

Topics in Physics - C. Mukku

Siddharth Bhat

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Chapter 1

Tensor algebra preliminaries

1.1 Raising and lowering of two indices simultaneously

Note that

$$a_i b^i = (a^j g_{ij}) b^i = (a^j g_{ij}) (b_k g^{ki})$$

In Minkowski space, we know that $g^{ij} = 0$ if $i \neq j$, and $(g^{ii} g_{ii})^2 = 1$, so we can rewrite the above expression as:

$$\begin{aligned} (a^j g_{ij}) (b_k g^{ki}) &= \\ (a^i g_{ii}) (b_i g^{ii}) &= \\ a^i b_i \end{aligned}$$

Chapter 2

Lagrangian, Hamiltonian mechanics

Mechanics in terms of generalized coords.

2.1 Lagrangian

Define a functional. L over the config. space of partibles q^i, \dot{q}^i . $L = L(q^i, \dot{q}^i)$. We have an explicit dependence on t .

$$L = KE - PE$$

Assuming a 1-particle system of unit mass,

$$L = \frac{1}{2}\dot{q}^2 - V(q)$$

Assuming an n-particle system of unit mass,

$$L = \sum_i \frac{1}{2}\dot{q}^{i2} - V(q^i)$$

2.2 Variational principle

Take a minimum path from A to B . Now notice that the path that is slightly different from this path will have some delta from the minimum.

Action

$$S(t_0, t_1) = \int_{t_0}^{t_1} L dt = \int_{t_0}^{t_1} L(q^i, \dot{q}^i) dt$$

. Least action: $\delta S = 0$

In physics, we try to minimise the action $L = T - V$ where T is the Kinetic energy (Travail), and V (Voltage) is the Potential energy.

So, the question is, why does minimising the lagrangian work, and how do we get the euler-lagrange equations from this?

Chapter 3

Functional calculus

this chapter develops a completely handwavy physics version of functional analysis.

Definition 1 A *functional* F is a function: $F : (\mathbb{R} \rightarrow \mathbb{R}) \rightarrow \mathbb{R}$

Notation 1 Evaluation of a functional F with respect to f is denoted by $F[f]$.

3.1 Functional Derivative - take 1

Consider a functional $F : (\mathbb{R} \rightarrow \mathbb{R}) \rightarrow \mathbb{R}$, a function $f : \mathbb{R} \rightarrow \mathbb{R}$, and a "test function" $\phi : \mathbb{R} \rightarrow \mathbb{R}$.

Consider a functional F . We only define the derivative of a functional F with respect to a function f by what happens under an integral sign as follows:

$$\int \frac{\delta F}{\delta f}(x) \phi(x) dx = \lim_{\epsilon \rightarrow 0} \frac{F[f + \epsilon \phi] - F[f]}{\epsilon}$$

Now, we can define a small variation in F as:

$$\begin{aligned} \delta F &: (\mathbb{R} \rightarrow \mathbb{R}) \times (\mathbb{R} \rightarrow \mathbb{R}) \rightarrow \mathbb{R} \\ \delta F(f, \phi) &\equiv \int \frac{\delta F}{\delta f}(x) \phi(x) dx \end{aligned}$$

Intuitively, δF tells us the variation of the function f along a test function ϕ . So, it encapsulates some kind of "directional derivative".

So, we can look at $\frac{\delta F}{\delta f}$ as a functional as follows:

$$\begin{aligned} \frac{\delta F}{\delta f} &: (\mathbb{R} \rightarrow \mathbb{R}) \rightarrow \mathbb{R} \\ \frac{\delta F}{\delta f}(\phi) &= \delta F(f, \phi) \end{aligned}$$

Wehre $\frac{\delta F}{\delta f}$ allows us to "test" the change of F with respect to f along a given "direction" ϕ .

3.2 Functional Derivative as taught in class

Substitute $\phi = \delta(x - p)$. Now, the quantity:

$$\frac{\delta F}{\delta f} \phi(x) = \delta F(f, \delta(x - p))$$

Rewriting δF by sticking it under an integral:

$$\begin{aligned} \int \frac{\delta F}{\delta f}(x) \delta(x - p) dx &= \lim_{\epsilon \rightarrow 0} \frac{F[f + \epsilon \delta(x - p)] - F[f]}{\epsilon} \\ \left. \frac{\delta F}{\delta f} \right|_p &= \lim_{\epsilon \rightarrow 0} \frac{F[f + \epsilon \delta(x - p)] - F[f]}{\epsilon} \end{aligned}$$

That is, we can start talking about "derivative of the functional F with respect to a function f at a point p " as long as we only test the functional F against δ -functions.

So, we can alternatively define this quantity as:

$$\left. \frac{\delta F}{\delta f} \right|_p \equiv \lim_{\epsilon \rightarrow 0} \frac{F[f + \epsilon \delta(x - p)] - F[f]}{\epsilon}$$

While this does not "look like a functional", it actually is, if we mentally replace:

$$p \rightarrow \int - \delta(x - p) dx$$

This is how mukku got that expression.

3.3 Common functional derivatives

3.3.1 $F[f] = \int_0^\infty f dx$

$$\begin{aligned} \frac{\delta F[f]}{\delta f(x_0)} &= \lim_{\epsilon \rightarrow 0} \frac{\int_0^\infty (f + \epsilon \delta(x - x_0)) dx - \int_0^\infty f dx}{\epsilon} \\ &= \int_0^\infty \delta(x - x_0) dx = 1 \end{aligned}$$

3.3.2 $F[f] = \int_0^\infty g[f] dx$

This does not actually type-check for me. $g : (\mathbb{R} \rightarrow \mathbb{R}) \rightarrow \text{reals}$, so I don't fully understand what we are "varying" where when we integrate with respect to dx .

So, there's something bizarre here that I don't understand — the integral doesn't really make sense.

