

# Topics in Physics - C. Mukku

Siddharth Bhat



# Contents

<b>1</b>	<b>Tensor algebra preliminaries</b>	<b>5</b>
1.1	Raising and lowering of two indeces simeltaneously . . . . .	5
<b>2</b>	<b>Lagrangian, Hamiltonian mechanics</b>	<b>7</b>
2.1	Lagrangian . . . . .	7
2.2	Variational principle . . . . .	7
<b>3</b>	<b>Functional calculus</b>	<b>9</b>
3.1	Functional Derivative - take 1 . . . . .	9
3.2	Functional Derivative as taught in class . . . . .	10
3.3	Common functional derivatives . . . . .	10
3.3.1	Derivative of $F[\phi] \equiv \int (\frac{\partial \phi}{\partial y})^2$ . . . . .	10
3.4	Deriving E-L from functional magic . . . . .	10
3.5	Weird things in Functional Analysis as taught in class . . . . .	10
<b>4</b>	<b>Maxwell's equations in Minkowski space</b>	<b>11</b>
4.1	Constructing $F$ , or Tensorifying Maxwell's equations . . . . .	11
4.2	Expressing $B$ , $E$ in terms of $F$ . . . . .	12
4.3	Rewriting Maxwell's equations in terms of $F$ . . . . .	14
4.3.1	Combining (1) $\nabla E = \frac{\rho}{\epsilon_0}$ , (4) $\nabla \times B = \mu_0 J + \frac{\partial E}{\partial t}$ . . . . .	14
4.3.2	Combining (2) $\nabla \times E = -\frac{\partial B}{\partial t}$ , (3) $\nabla B = 0$ . . . . .	16



# Chapter 1

## Tensor algebra preliminaries

### 1.1 Raising and lowering of two indices simultaneously

Note that

$$a_i b^i = (a^j g_{ij}) b^i = (a^j g_{ij}) (b_k g^{ki})$$

In Minkowski space, we know that  $g^{ij} = 0$  if  $i \neq j$ , and  $(g^{ii} g_{ii})^2 = 1$ , so we can rewrite the above expression as:

$$\begin{aligned} (a^j g_{ij}) (b_k g^{ki}) &= \\ (a^i g_{ii}) (b_i g^{ii}) &= \\ a^i b_i \end{aligned}$$



## Chapter 2

# Lagrangian, Hamiltonian mechanics

Mechanics in terms of generalized coords.

### 2.1 Lagrangian

Define a functional.  $L$  over the config. space of partibles  $q^i, \dot{q}^i$ .  $L = L(q^i, \dot{q}^i)$ . We have an explicit dependence on  $t$ .

$$L = KE - PE$$

Assuming a 1-particle system of unit mass,

$$L = \frac{1}{2}\dot{q}^2 - V(q)$$

Assuming an n-particle system of unit mass,

$$L = \sum_i \frac{1}{2}\dot{q}^{i2} - V(q^i)$$

### 2.2 Variational principle

Take a minimum path from  $A$  to  $B$ . Now notice that the path that is slightly different from this path will have some delta from the minimum.

Action

$$S(t_0, t_1) = \int_{t_0}^{t_1} L dt = \int_{t_0}^{t_1} L(q^i, \dot{q}^i) dt$$

. Least action:  $\delta S = 0$





## Chapter 3

# Functional calculus

this chapter develops a completely handwavy physics version of functional analysis.

**Definition 1** A *functional*  $F$  is a function:  $F : (\mathbb{R} \rightarrow \mathbb{R}) \rightarrow \mathbb{R}$

**Notation 1** Evaluation of a functional  $F$  with respect to  $f$  is denoted by  $F[f]$ .

### 3.1 Functional Derivative - take 1

Consider a functional  $F : (\mathbb{R} \rightarrow \mathbb{R}) \rightarrow \mathbb{R}$ , a function  $f : \mathbb{R} \rightarrow \mathbb{R}$ , and a "test function"  $\phi : \mathbb{R} \rightarrow \mathbb{R}$ .

