Topics in Physics - C. Mukku

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Contents

Chapter 1

Lagrangian, Hamiltonian mechanics

Mechanics in terms of generalized coords.

1.1 Lagrangian

Define a functional. L over the config. space of partibles q^i, $qdot^i$. $L = L(q^i, qdot^i)$. We have an explicit dependence on t.

$L = KE - PE$

Assuming a 1-particle system of unit mass,

$$L = \frac{1}{2}\dot{q}^2 - V(q)$$

Assuming an n-particle system of unit mass,

$$L = \sum_i \frac{1}{2} qdot^{i2} - V(q^i)$$

1.2 Variational principle

Take a minimum path from A to B. Now notice that the path that is slightly different from this path will have some delta from the minimum.

Action

$$S(t0, t1) = \int L \mathrm{d}t = \int_{t0}^{t1} L(q^i, qdot^i) \mathrm{d}t$$

. Least action: $\delta S = 0$

Chapter 2

Functional calculus

this chapter develops a completely handwavy physics version of functional analysis.

Definition 1 *A **functional** F is a function: $F : (\mathbb{R} \to \mathbb{R}) \to \mathbb{R}$*

Notation 1 *Evaluation of a functional F with respect to f is denoted by $F[f]$.*

2.1 Functional Derivative - take 1

Consider a functional $F : (\mathbb{R} \to \mathbb{R}) \to \mathbb{R}$, a function $f : \mathbb{R} \to \mathbb{R}$, and a "test function" $\phi : \mathbb{R} \to \mathbb{R}$.

Consider a functional F. We only define the derivative of a functional F with respect to a function f by what happens under an integral sign as follows:

$$\int \frac{\delta F}{\delta f}(x)\phi(x)\mathrm{d}x = \lim_{\epsilon \to 0} \frac{F[f + \epsilon\phi] - F[f]}{\epsilon}$$

Now, we can define a small variation in F as:

$$\delta F : (\mathbb{R} \to \mathbb{R}) \times (\mathbb{R} \to \mathbb{R}) \to \mathbb{R}$$
$$\delta F(f, \phi) \equiv \int \frac{\delta F}{\delta f}(x)\phi(x)\mathrm{d}x$$

Intuitively, δF tells us the variation of the function f along a test function ϕ. So, it encapsulates some kind of "directional derivative".

So, we can look at $\frac{\delta F}{\delta f}$ as a functional as follows:

$$\frac{\delta F}{\delta f} : (\mathbb{R} \to \mathbb{R}) \to \mathbb{R}$$
$$\frac{\delta F}{\delta f}(\phi) = \delta F(f, \phi)$$

Wehre $\frac{\delta F}{\delta f}$ allows us to "test" the change of F with respect to f along a given "direction" ϕ.

2.2 Functional Derivative as taught in class

Substitute $\phi = \delta(x-p)$. Now, the quantity:

$$\frac{\delta F}{\delta f}\phi(x) = \delta F(f, \delta(x-p))$$

Rewriting δF by sticking it under an integral:

$$\int \frac{\delta F}{\delta f}(x)\delta(x-p)\mathrm{d}x = \lim_{\epsilon\to 0} \frac{F[f+\epsilon\delta(x-p)] - F[f]}{\epsilon}$$

$$\left.\frac{\delta F}{\delta f}\right|_p = \lim_{\epsilon\to 0} \frac{F[f+\epsilon\delta(x-p)] - F[f]}{\epsilon}$$

That is, we can start talking about "derivative of the functional F with respect to a function f at a point p" as long as we only test the functional F against δ-functions.

So, we can alternatively define this quantity as:

$$\left.\frac{\delta F}{\delta f}\right|_p \equiv \lim_{\epsilon\to 0} \frac{F[f+\epsilon\delta(x-p)] - F[f]}{\epsilon}$$

While this does not "look like a functional", it actually is, if we mentally replace:

$$p \to \int \_\ \delta(x-p)\mathrm{d}x$$

This is how mukku got that expression.

2.3 Common functional derivatives

2.3.1 Derivative of $F[\phi] \equiv \int(\frac{\partial\phi}{\partial y})^2$

$\left.\frac{\delta F}{\delta f}\right|_p = \int(\frac{\partial\phi}{\partial y})^2$

2.4 Deriving E-L from functional magic

2.5 Weird things in Functional Analysis as taught in class

Consider the functional

$J[f] = \int g[f']dy$:

since g is a functional, it has a type $g : (\mathbb{R} \to \mathbb{R}) \to \mathbb{R}$. So, our integrand must be some function df, and not some **space component** dy. **I don't understand what the definition of J means**.