3.3.3 Derivative of $F[\phi] \equiv \int (\frac{\partial \phi}{\partial y})^2$

$$\left. \frac{\delta F}{\delta f} \right|_p = \int (\frac{\partial \phi}{\partial y})^2$$

3.4 Deriving E-L from functional magic

3.5 Weird things in Functional Analysis as taught in class

Consider the functional

$$J[f] = \int g[f'] dy:$$

since g is a functional, it has a type $g : (\mathbb{R} \rightarrow \mathbb{R}) \rightarrow \mathbb{R}$. So, our integrand must be some function df , and not some **space component** dy . **I don't understand what the definition of J means.**

Chapter 4

Maxwell's equations in Minkowski space

Let us first review Maxwell's equations:

$$\nabla \cdot E = \frac{\rho}{\epsilon_0} \text{ (Electric charges produce fields)}$$

$$\nabla \cdot B = 0 \text{ (Only magnetic dipoles exist)}$$

$$\nabla \times E = -\frac{\partial B}{\partial t} \text{ (Lenz Law / Faraday's law - time varying magnetic field induces current that opposes it)}$$

$$\nabla \times B = \mu_0 \left(J + \epsilon_0 \frac{\partial E}{\partial t} \right) \text{ (Ampere's law + fudge factor)}$$

4.1 Constructing F , or Tensorifying Maxwell's equations

Begin with the equation that $\nabla \cdot B = 0$. This tells that B can be written as the curl of some other field:

$$\boxed{B \equiv \nabla \times A} \tag{4.1}$$

Expanding this equation of B in tensorial form:

$$\boxed{B^i = \mathcal{E}^{ijk} \partial_j A^k} \tag{4.2}$$

Next, take $\nabla \times E = -\frac{\partial B}{\partial t}$.

$$\nabla \times E = -\frac{\partial B}{\partial t} = \frac{\partial(\nabla \times A)}{\partial t} = \nabla \times \frac{\partial A}{\partial t}$$

$$\nabla \times \left(E + \frac{\partial A}{\partial t} \right) = 0$$

writing this as the gradient of some field ϕ scaled by $\alpha : \mathbb{R}$

$$E + \frac{\partial A}{\partial t} = \alpha(\nabla\phi)$$

$$E = \alpha\nabla\phi - \frac{\partial A}{\partial t}$$

Since electrostatics is time-independent, we choose to think of $\alpha = -1$, so we can interpret ϕ as the potential.

$$E^i = -\frac{\partial\phi}{\partial x^k}g^{ik} - \frac{\partial A^i}{\partial t} \quad (4.3)$$

A slight reformulation (since we know that in Minkowski space, $\partial_t = \partial_0$) we get the equation:

$$\boxed{E^i = -g^{ik}\partial_k\phi - \partial_0 A^i} \quad (4.4)$$

We get the metric g^{ik} involved to raise the covariant $\frac{\partial\phi}{\partial x^k}$ into the contravariant E^i .

(Sid question: how does one justify switching $\nabla \times$ and ∂ ? It feels like some algebra)

Here be magic! We define A new rank-2 tensor in Minkowski space-time, called F (for Faraday),

$$\boxed{F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu} \quad (4.5)$$

(Sid question: why is this object $F_{\mu\nu}$ covariant? What does this *mean*?)

Lemma 1 $F_{\mu\nu}$ is antisymmetric.

Lemma 2 $F_{\mu\nu}$ has 6 degrees of freedom

Proof. Number of degrees of freedom of F :

$$\frac{4^2 \text{ (total)} - 4 \text{ (diagonal)}}{2 \text{ (anti-symmetry)}} = 6$$

Notice that F is a 1-form!

4.2 Expressing B , E in terms of F

We now wish to re-express B^{ij} and E^{ij} in terms of F , so that this F captures all of maxwell's equations.

$$B^i = \mathcal{E}^{ijk} \partial_j A^k = \mathcal{E}^{ikj} \partial_k A^j \quad \text{by } k, j \text{ being free variables}$$

$$B^i = \frac{1}{2} \left(\mathcal{E}^{ijk} \partial_j A^k + \mathcal{E}^{ikj} \partial_k A^j \right)$$

Substituting $\partial_j A_k - \partial_k A_j = F_{jk}$,

$$B^i = \frac{1}{2} \mathcal{E}^{ijk} F_{jk}$$

So, B in terms of F is:

$$\boxed{B^i = \frac{1}{2} \mathcal{E}^{ijk} F_{jk}} \quad (4.6)$$

Similarly, we wish to write E in terms of F . The algebra is as follows:

$$E^i = -g^{ik} \partial_k \phi - \partial_0 A^i$$

$$E^i = -g^{ik} \partial_k \phi - \partial_0 g^{ik} A_k$$

Is this allowed? Am I always allowed to insert the g_{ik} ?

$$E^i = -g^{ik} (\partial_k \phi + \partial_0 A_k)$$

Since $k = \{1, 2, 3\}$ (k is spacelike coordinates), and we would like to relate ϕ with A (to unify E), we **set**:

$$\boxed{A_0 \equiv -\phi} \quad (4.7)$$

Continuing the derivation,

$$E^i = -g^{ik} (\partial_k (-A_0) + \partial_0 A_k)$$

$$E^i = -g^{ik} (\partial_0 A_k - \partial_k A_0)$$

$$E^i = -g^{ik} F_{0k}$$

So, finally, the relation is:

$$\boxed{E^i = -g^{ik} F_{0k}} \quad (4.8)$$

Let us reconsider what we believed E to be. We had:

$$E = -\nabla \phi - \frac{\partial A}{\partial t}$$

However, comparing dimensions, space derivative of ϕ = time derivative of A . This means that $\frac{\delta \phi}{\delta x} = \frac{\delta A}{\delta y}$, and so $\frac{\delta \phi}{\frac{\delta x}{\delta t}} = \delta A$. We arbitrarily pick c as our measuring stick for $\frac{\delta x}{\delta t}$. Also, in minkowski space, our measuring stick is actually (ct, x, y, z) , so $\partial_0 = \partial_{ct}$. So, when we write the equation for E , we should actually write

$$E = c \left(-\frac{\nabla \phi}{c} - \frac{\partial A}{\partial ct} \right)$$

which becomes:

$$\boxed{E^i = c F^{i0}} \quad (4.9)$$

4.3 Rewriting Maxwell's equations in terms of F

Now that we have constructed the Faraday tensor F , we wish to re-express Maxwell's equations in terms of this object. This will give us a compact form of the laws which are invariant under coordinate transforms.