Consider a functional  $F$ . We only define the derivative of a functional  $F$  with respect to a function  $f$  by what happens under an integral sign as follows:

$$\int \frac{\delta F}{\delta f}(x) \phi(x) dx = \lim_{\epsilon \rightarrow 0} \frac{F[f + \epsilon \phi] - F[f]}{\epsilon}$$

Now, we can define a small variation in  $F$  as:

$$\begin{aligned} \delta F &: (\mathbb{R} \rightarrow \mathbb{R}) \times (\mathbb{R} \rightarrow \mathbb{R}) \rightarrow \mathbb{R} \\ \delta F(f, \phi) &\equiv \int \frac{\delta F}{\delta f}(x) \phi(x) dx \end{aligned}$$

Intuitively,  $\delta F$  tells us the variation of the function  $f$  along a test function  $\phi$ . So, it encapsulates some kind of "directional derivative".

So, we can look at  $\frac{\delta F}{\delta f}$  as a functional as follows:

$$\begin{aligned} \frac{\delta F}{\delta f} &: (\mathbb{R} \rightarrow \mathbb{R}) \rightarrow \mathbb{R} \\ \frac{\delta F}{\delta f}(\phi) &= \delta F(f, \phi) \end{aligned}$$

Wehre  $\frac{\delta F}{\delta f}$  allows us to "test" the change of  $F$  with respect to  $f$  along a given "direction"  $\phi$ .

### 3.2 Functional Derivative as taught in class

Substitute  $\phi = \delta(x - p)$ . Now, the quantity:

$$\frac{\delta F}{\delta f} \phi(x) = \delta F(f, \delta(x - p))$$

Rewriting  $\delta F$  by sticking it under an integral:

$$\int \frac{\delta F}{\delta f}(x) \delta(x - p) dx = \lim_{\epsilon \rightarrow 0} \frac{F[f + \epsilon \delta(x - p)] - F[f]}{\epsilon}$$

$$\left. \frac{\delta F}{\delta f} \right|_p = \lim_{\epsilon \rightarrow 0} \frac{F[f + \epsilon \delta(x - p)] - F[f]}{\epsilon}$$

That is, we can start talking about "derivative of the functional  $F$  with respect to a function  $f$  at a point  $p$ " as long as we only test the functional  $F$  against  $\delta$ -functions.

So, we can alternatively define this quantity as:

$$\left. \frac{\delta F}{\delta f} \right|_p \equiv \lim_{\epsilon \rightarrow 0} \frac{F[f + \epsilon \delta(x - p)] - F[f]}{\epsilon}$$

While this does not "look like a functional", it actually is, if we mentally replace:

$$p \rightarrow \int -\delta(x - p) dx$$

This is how mukku got that expression.

### 3.3 Common functional derivatives

#### 3.3.1 Derivative of $F[\phi] \equiv \int (\frac{\partial \phi}{\partial y})^2$

$$\left. \frac{\delta F}{\delta f} \right|_p = \int (\frac{\partial \phi}{\partial y})^2$$

### 3.4 Deriving E-L from functional magic

### 3.5 Weird things in Functional Analysis as taught in class

Consider the functional

$$J[f] = \int g[f'] dy:$$

since  $g$  is a functional, it has a type  $g : (\mathbb{R} \rightarrow \mathbb{R}) \rightarrow \mathbb{R}$ . So, our integrand must be some function  $df$ , and not some **space component**  $dy$ . **I don't understand what the definition of  $J$  means.**

## Chapter 4

# Maxwell's equations in Minkowski space

Let us first review Maxwell's equations:

$$\nabla \cdot E = \frac{\rho}{\epsilon_0} \text{ (Electric charges produce fields)}$$

$$\nabla \cdot B = 0 \text{ (Only magnetic dipoles exist)}$$

$$\nabla \times E = -\frac{\partial B}{\partial t} \text{ (Lenz Law / Faraday's law - time varying magnetic field induces current that opposes it)}$$

$$\nabla \times B = \mu_0 \left( J + \epsilon_0 \frac{\partial E}{\partial t} \right) \text{ (Ampere's law + fudge factor)}$$

### 4.1 Constructing $F$ , or Tensorifying Maxwell's equations

Begin with the equation that  $\nabla \cdot B = 0$ . This tells that  $B$  can be written as the curl of some other field:

$$\boxed{B \equiv \nabla \times A} \tag{4.1}$$

Expanding this equation of  $B$  in tensorial form:

$$\boxed{B^i = \mathcal{E}^{ijk} \partial_j A^k} \tag{4.2}$$

Next, take  $\nabla \times E = -\frac{\partial B}{\partial t}$ .

