Topics in Physics - C. Mukku

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Contents

Chapter 1

Tensor algebra preliminaries

1.1 Raising and lowering of two indeces simeltaneously

Note that

$$a_i b^i = (a^j g_{ij}) b^i = (a^j g_{ij})(b_k g^{ki})$$

In minkowski space, we know that $g^{ij} = 0$ if $i \neq j$, and $(g^{ii} g_{ii})^2 = 1$, so we can rewrite the above expression as:

$$\begin{aligned} (a^j g_{ij})(b_k g^{ki}) &= \\ (a^i g_{ii})(b_i g^{ii}) &= \\ a^i b_i & \end{aligned}$$

Chapter 2

Lagrangian, Hamiltonian mechanics

Mechanics in terms of generalized coords.

2.1 Lagrangian

Define a functional. L over the config. space of partibles q^i, $qdot^i$. $L = L(q^i, qdot^i)$. We have an explicit dependence on t.

$L = KE - PE$

Assuming a 1-particle system of unit mass,

$$L = \frac{1}{2}\dot{q}^2 - V(q)$$

Assuming an n-particle system of unit mass,

$$L = \sum_i \frac{1}{2} qdot^{i2} - V(q^i)$$

2.2 Variational principle

Take a minimum path from A to B. Now notice that the path that is slightly different from this path will have some delta from the minimum.

Action

$$S(t0, t1) = \int L \mathrm{d}t = \int_{t0}^{t1} L(q^i, qdot^i) \mathrm{d}t$$

. Least action: $\delta S = 0$

In physics, we try to minimise the action $L = T - V$ where T is the Kinetic energy (Travail), and V (Voltage) is the Potential energy.

So, the question is, why does minimising the lagrangian work, and how do we get the euler-lagrange equations from this?

Chapter 3

Functional calculus

this chapter develops a completely handwavy physics version of functional analysis.

Definition 1 *A **functional** F is a function: $F : (\mathbb{R} \to \mathbb{R}) \to \mathbb{R}$*

Notation 1 *Evaluation of a functional F with respect to f is denoted by $F[f]$.*

3.1 Functional Derivative - take 1

Consider a functional $F : (\mathbb{R} \to \mathbb{R}) \to \mathbb{R}$, a function $f : \mathbb{R} \to \mathbb{R}$, and a "test function" $\phi : \mathbb{R} \to \mathbb{R}$.

Consider a functional F. We only define the derivative of a functional F with respect to a function f by what happens under an integral sign as follows:

$$\int \frac{\delta F}{\delta f}(x)\phi(x)\mathrm{d}x = \lim_{\epsilon \to 0} \frac{F[f + \epsilon\phi] - F[f]}{\epsilon}$$

Now, we can define a small variation in F as:

$$\begin{aligned} \delta F &: (\mathbb{R} \to \mathbb{R}) \times (\mathbb{R} \to \mathbb{R}) \to \mathbb{R} \\ \delta F(f, \phi) &\equiv \int \frac{\delta F}{\delta f}(x)\phi(x)\mathrm{d}x \end{aligned}$$

Intuitively, δF tells us the variation of the function f along a test function ϕ. So, it encapsulates some kind of "directional derivative".

So, we can look at $\frac{\delta F}{\delta f}$ as a functional as follows:

$$\begin{aligned} \frac{\delta F}{\delta f} &: (\mathbb{R} \to \mathbb{R}) \to \mathbb{R} \\ \frac{\delta F}{\delta f}(\phi) &= \delta F(f, \phi) \end{aligned}$$

Wehre $\frac{\delta F}{\delta f}$ allows us to "test" the change of F with respect to f along a given "direction" ϕ.

3.2 Functional Derivative as taught in class

Substitute $\phi = \delta(x-p)$. Now, the quantity:

$$\frac{\delta F}{\delta f}\phi(x) = \delta F(f, \delta(x-p))$$

Rewriting δF by sticking it under an integral:

$$\int \frac{\delta F}{\delta f}(x)\delta(x-p)\mathrm{d}x = \lim_{\epsilon \to 0} \frac{F[f+\epsilon\delta(x-p)] - F[f]}{\epsilon}$$

$$\left.\frac{\delta F}{\delta f}\right|_p = \lim_{\epsilon \to 0} \frac{F[f+\epsilon\delta(x-p)] - F[f]}{\epsilon}$$

That is, we can start talking about "derivative of the functional F with respect to a function f at a point p" as long as we only test the functional F against δ-functions.

So, we can alternatively define this quantity as:

$$\left.\frac{\delta F}{\delta f}\right|_p \equiv \lim_{\epsilon \to 0} \frac{F[f+\epsilon\delta(x-p)] - F[f]}{\epsilon}$$

While this does not "look like a functional", it actually is, if we mentally replace:

$$p \to \int \_\ \delta(x-p)\mathrm{d}x$$

This is how mukku got that expression.

3.3 Common functional derivatives

3.3.1 $F[f] = \int_0^\infty f dx$

$$\frac{\delta F[f]}{\delta f(x_0)} = \lim_{\epsilon \to 0} \frac{\int_0^\infty (f+\epsilon\delta(x-x_0))\mathrm{d}x - \int_0^\infty f\mathrm{d}x}{\epsilon}$$
$$= \int_0^\infty \delta(x-x_0)\mathrm{d}x = 1$$

3.3.2 $F[f] = \int_0^\infty g[f] dx$

This does not actually type-check for me. $g : (\mathbb{R} \to \mathbb{R}) \to reals$, so I don't fully understand what we are "varying" where when we integrate with respect to dx.

So, there's something bizarre here that I don't understand — the integral doesn't really make sense.

3.3.3 Derivative of $F[\phi] \equiv \int (\frac{\partial \phi}{\partial y})^2$

$\frac{\delta F}{\delta f}\Big|_p = \int (\frac{\partial \phi}{\partial y})^2$

3.4 Deriving E-L from functional magic

3.5 Weird things in Functional Analysis as taught in class

Consider the functional

$J[f] = \int g[f']dy$:

since g is a functional, it has a type $g : (\mathbb{R} \to \mathbb{R}) \to \mathbb{R}$. So, our integrand must be some function df, and not some **space component** dy. **I don't understand what the definition of J means**.