4.3.1 Combining (1) $\nabla E = \frac{\rho}{\epsilon_0}$, (4) $\nabla \times B = \mu_0 J + \frac{\partial E}{\partial t}$

1. Using (4) $\nabla \times B = \mu_0 J + \frac{\partial E}{\partial t}$

We consider the 4th Maxwell equation:

$$\begin{aligned}\nabla \times B &= \mu_0 J + \epsilon_0 \mu_0 \frac{\partial E}{\partial t} \\ \nabla \times B &= \mu_0 J + \frac{1}{c^2} \frac{\partial E}{\partial t} \\ \text{Converting to indices,} \\ (\nabla \times B)^i &= \mu_0 J^i + \frac{1}{c} \frac{\partial E^i}{\partial ct} && \text{(From } \partial_{ct} = \frac{1}{c} \partial_t) \\ &= \mu_0 J^i + \frac{1}{c} \frac{\partial E^i}{\partial X^0} \\ &= \mu_0 J^i + \frac{\partial F^{i0}}{\partial X^0} && \text{(From } E^i = cF^{i0}) \\ &= \mu_0 J^i + \partial_0 F^{i0}\end{aligned}$$

Now, we start to simplify the LHS, $\nabla \times B$:

$$\begin{aligned}(\nabla \times B)^i &= \mathcal{E}^{ijk} \partial_j B_k \\ \text{Since } B^k &= \frac{1}{2} \mathcal{E}^{kmn} F_{mn}, \\ B_k &= \frac{1}{2} \mathcal{E}_{kmn} F^{mn}, && \text{(TODO: this is scam)} \\ (\nabla \times B)^i &= \mathcal{E}^{ijk} \partial_j \left(\frac{1}{2} \mathcal{E}_{kmn} F^{mn} \right) = \frac{1}{2} \mathcal{E}^{ijk} \mathcal{E}_{kmn} \partial_j F^{mn}\end{aligned}$$

Aside: We need to know how to evaluate $\mathcal{E}^{ijk} \mathcal{E}_{kmn}$:

$$\mathcal{E}_{i_1, i_2, \dots, i_n} \mathcal{E}_{j_1, j_2, \dots, j_n} = \det \left\{ \begin{vmatrix} \delta_{i_1 j_1} & \delta_{i_1 j_2} & \dots & \delta_{i_1 j_n} \\ \delta_{i_2 j_1} & \delta_{i_2 j_2} & \dots & \delta_{i_2 j_n} \\ \vdots & \vdots & \ddots & \vdots \\ \delta_{i_n j_1} & \delta_{i_n j_2} & \dots & \delta_{i_n j_n} \end{vmatrix} \right\}$$

$$\mathcal{E}^{ijk} \mathcal{E}^{imn} = -1(\delta_j^m \delta_k^n - \delta_j^n \delta_k^m)$$

He argued that we get a -1 factor here due to the presence of the metric. I'm not fully convinced, but I can handwave this using the magic words "tensor density".

Plugging both equations together,

$$\begin{aligned}
\frac{1}{2} \mathcal{E}^{ijk} \mathcal{E}_{kmn} \partial_j F^{mn} &= \mu_0 J^i + \partial_0 F^{i0} \\
(\text{Since } kij \text{ is an even permutation of } ijk): \\
\frac{1}{2} \mathcal{E}^{kij} \mathcal{E}_{kmn} \partial_j F^{mn} &= \mu_0 J^i + \partial_0 F^{i0} \\
(\text{Using } \mathcal{E}^{kij} \mathcal{E}^{kmn} &= -1(\delta_i^m \delta_j^n - \delta_i^n \delta_j^m)): \\
\frac{1}{2} [- (\delta_m^i \delta_n^j - \delta_n^i \delta_m^j)] \partial_j F^{mn} &= \mu_0 J^i + \partial_0 F^{i0} \\
-\frac{1}{2} [\partial_n F^{in} - \partial_m F^{mi}] &= \mu_0 J^i + \partial_0 F^{i0} \\
(F \text{ is anti-symmetric, so rewriting } \partial_m F^{mi} &= -\partial_m F^{im}): \\
-\frac{1}{2} [\partial_n F^{in} + \partial_m F^{im}] &= \mu_0 J^i + \partial_0 F^{i0} \\
(\text{Replacing } \partial_m F^{im} \equiv \partial_n F^{in} \text{ since } m \text{ is free}): \\
- [\partial_m F^{im}] &= \mu_0 J^i + \partial_0 F^{i0} \\
\mu_0 J^i + \partial_0 F^{i0} + \partial_m F^{im} &= 0 \\
\mu_0 J^i + \partial_\mu F^{i\mu} &= 0 \quad (\mu = \{0, 1, 2, 3\})
\end{aligned}$$

This gives us a continuity-style equation, linking the current density J to the rate of change of F .

$$\boxed{\mu_0 J^i + \partial_\mu F^{i\mu} = 0} \quad (\mu = \{0, 1, 2, 3\})$$

Second part, using 1st equation

$$\begin{aligned}
\nabla E &= \frac{\rho}{\epsilon_0} \\
\partial_i E^i &= \frac{\rho}{\epsilon_0} \\
(\text{Substituting } E^i &= cF^{i0}, c^2 = \frac{1}{\mu_0 \epsilon_0}): \\
c \partial_i F^{i0} &= \frac{\rho}{\epsilon_0} = \frac{\rho \mu_0}{\mu_0 \epsilon_0} = \rho \mu_0 c^2 \\
\partial_i F^{i0} &= \mu_0 c \rho \\
(\text{Since } F \text{ is anti-symmetric, } F^{00} &= 0, \text{ Hence}): \\
\partial_0 F^{00} + \partial_i F^{i0} &= \mu_0 c \rho \\
\partial_\mu F^{\mu 0} &= \mu_0 c \rho
\end{aligned}$$

$$\boxed{\partial_\mu F^{\mu 0} = \mu_0 c \rho} \quad (4.10)$$

Combining part 1 and part 2:

$$\begin{aligned}\mu_0 J^i + \partial_\mu F^{i\mu} &= 0 & (\text{From } B) \\ \partial_\mu F^{i\mu} &= -\mu_0 J^i \partial_\mu F^{\mu 0} = \mu_0 c \rho \\ \partial_\mu F^{0\mu} &= -\mu_0 c \rho\end{aligned}$$

