Two equal mass scalar fields :- φ_1, φ_2
mass : m

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi_1)^2 + \frac{1}{2}(\partial_\mu \varphi_2)^2 - \frac{1}{2}m^2\varphi_1^2 - \frac{1}{2}m^2\varphi_2^2 .$$

(i) Let $\varphi = \frac{1}{\sqrt{2}}(\varphi_1 + i\varphi_2) \Rightarrow \varphi^* = \frac{1}{\sqrt{2}}(\varphi_1 - i\varphi_2)$

$$\Rightarrow \quad \varphi + \varphi^* = \sqrt{2}\,\varphi_1 \; ; \quad \varphi - \varphi^* = i\sqrt{2}\,\varphi_2$$

$$\Rightarrow \quad \varphi_1 = \frac{1}{\sqrt{2}}(\varphi + \varphi^*) \; ; \quad \varphi_2 = \frac{i}{\sqrt{2}}(\varphi^* - \varphi) .$$

$$\varphi\varphi^* = \frac{1}{2}(\varphi_1^2 + \varphi_2^2)$$

$$\Rightarrow \quad -\frac{1}{2}m^2(\varphi_1^2 + \varphi_2^2) = -m^2\varphi\varphi^* .$$

$$\partial_\mu \varphi = \frac{1}{\sqrt{2}}(\partial_\mu \varphi_1 + i\,\partial_\mu \varphi_2)$$

$$\partial_\mu \varphi^* = \frac{1}{\sqrt{2}}(\partial_\mu \varphi_1 - i\,\partial_\mu \varphi_2) .$$

$$\Rightarrow \quad (\partial_\mu \varphi)(\partial^\mu \varphi^*) = \frac{1}{2}(\partial_\mu \varphi_1)^2 + \frac{1}{2}(\partial_\mu \varphi_2)^2 .$$

Hence,

$$\underline{\mathcal{L}(\varphi, \varphi^*, \partial_\mu \varphi, \partial_\mu \varphi^*) = (\partial_\mu \varphi)(\partial^\mu \varphi^*) - m^2\varphi\varphi^*} .$$

If $\varphi \rightarrow e^{i\theta}\varphi$; θ = constant,

then $\varphi^* \rightarrow e^{-i\theta}\varphi^*$

$$\Rightarrow \quad \varphi\varphi^* \longrightarrow \varphi\varphi^* \quad : \text{invariant.}$$

Since θ is constant,

$$\partial_\mu \varphi \longrightarrow e^{i\theta}\partial_\mu \varphi$$

&

$$\partial_\mu \varphi^* \longrightarrow e^{-i\theta}\partial_\mu \varphi^*$$

$$\Rightarrow \quad (\partial_\mu \varphi)(\partial^\mu \varphi^*) \longrightarrow (\partial_\mu \varphi)(\partial^\mu \varphi^*) \; : \text{invariant.}$$

$\Rightarrow \quad \mathcal{L} = \cancel{\tfrac{1}{2}} (\partial_\mu \varphi)(\partial^\mu \varphi^*) - m^2 \varphi \varphi^*$

is invariant under

$$\varphi \longrightarrow e^{i\theta} \varphi \quad \text{for } \theta = \text{Constant}.$$

Let $\theta \longrightarrow \theta(x)$: global symm $\longrightarrow$ local symm.

then $\varphi\varphi^* \longrightarrow \varphi\varphi^*$: invariant.

under $\varphi \longrightarrow e^{i\theta(x)} \varphi$.

however;

$$\partial_\mu \varphi \longrightarrow \partial_\mu \left(e^{i\theta(x)} \varphi\right)$$

$$= \cancel{\partial_\mu}\ e^{i\theta(x)} i(\partial_\mu \theta)\varphi + e^{i\theta(x)} \partial_\mu \varphi.$$

$$\partial_\mu \varphi^* \longrightarrow e^{-i\theta(x)} (-i\partial_\mu \theta)\varphi^* + e^{-i\theta(x)} \partial_\mu \varphi^*.$$

$$\Rightarrow (\partial_\mu \varphi)(\partial^\mu \varphi^*)$$

$$\not\Rightarrow (\partial_\mu \varphi)(\partial^\mu \varphi)^*$$

under $\varphi \longrightarrow e^{i\theta(x)} \varphi$.

Let $D_\mu = \partial_\mu - ieA_\mu$; e : constant parameter

A_μ : vector field

D_μ : Covariant derivative.

Under a local gauge transformation,

$$\varphi \rightarrow e^{i\theta(x)} \varphi(x) \equiv \tilde{\varphi}(x)$$

$$\varphi^* \rightarrow e^{-i\theta(x)} \varphi^*(x) \equiv \tilde{\varphi}^*$$

Then, $\varphi\varphi^* \longrightarrow \tilde{\varphi}\tilde{\varphi}^* = \varphi\varphi^*$

$\Rightarrow$ $-m^2\varphi\varphi^*$ is invariant (mass term in $\mathcal{L}$).