$$\nabla \times E = -\frac{\partial B}{\partial t} = \frac{\partial(\nabla \times A)}{\partial t} = \nabla \times \frac{\partial A}{\partial t}$$

$$\nabla \times \left( E + \frac{\partial A}{\partial t} \right) = 0$$

writing this as the divergence of some field  $\phi$  scaled by  $\alpha : \mathbb{R}$

$$E + \frac{\partial A}{\partial t} = \alpha(\nabla \cdot \phi)$$

$$E = \alpha \nabla \cdot \phi - \frac{\partial A}{\partial t}$$

Since electrostatics is time-independent, we choose to think of  $\alpha = -1$ , so we can interpret  $\phi$  as the potential.

$$E^i = -\frac{\partial \phi}{\partial x^k} g^{ik} - \frac{\partial A^i}{\partial t} \quad (4.3)$$

A slight reformulation (since we know that in Minkowski space,  $\partial_t = \partial_0$ ) we get the equation:

$$\boxed{E^i = -g^{ik} \partial_k \phi - \partial_0 A^i} \quad (4.4)$$

We get the metric  $g^{ik}$  involved to raise the covariant  $\frac{\partial \phi}{\partial x^k}$  into the contravariant  $E^i$ .

**(Sid question:** how does one justify switching  $\nabla \times$  and  $\partial$ ? It feels like some algebra)

**Here be magic!** We define A new rank-2 tensor in Minkowski space-time, called  $F$  (for Faraday),

$$\boxed{F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu} \quad (4.5)$$

**(Sid question:** why is this object  $F_{\mu\nu}$  covariant? What does this *mean*?)

**Lemma 1**  $F_{\mu\nu}$  is antisymmetric.

**Lemma 2**  $F_{\mu\nu}$  has 6 degrees of freedom

*Proof.* Number of degrees of freedom of  $F$ :

$$\frac{4^2 \text{ (total)} - 4 \text{ (diagonal)}}{2 \text{ (anti-symmetry)}} = 6$$

Notice that  $F$  is a 1-form!

## 4.2 Expressing $B$ , $E$ in terms of $F$

We now wish to re-express  $B^{ij}$  and  $E^{ij}$  in terms of  $F$ , so that this  $F$  captures all of maxwell's equations.

$$B^i = \mathcal{E}^{ijk} \partial_j A^k = \mathcal{E}^{ikj} \partial_k A^j \quad \text{by } k, j \text{ being free variables}$$

$$B^i = \frac{1}{2} \left( \mathcal{E}^{ijk} \partial_j A^k + \mathcal{E}^{ikj} \partial_k A^j \right)$$

Substituting  $\partial_j A_k - \partial_k A_j = F_{jk}$ ,

$$B^i = \frac{1}{2} \mathcal{E}^{ijk} F_{jk}$$

So,  $B$  in terms of  $F$  is:

$$\boxed{B^i = \frac{1}{2} \mathcal{E}^{ijk} F_{jk}} \quad (4.6)$$

Similarly, we wish to write  $E$  in terms of  $F$ . The algebra is as follows:

$$E^i = -g^{ik} \partial_k \phi - \partial_0 A^i$$

$$E^i = -g^{ik} \partial_k \phi - \partial_0 g^{ik} A_k \quad \text{Is this allowed? Am I always allowed to insert the } g_{ik}?$$

$$E^i = -g^{ik} (\partial_k \phi + \partial_0 A_k)$$

Since  $k = \{1, 2, 3\}$  ( $k$  is spacelike coordinates), and we would like to relate  $\phi$  with  $A$  (to unify  $E$ ), we **set**:

$$\boxed{A_0 \equiv -\phi} \quad (4.7)$$

Continuing the derivation,

$$E^i = -g^{ik} (\partial_k (-A_0) + \partial_0 A_k)$$

$$E^i = -g^{ik} (\partial_0 A_k - \partial_k A_0)$$

$$E^i = -g^{ik} F_{0k}$$

So, finally, the relation is:

$$\boxed{E^i = -g^{ik} F_{0k}} \quad (4.8)$$

Let us reconsider what we believed  $E$  to be. We had:

