IIIT Theory talks

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Chapter 1

Matroid theory

We follow the combinatorialists' tradition and use the notation $[m] = \{1, 2, \ldots, m\}$.

generalization of the notion of linear independence. Goal of the talk is to reach a decomposition theorem of Seymour, which simplifies a "regular matroid" into simpler matroids.

Exercise 1 *Let A be an $m \times n$ matrix, columns $[a_1, a_2, \ldots, a_n]$. Let I_1, I_2 be a collection of linearly independent column indeces of A such that $|I_1| < |I_2|$ (strictly less than). Show that there exists some column index $e \in I_2 \setminus I_1$ such that $I_1 \cup \{e\}$ is a set of linearly independent columns of A.*

Proof 1 *Space spanned by I_1 has dimension $|I_1|$. Space spanned by I_2 has dimension $|I_2|$. Hence, there must be at least one vector in I_2 which is independent of I_1. If this is not the case, then the dimension of I_2 will be equal to the dimension of I_1.*

So we pick the index of that vector in I_2 that is linearly independent of I_1.

Definition 1 *Let E be a finite set. Let $\mathcal{I} \subset \mathcal{P}(E)$ such that:*

- $\emptyset \in \mathcal{I}$
- $\forall I \in \mathcal{I},\ J \subset I \implies J \in \mathcal{I}$ *($\mathcal{I}$ is closed under subsets)*
- *Let $I_1, I_2 \in \mathcal{I}$ such that $|I_1| < |I_2|$. Then there exists an element $e \in I_2 \setminus I_1$ such that $I_1 \cup \{e\} \in \mathcal{I}$*

Then, $(E, \mathcal{I})$ is called as a matroid. $\mathcal{I}$ is called the collection of independent sets. An element of $\mathcal{I}$ is called as an independence set.

Example 1 *For a matrix $A_{m \times n}$ over an arbitrary field. Let E be the set of column indeces. Let $\mathcal{I}$ be all the subset of E such that the subset is linearly independent. Then $(E, \mathcal{I})$ is a matroid. This matroid is typically denoted by $M[A]$, where M is a mnemonic for "matroid".*