Chapter 3

Maxwell's equations in Minkowski space

Let us first review Maxwell's equations:

$\nabla \cdot E = \dfrac{\rho}{\epsilon_0}$ (Electric charges produce fields)
$\nabla \cdot B = 0$ (Only magnetic dipoles exist)
$\nabla \times E = -\dfrac{\partial B}{\partial t}$ (Lenz Law / Faraday's law - time varying magnetic field induces current that opposes it)
$\nabla \times B = \mu_0 \left(J + \epsilon_0 \dfrac{\partial E}{\partial t} \right)$ (Ampere's law + fudge factor)

3.1 Constructing F, or Tensorifying Maxwell's equations

Begin with the equation that $\nabla \cdot B = 0$. This tells that B can be written as the curl of some other field:

$$\boxed{B \equiv \nabla \times A} \tag{3.1}$$

Expanding this equation of B in tensorial form:

$$\boxed{B^i = \mathcal{E}^{ijk} \partial_j A^k} \tag{3.2}$$

Next, take $\nabla \times E = -\frac{\partial B}{\partial t}$.

$$\nabla \times E = -\frac{\partial B}{\partial t} = \frac{\partial(\nabla \times A)}{\partial t} = \nabla \times \frac{\partial A}{\partial t}$$
$$\nabla \times \left(E + \frac{\partial A}{\partial t}\right) = 0$$

writing this as the divergence of some field ϕ scaled by $\alpha : \mathbb{R}$

$$E + \frac{\partial A}{\partial t} = \alpha(\nabla \cdot \phi)$$
$$E = \alpha \nabla \cdot \phi - \frac{\partial A}{\partial t}$$

Since electrostatics is time-independent, we choose to think of $\alpha = -1$, so we can interpret ϕ as the potential.

$$E^i = -\frac{\partial \phi}{\partial x^k} g^{ik} - \frac{\partial A^i}{\partial t} \tag{3.3}$$

A slight reformulation (since we know that in Minkowski space, $\partial_t = \partial_0$) we get the equation:

$$\boxed{E^i = -g^{ik}\partial_k \phi - \partial_0 A^i} \tag{3.4}$$

We get the metric $g^i k$ involved to raise the covariant $\frac{\partial \phi}{\partial x^k}$ into the contravariant E^i.
(**Sid question:** how does one justify switching $\nabla \times$ and ∂? It feels like some algebra)

Here be magic! We define A new rank-2 tensor in Minkowski space-time, called F (for Faraday),

$$\boxed{F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu} \tag{3.5}$$

(**Sid question:** why is this object $F_{\mu\nu}$ covariant? What does this *mean*?)

Lemma 1 *$F_{\mu\nu}$ is antisymmetric.*

Lemma 2 *$F_{\mu\nu}$ has 6 degrees of freedom*

Proof. Number of degrees of freedom of F:

$$\frac{4^2 \text{ (total)} - 4 \text{ (diagonal)}}{2 \text{ (anti-symmetry)}} = 6$$

Notice that F is a 1-form!

3.2 Expressing B, E in terms of F

We now wish to re-expresss B^{ij} and E^{ij} in terms of F, so that this F captures all of maxwell's equations.

$$\begin{aligned}
B^i &= \mathcal{E}^{ijk}\partial_j A^k = \mathcal{E}^{ikj}\partial_k A^j && \text{by } k,\ j \text{ being free variables}\\
B^i &= \frac{1}{2}\Big(\mathcal{E}^{ijk}\partial_j A^k + \mathcal{E}^{ikj}\partial_k A^j\Big)\\
&\text{Substituting } \partial_j A_k - \partial_k A_j = F_{jk},\\
B^i &= \frac{1}{2}\mathcal{E}^{ijk}F_{jk}
\end{aligned}$$

So, B in terms of F is:

$$\boxed{B^i = \frac{1}{2}\mathcal{E}^{ijk}F_{jk}} \tag{3.6}$$

Similarly, we wish to write E in terms of F. The algebra is as follows:

$$\begin{aligned}
E^i &= -g^{ik}\partial_k\phi - \partial_0 A^i\\
E^i &= -g^{ik}\partial_k\phi - \partial_0 g^{ik}A_k && \text{Is this allowed? Am I always allowed to insert the } g_{ik}?\\
E^i &= -g^{ik}(\partial_k\phi + \partial_0 A_k)
\end{aligned}$$