$$E = -\nabla \phi - \frac{\partial A}{\partial t}$$

However, comparing dimensions, space derivative of  $\phi$  = time derivative of  $A$ . This means that  $\frac{\delta \phi}{\delta x} = \frac{\delta A}{\delta y}$ , and so  $\frac{\delta \phi}{\frac{\delta x}{\delta t}} = \delta A$ . We arbitrarily pick  $c$  as our measuring stick for  $\frac{\delta x}{\delta t}$ . Also, in minkowski space, our measuring stick is actually  $(ct, x, y, z)$ , so  $\partial_0 = \partial_{ct}$ . So, when we write the equation for  $E$ , we should actually write

$$E = c \left( -\frac{\nabla \phi}{c} - \frac{\partial A}{\partial ct} \right)$$

which becomes:

$$\boxed{E^i = c F^{i0}} \quad (4.9)$$

### 4.3 Rewriting Maxwell's equations in terms of $F$

Now that we have constructed the Faraday tensor  $F$ , we wish to re-express Maxwell's equations in terms of this object. This will give us a compact form of the laws which are invariant under coordinate transforms.

#### 4.3.1 Combining (1) $\nabla E = \frac{\rho}{\epsilon_0}$ , (4) $\nabla \times B = \mu_0 J + \frac{\partial E}{\partial t}$

##### 1. Using (4) $\nabla \times B = \mu_0 J + \frac{\partial E}{\partial t}$

We consider the 4th Maxwell equation:

$$\begin{aligned}\nabla \times B &= \mu_0 J + \epsilon_0 \mu_0 \frac{\partial E}{\partial t} \\ \nabla \times B &= \mu_0 J + \frac{1}{c^2} \frac{\partial E}{\partial t} \\ \text{Converting to indices,} \\ (\nabla \times B)^i &= \mu_0 J^i + \frac{1}{c} \frac{\partial E^i}{\partial ct} && \text{(From } \partial_{ct} = \frac{1}{c} \partial_t) \\ &= \mu_0 J^i + \frac{1}{c} \frac{\partial E^i}{\partial X^0} \\ &= \mu_0 J^i + \frac{\partial F^{i0}}{\partial X^0} && \text{(From } E^i = cF^{i0}) \\ &= \mu_0 J^i + \partial_0 F^{i0}\end{aligned}$$

Now, we start to simplify the LHS,  $\nabla \times B$ :

$$\begin{aligned}(\nabla \times B)^i &= \mathcal{E}^{ijk} \partial_j B_k \\ \text{Since } B^k &= \frac{1}{2} \mathcal{E}^{kmn} F_{mn}, \\ B_k &= \frac{1}{2} \mathcal{E}_{kmn} F^{mn}, && \text{(TODO: this is scam)} \\ (\nabla \times B)^i &= \mathcal{E}^{ijk} \partial_j \left( \frac{1}{2} \mathcal{E}_{kmn} F^{mn} \right) = \frac{1}{2} \mathcal{E}^{ijk} \mathcal{E}_{kmn} \partial_j F^{mn}\end{aligned}$$

**Aside:** We need to know how to evaluate  $\mathcal{E}^{ijk} \mathcal{E}_{kmn}$  :

$$\mathcal{E}_{i_1, i_2, \dots, i_n} \mathcal{E}_{j_1, j_2, \dots, j_n} = \det \left\{ \begin{array}{cccc} \delta_{i_1 j_1} & \delta_{i_1 j_2} & \dots & \delta_{i_1 j_n} \\ \delta_{i_2 j_1} & \delta_{i_2 j_2} & \dots & \delta_{i_2 j_n} \\ \vdots & \vdots & \ddots & \vdots \\ \delta_{i_n j_1} & \delta_{i_n j_2} & \dots & \delta_{i_n j_n} \end{array} \right\}$$

$$\mathcal{E}^{ijk} \mathcal{E}^{imn} = -1(\delta_j^m \delta_k^n - \delta_j^n \delta_k^m)$$

He argued that we get a  $-1$  factor here due to the presence of the metric. I'm not fully convinced, but I can handwave this using the magic words "tensor density".

