Introduction to Game Theory		Date:	*23rd March 2020*
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Lecture X: Quasi-Linear games

1 Recap

This is the a special class of environments where the GibbardSatterthwaite theorem does not hold. A popular example of quasi-linear games are actions.

2 Introduction

(We follow some of the exposition of Game Theory by Y. Narahari: The quasilinear environment). The structure of the quasi-linear setting is as follows:

$$X \equiv \left\{ (k, t_1, \ldots, t_n) : k \in K, t_i \in \mathbb{R}, \sum_i t_i \leq 0 \right\}.$$

where X is the space of alternatives, K is the set of possible allocations. $k \in K$ is the currently chosen allocation, and t_i are monetary transfer receives by agent i. By convention $t_i > 0$ implies that the agent *receives money*, and $t_i < 0$ implies that the agent *is paid money*. We assume that our agents have no external source of funding (the *weakly budget-balanced* condition). Hence, we stipulate that $\sum_i t_i \leq 0$.

A social choice function (henceforth abbreviated as SCF) in this setting is of the form $f : \Theta \to X$, where we write $f(\theta \in \Theta) \equiv (k(\theta), t_1(\theta), t_2(\theta), \ldots, t_n(\theta)) \in X$. That is, we require that $k : \Theta \to K$, $t_i : \Theta \to \mathbb{R}$ such that for alll $\theta \in \Theta, \sum_i t_i(\theta) \leq 0$.

This setting is known as quasi-linear since the agent's utility function is of the form:

$$u_i : X \times \Theta_i \to \mathbb{R}; u_i(x, \theta_i) \equiv u_i((k, t_1, t_2, \ldots, t_n), \theta_i) = v_i(k, \theta_i) + t_i$$
$$v_i : K \times \Theta_i \to \mathbb{R} \equiv \text{(Agent } i\text{'s valuation)} \quad t_i \equiv \text{amount paid to agent}$$

Here, $v_i : \Theta \to \mathbb{R}$ is the agent's valuation function, and t_i is the amount that is paid (or is to be paid) by the agent. This informs our choice of sign convention for t_i: if the agent i *is paid*, then it has earned money, t_i is positive, its utility is higher.

Definition 1. *Allocative Efficiency(AE) We say that a social choice function $f : \Theta \to X$ is allocatively efficient iff for all states of private information, the SCF causes us to choose the allocation that leads to the* maximum common good*. More formally, for all $(\theta_1, \theta_2, \ldots, \theta_n) \in \Theta$, we have that:*

$$k(\theta) \in \arg\max_{k \in K} \sum_{i=1}^{n} v_i(k, \theta_i).$$

Equivalently:

$$\sum_{i=1}^{n} v_i(k(\theta), \theta_i) = \arg\max_{k \in K} \sum_{i=1}^{n} v_i(k, \theta_i).$$

We can think about this as saying:

> "Every allocation is value-maximizing allocation. Allocations are given to those agents that covet them."

Definition 2. *Budget Balance(BB) Recall that a social choice function $f : \Theta \to X$ is said to be* budget-balanced *iff the total money is conserved for all states of private information. Formally:*

$$\forall \theta \in \Theta, \ \sum_i t_i(\theta) = 0$$

We first show that the class of quasi-linear functions is non-degenerate, in the sense that it is non-dictatorial.

Lemma 1. *All social choice function $f : \Theta \to X$ in the quasilinear setting are non-dictatorial.*

Let us assume we have a dictator who is player d (for dictator). For every $\theta \in \Theta$, we have that:

$$u_d(f(\theta), \theta_d) \geq u_d(x, \theta_d) \ \ \forall x \in X.$$

This models a dictator since this tells us that u_d gets what he wants for all scenarios. Written differently:

$$u_d(f(\theta), \theta_d) = \max_{x \in X} u_d(x, \theta_d)$$

Since our environment is quasi-linear, we have that $u_d(f(\theta), \theta_d) = v_d(k(\theta), \theta_d) + t_d(\theta)$. Hence, we can an alternative $f' : \Theta \to X$:

$$f(\theta) \left\{ (k(\theta), (t_{-d}(\theta), t_d \equiv t_d(\theta) - \textstyle\sum_i t_i(\theta))) \quad \sum_{i=1}^n t_i(\theta) < 0 \right.$$

For the following outcome, we have that $u_d(x, \theta) > u_d(f'(\theta), \theta_d)$ which contradicts the assumption that d is a dictator.