Chapter 4

Maxwell's equations in Minkowski space

Let us first review Maxwell's equations:

$\nabla \cdot E = \dfrac{\rho}{\epsilon_0}$ (Electric charges produce fields)
$\nabla \cdot B = 0$ (Only magnetic dipoles exist)
$\nabla \times E = -\dfrac{\partial B}{\partial t}$ (Lenz Law / Faraday's law - time varying magnetic field induces current that opposes it)
$\nabla \times B = \mu_0 \left(J + \epsilon_0 \dfrac{\partial E}{\partial t} \right)$ (Ampere's law + fudge factor)

4.1 Constructing F, or Tensorifying Maxwell's equations

Begin with the equation that $\nabla \cdot B = 0$. This tells that B can be written as the curl of some other field:

$$\boxed{B \equiv \nabla \times A} \tag{4.1}$$

Expanding this equation of B in tensorial form:

$$\boxed{B^i = \mathcal{E}^{ijk} \partial_j A^k} \tag{4.2}$$

Next, take $\nabla \times E = -\frac{\partial B}{\partial t}$.

$$
\begin{aligned}
&\boldsymbol{\nabla} \times E = -\frac{\partial B}{\partial t} = \frac{\partial(\boldsymbol{\nabla} \times A)}{\partial t} = \boldsymbol{\nabla} \times \frac{\partial A}{\partial t} \\
&\boldsymbol{\nabla} \times \left(E + \frac{\partial A}{\partial t}\right) = 0 \\
&\text{writing this as the gradient of some field } \phi \text{ scaled by } \alpha : \mathbb{R} \\
&E + \frac{\partial A}{\partial t} = \alpha(\boldsymbol{\nabla}\phi) \\
&E = \alpha \boldsymbol{\nabla}\phi - \frac{\partial A}{\partial t}
\end{aligned}
$$

Since electrostatics is time-independent, we choose to think of $\alpha = -1$, so we can interpret ϕ as the potential.

$$E^i = -\frac{\partial \phi}{\partial x^k} g^{ik} - \frac{\partial A^i}{\partial t} \tag{4.3}$$

A slight reformulation (since we know that in Minkowski space, $\partial_t = \partial_0$) we get the equation:

$$\boxed{E^i = -g^{ik}\partial_k \phi - \partial_0 A^i} \tag{4.4}$$

We get the metric $g^i k$ involved to raise the covariant $\frac{\partial \phi}{\partial x^k}$ into the contravariant E^i.
(**Sid question:** how does one justify switching $\boldsymbol{\nabla}\times$ and ∂? It feels like some algebra)
Here be magic! We define A new rank-2 tensor in Minkowski space-time, called F (for Faraday),

$$\boxed{F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu} \tag{4.5}$$

(**Sid question:** why is this object $F_{\mu\nu}$ covariant? What does this *mean*?)

Lemma 1 *$F_{\mu\nu}$ is antisymmetric.*

Lemma 2 *$F_{\mu\nu}$ has 6 degrees of freedom*

Proof. Number of degrees of freedom of F:

$$\frac{4^2 \text{ (total)} - 4 \text{ (diagonal)}}{2 \text{ (anti-symmetry)}} = 6$$

Notice that F is a 1-form!

4.2 Expressing B, E in terms of F

We now wish to re-expresss B^{ij} and E^{ij} in terms of F, so that this F captures all of maxwell's equations.

$$B^i = \mathcal{E}^{ijk}\partial_j A^k = \mathcal{E}^{ikj}\partial_k A^j \qquad \text{by } k,\ j \text{ being free variables}$$
$$B^i = \frac{1}{2}\Big(\mathcal{E}^{ijk}\partial_j A^k + \mathcal{E}^{ikj}\partial_k A^j\Big)$$
Substituting $\partial_j A_k - \partial_k A_j = F_{jk}$,
$$B^i = \frac{1}{2}\mathcal{E}^{ijk}F_{jk}$$

So, B in terms of F is:

$$\boxed{B^i = \frac{1}{2}\mathcal{E}^{ijk}F_{jk}} \tag{4.6}$$

Similarly, we wish to write E in terms of F. The algebra is as follows:

$$E^i = -g^{ik}\partial_k\phi - \partial_0 A^i$$
$$E^i = -g^{ik}\partial_k\phi - \partial_0 g^{ik}A_k \qquad \text{Is this allowed? Am I always allowed to insert the } g_{ik}\text{?}$$
$$E^i = -g^{ik}(\partial_k\phi + \partial_0 A_k)$$

Since $k = \{1, 2, 3\}$ (k is spacelike coordinates), and we would like to relate ϕ with A (to unify E), we **set**:

$$\boxed{A_0 \equiv -\phi} \tag{4.7}$$

Continuing the derivation,

$$E^i = -g^{ik}(\partial_k(-A_0) + \partial_0 A_k)$$
$$E^i = -g^{ik}(\partial_0 A_k - \partial_k A_0)$$
$$E^i = -g^{ik}F_{0k}$$

So, finally, the relation is:

$$\boxed{E^i = -g^{ik}F_{0k}} \tag{4.8}$$

Let us reconsider what we believed E to be. We had:

$$E = -\boldsymbol{\nabla}\phi - \frac{\partial A}{\partial t}$$

However, comparing dimensions, space derivative of ϕ = time derivative of A. This means that $\frac{\delta\phi}{\delta x} = \frac{\delta A}{\delta y}$, and so $\frac{\delta\phi}{\frac{\delta x}{\delta t}} = \delta A$. We arbitrarily pick c as our measuring stick for $\frac{\delta x}{\delta t}$. Also, in minkowski space, our measuring stick is actually (ct, x, y, z), so $\partial_0 = \partial_{ct}$ So, when we write the equation for E, we should actually write

$$E = c\bigg(-\frac{\boldsymbol{\nabla}\phi}{c} - \frac{\partial A}{\partial ct}\bigg)$$

which becomes:

$$\boxed{E^i = cF^{i0}} \tag{4.9}$$

4.3 Rewriting Maxwell's equations in terms of F

Now that we have constructed the Faraday tensor F, we wish to re-expresss Maxwell's equations in terms of this object. This will give us a compact form of the laws which are invariant under coordinate transforms.