Plugging both equations together,

$$\begin{aligned}
\frac{1}{2}\mathcal{E}^{ijk}\mathcal{E}_{kmn}\partial_j F^{mn} &= \mu_0 J^i + \partial_0 F^{i0} \\
(\text{Since } kij \text{ is an even permutation of } ijk): \\
\frac{1}{2}\mathcal{E}^{kij}\mathcal{E}_{kmn}\partial_j F^{mn} &= \mu_0 J^i + \partial_0 F^{i0} \\
(\text{Using } \mathcal{E}^{kij}\mathcal{E}^{kmn} = -1(\delta_i^m \delta_j^n - \delta_i^n \delta_j^m)): \\
\frac{1}{2}[-(\delta_m^i \delta_n^j - \delta_n^i \delta_m^j)]\partial_j F^{mn} &= \mu_0 J^i + \partial_0 F^{i0} \\
-\frac{1}{2}[\partial_n F^{in} - \partial_m F^{mi}] &= \mu_0 J^i + \partial_0 F^{i0} \\
(F \text{ is anti-symmetric, so rewriting } \partial_m F^{mi} = -\partial_m F^{im}): \\
-\frac{1}{2}[\partial_n F^{in} + \partial_m F^{im}] &= \mu_0 J^i + \partial_0 F^{i0} \\
(\text{Replacing } \partial_m F^{im} \equiv \partial_n F^{in} \text{ since } m \text{ is free}): \\
-[\partial_m F^{im}] &= \mu_0 J^i + \partial_0 F^{i0} \\
\mu_0 J^i + \partial_0 F^{i0} + \partial_m F^{im} &= 0 \\
\mu_0 J^i + \partial_\mu F^{i\mu} &= 0 \quad (\mu = \{0, 1, 2, 3\})
\end{aligned}$$

This gives us a continuity-style equation, linking the current density  $J$  to the rate of change of  $F$ .

$$\boxed{\mu_0 J^i + \partial_\mu F^{i\mu} = 0} \quad (\mu = \{0, 1, 2, 3\})$$

**Second part, using 1st equation**

$$\begin{aligned}
\nabla E &= \frac{\rho}{\epsilon_0} \\
\partial_i E^i &= \frac{\rho}{\epsilon_0} \\
(\text{Substituting } E^i = cF^{i0}): \\
c\partial_i F^{i0} &= \frac{\rho}{\epsilon_0} = \frac{\rho\mu_0}{\mu_0\epsilon_0} = \rho c^2 \\
\partial_i F^{i0} &= \mu_0 c\rho \\
(\text{Since } F \text{ is anti-symmetric, } F^{00} = 0, \text{ Hence}): \\
\partial_0 F^{00} + \partial_i F^{i0} &= \mu_0 c\rho \\
\partial_\mu F^{\mu 0} &= \mu_0 c\rho
\end{aligned}$$

$$\boxed{\partial_\mu F^{\mu 0} = \mu_0 c\rho} \quad (4.10)$$

**Combining part 1 and part 2:**

$$\begin{aligned}\mu_0 J^i + \partial_\mu F^{i\mu} &= 0 & (\text{From } B) \\ \partial_\mu F^{i\mu} &= -\mu_0 J^i \partial_\mu F^{\mu 0} = \mu_0 c \rho \\ \partial_\mu F^{0\mu} &= -\mu_0 c \rho\end{aligned}$$

To combine these equations, **we set:**

$$\boxed{J^0 \equiv c\rho} \quad (4.11)$$

We arrive at the unified equation:

$$\partial_\mu F^{\nu\mu} = -\mu_0 J^\nu$$

Choose units such that  $c = \frac{h}{2\pi} = G_n = 1$ , which gives us:

$$\begin{aligned}\partial_\mu F^{\nu\mu} &= -J^\nu \\ F &\text{ is antisymmetric, so flipping indices} \\ \partial_\mu F^{\mu\nu} &= J^\nu\end{aligned}$$

$$\boxed{\partial_\mu F^{\mu\nu} = J^\nu} \quad (4.12)$$

Note that this is **Ampere's law!**

**4.3.2 Combining (2)  $\nabla \times E = -\frac{\partial B}{\partial t}$ , (3)  $\nabla B = 0$**

$$(\nabla \times E)^i = \mathcal{E}^{ijk} \partial_j E_k$$