$\square$.

Definition 3. *Ex-post efficiency Recall that Ex-post efficiency is when the item is always allotted to the agents that value it the most. Formally, we state that a social choice function $f : \Theta \to X$ is said to be* Ex-post efficient *iff:*

$$\sum_{i=1}^n u_i(k(\theta), \theta_i) = \arg\max_{k \in K} \sum_{i=1}^n u_i(k, \theta_i).$$

Lemma 2. *A social choice function $f : \Theta \to X$ in the quasilinear setting is Ex-post efficient (EPE) iff it is budget-balanced.*

We can either relax DSIC or relax rich preference structure. We decided to look at quasi-linear environments where we relax preferences. A popular example of this is auctions.

$X = \{(k, t_1, \ldots, t_n) : k \in K, t_i \in \mathbb{R}, \sum_i t_i \leq 0\}$

t_i is monetary transfer receives by agent i.

$u_i(x, \theta_i) = v_i(k, \theta_i) + t_i$. Linear in t_i, hence the setting is quasi-linear. Often it is even $k_i \cdot \theta_i + t_i$ — these settings are known as linear settings.

3 Examples of SCF in quasi-linear settings

- **Players**: Seller and two buyers
- **Private information**: Seller $\Theta_0 = \{0\}$. Byers $= \theta_1 = \theta_2 = [0, 1]$.

4 Allocative efficiency

an SCF $f(\cdot)$ is allocative efficient if it maximises sum of valuations of agents. We assume such a maxima does exist. $k^\star(\theta) \in \arg\max_{k \in K} \sum_{i=1}^n v_i(k, \theta_i)$

We also want budget balance:

$\sum_{i=1}^n t_i(\theta) = 0$.

5 Properties of SCF(Social choice function) in quasi-linear settings

Lemma 3. *All SCFs in quasi-linear settings are non dictatorial.*

because $\sum_i t_i < 0$, we can increase payment for the dictator by using $t_i + \frac{e}{n-1}$ and decrease everyone else to $t_i - \frac{e}{n-1}$. So, there is always an outcome that is better for a dictator. Hence, the best outcome cannot have a dictator.

6 Ex-post efficiency

in quasi linear, scf is exp-post efficient iff if is allocatively efficient and strictly budget balanced. We have to prove that $EPE \implies AE + SBB$, and also $AE + SBB \implies EPE$.

Suppose $f = (k, t)$ is EPE but not SBB. So there exists a θ such that $\sum_i t_i(\theta) < 0$. Hence, there exists at least one agent j such that $t_j < 0$. (If everyone is positive, sum cannot be less than 0).

Now consider a new allocation $X' = (k, t')$ where

$t'_j(\theta) = \begin{cases} t_j(\theta) - \sum_i t_i(\theta)/n & \text{if } t_j(\theta) < 0 \\ t_j(\theta) & \text{otherwise} \end{cases}$

Hence, $u'_j(k, t') > u_j(k, t)$ for such j where $t_j(\theta) < 0$. For other agents, $u'_j(k, t') = u'_j(k, t)$.

This means that $(k.t')$ pareto dominates $(k.t)$. This is a contradiction to the assumption that f was EPE, since we constructed an outcome where one agent does better, and others don't do worse.

We now argue that f must be allocatively efficient, if f is EPE. For contradiction, let us assume that f is not AE. That means that there is a $k^\star$ such that $\sum_i v_i(k^\star, \theta) > v_i(k, \theta)$.