We require $D_\mu\varphi$ to transform like φ

i.e., $$D_\mu\varphi \longrightarrow \widetilde{D_\mu\varphi} = e^{i\theta(x)} D_\mu\varphi$$

So that

$$\mathcal{L} = (D_\mu\varphi)(D^\mu\varphi)^* - m^2\varphi\varphi^*$$

is invariant under the local gauge transf$^{\underline{n}}$.

Let us suppose $A_\mu \longrightarrow \tilde{A}_\mu$ under the transf$^{\underline{n}}$.

then

$$\widetilde{(D_\mu\varphi)} = \partial_\mu\tilde{\varphi} - ie\tilde{A}_\mu\tilde{\varphi}$$

$$= \partial_\mu\left(e^{i\theta(x)}\varphi\right) - ie\tilde{A}_\mu e^{i\theta(x)}\varphi$$

$$= ie^{i\theta}(\partial_\mu\theta)\varphi + e^{i\theta}\partial_\mu\varphi - ie\tilde{A}_\mu e^{i\theta}\varphi.$$

But since

$$\widetilde{(D_\mu\varphi)} = e^{i\theta} D_\mu\varphi,$$

we have

$$ie^{i\theta}(\partial_\mu\theta)\varphi + e^{i\theta}\partial_\mu\varphi - ie\tilde{A}_\mu e^{i\theta}\varphi$$

$$= e^{i\theta}\partial_\mu\varphi - ie\, e^{i\theta}A_\mu\varphi$$

$$\Rightarrow \quad -ie\tilde{A}_\mu + i\partial_\mu\theta = -ieA_\mu$$

$$\Rightarrow \quad \underline{\tilde{A}_\mu = A_\mu + \frac{1}{e}\partial_\mu\theta}$$

Hence under $\varphi \to e^{i\theta(x)} \varphi$,

$$D_\mu \varphi \to e^{i\theta(x)} D_\mu \varphi$$

and

$$A_\mu \to A_\mu + \frac{1}{e} \partial_\mu \theta ,$$

the Lagrangian,

$$\mathcal{L} = (D_\mu \varphi)(D^\mu \varphi)^* - m^2 \varphi \varphi^*$$

is invariant.

Since $A_\mu \to A_\mu + \frac{1}{e}\partial_\mu \theta$,

$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is invariant.

$$\Rightarrow \mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (D_\mu \varphi)(D^\mu \varphi)^* - m^2 \varphi \varphi^*$$

is a locally gauge invariant lagrangian for the fields, φ, φ^* and A_μ.

$$\begin{aligned}
\delta \mathcal{L} &= -\frac{1}{4} (\delta F_{\mu\nu}) F^{\mu\nu} - \frac{1}{4} F_{\mu\nu} (\delta F^{\mu\nu}) \\
&\quad + \delta (D_\mu \varphi)(D^\mu \varphi)^* + (D_\mu \varphi)\, \delta (D^\mu \varphi)^* \\
&\quad - m^2 (\delta \varphi) \varphi^* - m^2 \varphi (\delta \varphi^*) .
\end{aligned}$$

$$\begin{aligned}
&= -\frac{1}{2} (\delta F_{\mu\nu}) F^{\mu\nu} + \delta (D_\mu \varphi)(D^\mu \varphi)^* \\
&\quad + (D_\mu \varphi)\, \delta (D^\mu \varphi)^* - m^2 (\delta \varphi) \varphi^* - m^2 \varphi (\delta \varphi)^* .
\end{aligned}$$

Now,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$\Rightarrow \delta F_{\mu\nu} = \partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu .$$

$$\Rightarrow (\delta F_{\mu\nu}) F^{\mu\nu} = (\partial_\mu \delta A_\nu) F^{\mu\nu} - (\partial_\nu \delta A_\mu) F^{\mu\nu}$$

$$= \underline{2 (\partial_\mu \delta A_\nu) F^{\mu\nu}}$$

$$\delta(D_\mu \varphi) = \delta(\partial_\mu \varphi - ie A_\mu \varphi)$$

$$= \partial_\mu(\delta\varphi) - ie(\delta A_\mu)\varphi - ie A_\mu(\delta\varphi) .$$

$$= (\partial_\mu - ie A_\mu)\delta\varphi - ie(\delta A_\mu)\varphi$$

$$\cancel{= D_\mu \delta}$$

$$\delta \mathcal{L} = -(\partial_\mu \delta A_\nu) F^{\mu\nu} + \{ (D_\mu \delta\varphi)(D^\mu\varphi)^* - m^2(\delta\varphi)\varphi^* + \text{Complex Conjugates} \} .$$