4.3.1 Combining (1) $\nabla E = \frac{\rho}{\epsilon_0}$, (4) $\nabla \times B = \mu_0 J + \frac{\partial E}{\partial t}$

1. Using (4) $\nabla \times B = \mu_0 J + \frac{\partial E}{\partial t}$

We consider the 4th Maxwell equation:

$$\begin{aligned}
\nabla \times B &= \mu_0 J + \epsilon_0 \mu_0 \frac{\partial E}{\partial t} \\
\nabla \times B &= \mu_0 J + \frac{1}{c^2} \frac{\partial E}{\partial t} \\
&\text{Converting to indices,} \\
(\nabla \times B)^i &= \mu_0 J^i + \frac{1}{c} \frac{\partial E^i}{\partial ct} && \text{(From } \partial_{ct} = \tfrac{1}{c}\partial_t\text{)} \\
&= \mu_0 J^i + \frac{1}{c} \frac{\partial E^i}{\partial X^0} \\
&= \mu_0 J^i + \frac{\partial F^{i0}}{\partial X^0} && \text{(From } E^i = cF^{i0}\text{)} \\
&= \mu_0 J^i + \partial_0 F^{i0}
\end{aligned}$$

Now, we start to simplify the LHS, $\nabla \times B$:

$$\begin{aligned}
&(\nabla \times B)^i = \mathcal{E}^{ijk} \partial_j B_k \\
&\text{Since } B^k = \frac{1}{2} \mathcal{E}^{kmn} F_{mn}, \\
&B_k = \frac{1}{2} \mathcal{E}_{kmn} F^{mn}, && \text{(\textbf{TODO:} this is scam)} \\
&(\nabla \times B)^i = \mathcal{E}^{ijk} \partial_j \left(\frac{1}{2} \mathcal{E}_{kmn} F^{mn} \right) = \frac{1}{2} \mathcal{E}^{ijk} \mathcal{E}_{kmn} \partial_j F^{mn}
\end{aligned}$$

Aside: We need to know how to evaluate $\mathcal{E}^{ijk}\mathcal{E}_{kmn}$:

$$\mathcal{E}_{i_1,i_2,\ldots,i_n} \mathcal{E}_{j_1,j_2,\ldots j_n} = \det \left\{ \begin{vmatrix} \delta_{i_1 j_1} & \delta_{i_1 j_2} & \ldots & \delta_{i_1 j_n} \\ \delta_{i_2 j_1} & \delta_{i_2 j_2} & \ldots & \delta_{i_2 j_n} \\ \vdots & \vdots & \ddots & \vdots \\ \delta_{i_n j_1} & \delta_{i_n j_2} & \ldots & \delta_{i_n j_n} \end{vmatrix} \right\}$$

$\mathcal{E}^{ijk}\mathcal{E}^{imn} = -1(\delta_j^m \delta_k^n - \delta_j^n \delta_k^m)$

He argued that we get a -1 factor here due to the presence of the metric. I'm not fully convinced, but I can handwave this using the magic words "tensor density".

Plugging both equations together,

$$\frac{1}{2}\mathcal{E}^{ijk}\mathcal{E}_{kmn}\partial_j F^{mn} = \mu_0 J^i + \partial_0 F^{i0}$$
(Since kij is an even permutation of ijk):
$$\frac{1}{2}\mathcal{E}^{kij}\mathcal{E}_{kmn}\partial_j F^{mn} = \mu_0 J^i + \partial_0 F^{i0}$$
(Using $\mathcal{E}^{kij}\mathcal{E}^{kmn} = -1(\delta_i^m\delta_j^n - \delta_i^n\delta_j^m)$):
$$\frac{1}{2}\big[-\big(\delta_m^i\delta_n^j - \delta_n^i\delta_m^j\big)\big]\partial_j F^{mn} = \mu_0 J^i + \partial_0 F^{i0}$$
$$-\frac{1}{2}\big[\partial_n F^{in} - \partial_m F^{mi}\big] = \mu_0 J^i + \partial_0 F^{i0}$$
(F is anti-symmetric, so rewriting $\partial_m F^{mi} = -\partial_m F^{im}$):
$$-\frac{1}{2}\big[\partial_n F^{in} + \partial_m F^{im}\big] = \mu_0 J^i + \partial_0 F^{i0}$$
(Replacing $\partial_m F^{im} \equiv \partial_n F^{in}$ since m is free):
$$-\big[\partial_m F^{im}\big] = \mu_0 J^i + \partial_0 F^{i0}$$
$$\mu_0 J^i + \partial_0 F^{i0} + \partial_m F^{im} = 0$$
$$\mu_0 J^i + \partial_\mu F^{i\mu} = 0 \qquad (\mu = \{0, 1, 2, 3\})$$

This gives us a continuity-style equation, linking the current density J to the rate of change of F.

$$\boxed{\mu_0 J^i + \partial_\mu F^{i\mu} = 0} \qquad (\mu = \{0, 1, 2, 3\})$$

Second part, using 1st equation

$$\boldsymbol{\nabla} E = \frac{\rho}{\epsilon_0}$$
$$\partial_i E^i = \frac{\rho}{\epsilon_0}$$
(Substituting $E^i = cF^{i0}$, $c^2 = \frac{1}{\mu_0\epsilon_0}$):
$$c\partial_i F^{i0} = \frac{\rho}{\epsilon_0} = \frac{\rho\mu_0}{\mu_0\epsilon_0} = \rho\mu_0 c^2$$
$$\partial_i F^{i0} = \mu_0 c\rho$$
(Since F is anti-symmetric, $F^{00} = 0$, Hence):
$$\partial_0 F^{00} + \partial_i F^{i0} = \mu_0 c\rho$$
$$\partial_\mu F^{\mu 0} = \mu_0 c\rho$$

$$\boxed{\partial_\mu F^{\mu 0} = \mu_0 c\rho} \qquad (4.10)$$

Combining part 1 and part 2:

$$\mu_0 J^i + \partial_\mu F^{i\mu} = 0 \qquad \text{(From } B\text{)}$$
$$\partial_\mu F^{i\mu} = -\mu_0 J^i \partial_\mu F^{\mu 0} = \mu_0 c\rho$$
$$\partial_\mu F^{0\mu} = -\mu_0 c\rho$$

