Game theory

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Contents

Chapter 1

Introduction

TODO: find out and write about: Extrinsic form representation of a game $\Gamma \equiv \langle N, T, Z, o, A, s, u_i, \mathcal{H} \rangle$ where N is number of players, T is game tree, Z is leaves, o is owner function, A is actions, s is transition, u_i is player function, $\mathcal{H}$ is information sets. An information set contains equivalence classes of states that the player cannot distinguish between. There can be ambiguity due to missing information. Perfect information game is one where all information sets are singleton. For example, chess is perfect information. Example of partial information is card games.

Next, we look at strategies. A strategy is a computable function by which each player selects their actions.

Game tree for matching coins with observation: $S_A : \{1\} \to \{H, T\}$. $S_B : \{2, 3\} \to \{H, T\}$.

Game tree for matching coins without observation: $S_A : \{1\} \to \{H, T\}$. $S_B : \{\{2, 3\}\} \to \{H, T\}$.

We let $S \equiv S_1 \times S_2 \times \cdots \times S_n$. This a set containing all possible strategies, called as a "strategy profile", $s \in S$. We write s as $s \equiv (s_i, s_{-i})$, where s_{-i} is cute notation for "the rest of the players". For example, if $s \equiv (s_1, s_2, s_3)$ we can notate $s = (s_2, s_{-2}) = (s_2, s_1, s_3)$.

We will currently focus on pure strategies, where we have a deterministic function per strategy.

1.1 Normal form games

Another representation for games is called as strategic form / matrix form / normal form games. Here, a game $\Gamma = \langle N, (S_i)_{i \in N}, (u_i : S \to \mathbb{R})_{i \in N} \rangle$. N is the number of players, S_i are strategies for each player, u_i are the utility functions / payoffs for each player. u_i maps each strategy profile s how worth it it is for player i if the game proceeds with strategy profile s.

1.1.1 Normal form for matching coins with observation

$$
\begin{aligned}
& S_a \equiv \{S_a^1(H), S_a^2(T)\} \\
& S_b \equiv \{S_b^1(HH), S_b^2(HT), S_b^3(TH), S_b^4(TT)\} \\
& u_a : S \to \mathbb{R} \\
& u_a((s_a^1, s_b^1)) = +10 \qquad (H, HH) \\
& u_a((s_a^1, s_b^2)) = +10 \qquad (H, HT) \\
& u_a((s_a^1, s_b^3)) = -10 \qquad (H, TH) \\
& u_a((s_a^1, s_b^4)) = -10 \qquad (H, TT) \\
& u_a((s_a^2, s_b^1)) = -10 \qquad (T, HH) \\
& u_a((s_a^2, s_b^2)) = +10 \qquad (T, HT) \\
& u_a((s_a^2, s_b^3)) = -10 \qquad (T, TH) \\
& u_a((s_a^2, s_b^4)) = +10 \qquad (T, TT)
\end{aligned}
$$

Alternate representation of same game:

$s_b \to$	HH	HT	TH	TT
H	(10, -10)	(10, -10)	(-10, +10)	(-10, +10)
T	(-10, 10)	(10, -10)	(-10, 10)	(10, -10)

1.1.2 Normal form for matching coins without observation

$s_a \to$	H	T
H	(10, -10)	(-10, 10)
T	(-10, 10)	(10, -10)

1.1.3 Normal form for prisoners dilemma

C for confess, NC for not confess.

	C	NC
C	(-5, -5)	(-1, -10)
NC	(-10, -1)	(-2, -2)

Chapter 2

Game analysis

Rationality implies that each player is motivated to maximise his own payoff. Intelligent implies that player can take into account all available information. An intelligent and rational player implies that every player will attempt to maximise their utility.

2.0.1 Common knowledge and puzzles about common knowledge

Definition 1 Common knowledge — *Player knows it. Every player knows that every player knows it. Every player knows that every player knows that every player knows it.* $\forall k \in \mathbb{N}$, (*Every player knows that*)^k *every player knows it.*

If we have an island with two water streams and all humans and intelligent, rational and cannot speak. They have a rule that says that if a person has a blue mark on their forehead, the drink water from a stream farther away. One day, a visitor, who knows the above fact, shouts "why is a person with a blue mark drinking water here?" The next day, no one comes to the stream. What changed? The only difference before and after is that it is now common knowledge that there is one person with a blue mark drinking water at the stream. This

Some one imagined two positive whole numbers $1 \leqslant a, b \leqslant 20$. He tells the sum of these two numbers to mathematician A, the product of these numbers to mathematician B. A tells B that there is no way for B to know the sum. Then B exclaims "But I know the sum now!", to which A exclaims "and now I know the product".