partial integration :

$$\delta S = \delta \int \mathcal{L} = \int \delta \mathcal{L} .$$

$$= \int \Big[(\partial_\mu F^{\mu\nu}) \delta A_\nu + \text{total div. term} - ie(\delta A_\mu)\varphi (D^\mu\varphi)^* + ie(\delta A_\mu)\varphi^*(D^\mu\varphi) \Big]$$

$$+ \int \Big\{ \big[(\partial_\mu \delta\varphi)(D^\mu\varphi)^* - ie\, \delta\varphi A_\mu (D^\mu\varphi)^* - m^2(\delta\varphi)\varphi^* \big] + [\text{Complex Conjugates}] \Big\}$$

$$= \int (\partial_\mu F^{\mu\nu}) \delta A_\nu$$

$$- \int \Big\{ \delta\varphi \big[\partial_\mu (D^\mu\varphi)^* + ie (D^\mu\varphi)^* A_\mu + m^2 \varphi^* \big] + \delta\varphi^* \big[\partial_\mu (D^\mu\varphi) - ie (D^\mu\varphi) A_\mu + m^2 \varphi \big] \Big\}$$

Since $\delta S = 0$

& variations, $\delta\varphi, \delta\varphi^*, \delta A_\mu$ are all arbitrary,

$$\Rightarrow 0 = \int \delta A_\mu \left\{ \partial_\nu F^{\nu\mu} - ie\,\varphi (D^\mu\varphi)^* + ie\,\varphi^* (D^\mu \varphi) \right\}$$

$$+ \int \delta\varphi \left\{ (D_\mu D^\mu \varphi)^* + m^2 \varphi^* \right\}$$

$$+ \int \delta\varphi^* \left\{ D_\mu D^\mu \varphi + m^2 \varphi \right\}$$

Hence,

$$\partial_\nu F^{\nu\mu} = ie\,\varphi (D^\mu\varphi)^* - ie\,\varphi^* (D^\mu\varphi) = J^\mu$$

Maxwell's eq^ns. with current J^μ;

$$(D_\mu D^\mu + m^2)\varphi = 0 \quad \text{\& Complex Conjugate.}$$

Covariant charged Scalar field eq^n. interacting with gauge field A_μ

(vi) done in Class.

$$S = \int \mathcal{L}\, d^4x = \int \left[\frac{1}{2} (\partial_\mu\varphi)(\partial^\mu\varphi) - \frac{1}{2} m^2 \varphi^2 \right] d^4x.$$

$$\delta S = 0 = \int \delta \mathcal{L}\, d^4x$$

$$= \int \left[\frac{\partial \mathcal{L}}{\partial \varphi} \delta\varphi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \delta(\partial_\mu \varphi) \right] d^4x$$

$$= \int \left[\frac{\partial \mathcal{L}}{\partial \varphi} \delta\varphi + \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \delta\varphi \right] - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \right) \delta\varphi \right] d^4x$$

$$= \int \left[\frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \right) \delta\varphi \right] d^4x + \int \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \delta\varphi \right] d^4x.$$

Applying Euler-Lagrange eq^{ns} of motion,

$$\delta S = 0 = \int \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \delta\varphi \right] d^4x$$

- Let symmetry be

$$\varphi \rightarrow \tilde{\varphi} = U(\epsilon)\varphi \simeq \varphi + \epsilon\varphi.$$

∞'simal parameter ϵ

then $\delta\varphi = \tilde{\varphi} - \varphi = \epsilon\varphi.$

$$\Rightarrow \int \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \epsilon\varphi \right] d^4x = 0$$

$$\Rightarrow \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \varphi \right) = 0$$

This is a Continuity eq^n;

$$\partial_\mu J^\mu = 0 \Rightarrow \partial_0 J^0 + \partial_i J^i = 0$$

$$J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)}\varphi$$

for $\mathcal{L} = \frac{1}{2}(\partial_\mu\varphi)(\partial^\mu\varphi) - \frac{1}{2}m^2\varphi^2$,

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu\varphi)} = \partial^\mu\varphi.$$

$$\Rightarrow \quad J^\mu = \varphi\,\partial^\mu\varphi$$

$$\Rightarrow \quad J^0 = \varphi\,\partial^0\varphi = \varphi\dot{\varphi}$$

$$\partial_0 J^0 + \partial_i J^i = 0$$

$$\Rightarrow \quad \int_{d^3x} \partial_0 J^0 + \int_{d^3x} \partial_i J^i = 0$$

$$\Rightarrow \quad \frac{d}{dt}\int J^0\, d^3x = 0$$

$$\Rightarrow \quad \frac{d}{dt}\int \varphi\dot{\varphi}\, d^3x = 0$$

Total conserved charge is $\int J^0\, d^3x$.