To combine these equations, **we set:**

$$\boxed{J^0 \equiv c\rho} \tag{4.11}$$

We arrive at the unified equation:

$$\partial_\mu F^{\nu\mu} = -\mu_0 J^\nu$$

Choose units such that $c = \frac{h}{2\pi} = G_n = 1$, which gives us:

$$\partial_\mu F^{\nu\mu} = -J^\nu$$
F is antisymmetric, so flipping indices
$$\partial_\mu F^{\mu\nu} = J^\nu$$

$$\boxed{\partial_\mu F^{\mu\nu} = J^\nu} \tag{4.12}$$

Note that this is **Ampere's law!**

4.3.2 Combining (2) $\nabla \times E = -\frac{\partial B}{\partial t}$, (3) $\nabla B = 0$

$$\nabla \times E = -\frac{\partial B}{\partial t}$$
$$(\nabla \times E)^i = \mathcal{E}^{ijk}\partial_j E_k = -\partial_0 B$$
$$\mathcal{E}^{ijk}\partial_j E_k = -\partial_0(\frac{1}{2}\mathcal{E}^{ijk}F_{jk})$$
$$\mathcal{E}^{ijk}\partial_j E_k + \partial_0(\frac{1}{2}\mathcal{E}^{ijk}F_{jk}) = 0$$
$$2\mathcal{E}^{ijk}\partial_j E_k + \partial_0(\mathcal{E}^{ijk}F_{jk}) = 0$$

Now we begin from the other direction, and start the derivation.
We know that the equation we want is:

$$\boxed{\mathcal{E}^{\alpha\beta\mu\nu}\partial_\beta F_{\mu\nu} = 0} \tag{4.13}$$

$\alpha = 0$ **case:**

First, set $\alpha = 0$. So now, the other β, μ, ν are forced to be become space components — (i, j, k).

Therefore, the equation now becomes:

$$\mathcal{E}^{0ijk}\partial_i F_{jk} = 0$$

However, note that $\mathcal{E}0ijk = \mathcal{E}ijk$, because if (ijk) is an even permutation, so will $(0ijk)$, and vice versa for odd (since $0 < i, j, k$).

Using this, the equation becomes

$$\begin{aligned}
\mathcal{E}^{ijk}\partial_i F_{jk} &= 0 \\
\partial_i(\mathcal{E}^{ijk}F_{jk}) &= 0 \\
\text{Since } B^i = \frac{1}{2}\mathcal{E}^{ijk}F_{jk}&: \\
\partial_i\left(\frac{B^i}{2}\right) &= 0 \\
\partial_i B^i &= 0 \\
\boldsymbol{\nabla} B &= 0
\end{aligned}$$

Hence, the above equation does encode $\boldsymbol{\nabla} B = 0$.

$\alpha = m$ **case:**

Let α be a spatial dimension $m = \{1, 2, 3\}$.

$$\begin{aligned}
\mathcal{E}^{\alpha\beta\mu\nu}\partial_\beta F_{\mu\nu} &= 0 \\
\mathcal{E}^{m\beta\mu\nu}\partial_\beta F_{\mu\nu} &= 0
\end{aligned}$$

Once again, we get two cases, one where $\beta = 0$, and one where $\beta = n$ where n is a spatial dimension. If $\beta = 0$, then the other dimensions are forced to be spatial dimensions, which we shall denote as $\mu \equiv x$, $\nu \equiv y$

$$\begin{aligned}
\mathcal{E}^{m\beta\mu\nu}\partial_\beta F_{\mu\nu} &= 0 \\
\mathcal{E}^{m0xy}\partial_0 F_{xy} + \mathcal{E}^{mn\mu\nu}\partial_n F_{\mu\nu} &= 0
\end{aligned}$$

Now note that $\mathcal{E}^{m0\mu\nu} = -\mathcal{E}0m\mu\nu = -\mathcal{E}m\mu\nu$.

Using this, we can rewrite the above equation as:

$$\begin{aligned}
\mathcal{E}^{m0xy}\partial_0 F_{xy} + \mathcal{E}^{mn\mu\nu}\partial_n F_{\mu\nu} &= 0 \\
-\mathcal{E}^{mxy}\partial_0 F_{xy} + \mathcal{E}^{mn\mu\nu}\partial_n F_{\mu\nu} &= 0
\end{aligned}$$

We now consider cases for μ in the second term, where either $\mu = 0$ or $\mu = o \in \{1, 2, 3\}$

If $\mu = 0$, then the other dimension ν must be a spatial dimension p. If $\mu = q$, then the other dimension ν must be a time dimension 0 (This is because we are not allowed to have 4 spatial dimensions, since the $\mathcal{E}$ evaluates to 0 on repeated dimensions).

$$-\mathcal{E}^{mxy}\partial_0 F_{xy} + \mathcal{E}^{mn\mu\nu}\partial_n F_{\mu\nu} = 0$$

$$\begin{aligned} -\mathcal{E}^{mxy}\partial_0 F_{xy} + & \\ \mathcal{E}^{mn0p}\partial_n F_{0p} & \qquad (\mu = 0,\ \nu = p) \\ \mathcal{E}^{mnq0}\partial_n F_{q0} & \qquad (\mu = q,\ \nu = 0) \\ = 0 & \end{aligned}$$

Rearranging, and using the fact that $F_{0p} = -Fp0$, $\mathcal{E}mn0p = \mathcal{E}0mnp = \mathcal{E}mnp$, $\mathcal{E}mnq0 = -\mathcal{E}0mnq = -\mathcal{E}mnq$,

$$-\mathcal{E}^{mxy}\partial_0 F_{xy} + \mathcal{E}^{mnp}(-\partial_n F_{p0}) + (-\mathcal{E}^{mnq})\partial_n F_{q0} = 0$$

Multiplying throughout by -1, and noticing that since p, q are dummy indeces, we can set $p = q$. This allows us to get:

$$\mathcal{E}^{mxy}\partial_0 F_{xy} + 2\mathcal{E}^{mnp}\partial_n F_{p0} = 0$$

First, remember that $E_p = F_{p0}$. So, we can replace the term F_{p0} (upto fudging of constant factors that we have always done), with E_p.

