

IIIT Theory talks

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Monsoon 2019

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Chapter 1

Matroid theory

We follow the combinatorialists' tradition and use the notation $[m] = \{1, 2, \dots, m\}$.

generalization of the notion of linear independence. Goal of the talk is to reach a decomposition theorem of Seymour, which simplifies a "regular matroid" into simpler matroids.

Exercise 1 Let A be an $m \times n$ matrix, columns $[a_1, a_2, \dots, a_n]$. Let I_1, I_2 be a collection of linearly independent column indices of A such that $|I_1| < |I_2|$ (strictly less than). Show that there exists some column index $e \in I_2 \setminus I_1$ such that $I_1 \cup \{e\}$ is a set of linearly independent columns of A .

Proof 1 Space spanned by I_1 has dimension $|I_1|$. Space spanned by I_2 has dimension $|I_2|$. Hence, there must be at least one vector in I_2 which is independent of I_1 . If this is not the case, then the dimension of I_2 will be equal to the dimension of I_1 .

So we pick the index of that vector in I_2 that is linearly independent of I_1 .

Definition 1 Let E be a finite set. Let $\mathcal{I} \subset \mathcal{P}(E)$ such that:

- $\emptyset \in \mathcal{I}$
- $\forall I \in \mathcal{I}, J \subset I \implies J \in \mathcal{I}$ (\mathcal{I} is closed under subsets)
- Let $I_1, I_2 \in \mathcal{I}$ such that $|I_1| < |I_2|$. Then there exists an element $e \in I_2 \setminus I_1$ such that $I_1 \cup \{e\} \in \mathcal{I}$

Then, (E, \mathcal{I}) is called as a matroid. \mathcal{I} is called the collection of independent sets. An element of \mathcal{I} is called as an independence set.

Example 1 For a matrix $A_{m \times n}$ over an arbitrary field. Let E be the set of column indices. Let \mathcal{I} be all the subset of E such that the subset is linearly independent. Then (E, \mathcal{I}) is a matroid.