To combine these equations, **we set:**

$$\boxed{J^0 \equiv c\rho} \quad (4.11)$$

We arrive at the unified equation:

$$\partial_\mu F^{\nu\mu} = -\mu_0 J^\nu$$

Choose units such that $c = \frac{h}{2\pi} = G_n = 1$, which gives us:

$$\begin{aligned}\partial_\mu F^{\nu\mu} &= -J^\nu \\ F &\text{ is antisymmetric, so flipping indices} \\ \partial_\mu F^{\mu\nu} &= J^\nu\end{aligned}$$

$$\boxed{\partial_\mu F^{\mu\nu} = J^\nu} \quad (4.12)$$

Note that this is **Ampere's law!**

4.3.2 Combining (2) $\nabla \times E = -\frac{\partial B}{\partial t}$, (3) $\nabla B = 0$

$$\begin{aligned}\nabla \times E &= -\frac{\partial B}{\partial t} \\ (\nabla \times E)^i &= \mathcal{E}^{ijk} \partial_j E_k = -\partial_0 B \\ \mathcal{E}^{ijk} \partial_j E_k &= -\partial_0 \left(\frac{1}{2} \mathcal{E}^{ijk} F_{jk} \right) \\ \mathcal{E}^{ijk} \partial_j E_k + \partial_0 \left(\frac{1}{2} \mathcal{E}^{ijk} F_{jk} \right) &= 0 \\ 2\mathcal{E}^{ijk} \partial_j E_k + \partial_0 (\mathcal{E}^{ijk} F_{jk}) &= 0\end{aligned}$$

Now we begin from the other direction, and start the derivation.

We know that the equation we want is:

$$\boxed{\mathcal{E}^{\alpha\beta\mu\nu} \partial_\beta F_{\mu\nu} = 0} \quad (4.13)$$

$\alpha = 0$ **case:**

First, set $\alpha = 0$. So now, the other β, μ, ν are forced to be become space components — (i, j, k) .

Therefore, the equation now becomes:

$$\mathcal{E}^{0ijk} \partial_i F_{jk} = 0$$

However, note that $\mathcal{E}^{0ijk} = \mathcal{E}^{ijk}$, because if (ijk) is an even permutation, so will $(0ijk)$, and vice versa for odd (since $0 < i, j, k$).

Using this, the equation becomes

$$\begin{aligned} \mathcal{E}^{ijk} \partial_i F_{jk} &= 0 \\ \partial_i (\mathcal{E}^{ijk} F_{jk}) &= 0 \\ \text{Since } B^i &= \frac{1}{2} \mathcal{E}^{ijk} F_{jk}: \\ \partial_i \left(\frac{B^i}{2} \right) &= 0 \\ \partial_i B^i &= 0 \\ \nabla B &= 0 \end{aligned}$$

Hence, the above equation does encode $\nabla B = 0$.

$\alpha = m$ **case:**

Let α be a spatial dimension $m = \{1, 2, 3\}$.

$$\begin{aligned} \mathcal{E}^{\alpha\beta\mu\nu} \partial_\beta F_{\mu\nu} &= 0 \\ \mathcal{E}^{m\beta\mu\nu} \partial_\beta F_{\mu\nu} &= 0 \end{aligned}$$

Once again, we get two cases, one where $\beta = 0$, and one where $\beta = n$ where n is a spatial dimension. If $\beta = 0$, then the other dimensions are forced to be spatial dimensions, which we shall denote as $\mu \equiv x, \nu \equiv y$

$$\begin{aligned} \mathcal{E}^{m\beta\mu\nu} \partial_\beta F_{\mu\nu} &= 0 \\ \mathcal{E}^{m0xy} \partial_0 F_{xy} + \mathcal{E}^{mn\mu\nu} \partial_n F_{\mu\nu} &= 0 \end{aligned}$$

Now note that $\mathcal{E}^{m0\mu\nu} = -\mathcal{E}^{0m\mu\nu} = -\mathcal{E}^{m\mu\nu}$.

Using this, we can rewrite the above equation as:

$$\begin{aligned} \mathcal{E}^{m0xy} \partial_0 F_{xy} + \mathcal{E}^{mn\mu\nu} \partial_n F_{\mu\nu} &= 0 \\ -\mathcal{E}^{mxy} \partial_0 F_{xy} + \mathcal{E}^{mn\mu\nu} \partial_n F_{\mu\nu} &= 0 \end{aligned}$$

We now consider cases for μ in the second term, where either $\mu = 0$ or $\mu = o \in \{1, 2, 3\}$

If $\mu = 0$, then the other dimension ν must be a spatial dimension p . If $\mu = q$, then the other dimension ν must be a time dimension 0 (This is because we are not allowed to have 4 spatial dimensions, since the \mathcal{E} evaluates to 0 on repeated dimensions).

$$\begin{aligned}
 -\mathcal{E}^{mxy}\partial_0 F_{xy} + \mathcal{E}^{mn\mu\nu}\partial_n F_{\mu\nu} &= 0 \\
 -\mathcal{E}^{mxy}\partial_0 F_{xy} + & \\
 \mathcal{E}^{mn0p}\partial_n F_{0p} & \quad (\mu = 0, \nu = p) \\
 \mathcal{E}^{mnq0}\partial_n F_{q0} & \quad (\mu = q, \nu = 0) \\
 &= 0
 \end{aligned}$$

Rearranging, and using the fact that $F_{0p} = -F_{p0}$, $\mathcal{E}^{mn0p} = \mathcal{E}^{0mnp} = \mathcal{E}^{mnp}$, $\mathcal{E}^{mnq0} = -\mathcal{E}^{0mnq} = -\mathcal{E}^{mnq}$,

$$-\mathcal{E}^{mxy}\partial_0 F_{xy} + \mathcal{E}^{mnp}(-\partial_n F_{p0}) + (-\mathcal{E}^{mnq})\partial_n F_{q0} = 0$$

Multiplying throughout by -1 , and noticing that since p, q are dummy indices, we can set $p = q$. This allows us to get:

$$\mathcal{E}^{mxy}\partial_0 F_{xy} + 2\mathcal{E}^{mnp}\partial_n F_{p0} = 0$$

First, remember that $E_p = F_{p0}$. So, we can replace the term F_{p0} (upto fudging of constant factors that we have always done), with E_p .

