Topics in Physics - C. Mukku

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Contents

Chapter 1

Tensor algebra preliminaries

1.1 Raising and lowering of two indeces simeltaneously

Note that

$$a_i b^i = (a^j g_{ij}) b^i = (a^j g_{ij})(b_k g^{ki})$$

In minkowski space, we know that $g^{ij} = 0$ if $i \neq j$, and $(g^{ii} g_{ii})^2 = 1$, so we can rewrite the above expression as:

$$\begin{aligned}(a^j g_{ij})(b_k g^{ki}) =\\ (a^i g_{ii})(b_i g^{ii}) =\\ a^i b_i\end{aligned}$$

Chapter 2

Lagrangian, Hamiltonian mechanics

Mechanics in terms of generalized coords.

2.1 Lagrangian

Define a functional. L over the config. space of partibles q^i, $qdot^i$. $L = L(q^i, qdot^i)$. We have an explicit dependence on t.

$L = KE - PE$

Assuming a 1-particle system of unit mass,

$$L = \frac{1}{2}\dot{q}^2 - V(q)$$

Assuming an n-particle system of unit mass,

$$L = \sum_i \frac{1}{2} qdot^{i2} - V(q^i)$$

2.2 Variational principle

Take a minimum path from A to B. Now notice that the path that is slightly different from this path will have some delta from the minimum.

Action

$$S(t0, t1) = \int L \mathrm{d}t = \int_{t0}^{t1} L(q^i, qdot^i) \mathrm{d}t$$

. Least action: $\delta S = 0$

Chapter 3

Functional calculus

this chapter develops a completely handwavy physics version of functional analysis.

Definition 1 *A **functional** F is a function: $F : (\mathbb{R} \to \mathbb{R}) \to \mathbb{R}$*

Notation 1 *Evaluation of a functional F with respect to f is denoted by $F[f]$.*

3.1 Functional Derivative - take 1

Consider a functional $F : (\mathbb{R} \to \mathbb{R}) \to \mathbb{R}$, a function $f : \mathbb{R} \to \mathbb{R}$, and a "test function" $\phi : \mathbb{R} \to \mathbb{R}$.

Consider a functional F. We only define the derivative of a functional F with respect to a function f by what happens under an integral sign as follows:

$$\int \frac{\delta F}{\delta f}(x)\phi(x)\mathrm{d}x = \lim_{\epsilon \to 0} \frac{F[f + \epsilon\phi] - F[f]}{\epsilon}$$

Now, we can define a small variation in F as:

$$\delta F : (\mathbb{R} \to \mathbb{R}) \times (\mathbb{R} \to \mathbb{R}) \to \mathbb{R}$$
$$\delta F(f, \phi) \equiv \int \frac{\delta F}{\delta f}(x)\phi(x)\mathrm{d}x$$

Intuitively, δF tells us the variation of the function f along a test function ϕ. So, it encapsulates some kind of "directional derivative".

So, we can look at $\frac{\delta F}{\delta f}$ as a functional as follows:

$$\frac{\delta F}{\delta f} : (\mathbb{R} \to \mathbb{R}) \to \mathbb{R}$$
$$\frac{\delta F}{\delta f}(\phi) = \delta F(f, \phi)$$

Wehre $\frac{\delta F}{\delta f}$ allows us to "test" the change of F with respect to f along a given "direction" ϕ.

3.2 Functional Derivative as taught in class

Substitute $\phi = \delta(x-p)$. Now, the quantity:

$$\frac{\delta F}{\delta f}\phi(x) = \delta F(f, \delta(x-p))$$

Rewriting δF by sticking it under an integral:

$$\int \frac{\delta F}{\delta f}(x)\delta(x-p)\mathrm{d}x = \lim_{\epsilon\to 0}\frac{F[f+\epsilon\delta(x-p)] - F[f]}{\epsilon}$$

$$\left.\frac{\delta F}{\delta f}\right|_p = \lim_{\epsilon\to 0}\frac{F[f+\epsilon\delta(x-p)] - F[f]}{\epsilon}$$

That is, we can start talking about "derivative of the functional F with respect to a function f at a point p" as long as we only test the functional F against δ-functions.