Define $t'_i(\theta) = v_i(k, \theta) - t_i(\theta) - \sum_j \theta_j(k^\star, \theta) + \epsilon$ where $\epsilon < \sum_j v_j(k^s tar, \theta) - theta_j(k, \theta)$.

Note that $v_i(k, \theta) - t_i(\theta) = u_i(k, t)$. Now note that $u_i(k^\star, t') = u_i(k, t) + \epsilon/n$, where ϵ is positive. Hence, $u_i(k^\star, t') > u_i(k, t)$.

We need to check that t' is feasible: ie, $\sum_i t'_i < 0$.

$$\sum_i t'_i = \sum_i v_i(k, theta) - \sum v_j(k^\star, \theta) + \sum_i t_i(\theta) \leq 0??$$

Also note that for all i, $u_i(k^\star, t') > u_i(k, t)$. This is contradiction to the fact that f is APE. Hence, f must be AE.

7 Other way round: if f is AE + SBB, then it is EPE

For this, we will need to prove a lemma:

Lemma 4. *If $f : \Theta \to X$ st $\forall \theta \in \Theta$,*

$$\sum_i u_i(f(\theta), \theta_i) \geq \sum_i u_i(x, \theta_i) \forall x \in X$$

then f is EPE.

The key idea is to write $u_i = v_i + t_i$, an we can get rid of t_i since f is SBB.

8 First price versus second price auction

First price: reporting valuation truthfully is not an equilibrium. Second price: truthful reporting is equilibrium.

How do we generalize this to more situations? The key idea is that in a second price auction, our payment is independent of what we report. The allocation might depend on our payment, but payment does not. How can we have more DSIC mechanisms?

9 Groves theorem

TODO: fill up groves theorem

Three families A B C, can go to Munnar or Simla.

	Manali	Shimoga	
Alice	-1	10	
Bob	5	-2	
Claire	5	4	(Claire is a kid, loves vacations)

We want to get this information truthfully, by using VCG/Groves mechanism.

there are two outcoomes, M or S . If we go to M, the tuility is 5+5-1=9. If we choose S, it is 10-2+4=12. so S is allocatively efficient.

	$\{A\}$	$\{B\}$	$\{C\}$	$\{A,B\}$	$\{A,C\}$	$\{B,C\}$	$\{A,B,C\}$
P_1	10	0	5	10	20	5	20
P_2	0	9	15	9	15	20	20
P_3	10	2	2	10	12	2	28
P_4	8	3	3	8	8	3	8

Giving A to P_1 and BC to P_2 gives $10 + 20 = 30$.

A direct revelation mechanism in which f satisfies allocative efficiency and the groves payment scheme is knows as the groves mechanism.

before this, there is another mechanism called as Clarke's mechanism

10 Clarke's mechanism

$h_i(\theta_i) = \sum_{j \neq i} v_j(k^\star_{-i}(\theta_{-i}, \theta_j)) \forall \theta_{-i} \in \Theta_{-i}$

That is, each agent i receives

$$t_i(\theta) = \sum_{j \neq i} (v_j(k^\star(\theta), \theta_j)) - \sum_{j \neq i} v_j(k^\star_{-i}(\theta_{-i}), \theta_j))$$

This works for combinatorial auctions as well. It's a generalization of second-price auction.

	M	S	
A	-1	10	
B	5	-2	
C	5	4	(C is a kid, loves vacations)

For player A, first consider:

	M	S	
A	-	-	
B	5	-2	
C	5	4	(C is a kid, loves vacations)

AE is M.

Following Clarke Mechanism:

$$\begin{aligned} t_A =& [\text{valuation of remaining agents at allocatively efficient outcome without A}](-2+4) \\ & - [\text{valuation of remaining agents at allocatively efficient outcome with A}][5+5] \\ =& \, 8 \end{aligned}$$

for player B, first consider:

A	-1	10	
B	-	-1	-
C	5	-1	4
	M	-1	S

AE is S. So, $t_B = 0$. Similarly, $t_C = 0$.