Now, compare

$$\mathcal{E}^{mxy}\partial_0 F_{xy} + 2\mathcal{E}^{mnp}\partial_n E_p = 0 \quad (\text{Our equation})$$

$$2\mathcal{E}^{ijk}\partial_j E_k + \partial_0(\mathcal{E}^{ijk}F_{jk}) = 0 \quad (\text{Previous equation})$$

Note that the two equations are identical upto variable naming, and are hence considered equal. So, we have encoded both of Maxwell's laws into this particular equation:

$$\boxed{\mathcal{E}^{\alpha\beta\mu\nu}\partial_\beta F_{\mu\nu} = 0} \quad (4.14)$$

Chapter 5

Gauge theories

We construct a 1-dimensional guage theory and study its symmetries.

5.1 Euler-Lagrange equations for a field

Consider a Lagrangian:

$$\mathcal{L}(\phi, \phi', \dot{\phi}) = \dot{\phi}^2 - \phi'^2$$

When written in terms of M^4 (minkowski space), we know that

$$\partial_\mu \equiv (\partial_t, -\nabla)$$

Hence, in minkowski space, the lagrangian becomes a function of *only*:

$$\mathcal{L}(\phi, \partial_\mu \phi) \equiv \dots$$

So we managed to unify the space derivative and the time derivative (yay).

Now, we can consider the action of this Lagrangian:

$$S[\phi] = \int L(\phi, \partial_\mu \phi) \, \mathrm{d}^4x$$

Minimising the functional S ,

$$\delta S[\phi] = 0$$

$$\delta \int L(\phi, \partial_\mu \phi) d^4x = 0$$

For an analogy, consider $dL = \frac{\partial L}{\partial \phi} d\phi + \frac{\partial L}{\partial \psi} d\psi + \dots$

$$\int \left[\frac{\partial L}{\partial \phi} \delta\phi + \frac{\partial L}{\partial \partial_\mu \phi} \delta(\partial_\mu \phi) \right] d^4x = 0$$

Using linearity of integration, and commuting of δ, ∂_μ

$$\int \frac{\partial L}{\partial \phi} (\delta\phi) d^4x + \int \frac{\partial L}{\partial \partial_\mu \phi} (\partial_\mu \delta\phi) d^4x = 0$$

Using $\int U dV = UV - \int V dU$, $V = \delta\phi$, $U = \frac{\partial L}{\partial \partial_\mu \phi}$

$$\int \frac{\partial L}{\partial \phi} (\delta\phi) d^4x + \frac{\partial L}{\partial \partial_\mu \phi} (\delta\phi) \Big|_{\text{endpoints}} - \int \delta\phi \left(\partial_\mu \frac{\partial L}{\partial \partial_\mu \phi} \right) d^4x = 0$$

Since fields decay at endpoints, forget the integral

$$\int \frac{\partial L}{\partial \phi} (\delta\phi) d^4x - \int \delta\phi \left(\partial_\mu \frac{\partial L}{\partial \partial_\mu \phi} \right) d^4x = 0$$

Refactoring to pull the common $\delta\phi$,

$$\int \left(\frac{\partial L}{\partial \phi} - \partial_\mu \frac{\partial L}{\partial \partial_\mu \phi} \right) (\delta\phi) d^4x = 0$$

Since this is true for all perturbations $\delta\phi$ implies that:

$$\frac{\partial L}{\partial \phi} - \partial_\mu \frac{\partial L}{\partial \partial_\mu \phi} = 0$$

Hence, we have the Euler-Lagrange equation for **scalar fields**:

$$\boxed{\frac{\partial L}{\partial \phi} - \partial_\mu \frac{\partial L}{\partial (\partial_\mu \phi)} = 0} \quad (5.1)$$

So next, let's consider a Lagrangian (which is supposedly the free particle analogue): (**TODO: find out why this is free particle KE**)

$$(\partial_\mu \phi)^2 \equiv \partial_\mu \phi \partial^\mu \phi \text{ (This is notation)}$$

Note that:

$$\partial_\mu \phi \partial^\mu \phi = \partial_\mu \phi \eta^{\mu\nu} \partial_\nu \phi \text{ (where } \eta^{\mu\nu} \text{ is the metric; lowering indices)}$$

We define the Lagrangian as:

$$\mathcal{L}(\phi, \partial_\mu \phi) = \frac{1}{2} (\partial_\mu \phi)^2$$

Solving EL equations,

$$\frac{\partial L}{\partial \phi} = 0$$

$$\begin{aligned} & \frac{\partial L}{\partial(\partial_\sigma \phi)} \\ &= \frac{\partial}{\partial(\partial_\sigma \phi)} \left(\frac{1}{2} (\partial_\mu \phi)^2 \right) \\ &= \frac{\partial}{\partial(\partial_\sigma \phi)} \left(\frac{1}{2} \eta^{\mu\nu} (\partial_\mu \phi) (\partial_\nu \phi) \right) \\ &= \frac{1}{2} \eta^{\mu\nu} \frac{\partial(\partial_\mu \phi)}{\partial(\partial_\sigma \phi)} (\partial_\nu \phi) + \frac{1}{2} \eta^{\mu\nu} (\partial_\mu \phi) \frac{\partial(\partial_\nu \phi)}{\partial(\partial_\sigma \phi)} \\ & \text{(TODO: understand why } \frac{\partial(\partial_\mu \phi)}{\partial(\partial_\sigma \phi)} = \delta_\mu^\sigma) \\ &= \frac{1}{2} \eta^{\mu\nu} \delta_\mu^\sigma (\partial_\nu \phi) + \frac{1}{2} \eta^{\mu\nu} (\partial_\mu \phi) \delta_\nu^\sigma \\ & \text{(Contracting on } \mu) \\ &= \frac{1}{2} \eta^{\sigma\nu} (\partial_\nu \phi) + \frac{1}{2} \eta^{\mu\sigma} (\partial_\mu \phi) \\ & \text{(Replacing dummy index } \mu \equiv \nu) \\ &= \eta^{\nu\sigma} \partial_{\nu\sigma} \phi \\ & \text{(TODO: understand how this happens, something about } \eta \text{'s signature)} \\ &= \partial^\sigma \phi \end{aligned}$$

5.2 Klein-gordon equations

5.3 Lagrangian for a massive scalar field

5.4 Symmetries of a scalar field Lagrangian

5.5 Dervng the force of the EM-field from the Lanrangian

Recall that $B = \nabla \times A$, $E = \nabla \phi - \frac{\partial A}{\partial t}$, and the force on a particle is $\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$.

$$\begin{aligned} ma &= e \left(\nabla \phi - \frac{\partial A}{\partial t} \right) + e(v \times (\nabla \times A)) \\ ma &= e \left(\nabla \phi - \frac{\partial A}{\partial t} \right) + e(\nabla \cdot (v \cdot A) - (v \cdot \nabla)A) \end{aligned}$$

Note that $(v \cdot \nabla)A$ is:

$$\begin{aligned} v \cdot \nabla &= \frac{dx}{dt} \frac{\partial}{\partial x} + \frac{dy}{dt} \frac{\partial}{\partial y} + \frac{dz}{dt} \frac{\partial}{\partial z} \\ (v \cdot \nabla)A &= \frac{dx}{dt} \frac{\partial A}{\partial x} + \frac{dy}{dt} \frac{\partial A}{\partial y} + \frac{dz}{dt} \frac{\partial A}{\partial z} \end{aligned}$$

However, now let us compare $\frac{dA}{dt}$ and $(v \cdot \nabla)A$:

$$\begin{aligned} \frac{dA}{dt} &= \frac{\partial A}{\partial t} + \frac{dx}{dt} \frac{\partial A}{\partial x} + \frac{dy}{dt} \frac{\partial A}{\partial y} + \frac{dz}{dt} \frac{\partial A}{\partial z} \\ \frac{dA}{dt} &= \frac{\partial A}{\partial t} + (v \cdot \nabla)A \end{aligned}$$

Now, rewriting ma ,

$$\begin{aligned} ma &= e\left(\nabla\phi - \frac{\partial A}{\partial t}\right) + e(\nabla \cdot (v \cdot A) - (v \cdot \nabla)A) \\ ma &= e\left(\nabla\phi - \frac{\partial A}{\partial t}\right) + e(\nabla \cdot (v \cdot A) - (v \cdot \nabla)A) \end{aligned}$$

5.6 Local and global symmetries

Chapter 6

Charged particle interaction in fields, or, how maxwell's equations have $U(1)$ symmetry

(written in red ink pen)

Consider a scalar field ϕ , and the lagrangian:

$$\mathcal{L}(\phi, \partial_\mu \phi) = \frac{1}{2}(\partial_\mu \phi)(\partial_\mu \phi) - \frac{1}{2}m^2\phi^2$$

Next, we want to consider charged particle interactions, which comes from $H = \frac{(p-eA)^2}{2m}$, which creates the lorentz force $e\vec{v} \times \vec{B} = e\vec{v} \cdot \vec{A}$.

We know that \mathcal{L} is invariant under global rotation $e^{i\theta}$. Now we study local gauge invariance by making θ a function of space. That is, $\theta \rightarrow \theta(x)$.

We use the complex form of the lagrangian:

$$\mathcal{L} = (\partial_\mu \phi)(\partial^\mu \phi)^* - m^2|\phi|^2$$

We now consider the transform:

$$\phi \rightarrow e^{ie\theta(x)}\phi$$

This implies the transforms:

$$\begin{aligned}\partial_\mu \phi &= \partial_\mu (e^{ie\theta(x)}\phi) = e^{ie\theta(x)}\partial_\mu \phi + ie e^{ie\theta(x)}\phi \partial_\mu \theta = \\ &e^{ie\theta(x)}(\partial_\mu \phi + ie\phi \partial_\mu \theta)\end{aligned}$$

Hence, for invariance, we get:

$$\begin{aligned}
& (\partial_\mu \phi)(\partial^\mu \phi)^* \\
&= (e^{ie\theta(x)}(\partial_\mu \phi + ie\phi \partial_\mu \theta))(e^{ie\theta(x)}(\partial^\mu \phi + ie\phi \partial^\mu \theta))^* \\
&= (\partial_\mu \phi + ie\phi \partial_\mu \theta)(\partial^\mu \phi^* - ie\phi^* \partial^\mu \theta)
\end{aligned}$$