Since $k = \{1, 2, 3\}$ (k is spacelike coordinates), and we would like to relate ϕ with A (to unify E), we **set**:

$$\boxed{A_0 \equiv -\phi} \tag{3.7}$$

Continuing the derivation,

$$\begin{aligned}
E^i &= -g^{ik}(\partial_k(-A_0) + \partial_0 A_k)\\
E^i &= -g^{ik}(\partial_0 A_k - \partial_k A_0)\\
E^i &= -g^{ik}F_{0k}
\end{aligned}$$

So, finally, the relation is:

$$\boxed{E^i = -g^{ik}F_{0k}} \tag{3.8}$$

Let us reconsider what we believed E to be. We had:

$$E = -\boldsymbol{\nabla}\phi - \frac{\partial A}{\partial t}$$

However, comparing dimensions, space derivative of ϕ = time derivative of A. This means that $\frac{\delta\phi}{\delta x} = \frac{\delta A}{\delta y}$, and so $\frac{\delta\phi}{\frac{\delta x}{\delta t}} = \delta A$. We arbitrarily pick c as our measuring stick for $\frac{\delta x}{\delta t}$. Also, in minkowski space, our measuring stick is actually (ct, x, y, z), so $\partial_0 = \partial_{ct}$ So, when we write the equation for E, we should actually write

$$E = c\Big(-\frac{\boldsymbol{\nabla}\phi}{c} - \frac{\partial A}{\partial ct}\Big)$$

which becomes:

$$\boxed{E^i = cF^{i0}} \tag{3.9}$$

3.3 Rewriting Maxwell's equations in terms of F

Now that we have constructed the Faraday tensor F, we wish to re-expresss Maxwell's equations in terms of this object. This will give us a compact form of the laws which are invariant under coordinate transforms.

3.3.1 Combining (1) $\nabla E = \frac{\rho}{\epsilon_0}$, (4) $\nabla \times B = \mu_0 J + \frac{\partial E}{\partial t}$

1. Using (4) $\nabla \times B = \mu_0 J + \frac{\partial E}{\partial t}$

We consider the 4th Maxwell equation:

$$\begin{aligned}
\nabla \times B &= \mu_0 J + \epsilon_0 \mu_0 \frac{\partial E}{\partial t} \\
\nabla \times B &= \mu_0 J + \frac{1}{c^2}\frac{\partial E}{\partial t} \\
&\text{Converting to indices,} \\
(\nabla \times B)^i &= \mu_0 J^i + \frac{1}{c}\frac{\partial E^i}{\partial ct} && \text{(From } \partial_{ct} = \tfrac{1}{c}\partial_t\text{)} \\
&= \mu_0 J^i + \frac{1}{c}\frac{\partial E^i}{\partial X^0} \\
&= \mu_0 J^i + \frac{\partial F^{i0}}{\partial X^0} && \text{(From } E^i = cF^{i0}\text{)} \\
&= \mu_0 J^i + \partial_0 F^{i0}
\end{aligned}$$

Now, we start to simplify the LHS, $\nabla \times B$:

$$\begin{aligned}
&(\nabla \times B)^i = \mathcal{E}^{ijk}\partial_j B_k \\
&\text{Since } B^k = \frac{1}{2}\mathcal{E}^{kmn}F_{mn}, \\
&B_k = \frac{1}{2}\mathcal{E}_{kmn}F^{mn}, && (\textbf{TODO:} \text{ this is scam}) \\
&(\nabla \times B)^i = \mathcal{E}^{ijk}\partial_j\left(\frac{1}{2}\mathcal{E}_{kmn}F^{mn}\right) = \frac{1}{2}\mathcal{E}^{ijk}\mathcal{E}_{kmn}\partial_j F^{mn}
\end{aligned}$$

Aside: We need to know how to evaluate $\mathcal{E}^{ijk}\mathcal{E}_{kmn}$:

$$\mathcal{E}_{i_1,i_2,\ldots,i_n}\mathcal{E}_{j_1,j_2,\ldots j_n} = \det\left\{\begin{vmatrix} \delta_{i_1j_1} & \delta_{i_1j_2} & \dots & \delta_{i_1j_n} \\ \delta_{i_2j_1} & \delta_{i_2j_2} & \dots & \delta_{i_2j_n} \\ \vdots & \vdots & \ddots & \vdots \\ \delta_{i_nj_1} & \delta_{i_nj_2} & \dots & \delta_{i_nj_n} \end{vmatrix}\right\}$$

$\mathcal{E}^{ijk}\mathcal{E}^{imn} = -1(\delta^m_j\delta^n_k - \delta^n_j\delta^m_k)$

He argued that we get a -1 factor here due to the presence of the metric. I'm not fully convinced, but I can handwave this using the magic words "tensor density".