Now, compare

$$\mathcal{E}^{mxy}\partial_0 F_{xy} + 2\mathcal{E}^{mnp}\partial_n E_p = 0 \qquad \text{(Our equation)}$$

$$2\mathcal{E}^{ijk}\partial_j E_k + \partial_0(\mathcal{E}^{ijk}F_{jk}) = 0 \qquad \text{(Previous equation)}$$

Note that the two equations are identical upto variable naming, and are hence considered equal. So, we have encoded both of Maxwell's laws into this particular equation:

$$\boxed{\mathcal{E}^{\alpha\beta\mu\nu}\partial_\beta F_{\mu\nu} = 0} \tag{4.14}$$

Chapter 5

Gauge theories

We construct a 1-dimensional guage theory and study its symmetries.

5.1 Euler-Lagrange equations for a field

Consider a Lagrangian:

$$\mathcal{L}(\phi, \phi', \dot{\phi}) = \dot{\phi}^2 - \phi'^2$$

When written in terms of M^4 (minkowski space), we know that

$$\partial_\mu \equiv (\partial_t, -\boldsymbol{\nabla})$$

Hence, in minkowski space, the lagrangian becomes a function of *only*:

$$\mathcal{L}(\phi, \partial_\mu \phi) \equiv \dots$$

So we managed to unify the space derivative and the time derivative (yay).

Now, we can consider the action of this Lagrangian:

$$S[\phi] = \int L(\phi, \partial_\mu \phi) \; \mathrm{d}^4 x$$

Minimising the functional S,

$$\delta S[\phi] = 0$$

$$\delta \int L(\phi, \partial_\mu \phi)\ \mathrm{d}^4 x = 0$$

For an analogy, consider $dL = \frac{\partial L}{\partial \phi}\mathrm{d}\phi + \frac{\partial L}{\partial \psi}\mathrm{d}\psi + \dots$

$$\int \left[\frac{\partial L}{\partial \phi}\delta\phi + \frac{\partial L}{\partial \partial_\mu \phi}\ \delta(\partial_\mu \phi)\right]\mathrm{d}^4 x = 0$$

Using linearity of integration, and commuting of δ, ∂_μ

$$\int \frac{\partial L}{\partial \phi}(\delta\phi)\mathrm{d}^4 x + \int \frac{\partial L}{\partial \partial_\mu \phi}(\partial_\mu \delta\phi)\mathrm{d}^4 x = 0$$

Using $\int U \mathrm{d}V = UV - \int V \mathrm{d}U$, $V = \delta\phi$, $U = \frac{\partial L}{\partial \partial_\mu \phi}$

$$\int \frac{\partial L}{\partial \phi}(\delta\phi)\mathrm{d}^4 x + \frac{\partial L}{\partial \partial_\mu \phi}(\delta\phi)\Big|_{endpoints} - \int \delta\phi\left(\partial_\mu \frac{\delta L}{\delta \partial_\mu \phi}\right)\mathrm{d}^4 x = 0$$

Since fields decay at endpoints, forget the integral

$$\int \frac{\partial L}{\partial \phi}(\delta\phi)\mathrm{d}^4 x - \int \delta\phi\left(\partial_\mu \frac{\partial L}{\partial \partial_\mu \phi}\right)\mathrm{d}^4 x = 0$$

Refactoring to pull the common $\delta\phi$,

$$\int \left(\frac{\partial L}{\partial \phi} - \partial_\mu \frac{\partial L}{\partial \partial_\mu \phi}\right)(\delta\phi)\mathrm{d}^4 x = 0$$

Since this is true for all perturbations $\delta\phi$ implies that:

$$\frac{\partial L}{\partial \phi} - \partial_\mu \frac{\partial L}{\partial \partial_\mu \phi} = 0$$

Hence, we have the Euler-Lagrange equation for **scalar fields**:

$$\boxed{\frac{\partial L}{\partial \phi} - \partial_\mu \frac{\partial L}{\partial(\partial_\mu \phi)} = 0} \tag{5.1}$$

So next, let's consider a Lagrangian (which is supposedly the free particle analogue): (**TODO: find out why this is free particle KE**)

$(\partial_\mu \phi)^2 \equiv \partial_\mu \phi \partial^\mu \phi$ (This is notation)

Note that:

$\partial_\mu \phi \partial^\mu \phi = \partial_\mu \phi \eta^{\mu\nu} \partial_\nu \phi$ (where $\eta^{\mu\nu}$ is the metric; lowering indeces)

We define the Lagrangian as:

$$\mathcal{L}(\phi, \partial_\mu \phi) = \frac{1}{2}(\partial_\mu \phi)^2$$

Calculating the terms in the EL equation:

$$\frac{\partial L}{\partial \phi} = 0$$

$$\frac{\partial L}{\partial(\partial_\sigma \phi)} = \frac{\partial}{\partial(\partial_\sigma \phi)}\left(\frac{1}{2}(\partial_\mu \phi)^2\right)$$

$$= \frac{\partial}{\partial(\partial_\sigma \phi)}\left(\frac{1}{2}\eta^{\mu\nu}(\partial_\mu \phi)(\partial_\nu \phi)\right)$$

$$= \frac{1}{2}\eta^{\mu\nu}\frac{\partial(\partial_\mu \phi)}{\partial(\partial_\sigma \phi)}(\partial_\nu \phi) + \frac{1}{2}\eta^{\mu\nu}(\partial_\mu \phi)\frac{\partial(\partial_\nu \phi)}{\partial(\partial_\sigma \phi)}$$

(TODO: understand why $\frac{\partial(\partial_\mu \phi)}{\partial(\partial_\sigma \phi)} = \delta_\mu^\sigma$)

$$= \frac{1}{2}\eta^{\mu\nu}\delta_\mu^\sigma(\partial_\nu \phi) + \frac{1}{2}\eta^{\mu\nu}(\partial_\mu \phi)\delta_\nu^\sigma$$

(Contracting on μ)

$$= \frac{1}{2}\eta^{\sigma\nu}(\partial_\nu \phi) + \frac{1}{2}\eta^{\mu\sigma}(\partial_\mu \phi)$$