We introduce:

$$D_\mu \phi \equiv (\partial_\mu - iA_\mu)\phi(\partial^\mu + iA^\mu)\phi^*$$

We shall check that this transforms in a lorentzian way. In the new coordinate system:

$$\begin{aligned}
\bar{D}_\mu \bar{\phi} &\equiv ([\partial_\mu - i\bar{A}_\mu]\bar{\phi})[(\partial^\mu + i\bar{A}^\mu)\bar{\phi}^*] \\
&= [(\partial_\mu - i\bar{A}_\mu)(e^{ie\theta(x)}\phi)][(\partial^\mu + i\bar{A}^\mu)(e^{ie\theta(x)}\phi)^*] \\
&= [\partial_\mu(e^{ie\theta(x)}\phi) - i\bar{A}_\mu e^{ie\theta(x)}\phi][\partial_\mu(e^{-ie\theta(x)}\phi^*) + i\bar{A}_\mu e^{-ie\theta(x)}\phi^*] \\
&\quad \text{evaluate } \partial_\mu(UV) \\
&= [e^{ie\theta(x)}(\partial_\mu \phi) + ie e^{ie\theta(x)}(\partial_\mu \theta)\phi - i\bar{A}_\mu e^{ie\theta(x)}\phi][e^{-ie\theta(x)}(\partial_\mu \phi^*) - ie e^{-ie\theta(x)}(\partial_\mu \theta)\phi^* + i\bar{A}_\mu e^{-ie\theta(x)}\phi^*] \\
&= [e^{ie\theta(x)}(\partial_\mu \phi + ie(\partial_\mu \theta)\phi - i\bar{A}_\mu \phi)][e^{-ie\theta(x)}(\partial_\mu \phi^* - ie(\partial_\mu \theta)\phi^* + i\bar{A}_\mu \phi^*)] \\
&\quad \text{cancelling } e^{ie\theta(x)} \text{ with } e^{-ie\theta(x)} \\
&= [\partial_\mu \phi + ie(\partial_\mu \theta)\phi - i\bar{A}_\mu \phi][\partial_\mu \phi^* - ie(\partial_\mu \theta)\phi^* + i\bar{A}_\mu \phi^*] \\
&= [\partial_\mu + ie(\partial_\mu \theta) - i\bar{A}_\mu]\phi[\partial_\mu - ie(\partial_\mu \theta) + i\bar{A}_\mu]\phi^*
\end{aligned}$$

Comparing:

$$\begin{aligned}
D_\mu \phi &\equiv (\partial_\mu - iA_\mu)\phi(\partial^\mu + iA^\mu)\phi^* \\
\bar{D}_\mu \bar{\phi} &\equiv [\partial_\mu + ie(\partial_\mu \theta) - i\bar{A}_\mu]\phi[\partial_\mu - ie(\partial_\mu \theta) + i\bar{A}_\mu]\phi^* \\
-iA_\mu &= +ie(\partial_\mu \theta) - i\bar{A}_\mu \\
-A_\mu &= e(\partial_\mu \theta) - \bar{A}_\mu \\
\bar{A}_\mu &= e(\partial_\mu \theta) + A_\mu
\end{aligned}$$

TODO: This is not what mukku got! mukku got $\bar{A}_\mu = A_\mu - e(\partial_\mu \theta)$. HOW?

So now, we know how A_μ transforms:

$$\boxed{\bar{A}_\mu = A_\mu - e(\partial_\mu \theta)} \quad (6.1)$$

So, now, our modified lagrangian is:

$$\mathcal{L}(\phi, \phi^*, \partial_\mu \phi, \partial_\mu \phi^*, A_\mu) = [(\partial_\mu - iA_\mu)\phi][(\partial^\mu - iA^\mu)\phi]^* - \frac{1}{2}m|\phi|^2$$

We want to understand if we can add interaction terms for A_μ . If not, what the obstacles are. We would like the interaction terms to be a Lorentz tensor, and we would like it to respect local symmetries.

$$\mu^2 A_\mu A^\mu \rightarrow \mu^2 \bar{A}_\mu \bar{A}^\mu = \mu^2 (A_\mu + e \partial_\mu \theta) (A^\mu + e \partial^\mu \theta)$$

So clearly, we cannot have mass terms of the form $\mu^2 A_\mu A^\mu$. We must try other things and check if they are interesting.

We now try to inspect some invariants of A_μ . For example, we can construct:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

We already know that F is a Lorentz tensor. Let's check if it's gauge invariant.

$$\begin{aligned} \bar{F}_{\mu\nu} &= \partial_\nu \bar{A}_\mu - \partial_\mu A_\nu \\ &= \partial_\nu (A_\mu - e(\partial_\mu \theta)) - \partial_\mu (A_\nu - e \partial_\nu \theta) \\ &\quad \text{Since partial derivatives commute:} \\ &= \partial_\nu A_\mu - \cancel{e \partial_\nu \partial_\mu \theta} - \partial_\mu A_\nu - \cancel{e \partial_\mu \partial_\nu \theta} \\ &= \partial_\nu A_\mu - \partial_\mu A_\nu = F_{\mu\nu} \end{aligned}$$

Hence, $F_{\mu\nu}$ is also gauge invariant.

Here, there is some sentence of how "there are only two invariants possible for $F_{\mu\nu}$:"

- $F^{\mu\nu} F_{\mu\nu}$
- $e^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta}$ (what is e in this context?) Supposedly, we can ignore this since it relates to divergence (??) **TODO: figure this out**

So, extending the lagrangian with the $F_{\mu\nu}$ term gives us:

$$\mathcal{L}(\phi, \phi^*, \partial_\mu \phi, \partial_\mu \phi^*, A_\mu) = \frac{1}{4} F^{\mu\nu} F_{\mu\nu} + [D_\mu \phi][D^\mu \phi]^* - \frac{1}{2} m |\phi|^2$$

where:

$$D_\mu \equiv (\partial^\mu - i A^\mu)$$

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \text{partial}_\nu A_\mu$$

Now, performing variational calculus on this lagrangian, we should theoretically regain maxwell's equations:

TODO: follow the last part of the stuff written in red ink pen to understand what in the actual fuck happened

Chapter 7

Non abelian gauge theories

Let $\phi_i = (\phi_1, \phi_2, \phi_3)$ be a vector of scalar fields.

Consider the "usual" lagrangian:

$$\mathcal{L} = (\partial_\mu \phi)^\dagger (\partial^\mu \phi) - \frac{1}{2} m \phi^\dagger \phi$$

This clearly has a global symmetry $\phi \rightarrow U(\theta)\phi$ where $U \in SU(n)$

We enlarge the global symmetry to a local symmetry $\phi \rightarrow U(\theta(x))\phi$. Note that $\phi^\dagger \phi$ is still invariant, but we need to check the first term of the Lagrangian.