Plugging both equations together,

$$\frac{1}{2}\mathcal{E}^{ijk}\mathcal{E}_{kmn}\partial_j F^{mn} = \mu_0 J^i + \partial_0 F^{i0}$$

(Since kij is an even permutation of ijk):

$$\frac{1}{2}\mathcal{E}^{kij}\mathcal{E}_{kmn}\partial_j F^{mn} = \mu_0 J^i + \partial_0 F^{i0}$$

(Using $\mathcal{E}^{kij}\mathcal{E}^{kmn} = -1(\delta_i^m \delta_j^n - \delta_i^n \delta_j^m)$):

$$\frac{1}{2}\big[-\big(\delta^i_m \delta^j_n - \delta^i_n \delta^j_m\big)\big]\partial_j F^{mn} = \mu_0 J^i + \partial_0 F^{i0}$$

$$-\frac{1}{2}\big[\partial_n F^{in} - \partial_m F^{mi}\big] = \mu_0 J^i + \partial_0 F^{i0}$$

(F is anti-symmetric, so rewriting $\partial_m F^{mi} = -\partial_m F^{im}$):

$$-\frac{1}{2}\big[\partial_n F^{in} + \partial_m F^{im}\big] = \mu_0 J^i + \partial_0 F^{i0}$$

(Replacing $\partial_m F^{im} \equiv \partial_n F^{in}$ since m is free):

$$-\big[\partial_m F^{im}\big] = \mu_0 J^i + \partial_0 F^{i0}$$

$$\mu_0 J^i + \partial_0 F^{i0} + \partial_m F^{im} = 0$$

$$\mu_0 J^i + \partial_\mu F^{i\mu} = 0 \qquad (\mu = \{0,1,2,3\})$$

This gives us a continuity-style equation, linking the current density J to the rate of change of F.

$$\boxed{\mu_0 J^i + \partial_\mu F^{i\mu} = 0} \qquad (\mu = \{0,1,2,3\})$$

Second part, using 1st equation

$$\boldsymbol{\nabla} E = \frac{\rho}{\epsilon_0}$$

$$\partial_i E^i = \frac{\rho}{\epsilon_0}$$

(Substituting $E^i = cF^{i0}$):

$$c\partial_i F^{i0} = \frac{\rho}{\epsilon_0} = \frac{\rho\mu_0}{\mu_0\epsilon_0} = \rho c^2$$

$$\partial_i F^{i0} = \mu_0 c\rho$$

(Since F is anti-symmetric, $F^{00} = 0$, Hence):

$$\partial_0 F^{00} + \partial_i F^{i0} = \mu_0 c\rho$$

$$\partial_\mu F^{\mu 0} = \mu_0 c\rho$$

$$\boxed{\partial_\mu F^{\mu 0} = \mu_0 c\rho} \qquad (3.10)$$

Combining part 1 and part 2:

$$\begin{aligned} \mu_0 J^i + \partial_\mu F^{i\mu} &= 0 && \text{(From } B\text{)} \\ \partial_\mu F^{i\mu} = -\mu_0 J^i \partial_\mu F^{\mu 0} &= \mu_0 c\rho \\ \partial_\mu F^{0\mu} &= -\mu_0 c\rho \end{aligned}$$

To combine these equations, **we set:**

$$\boxed{J^0 \equiv c\rho} \tag{3.11}$$

We arrive at the unified equation:

$$\partial_\mu F^{\nu\mu} = -\mu_0 J^\nu$$

Choose units such that $c = \frac{h}{2\pi} = G_n = 1$, which gives us:

$$\begin{aligned} &\partial_\mu F^{\nu\mu} = -J^\nu \\ &F \text{ is antisymmetric, so flipping indices} \\ &\partial_\mu F^{\mu\nu} = J^\nu \end{aligned}$$

$$\boxed{\partial_\mu F^{\mu\nu} = J^\nu} \tag{3.12}$$

Note that this is **Ampere's law!**

3.3.2 Combining (2) $\nabla \times E = -\frac{\partial B}{\partial t}$, (3) $\nabla B = 0$