So, we can alternatively define this quantity as:

$$\left.\frac{\delta F}{\delta f}\right|_p \equiv \lim_{\epsilon\to 0}\frac{F[f+\epsilon\delta(x-p)] - F[f]}{\epsilon}$$

While this does not "look like a functional", it actually is, if we mentally replace:

$$p \to \int \_\ \delta(x-p)\mathrm{d}x$$

This is how mukku got that expression.

3.3 Common functional derivatives

3.3.1 $F[f] = \int_0^\infty f dx$

$$\begin{aligned}\frac{\delta F[f]}{\delta f(x_0)} &= \lim_{\epsilon\to 0}\frac{\int_0^\infty (f+\epsilon\delta(x-x_0))\mathrm{d}x - \int_0^\infty f\mathrm{d}x}{\epsilon} \\ &= \int_0^\infty \delta(x-x_0)\mathrm{d}x = 1\end{aligned}$$

3.3.2 $F[f] = \int_0^\infty g[f] dx$

This does not actually type-check for me. $g : (\mathbb{R} \to \mathbb{R}) \to reals$, so I don't fully understand what we are "varying" where when we integrate with respect to dx.

So, there's something bizarre here that I don't understand — the integral doesn't really make sense.

3.3.3 Derivative of $F[\phi] \equiv \int (\frac{\partial \phi}{\partial y})^2$

$\frac{\delta F}{\delta f}\Big|_p = \int (\frac{\partial \phi}{\partial y})^2$

3.4 Deriving E-L from functional magic

3.5 Weird things in Functional Analysis as taught in class

Consider the functional

$J[f] = \int g[f'] dy$:

since g is a functional, it has a type $g : (\mathbb{R} \to \mathbb{R}) \to \mathbb{R}$. So, our integrand must be some function df, and not some **space component** dy. **I don't understand what the definition of J means**.

Chapter 4

Maxwell's equations in Minkowski space

Let us first review Maxwell's equations:

$\nabla \cdot E = \dfrac{\rho}{\epsilon_0}$ (Electric charges produce fields)
$\nabla \cdot B = 0$ (Only magnetic dipoles exist)
$\nabla \times E = -\dfrac{\partial B}{\partial t}$ (Lenz Law / Faraday's law - time varying magnetic field induces current that opposes it)
$\nabla \times B = \mu_0 \left(J + \epsilon_0 \dfrac{\partial E}{\partial t} \right)$ (Ampere's law + fudge factor)

4.1 Constructing F, or Tensorifying Maxwell's equations

Begin with the equation that $\nabla \cdot B = 0$. This tells that B can be written as the curl of some other field:

$$\boxed{B \equiv \nabla \times A} \tag{4.1}$$

Expanding this equation of B in tensorial form:

$$\boxed{B^i = \mathcal{E}^{ijk} \partial_j A^k} \tag{4.2}$$

Next, take $\nabla \times E = -\frac{\partial B}{\partial t}$.

$$\nabla \times E = -\frac{\partial B}{\partial t} = \frac{\partial(\nabla \times A)}{\partial t} = \nabla \times \frac{\partial A}{\partial t}$$
$$\nabla \times \left(E + \frac{\partial A}{\partial t}\right) = 0$$

writing this as the divergence of some field ϕ scaled by $\alpha : \mathbb{R}$

$$E + \frac{\partial A}{\partial t} = \alpha(\nabla \cdot \phi)$$
$$E = \alpha \nabla \cdot \phi - \frac{\partial A}{\partial t}$$

Since electrostatics is time-independent, we choose to think of $\alpha = -1$, so we can interpret ϕ as the potential.

$$E^i = -\frac{\partial \phi}{\partial x^k} g^{ik} - \frac{\partial A^i}{\partial t} \tag{4.3}$$

A slight reformulation (since we know that in Minkowski space, $\partial_t = \partial_0$) we get the equation:

$$\boxed{E^i = -g^{ik}\partial_k \phi - \partial_0 A^i} \tag{4.4}$$

We get the metric $g^i k$ involved to raise the covariant $\frac{\partial \phi}{\partial x^k}$ into the contravariant E^i.