Working out the changes:

$$\partial_\mu \bar{\phi} = \partial_\mu (U\phi) = (\partial_\mu U)\phi + (U\partial_\mu \phi)$$

$$\partial_\mu \bar{\phi}^\dagger = \partial_\mu (U\phi)^\dagger = \partial_\mu (\phi^\dagger U^\dagger) = (\partial_\mu \phi^\dagger)U^\dagger + \phi^\dagger (\partial_\mu U^\dagger)$$

So, the term to be invariant is:

$$\begin{aligned} & (\partial_\mu \bar{\phi})^\dagger (\partial^\mu \bar{\phi}) = \\ & [(\partial_\mu \phi^\dagger)U^\dagger + \phi^\dagger (\partial_\mu U^\dagger)][(\partial^\mu U)\phi + (U\partial^\mu \phi)] = \\ & (\partial_\mu \phi^\dagger)U^\dagger (\partial^\mu U)\phi + (\partial_\mu \phi^\dagger)U^\dagger (U\partial^\mu \phi) + \\ & \phi^\dagger (\partial_\mu U^\dagger) (\partial^\mu U)\phi + \phi^\dagger (\partial_\mu U^\dagger) (U\partial^\mu \phi) \end{aligned}$$

This mess of equations clearly does not look like $(\partial_\mu \phi)(\partial^\mu \phi)$, even after using the simplification $UU^\dagger = U^\dagger U = I$, so this is not invariant.

So let's define a new covariant derivative (I wish I knew what those words mean):

$$(D_\mu)_{\alpha,\beta} = \partial_\mu \delta_{\alpha\beta} - ig(A_\mu)_{\alpha,\beta}$$

Where g is some kind of coupling coefficient (more on this later), and A_μ is some arbitrary quantity on which we will use the symmetries we expect to give some structure.

We need $D_\mu \phi$ to transform reasonably, hence, we stipulate that:

$$(D_\mu \bar{\phi}) \rightarrow U D_\mu \phi$$

Assuming that transformation law holds, we show that $D_\mu\phi$ is invariant:

$$\begin{aligned} (D_\mu\bar{\phi})^\dagger(D^\mu\bar{\phi}) &= (U(D_\mu\phi))^\dagger(U(D_\mu\phi)) = ((D_\mu\phi)^\dagger U^\dagger)(U(D_\mu\phi)) = \\ &= (D_\mu\phi)^\dagger(D^\mu\phi) \text{ since } UU^\dagger = I \\ \text{Hence, we showed that:} \\ (D_\mu\bar{\phi})^\dagger(D^\mu\bar{\phi}) &\rightarrow (D_\mu\phi)^\dagger(D^\mu\phi) \end{aligned}$$

Now, we need to ensure that the law we took actually works. For this law to hold, we will derive conditions that govern A :

$$\begin{aligned} (D_\mu\bar{\phi}) &= U D_\mu\phi \\ \partial_\mu\bar{\phi} - ig\bar{A}_\mu\bar{\phi} &= U(\partial_\mu\phi - igA_\mu\phi) \\ \partial_\mu(U\phi) - ig\bar{A}_\mu\bar{\phi} &= U(\partial_\mu\phi - igA_\mu\phi) \\ (\partial_\mu U)\phi + \cancel{U(\partial_\mu\phi)} - ig\bar{A}_\mu\bar{\phi} &= \cancel{U\partial_\mu\phi} - igU A_\mu\phi \\ (\partial_\mu U)\phi - ig\bar{A}_\mu\bar{\phi} &= -igU A_\mu\phi \\ -ig\bar{A}_\mu(U\phi) &= -igU A_\mu\phi - (\partial_\mu U)\phi \\ (ig\bar{A}_\mu U)\phi &= (igU A_\mu + (\partial_\mu U))\phi \\ ig\bar{A}_\mu U &= igU A_\mu + (\partial_\mu U) \\ A_\mu &= U A_\mu U^{-1} + \frac{(\partial_\mu U)U^{-1}}{ig} \end{aligned}$$

So, we now know what the correction term is for the D_μ for the non-abelian gauge theory. Notice that $(\partial_\mu U)U^{-1}$ is a function of θ , the parameter.

$$\boxed{A_\mu = U A_\mu U^{-1} + \frac{(\partial_\mu U)U^{-1}}{ig}} \quad (7.1)$$

Supposedly, A_μ is a *connection*, and the field strength tensor F is the *curvature of the connection*:

$$\begin{aligned} [D_\mu, D_\nu] &= [\partial_\mu - ieA_\mu, \partial_\nu - ieA_\nu] \\ &= (\partial_\mu - ieA_\mu)(\partial_\nu - ieA_\nu) - (\partial_\nu - ieA_\nu)(\partial_\mu - ieA_\mu) \\ &= (\partial_\mu\partial_\nu - \partial_\mu(ieA_\nu) + (-ieA_\mu)(\partial_\nu) + i^2e^2A_\mu A_\nu) - \\ &\quad (\partial_\nu\partial_\mu - \partial_\nu(ieA_\mu) + (-ieA_\nu)(\partial_\mu) + i^2e^2A_\nu A_\mu) \\ &= (\cancel{\partial_\mu\partial_\nu} - \partial_\mu(ieA_\nu) + (-ieA_\mu)(\partial_\nu) + \cancel{i^2e^2A_\mu A_\nu}) - \\ &\quad (\cancel{\partial_\nu\partial_\mu} - \partial_\nu(ieA_\mu) + (-ieA_\nu)(\partial_\mu) + \cancel{i^2e^2A_\nu A_\mu}) \\ &\quad \text{TODO: understand how } (-ieA_\mu)(\partial_\nu) \text{ terms get cancelled} \\ &= -ie(\partial_\mu A_\nu - \partial_\nu A_\mu) \end{aligned}$$

which is indeed F .

The group that governs the symmetries of ϕ is $SU(2)$, since $SU(2)$ has a dimension (as a manifold) of 3 ($SU(n)$ has $n^2 - 1$ degrees of freedom).