2.0.2 Strongly dominated strategy

Given a game $\Gamma \equiv \langle N, (S_i), (u_i) \rangle$, a strategy $s_i \in S_i$ is said to be strongly dominated by a strategy $s_i' \in S_i$ if:

$$u_i(s_i, s_{-1}) < u_i(s_i', s_{-i}) \ \forall s_{-i} \in S_{-i}$$

2.0.3 Strongly dominant strategy

Given a game $\Gamma \equiv \langle N, (S_i), (u_i) \rangle$, a strategy s_i^* is said to be strongly dominant if it strongly dominates every other strategy $s_i \in S_i$.

$$\forall s_i \in S_i, s_i \neq s_i^\star \implies u_i(s_i, s_{-1}) < u_i(s_i^\star, s_{-i}) \; \forall s_{-i} \in S_{-i},$$

Note that to analyze strongly dominated and strongly dominant strategies, we only need u_i, the utility of the ith player. Hence, to analyze dominance of strategies, we can get away with writing the utility of just a single player.

	L	R
A	6, -	7, -
B	5, -	6, -
C	4, -	6, -

L, R are moves of the player. A, B, C are stratgies with utilities filled in.

A strongly dominate C, A strongly dominates B. B does *not* strongly dominate C, since on the R action, we have 6 for both B and C. Hence, A is the strongly dominant strategy. Note that there need not always exist a strongly dominant strategy:

	L	R
A	6, -	7, -
B	7, -	6, -

Neither A nor B are strictly better than the other.

2.0.4 Weakly dominated strategy

Given a game $\Gamma \equiv \langle N, (S_i), (u_i) \rangle$, a strategy $s_i \in S_i$ is said to be weakly dominated by a strategy $s_i' \in S_i$ if:

$$u_i(s_i, s_{-1}) \leqslant u_i(s_i', s_{-i}) \; \forall s_{-i} \in S_{-i}$$

with strict inequality for at least one s_{-i}.

2.0.5 Weakly dominant strategy

Given a game $\Gamma \equiv \langle N, (S_i), (u_i) \rangle$, a strategy $s_i^\star$ is said to be weakly dominant if it weakly dominates every other strategy $s_i \in S_i$.

$$\forall s_i \in S_i, s_i \neq s_i^\star \implies u_i(s_i, s_{-1}) \leqslant u_i(s_i^\star, s_{-i}) \; \forall s_{-i} \in S_{-i},$$

with strict inequality for at least one s_{-i}.

Once again, a weakly dominant strategy need not always exist:

	L	R
A	6, -	7, -
B	7, -	6, -

2.0.6 Very weakly dominated strategy

Given a game $\Gamma \equiv \langle N, (S_i), (u_i) \rangle$, a strategy $s_i \in S_i$ is said to be very weakly dominated by a strategy $s_i' \in S_i$ if:

$$u_i(s_i, s_{-1}) \leqslant u_i(s_i', s_{-i}) \ \forall s_{-i} \in S_{-i}$$

Note that we do not have the strict inequality requirement anymore. This is now a true partial order.

2.0.7 Example of strong dominance in prisoners dilemma

	C	NC
C	(-5, -5)	(-1, -10)
NC	(-10, -1)	(-2, -2)

Here, C is the strongly dominant strategy for both players.

2.0.8 Another game

	a	b	c
A	5	5	5
B	4	5	5
C	4	4	4

There does not exist a strongly dominant strategy. A is weakly dominant. C is weakly dominated by both A, B.

2.0.9 Can there exist two weakly dominant strategies?

No there cannot. If A is a weakly dominant strategies, then assume $A[i] > B[i]$.If $A[i] > B[i]$, then B cannot weakly dominate A, since for B to dominate A we need $B[j] \geqslant A[j] \ \forall j$, but $B[i] < A[i]$.

2.0.10 Strongly (Weakly) Dominant Strategy Equilibrium

A strategy profile $(s_1^\star, s_2^\star, \dots, s_n^\star)$ is called as a strongly dominant strategy equilibrium of the game $\Gamma \equiv \langle N, (S_i), (U_i) \rangle$ iff the strategy $s_i^\star$ is a strongly dominating strategy for player i.

Recall the example of Prisoners dilemma —

	C	NC
C	(-5, -5)	(-1, -10)
NC	(-10, -1)	(-2, -2)

Chapter 3

Lecture 2

Missed! write this down. TODO!

3.1 Review of lecture 2

Two player zero sum games. $N = \{1, 2\}$. $u_1 = -u_2$. Also called matrix games since the game can be represented as a matrix. a_{ij} is the utility for row player playing row i, column player is playing column j.

We showed that we are not guaranteed a dominant strategy. Hence, we need to look at other types of equilibria.

Here, we define saddle points.

Chapter 4

Equilibrium in two-player zero-sum games

4.1 Saddle points

Given a matrix A, (i^*, j^*) is a saddle point iff:

$$
\begin{aligned}
a_{i^\star j^\star} &\leqslant a_{i^\star K} && \forall K = 1, 2, \ldots, n \\
a_{i^\star j^\star} &\geqslant a_{L j^\star} && \forall L = 1, 2, \ldots, m
\end{aligned}
$$

That is, $a_{i^\star j^\star}$ is the maximum in the column $j^\star$, minimum in the column $i^\star$.