(**Sid question:** how does one justify switching $\nabla\times$ and ∂? It feels like some algebra)

Here be magic! We define A new rank-2 tensor in Minkowski space-time, called F (for Faraday),

$$\boxed{F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu} \tag{4.5}$$

(**Sid question:** why is this object $F_{\mu\nu}$ covariant? What does this *mean*?)

Lemma 1 *$F_{\mu\nu}$ is antisymmetric.*

Lemma 2 *$F_{\mu\nu}$ has 6 degrees of freedom*

Proof. Number of degrees of freedom of F:

$$\frac{4^2 \text{ (total)} - 4 \text{ (diagonal)}}{2 \text{ (anti-symmetry)}} = 6$$

Notice that F is a 1-form!

4.2 Expressing B, E in terms of F

We now wish to re-expresss B^{ij} and E^{ij} in terms of F, so that this F captures all of maxwell's equations.

$$
\begin{aligned}
B^i &= \mathcal{E}^{ijk}\partial_j A^k = \mathcal{E}^{ikj}\partial_k A^j && \text{by } k,\, j \text{ being free variables}\\
B^i &= \frac{1}{2}\Big(\mathcal{E}^{ijk}\partial_j A^k + \mathcal{E}^{ikj}\partial_k A^j\Big)\\
&\quad \text{Substituting } \partial_j A_k - \partial_k A_j = F_{jk},\\
B^i &= \frac{1}{2}\mathcal{E}^{ijk}F_{jk}
\end{aligned}
$$

So, B in terms of F is:

$$\boxed{B^i = \frac{1}{2}\mathcal{E}^{ijk}F_{jk}} \tag{4.6}$$

Similarly, we wish to write E in terms of F. The algebra is as follows:

$$
\begin{aligned}
E^i &= -g^{ik}\partial_k\phi - \partial_0 A^i\\
E^i &= -g^{ik}\partial_k\phi - \partial_0 g^{ik}A_k && \text{Is this allowed? Am I always allowed to insert the } g_{ik}?\\
E^i &= -g^{ik}(\partial_k\phi + \partial_0 A_k)
\end{aligned}
$$

Since $k = \{1, 2, 3\}$ (k is spacelike coordinates), and we would like to relate ϕ with A (to unify E), we **set**:

$$\boxed{A_0 \equiv -\phi} \tag{4.7}$$

Continuing the derivation,

$$
\begin{aligned}
E^i &= -g^{ik}(\partial_k(-A_0) + \partial_0 A_k)\\
E^i &= -g^{ik}(\partial_0 A_k - \partial_k A_0)\\
E^i &= -g^{ik}F_{0k}
\end{aligned}
$$

So, finally, the relation is:

$$\boxed{E^i = -g^{ik}F_{0k}} \tag{4.8}$$

Let us reconsider what we believed E to be. We had:

$$E = -\nabla\phi - \frac{\partial A}{\partial t}$$

However, comparing dimensions, space derivative of ϕ = time derivative of A. This means that $\frac{\delta\phi}{\delta x} = \frac{\delta A}{\delta y}$, and so $\frac{\delta\phi}{\frac{\delta x}{\delta t}} = \delta A$. We arbitrarily pick c as our measuring stick for $\frac{\delta x}{\delta t}$. Also, in minkowski space, our measuring stick is actually (ct, x, y, z), so $\partial_0 = \partial_{ct}$ So, when we write the equation for E, we should actually write

$$E = c\bigg(-\frac{\nabla\phi}{c} - \frac{\partial A}{\partial ct}\bigg)$$

which becomes:

$$\boxed{E^i = cF^{i0}} \tag{4.9}$$

4.3 Rewriting Maxwell's equations in terms of F

Now that we have constructed the Faraday tensor F, we wish to re-expresss Maxwell's equations in terms of this object. This will give us a compact form of the laws which are invariant under coordinate transforms.