(Replacing dummy index $\mu \equiv \nu$)

$$= \eta^{\nu\sigma}\partial_{mu}\phi$$

(TODO: understand how this happens, something about η's signature)

$$= \partial^\sigma \phi$$

Hence, the inner part of the second term in the EL equation is:

$$\frac{\partial L}{\partial(\partial_\sigma \phi)} = \partial^\sigma \phi \tag{5.2}$$

Now, the second term of the Lagrangian is:

$$\partial_\mu \frac{\partial L}{\partial(\partial_\mu \phi)} = \partial_\mu(\partial^\mu \phi) = \ \partial_0 \partial^0 \phi - \nabla^2 \phi = \Box \phi$$

Hence, finally, our Euler-Lagrange equation is:

$$\frac{\partial L}{\partial \phi} - \partial_\mu \frac{\partial L}{\partial(\partial_\mu \phi)} = 0$$
$$0 + \Box \phi = 0$$

So, finally, The Euler-Lagrange equations for a scalar field governed by $L = \frac{1}{2}(\partial_\mu \phi)^2$ is:

$$\boxed{\Box \phi = 0} \tag{5.3}$$

5.2 Klein-gordon equations

5.3 Lagrangian for a massive scalar field

5.4 Symmetries of a scalar field Lagrangian

5.5 Derving the force of the EM-field from the Lanrangian

Recall that $B = \nabla \times A$, $E = \nabla\phi - \frac{\partial A}{\partial t}$, and the force on a particle is $\vec{\mathbf{F}} = q(\vec{\mathbf{E}} + \vec{\mathbf{v}} \times \vec{\mathbf{B}})$.

$$ma = e(\nabla\phi - \frac{\partial A}{\partial t}) + e(v \times (\nabla \times A))$$
$$ma = e(\nabla\phi - \frac{\partial A}{\partial t}) + e(\nabla \cdot (v \cdot A) - (v \cdot \nabla)A)$$

Note that $(v \cdot \nabla)A$ is:

$$v \cdot \nabla = \frac{\mathrm{d}x}{\mathrm{d}t}\frac{\partial}{\partial x} + \frac{\mathrm{d}y}{\mathrm{d}t}\frac{\partial}{\partial y} + \frac{\mathrm{d}z}{\mathrm{d}t}\frac{\partial}{\partial z}$$
$$(v \cdot \nabla)A = \frac{\mathrm{d}x}{\mathrm{d}t}\frac{\partial A}{\partial x} + \frac{\mathrm{d}y}{\mathrm{d}t}\frac{\partial A}{\partial y} + \frac{\mathrm{d}z}{\mathrm{d}t}\frac{\partial A}{\partial z}$$

However, now let us compare $\frac{\mathrm{d}A}{\mathrm{d}t}$ and $(v \cdot \nabla)A$:

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{\partial A}{\partial t} + \frac{\mathrm{d}x}{\mathrm{d}t}\frac{\partial A}{\partial x} + \frac{\mathrm{d}y}{\mathrm{d}t}\frac{\partial A}{\partial y}\frac{\mathrm{d}z}{\mathrm{d}t}\frac{\partial A}{\partial z}$$
$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{\partial A}{\partial t} + (v \cdot \nabla)A$$

Now, rewriting ma,

$$ma = e(\nabla\phi - \frac{\partial A}{\partial t}) + e(\nabla \cdot (v \cdot A) - (v \cdot \nabla)A)$$
$$ma = e(\nabla\phi - \frac{\partial A}{\partial t}) + e(\nabla \cdot (v \cdot A) - (v \cdot \nabla)A)$$

5.6 Local and global symmetries

Chapter 6

Charged particle interaction in fields, or, how maxwell's equations have $U(1)$ symmetry

(written in red ink pen)

Consider a scalar field ϕ, and the lagrangian:

$$\mathcal{L}(\phi, \partial_\mu \phi) = \frac{1}{2}(\partial_\mu \phi)(\partial_\mu \phi) - \frac{1}{2}m^2\phi^2$$

Next, we want to consider charged particle interactions, which comes from $H = \frac{(p-eA)^2}{2m}$, which creates the lorentz force $e\vec{\mathbf{v}} \times \vec{\mathbf{B}} = e\vec{\mathbf{v}} \cdot \vec{\mathbf{A}}$.

We know that $\mathcal{L}$ is invariant under global rotation $e^{i\theta}$. Now we study local gauge invariance by making θ a function of space. That is, $\theta \to \theta(x)$.

We use the complex form of the lagrangian:

$$\mathcal{L} = (\partial_\mu \phi)(\partial^\mu \phi)^* - m^2|\phi|^2$$

We now consider the transform:

$$phi \to e^{ie\theta(x)}\phi$$

This implies the transforms:

$$\partial_\mu \phi = \partial_\mu (e^{ie\theta(x)}\phi) = e^{ie\theta(x)}\partial_\mu \phi + iee^{ie\theta(x)}\phi\partial_\mu \theta = \\ e^{ie\theta(x)}(\partial_\mu \phi + ie\phi\partial_\mu \theta)$$

Hence, for invariance, we get:

$$\begin{aligned}
&(\partial_\mu \phi)(\partial^\mu \phi)^* \\
=\ &(e^{ie\theta(x)}(\partial_\mu \phi + ie\phi\partial_\mu\theta))(e^{ie\theta(x)}(\partial^\mu \phi + ie\phi\partial^\mu\theta))^* \\
=\ &(\partial_\mu \phi + ie\phi\partial_\mu\theta)(\partial^\mu \phi^* - ie\phi^*\partial^\mu\theta)
\end{aligned}$$