Let's now analyze the setting and try to deduce when a saddle point exists from the entries of the matrix. For this, we define:

$$
\begin{aligned}
u_R &\equiv \max_i \min_j a_{ij} \\
u_c &\equiv \min_j \max_i a_{ij}
\end{aligned}
$$

Lemma 2 *If there are multiple saddle points, (i_1, j_1) and (i_2, j_2), then $a_{i_1 j_1} = a_{i_2 j_2}$*

Proof 3 *TODO*

Lemma 4 $u_r \leqslant u_c$

Proof 5 *TODO*

Lemma 6 $u_r = u_c \implies$ *saddle point exists for* A*.*

Proof 7 *TODO*

Lemma 8 $u_r = u_c \iff$ *saddle point exists for* A*.*

Proof 9 *TODO*

4.1.1 Analysis of saddle points

If a saddle points exists, then the row player is maximising her minimum assured gain. The column player is minimising her worst loss.

If such a saddle point exists, it is called as an equilibrium. The strategy that achieves this is the **mini-max** strategy.

Single saddle point

A ↓ B →	L	M	R
T	1	2	1
M	0	-1	2
B	-1	0	-2

$(u_r \equiv 2, u_c \equiv 2)$.

Multiple saddle points

	L	M	R
T	0	2	0
M	-1	5	2
B	0	5	0

There are 4 saddle points: $(T, L), (T, R), (B, L), (B, R)$.

Matching coins — no saddle point

	H	T
H	10	-10
T	-10	10

$(u_R \equiv -10, u_c \equiv 10)$ — no saddle point!

Here, notice that no deterministic strategy can win. What we should do is to use a randomized strategy where we play H or T with equal probability.

4.2 Mixed strategies

As the matching coins example shows, we need to use a randomized strategy so we are not exploited. What we can do is to maximise expected utility:

The expected utility of the row player is going to be $\mathbb{P}[H,H] - \mathbb{P}[H,T] - \mathbb{P}[T,H] + \mathbb{P}[T,T]$.

The mixed strategy space consists of probability distributions over pure strategies.

$$\Delta(S) \equiv \left\{ (p_1, p_2, \dots p_k) \mid |S| = k \quad p_i \geqslant 0 \quad \sum_i p_i = 1 \right\}$$

We denote mixed strategies with $\sigma_i \in \Delta(S_i)$, and mixed strategy profiles as $\sigma \equiv (\sigma_1, \sigma_2, \dots \sigma_n) \in \prod_i \Delta(S_i)$

We calculate the expected utility as:

$$U_i(\sigma_i, \sigma_{-i}) \equiv \sum_{s_{-1} \in S_{-i}} p_{i_1} p(s_{-i}) u(a_{i_1}, s_{-i}) + p_{i_2} p(s_{-i}) u(a_{i_2}, s_{-i}) + \dots + p_{i_k} p(s_{-i}) u(a_{i_k}, s_{-i})$$

For two-player zero-sum games, we refer to mixed strategies as $p \equiv (p_1, \dots, p_m)$, $q \equiv (q_1, \dots q_n)^T$. This leads us to **Utility theory**.

4.3 Axiomatic description of utility theory

Let X be the set of outcomes. $\geqslant$ (TODO: curly, need to find this) be the preference of the player over the set of outcomes. We define $(x_1 \sim x_2 \equiv x_1 \geqslant x_2 \wedge x_2 \geqslant x_1)$.

TODO: find formal definition of a lottery.

- Completeness: every pair of outcomes is ranked.
- Transitivity: $x_1 \geqslant x_2 \wedge x_2 \geqslant x_3 \implies x_1 \geqslant x_3$.
- Substitutability: if $x_1 \sim x_2$, then any lottery in which x_1 is substituted by x_2 is equally preferred.
- Decomposability: If two lotteries assign the same probability to each outcome, then the player is indifferent between these two lotteries.
- Monotonicity: If $x_1 > x_2$ and $p > q$, then $[x_1 : p, x_2 : 1-p] \geqslant [x_1 : q, x_2 : 1-q]$.
- Continuity: If $x_1 > x_2 > x_3$, then there exists $p \in [0,1]$ such that $x_2 \sim [x_1 : p, x_3 : 1-p]$.

Theorem 10 *Von Neumann and Morgenstern: Given a set of outcomes X has a preference relation on X that satisfies the above axioms, there exists a utility function $u : X \to [0,1]$ such that:*

- $u(x_1) \geqslant u(x_2) \iff x_1 \geqslant x_2$
- $U([x_1 : p_1, x_2 : p_2, \ldots x_m : p_m]) = \sum_{j=1}^{m} p_j u(x_j)$