4.3.1 Combining (1) $\nabla E = \frac{\rho}{\epsilon_0}$, (4) $\nabla \times B = \mu_0 J + \frac{\partial E}{\partial t}$

1. Using (4) $\nabla \times B = \mu_0 J + \frac{\partial E}{\partial t}$

We consider the 4th Maxwell equation:

$$
\begin{aligned}
\nabla \times B &= \mu_0 J + \epsilon_0 \mu_0 \frac{\partial E}{\partial t} \\
\nabla \times B &= \mu_0 J + \frac{1}{c^2}\frac{\partial E}{\partial t} \\
&\text{Converting to indices,} \\
(\nabla \times B)^i &= \mu_0 J^i + \frac{1}{c}\frac{\partial E^i}{\partial ct} && \text{(From } \partial_{ct} = \tfrac{1}{c}\partial_t\text{)} \\
&= \mu_0 J^i + \frac{1}{c}\frac{\partial E^i}{\partial X^0} \\
&= \mu_0 J^i + \frac{\partial F^{i0}}{\partial X^0} && \text{(From } E^i = cF^{i0}\text{)} \\
&= \mu_0 J^i + \partial_0 F^{i0}
\end{aligned}
$$

Now, we start to simplify the LHS, $\nabla \times B$:

$$
\begin{aligned}
&(\nabla \times B)^i = \mathcal{E}^{ijk}\partial_j B_k \\
&\text{Since } B^k = \frac{1}{2}\mathcal{E}^{kmn}F_{mn}, \\
&B_k = \frac{1}{2}\mathcal{E}_{kmn}F^{mn}, && \text{(}\textbf{TODO:}\text{ this is scam)} \\
&(\nabla \times B)^i = \mathcal{E}^{ijk}\partial_j\left(\frac{1}{2}\mathcal{E}_{kmn}F^{mn}\right) = \frac{1}{2}\mathcal{E}^{ijk}\mathcal{E}_{kmn}\partial_j F^{mn}
\end{aligned}
$$

Aside: We need to know how to evaluate $\mathcal{E}^{ijk}\mathcal{E}_{kmn}$:

$$
\mathcal{E}_{i_1,i_2,\ldots,i_n}\mathcal{E}_{j_1,j_2,\ldots j_n} = \det\left\{\begin{vmatrix} \delta_{i_1j_1} & \delta_{i_1j_2} & \ldots & \delta_{i_1j_n} \\ \delta_{i_2j_1} & \delta_{i_2j_2} & \ldots & \delta_{i_2j_n} \\ \vdots & \vdots & \ddots & \vdots \\ \delta_{i_nj_1} & \delta_{i_nj_2} & \ldots & \delta_{i_nj_n} \end{vmatrix}\right\}
$$

$\mathcal{E}^{ijk}\mathcal{E}^{imn} = -1(\delta^m_j\delta^n_k - \delta^n_j\delta^m_k)$

He argued that we get a -1 factor here due to the presence of the metric. I'm not fully convinced, but I can handwave this using the magic words "tensor density".

Plugging both equations together,

$$\frac{1}{2}\mathcal{E}^{ijk}\mathcal{E}_{kmn}\partial_j F^{mn} = \mu_0 J^i + \partial_0 F^{i0}$$

(Since kij is an even permutation of ijk):

$$\frac{1}{2}\mathcal{E}^{kij}\mathcal{E}_{kmn}\partial_j F^{mn} = \mu_0 J^i + \partial_0 F^{i0}$$

(Using $\mathcal{E}^{kij}\mathcal{E}^{kmn} = -1(\delta_i^m\delta_j^n - \delta_i^n\delta_j^m)$):

$$\frac{1}{2}\big[-\big(\delta^i_m\delta^j_n - \delta^i_n\delta^j_m\big)\big]\partial_j F^{mn} = \mu_0 J^i + \partial_0 F^{i0}$$

$$-\frac{1}{2}\big[\partial_n F^{in} - \partial_m F^{mi}\big] = \mu_0 J^i + \partial_0 F^{i0}$$

(F is anti-symmetric, so rewriting $\partial_m F^{mi} = -\partial_m F^{im}$):

$$-\frac{1}{2}\big[\partial_n F^{in} + \partial_m F^{im}\big] = \mu_0 J^i + \partial_0 F^{i0}$$

(Replacing $\partial_m F^{im} \equiv \partial_n F^{in}$ since m is free):

$$-\big[\partial_m F^{im}\big] = \mu_0 J^i + \partial_0 F^{i0}$$

$$\mu_0 J^i + \partial_0 F^{i0} + \partial_m F^{im} = 0$$

$$\mu_0 J^i + \partial_\mu F^{i\mu} = 0 \qquad (\mu = \{0,1,2,3\})$$

This gives us a continuity-style equation, linking the current density J to the rate of change of F.