We introduce:

$$D_\mu \phi \equiv (\partial_\mu - iA_\mu)\phi(\partial^\mu + iA^\mu)\phi^*$$

We shall check that this transforms in a lorentzian way. In the new coordinate system:

$$\begin{aligned}
\bar{D}_\mu \bar{\phi} \equiv &([\partial_\mu - i\bar{A}_\mu)\bar{\phi}][(\partial^\mu + i\bar{A}^\mu)\bar{\phi}^*] \\
= \ &[(\partial_\mu - i\bar{A}_\mu)(e^{ie\theta(x)})\phi][(\partial^\mu + i\bar{A}^\mu)(e^{ie\theta(x)}\phi)^*] \\
= \ &[\partial_\mu(e^{ie\theta(x)}\phi) - i\bar{A}_\mu e^{ie\theta(x)}\phi][\partial_\mu(e^{-ie\theta(x)}\phi^*) + i\bar{A}_\mu e^{-ie\theta(x)}\phi^*] \\
&\text{evaluate } \partial_\mu(UV) \\
= \ &[e^{ie\theta(x)}(\partial_\mu\phi) + iee^{ie\theta(x)}(\partial_\mu\theta)\phi - i\bar{A}_\mu e^{ie\theta(x)}\phi][e^{-ie\theta(x)}(\partial_\mu\phi^*) - iee^{-ie\theta(x)}(\partial_\mu\theta)\phi^* + i\bar{A}_\mu e^{-ie\theta(x)}\phi^*] \\
= \ &[e^{ie\theta(x)}(\partial_\mu\phi + ie(\partial_\mu\theta)\phi - i\bar{A}_\mu\phi)][e^{-ie\theta(x)}(\partial_\mu\phi^* - ie(\partial_\mu\theta)\phi^* + i\bar{A}_\mu\phi^*)] \\
&\text{cancelling } e^{ie\theta(x)} \text{ with } e^{-ie\theta(x)} \\
= \ &[\partial_\mu\phi + ie(\partial_\mu\theta)\phi - i\bar{A}_\mu\phi][\partial_\mu\phi^* - ie(\partial_\mu\theta)\phi^* + i\bar{A}_\mu\phi^*] \\
= \ &[\partial_\mu + ie(\partial_\mu\theta) - i\bar{A}_\mu]\phi[\partial_\mu - ie(\partial_\mu\theta) + i\bar{A}_\mu]\phi^*
\end{aligned}$$

Comparing:

$$\begin{aligned}
D_\mu \phi &\equiv (\partial_\mu - iA_\mu)\phi(\partial^\mu + iA^\mu)\phi^* \\
\bar{D}_\mu \bar{\phi} &\equiv [\partial_\mu + ie(\partial_\mu\theta) - i\bar{A}_\mu]\phi[\partial_\mu - ie(\partial_\mu\theta) + i\bar{A}_\mu]\phi^* \\
-iA_\mu &= +ie(\partial_\mu\theta) - i\bar{A}_\mu \\
-A_\mu &= e(\partial_\mu\theta) - \bar{A}_\mu \\
\bar{A}_\mu &= e(\partial_\mu\theta) + A_\mu
\end{aligned}$$

TODO: This is not what mukku got! mukku got $\bar{A}_\mu = A_\mu - e(\partial_\mu\theta)$. HOW?

So now, we know how A_μ transforms:

$$\boxed{\bar{A}_\mu = A_\mu - e(\partial_\mu\theta)} \tag{6.1}$$

So, now, our modified lagrangian is:

$$\mathcal{L}(\phi, \phi^*, \partial_\mu\phi, \partial_\mu\phi^*, A_\mu) = [(\partial_\mu - iA_\mu)\phi][(\partial^\mu - iA^\mu)\phi]^* - \frac{1}{2}m|\phi|^2$$

We want to understand if we can add interaction terms for A_μ. If not, what the obstacles are. We would like the interaction terms to be a Lorentz tensor, and we would like it to respect local symmetrics.

$$\mu^2 A_\mu A^\mu \to \mu^2 \bar{A}_\mu \bar{A}^\mu = \mu^2 (A_\mu + e\partial_\mu \theta)(A^\mu + e\partial^\mu \theta)$$

So clearly, we cannot have mass terms of the form $\mu^2 A_\mu A^\mu$. We must try other things and check if they are interesting.

We now try to inspect some invariants of A_μ. For example, we can construct:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

We already know that F is a Lorentz tensor. Let's check if it's gauge invariant.

$$\begin{aligned}
\bar{F}_{\mu\nu} &= \partial_\nu \bar{A}_\mu - \partial_\mu A_\nu \\
&= \partial_\nu (A_\mu - e(\partial_\mu \theta)) - \partial_\mu (A_\nu - e\partial_\nu \theta)) \\
&\quad \text{Since partial derivatives commute:} \\
&= \partial_\nu A_\mu - \cancel{e\partial_\nu \partial_\mu \theta} - \partial_\mu A_\nu - \cancel{e\partial_\mu \partial_\nu \theta})) \\
&= \partial_\nu A_\mu - \partial_\mu A_\nu = F_{\mu\nu}
\end{aligned}$$

Hence, $F_{\mu\nu}$ is also gauge invariant.
Here, there is some sentence of how "there are only two invariants possible for $F_{\mu\nu}$:

- $F^{\mu\nu} F_{\mu\nu}$
- $e^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta}$ (what is e in this context?) Supposedly, we can ignore this since it relates to divergence (??) **TODO: figure this out**

So, extending the lagrangian with the $F_{\mu\nu}$ term gives us:

$$\begin{aligned}
&\mathcal{L}(\phi, \phi^*, \partial_\mu \phi, \partial_\mu \phi^*, A_\mu) = \frac{1}{4} F^{\mu\nu} F_{\mu\nu} + [D_\mu \phi][D^\mu \phi]^* - \frac{1}{2} m |\phi|^2 \\
&\text{where:} \\
&D_\mu \equiv (\partial^\mu - iA^\mu) \\
&F_{\mu\nu} \equiv \partial_\mu A_\nu - partial_\nu A_\mu
\end{aligned}$$

Now, performing variational calculus on this lagrangian, we should theoretically regain maxwell's equations:

TODO: follow the last part of the stuff written in red ink pen to understand what in the actual fuck happened

Chapter 7

Non abelian gauge theories

Let $\phi_i = (\phi_1, \phi_2, \phi_3)$ be a vector of scalar fields.

Consider the "usual" lagrangian:

$$\mathcal{L} = (\partial_\mu \phi)^\dagger (\partial^\mu \phi) - \frac{1}{2} m \phi^\dagger \phi$$

This clearly has a global symmetry $\phi \to U(\theta)\phi$ where $U \in SU(n)$

We enlarge the global symmetry to a local symmetry $\phi \to U(\theta(x))\phi$. Note that $\phi^\dagger \phi$ is still invariant, but we need to check the first term of the Lagrangian.

