Optimization assignment — Basic descriptive 6

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Yes, linear programming is an example of convex programming, since we are optimizing a linear function (which is convex), over a polyhedra (which is convex).

Theorem 1. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a linear function. We prove it is convex*

Proof.

$$\forall \lambda \in [0, 1], \forall x, y \in \mathbb{R}^n, f(\lambda x + (1 - \lambda) y)) = \lambda f(x) + (1 - \lambda) f(y) \implies$$
$$\forall \lambda \in [0, 1], \forall x, y \in \mathbb{R}^n, f(\lambda x + (1 - \lambda) y)) \leqslant \lambda f(x) + (1 - \lambda) f(y)$$

□

Theorem 2. *Let $P = \{\vec{x} \in \mathbb{R}^n \mid A\vec{x} \leqslant \vec{b}\}$. We prove that it is convex. That is:*

$$\forall \lambda \in [0, 1], \forall x, y \in P, \lambda x + (1 - \lambda y) \in P$$

Proof. Since $x, y \in P$, we know that $Ax \leqslant b$, $Ay \leqslant b$. Hence,

$$\lambda(Ax) + (1 - \lambda)(Ay) \leqslant (\lambda b) + (1 - \lambda) b \implies \lambda(Ax) + (1 - \lambda)(Ay) \leqslant b$$
$$\text{By linearity of } A, \quad A(\lambda x) + A((1 - \lambda) y) \leqslant b$$
$$A[\lambda x + (1 - \lambda) y)] \leqslant b$$

Hence, $\lambda x + (1 - \lambda y) \in P$, since it satisfies $Ax \leqslant b$. Therefore, polyhedra are convex, and our domain is a convex domain. □

However, in the case of ILP, the domain is some subset of $\mathbb{Z}^n$, which is not convex. For example, $0, 1 \in \mathbb{Z}$ but $(0.5 \times 0 + (1 - 0.5) \times 1 = 0.5 \notin Z)$. Hence, ILP does not fall under the class of convex optimization.

Another way to see that this is not possible is that there are efficient algorithms for convex optimization, but is not the case for ILP (which is NP-hard). Hence, we should suspect that ILP does not fall under convex optimization (unless `P = NP`)