$$\boxed{\mu_0 J^i + \partial_\mu F^{i\mu} = 0} \qquad (\mu = \{0,1,2,3\})$$

Second part, using 1st equation

$$\boldsymbol{\nabla} E = \frac{\rho}{\epsilon_0}$$

$$\partial_i E^i = \frac{\rho}{\epsilon_0}$$

(Substituting $E^i = cF^{i0}$):

$$c\partial_i F^{i0} = \frac{\rho}{\epsilon_0} = \frac{\rho\mu_0}{\mu_0\epsilon_0} = \rho c^2$$

$$\partial_i F^{i0} = \mu_0 c\rho$$

(Since F is anti-symmetric, $F^{00} = 0$, Hence):

$$\partial_0 F^{00} + \partial_i F^{i0} = \mu_0 c\rho$$

$$\partial_\mu F^{\mu 0} = \mu_0 c\rho$$

$$\boxed{\partial_\mu F^{\mu 0} = \mu_0 c\rho} \tag{4.10}$$

Combining part 1 and part 2:

$$\mu_0 J^i + \partial_\mu F^{i\mu} = 0 \qquad \text{(From } B\text{)}$$
$$\partial_\mu F^{i\mu} = -\mu_0 J^i \partial_\mu F^{\mu 0} = \mu_0 c\rho$$
$$\partial_\mu F^{0\mu} = -\mu_0 c\rho$$

To combine these equations, **we set:**

$$\boxed{J^0 \equiv c\rho} \tag{4.11}$$

We arrive at the unified equation:

$$\partial_\mu F^{\nu\mu} = -\mu_0 J^\nu$$

Choose units such that $c = \frac{h}{2\pi} = G_n = 1$, which gives us:

$$\partial_\mu F^{\nu\mu} = -J^\nu$$
F is antisymmetric, so flipping indices
$$\partial_\mu F^{\mu\nu} = J^\nu$$

$$\boxed{\partial_\mu F^{\mu\nu} = J^\nu} \tag{4.12}$$

Note that this is **Ampere's law!**

4.3.2 Combining (2) $\nabla \times E = -\frac{\partial B}{\partial t}$, (3) $\nabla B = 0$

$$\nabla \times E = -\frac{\partial B}{\partial t}$$
$$(\nabla \times E)^i = \mathcal{E}^{ijk}\partial_j E_k = -\partial_0 B$$
$$\mathcal{E}^{ijk}\partial_j E_k = -\partial_0(\frac{1}{2}\mathcal{E}^{ijk}F_{jk})$$
$$\mathcal{E}^{ijk}\partial_j E_k + \partial_0(\frac{1}{2}\mathcal{E}^{ijk}F_{jk}) = 0$$
$$2\mathcal{E}^{ijk}\partial_j E_k + \partial_0(\mathcal{E}^{ijk}F_{jk}) = 0$$

Now we begin from the other direction, and start the derivation.
We know that the equation we want is:

$$\boxed{\mathcal{E}^{\alpha\beta\mu\nu}\partial_\beta F_{\mu\nu} = 0} \tag{4.13}$$

$\alpha = 0$ **case:**

First, set $\alpha = 0$. So now, the other β, μ, ν are forced to be become space components — (i, j, k).

Therefore, the equation now becomes:

$$\mathcal{E}^{0ijk}\partial_i F_{jk} = 0$$

However, note that $\mathcal{E}0ijk = \mathcal{E}ijk$, because if (ijk) is an even permutation, so will $(0ijk)$, and vice versa for odd (since $0 < i, j, k$).

Using this, the equation becomes

$$\begin{aligned}
\mathcal{E}^{ijk}\partial_i F_{jk} &= 0 \\
\partial_i(\mathcal{E}^{ijk} F_{jk}) &= 0 \\
\text{Since } B^i = \frac{1}{2}\mathcal{E}^{ijk} F_{jk}&: \\
\partial_i\left(\frac{B^i}{2}\right) &= 0 \\
\partial_i B^i &= 0 \\
\boldsymbol{\nabla} B &= 0
\end{aligned}$$

Hence, the above equation does encode $\boldsymbol{\nabla} B = 0$.