Working out the changes:
$\partial_\mu \bar{\phi} = \partial_\mu (U\phi) = (\partial_\mu U)\phi + (U \partial_\mu \phi)$
$\partial_\mu \bar{\phi}^\dagger = \partial_\mu (U\phi)^\dagger = \partial_\mu (\phi^\dagger U^\dagger) = (\partial_\mu \phi^\dagger) U^\dagger + \phi^\dagger (\partial_\mu U^\dagger)$
So, the term to be invariant is:
$(\partial_\mu \bar{\phi})^\dagger (\partial^\mu \bar{\phi}) =$
$[(\partial_\mu \phi^\dagger) U^\dagger + \phi^\dagger (\partial_\mu U^\dagger)][(\partial^\mu U)\phi + (U \partial^\mu \phi)] =$
$(\partial_\mu \phi^\dagger) U^\dagger (\partial^\mu U)\phi + (\partial_\mu \phi^\dagger) U^\dagger (U \partial^\mu \phi) +$
$\phi^\dagger (\partial_\mu U^\dagger)(\partial^\mu U)\phi + \phi^\dagger (\partial_\mu U^\dagger)(U \partial^\mu \phi)$

This mess of equations clearly does not look like $(\partial_\mu \phi)(\partial^\mu \phi)$, even after using the simplification $UU^\dagger = U^\dagger U = I$, so this is not invariant.

So let's define a new covariant derivative (I wish I knew what those words mean):

$$(D_\mu)_{\alpha,\beta} = \partial_\mu \delta_{\alpha\beta} - ig(A_\mu)_{\alpha,\beta}$$

Where g is some kind of coupling coefficient (more on this later), and A_μ is some arbitrary quantity on which we will use the symmetries we expect to give some structure.

We need $D_\mu \phi$ to transform reasonably, hence, we stipulate that:

$$(D_\mu \bar{\phi}) \to U D_\mu \phi$$

Assuming that transformation law holds, we show that $D_\mu \phi$ is invariant:

$$
\begin{aligned}
&(D_\mu \bar{\phi})^\dagger (D^\mu \bar{\phi}) = (U(D_\mu \phi))^\dagger (U(D_\mu \phi)) = ((D_\mu \phi^\dagger) U^\dagger)(U(D_\mu \phi)) = \\
&(D_\mu \phi)^\dagger (D^\mu \phi) \text{ since } UU^\dagger = I \\
&\text{Hence, we showed that:} \\
&(D_\mu \bar{\phi})^\dagger (D^\mu \bar{\phi}) \to (D_\mu \phi)^\dagger (D^\mu \phi)
\end{aligned}
$$

Now, we need to ensure that the law we took actually works. For this law to hold, we will derive conditions that govern A:

$$
\begin{aligned}
&(D_\mu \bar{\phi}) = U D_\mu \phi \\
&\partial_\mu \bar{\phi} - ig\bar{A}_\mu \bar{\phi} = U(\partial_\mu \phi - igA_\mu \phi) \\
&\partial_\mu (U\phi) - ig\bar{A}_\mu \bar{\phi} = U(\partial_\mu \phi - igA_\mu \phi) \\
&(\partial_\mu U)\phi + \cancel{U(\partial_\mu \phi)} - ig\bar{A}_\mu \bar{\phi} = \cancel{U\partial_\mu \phi} - igUA_\mu \phi \\
&(\partial_\mu U)\phi - ig\bar{A}_\mu \bar{\phi} = -igUA_\mu \phi \\
&-ig\bar{A}_\mu (U\phi) = -igUA_\mu \phi - (\partial_\mu U)\phi \\
&(ig\bar{A}_\mu U)\phi = (igUA_\mu + (\partial_\mu U))\phi \\
&ig\bar{A}_\mu U = igUA_\mu + (\partial_\mu U) \\
&A_\mu = UA_\mu U^{-1} + \frac{(\partial_\mu U)U^{-1}}{ig}
\end{aligned}
$$

So, we now know what the correction term is for the D_μ for the non-abelian gauge theory. Notice that $(\partial_\mu U)U^{-1}$ is a function of θ, the parameter.

$$
\boxed{A_\mu = UA_\mu U^{-1} + \frac{(\partial_\mu U)U^{-1}}{ig}} \tag{7.1}
$$

Supposedly, A_μ is a *connection*, and the field strength tensor F is the *curvature of the connection*:

$$
\begin{aligned}
& [D_\mu, D_\nu] = [\partial_\mu - ieA_\mu, \partial_\nu - ieA_\nu] \\
= \;& (\partial_\mu - ieA_\mu)(\partial_\nu - ieA_\nu) - (\partial_\nu - ieA_\nu)(\partial_\mu - ieA_\mu) \\
= \;& (\partial_\mu \partial_\nu - \partial_\mu (ieA_\nu) + (-ieA_\mu)(\partial_\nu) + i^2 e^2 A_\mu A_\nu) - \\
& (\partial_\nu \partial_\mu - \partial_\nu (ieA_\mu) + (-ieA_\nu)(\partial_\mu) + i^2 e^2 A_\mu A_\nu) \\
= \;& (\cancel{\partial_\mu \partial_\nu} - \partial_\mu (ieA_\nu) + (-ieA_\mu)(\partial_\nu) + \cancel{i^2 e^2 A_\mu A_\nu}) - \\
& (\cancel{\partial_\nu \partial_\mu} - \partial_\nu (ieA_\mu) + (-ieA_\nu)(\partial_\mu) + \cancel{i^2 e^2 A_\mu A_\nu}) \\
& \textbf{TODO: understand how } (-ieA_\mu)(\partial_\nu) \textbf{ terms get cancelled} \\
= \;& -ie(\partial_\mu A_\nu - \partial_\nu A_\mu)
\end{aligned}
$$

which is indeed F.

The group that governs the symmetries of ϕ is $SU(2)$, since $SU(2)$ has a dimension (as a manifold) of 3 ($SU(n)$ has $n^2 - 1$ degrees of freedom).