Topoi, Sheaves, Logic

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Chapter 1

Introduction

I'm following Seven sketches of compositionality. This is a transcription of the four parts of lecture 7, which promises to cover sheaves, topoi, and internal logics.

Consider a plane, that has a relationship $\texttt{dial} \mapsto \texttt{thruster}$. We want this to represent some logical statement $\forall t \in Time, @_t(\texttt{dial} = \texttt{bad}), \exists r \in \mathbb{R}, 0 < r < 1, st. @_{t+r}(\texttt{thursters} = \texttt{on})$ We need some category where this logical statement lives. How do we define this @ operator and things like that in the category?

We begin by studying the category of sets, where we normally interpret logic. Then, we move to richer logics, for example, those with truth values other than $\top, \bot$. For example, we can have things like $(\top$ for 0 ¡ t ¡ 10$)$.

Chapter 2

Logic in the category Set

Objects are all sets, arrows are functions.

2.1 Properties of Set

- 1. **Set** has limits and colimits. (eg. empty set / initial object, disjoint union / coproduct, pushouts)

- 1.1 **Set** has terminal objects: the single element set. eg: $1 = \{\star\}$. (it's unique upto unique isomorphism).

- 1.2 **Set** has products: $X, Y \in Set \implies X \times Y \in Set$. $X \times Y \equiv \{(x, y) | x \in X, y \in Y\}$

- 1.3 **Set** has pullbacks: Given functions $f : X \to A, g : Y \to A$, we can create $X \times_A Y \equiv \{(x, y) | f(x) = g(y)\}$

  $$\begin{tikzcd} X \times_A Y \arrow[r, dotted, "!"] \arrow[d, dotted, "!"] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ Y \arrow[r, "g"] & A \end{tikzcd}$$ where the $\lrcorner$ means that the square is a pullback square.

- 2. Set has epi-mono factorizations: given $f : X \to Y$, we can get $epi : X \twoheadrightarrow Im(f)$, $mono : Im(f) \hookrightarrow Y$. epi is surjective, $mono$ is injective.

  $$\begin{tikzcd} X \arrow[r, two heads] \arrow[dr, "f"] & Im(f) \arrow[d, hook] \\ & Y \end{tikzcd}$$

- 3. Internal hom: **Set**$(A \times B, C) \simeq$ **Set**(A, C^B) ($C^B \equiv$ functions $B \to C$), where $Set(X, Y)$ is the hom-set.

- 4. Subobject classifiers: a **subobject** of $X \in C$ is (a subset in Set), is an equivalence class of monomorphisms $A \hookrightarrow X$. Given two monomorphisms $f : A \hookrightarrow X, g : B \hookrightarrow X$, The subobject will specify if $(f \sim g)$.

$$\begin{tikzcd} A \arrow[d, equal] \arrow[dr, hook] & \\ B \arrow[r, hook] & X \end{tikzcd}$$

A **subobject classifier** in C is an object Ω and a map $1 \to \Omega$, where 1 is the terminal object, such that for all subobjects, $m : A \hookrightarrow X$, there exists $\lceil m \rceil : X \to \Omega$ such that:

$$\begin{tikzcd} A \arrow[r, "!"] \arrow[d, hook, "m"] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, "\mathtt{true}"] \\ X \arrow[r, "\lceil m \rceil"] & \Omega \end{tikzcd}$$

Where $\lceil m \rceil$ is called as the *classifier* of $m : A \to X$

The subobject classifier in **Set** is $\Omega = \{\mathtt{true}, \mathtt{false}\}$. $(\mathtt{true} : 1 \to \Omega; * \mapsto \mathtt{true})$

We now classify a subobject, the even numbers of $\mathbb{N}$. let $E = \{0, 2 \ldots\} \subseteq \mathbb{N}$. To classify this, we have the commuting square:

$$\begin{tikzcd} E \arrow[r, "!"] \arrow[d, hook, "m"] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, "\mathtt{true}"] \\ \mathbb{N} \arrow[r, "\lceil m \rceil"] & \Omega \end{tikzcd}$$

What is $\lceil m \rceil : \mathbb{N} \to \Omega$? It's going be $\lceil e \rceil = \begin{cases} \mathtt{true} & e \mod 2 = 0 \\ \mathtt{false} & \text{otherwise} \end{cases}$.

We need to check that this is indeed a pullback. This clearly commutes, but we need to check that it's the most general solution.

(It needs to be a pullback so we have a one-to-one correspondence between E and $\lceil e \rceil$, apparently. I don't see it.)

We call a morphism $(X \to \Omega)$ as a **predicate**. (This is clear in **Set**). Intuitively, previously, $\lceil e \rceil$ was a predicate. The subobject classifier allows us to find E given the $\lceil e \rceil$, thereby find the semantics (as it were) from the predicate.

2.2 Logical operations

2.2.1 And

$$\begin{tikzcd} 1 \arrow[r, equal] \arrow[d, hook, "(\mathtt{true},\mathtt{true})"] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, "\mathtt{true}"] \\ \Omega \times \Omega \arrow[r, "\wedge \equiv \lceil (\mathtt{true},\mathtt{true}) \rceil"] & \Omega \end{tikzcd}$$

We first draw the LHS, and we get the RHS by apply the subobject classifier onto the LHS! $(\wedge : \Omega \times \Omega \to \Omega)$.

Question: what is $\wedge(\mathtt{false}, \mathtt{false})$?