$\alpha = m$ **case:**

Let α be a spatial dimension $m = \{1, 2, 3\}$.

$$\begin{aligned}
\mathcal{E}^{\alpha\beta\mu\nu}\partial_\beta F_{\mu\nu} &= 0 \\
\mathcal{E}^{m\beta\mu\nu}\partial_\beta F_{\mu\nu} &= 0
\end{aligned}$$

Once again, we get two cases, one where $\beta = 0$, and one where $\beta = n$ where n is a spatial dimension. If $\beta = 0$, then the other dimensions are forced to be spatial dimensions, which we shall denote as $\mu \equiv x$, $\nu \equiv y$

$$\begin{aligned}
\mathcal{E}^{m\beta\mu\nu}\partial_\beta F_{\mu\nu} &= 0 \\
\mathcal{E}^{m0xy}\partial_0 F_{xy} + \mathcal{E}^{mn\mu\nu}\partial_n F_{\mu\nu} &= 0
\end{aligned}$$

Now note that $\mathcal{E}^{m0\mu\nu} = -\mathcal{E}0m\mu\nu = -\mathcal{E}m\mu\nu$.

Using this, we can rewrite the above equation as:

$$\begin{aligned}
\mathcal{E}^{m0xy}\partial_0 F_{xy} + \mathcal{E}^{mn\mu\nu}\partial_n F_{\mu\nu} &= 0 \\
-\mathcal{E}^{mxy}\partial_0 F_{xy} + \mathcal{E}^{mn\mu\nu}\partial_n F_{\mu\nu} &= 0
\end{aligned}$$

We now consider cases for μ in the second term, where either $\mu = 0$ or $\mu = o \in \{1, 2, 3\}$

If $\mu = 0$, then the other dimension ν must be a spatial dimension p. If $\mu = q$, then the other dimension ν must be a time dimension 0 (This is because we are not allowed to have 4 spatial dimensions, since the $\mathcal{E}$ evaluates to 0 on repeated dimensions).

$$-\mathcal{E}^{mxy}\partial_0 F_{xy} + \mathcal{E}^{mn\mu\nu}\partial_n F_{\mu\nu} = 0$$

$$\begin{aligned} -\mathcal{E}^{mxy}\partial_0 F_{xy} + & \\ \mathcal{E}^{mn0p}\partial_n F_{0p} & \qquad (\mu = 0,\, \nu = p) \\ \mathcal{E}^{mnq0}\partial_n F_{q0} & \qquad (\mu = q,\, \nu = 0) \\ = 0 & \end{aligned}$$

Rearranging, and using the fact that $F_{0p} = -Fp0$, $\mathcal{E}mn0p = \mathcal{E}0mnp = \mathcal{E}mnp$, $\mathcal{E}mnq0 = -\mathcal{E}0mnq = -\mathcal{E}mnq$,

$$-\mathcal{E}^{mxy}\partial_0 F_{xy} + \mathcal{E}^{mnp}(-\partial_n F_{p0}) + (-\mathcal{E}^{mnq})\partial_n F_{q0} = 0$$

Multiplying throughout by -1, and noticing that since p, q are dummy indeces, we can set $p = q$. This allows us to get:

$$\mathcal{E}^{mxy}\partial_0 F_{xy} + 2\mathcal{E}^{mnp}\partial_n F_{p0} = 0$$

First, remember that $E_p = F_{p0}$. So, we can replace the term F_{p0} (upto fudging of constant factors that we have always done), with E_p.

Now, compare

$$\mathcal{E}^{mxy}\partial_0 F_{xy} + 2\mathcal{E}^{mnp}\partial_n E_p = 0 \qquad \text{(Our equation)}$$

$$2\mathcal{E}^{ijk}\partial_j E_k + \partial_0(\mathcal{E}^{ijk}F_{jk}) = 0 \qquad \text{(Previous equation)}$$

Note that the two equations are identical upto variable naming, and are hence considered equal. So, we have encoded both of Maxwell's laws into this particular equation:

$$\boxed{\mathcal{E}^{\alpha\beta\mu\nu}\partial_\beta F_{\mu\nu} = 0} \tag{4.14}$$