2x+3)\delta(3x)dx$,

*To move a distance $d\vec{l}$, the work done by a force $\vec{F}$ is $dW=\vec{F}\cdot d\vec{l}$

(d) $\int_0^2 (x^3+3x+2)\delta(1-x)dx$,

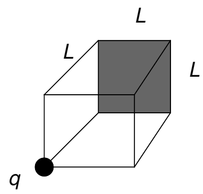
(e) $\int_{-1}^{1} 9x^2\delta(3x+1)dx$.

**Q 29.** The electric field for a point charge $q$ at the origin is

$$\vec{E}(\vec{r}) = \frac{q}{4\pi\varepsilon_0 r^2}\hat{r} \tag{2.15}$$

Compute the left hand side (LHS) of the Gauss's law [Eq. (2.3)] for this field with a sphere of radius $R$ whose centre is at the origin. You may assume,

$$\vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right) = 4\pi\,\left[\delta^{(3)}(\vec{r})\right] \stackrel{\text{def.}}{=} 4\pi\delta(x)\delta(y)\delta(z). \tag{2.16}$$




Do you see how the right hand side (RHS) also gives the same quantity (was done in the class!)? Notice that both the LHS & RHS are independent of $R$! In fact, this quantity (the total flux of the electric field through a surface enclosing the charge) is even independent of the shape of the enclosing surface – it need not be a sphere. Now, given this clue, can you figure out the flux though the shaded surface of the cube due the charge q (as shown in the figure)?