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Problem Set 3

Topic: Orthogonal Coordinates

$$q_i = q_i(x_1, x_2, x_3) \quad \Leftrightarrow \quad x_i = x_i(q_1, q_2, q_3) \tag{3.1}$$

$$\vec{V} = \hat{e}_i V_i = \hat{e}_1 V_1 + \hat{e}_2 V_2 + \hat{e}_3 V_3, \text{ with } e_i^2 = 1, \text{ and } \hat{e}_1 \cdot (\hat{e}_2 \times \hat{e}_3) > 0 \tag{3.2}$$

$$ds^2 = g_{ij}\, dq_i dq_j, \text{ with, } g_{ij} = \frac{\partial x_k}{\partial q_i}\frac{\partial x_k}{\partial q_j} \tag{3.3}$$

$$\text{For orthogonal coordinates, } g_{ij} = 0, \text{ if } i \neq j \tag{3.4}$$

$$\text{Let, } h_i^2 = g_{ii} = \sum_k \frac{\partial x_k}{\partial q_i}\frac{\partial x_k}{\partial q_i} \text{ [no summation over } i\text{] with } h_i > 0 \tag{3.5}$$

$$ds_i = h_i dq_i \text{ [no summation over } i\text{]} \tag{3.6}$$

$$ds^2 = \sum_i ds_i^2 = \sum_i (h_i dq_i)^2 \tag{3.7}$$

$$\text{Line element: } d\vec{r} = \sum_i \hat{e}_i ds_i = \sum_i \hat{e}_i\, h_i dq_i = \hat{e}_1\, h_1 dq_1 + \hat{e}_2\, h_2 dq_2 + \hat{e}_3\, h_3 dq_3 \tag{3.8}$$

$$\Rightarrow \hat{e}_i = \frac{1}{h_i}\frac{\partial \vec{r}}{\partial q_i} \quad \text{[no summation over } i\text{]} \tag{3.9}$$

$$\text{or, } h_i = \left|\frac{\partial \vec{r}}{\partial q_i}\right| \;\&\; \hat{e}_i = \frac{\frac{\partial \vec{r}}{\partial q_i}}{\left|\frac{\partial \vec{r}}{\partial q_i}\right|} \text{ [no summation over } i\text{]} \tag{3.10}$$

$$\text{Area element: } d\vec{\sigma}_{ij} = \left(\frac{\partial \vec{r}}{\partial q_i} \times \frac{\partial \vec{r}}{\partial q_j}\right) dq_i dq_j = h_i h_j dq_i dq_j\, (\hat{e}_i \times \hat{e}_j) \text{ [no summation over } i, j\text{]} \tag{3.11}$$

$$\text{Volume element: } d\tau = \left(\frac{\partial \vec{r}}{\partial q_1} \cdot \frac{\partial \vec{r}}{\partial q_2} \times \frac{\partial \vec{r}}{\partial q_3}\right) dq_1 dq_2 dq_3 = h_1 h_2 h_3\, dq_1 dq_2 dq_3 \tag{3.12}$$

$$\text{Gradient: } \vec{\nabla}\psi = \hat{e}_i \frac{\partial \psi}{\partial s_i} = \hat{e}_1 \frac{1}{h_1}\frac{\partial \psi}{\partial q_1} + \hat{e}_2 \frac{1}{h_2}\frac{\partial \psi}{\partial q_2} + \hat{e}_3 \frac{1}{h_3}\frac{\partial \psi}{\partial q_3} \tag{3.13}$$

$$\text{Divergence: } \vec{\nabla}\cdot\vec{V} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}(h_2 h_3 V_1) + \frac{\partial}{\partial q_2}(h_1 h_3 V_2) + \frac{\partial}{\partial q_3}(h_1 h_2 V_3)\right] \tag{3.14}$$

$$\begin{aligned} \text{Laplacian: } \nabla^2\psi &= \vec{\nabla}\cdot\left(\vec{\nabla}\psi\right) \\ &= \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial \psi}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{h_1 h_3}{h_2}\frac{\partial \psi}{\partial q_2}\right) + \frac{\partial}{\partial q_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial \psi}{\partial q_3}\right)\right] \end{aligned} \tag{3.15}$$

$$\text{Curl: } \vec{\nabla}\times\vec{V} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} \hat{e}_1 h_1 & \hat{e}_2 h_2 & \hat{e}_3 h_3 \\ \partial_1 & \partial_2 & \partial_3 \\ h_1 V_1 & h_2 V_2 & h_3 V_3 \end{vmatrix} \text{ where, } \partial_i \overset{\text{def.}}{=} \frac{\partial}{\partial q_i} \tag{3.16}$$

Q 30. Convince yourself that Eqs. (3.14) & (3.16) are correct. Evaluate $\vec{\nabla}\cdot\hat{e}_1$ and $\vec{\nabla}\times\hat{e}_1$. Notice the unit vectors need not be constant always (though their magnitudes are constant by definition).

Q 31. The sets of vectors $\{\vec{A}, \vec{B}, \vec{C}\}$ and $\{\vec{a}, \vec{b}, \vec{c}\}$ are called **reciprocal sets of vectors** if,

$$\vec{A} = \frac{\vec{b}\times\vec{c}}{\vec{a}\cdot\vec{b}\times\vec{c}}; \quad \vec{B} = \frac{\vec{c}\times\vec{a}}{\vec{a}\cdot\vec{b}\times\vec{c}}; \quad \vec{C} = \frac{\vec{a}\times\vec{b}}{\vec{a}\cdot\vec{b}\times\vec{c}}, \tag{3.17}$$

$$\Rightarrow \vec{A}\cdot\vec{a} = \vec{B}\cdot\vec{b} = \vec{C}\cdot\vec{c} = 1, \quad \text{and} \quad \vec{A}\cdot\vec{b} = \vec{A}\cdot\vec{c} = \vec{B}\cdot\vec{a} = \vec{B}\cdot\vec{c} = \vec{C}\cdot\vec{a} = \vec{C}\cdot\vec{b} = 0 \tag{3.18}$$

If the volume contained by $\{\vec{A}, \vec{B}, \vec{C}\}$ is $V = \vec{A}\cdot\vec{B}\times\vec{C}$ then the volume contained by $\{\vec{a}, \vec{b}, \vec{c}\}$ is $v = \vec{a}\cdot\vec{b}\times\vec{c} = 1/V$. Actually $\{\partial\vec{r}/\partial q_i = h_i\hat{e}_i\}$ and $\{\vec{\nabla}q_j\}$ are reciprocal systems of vectors. Show that

(a) $\dfrac{\partial \vec{r}}{\partial q_i}\cdot\vec{\nabla}q_j = \delta_{ij}$, and

(b) $\left(\dfrac{\partial \vec{r}}{\partial q_1}\cdot\dfrac{\partial \vec{r}}{\partial q_2}\times\dfrac{\partial \vec{r}}{\partial q_3}\right)\left(\vec{\nabla}q_1\cdot\vec{\nabla}q_2\times\vec{\nabla}q_3\right) = 1.$

Q 32. When we change from $\{q_i, \ldots, q_n\}$ coordinates to $\{q'_i \ldots, q'_n\}$ coordinates, the volume element changes by what is called a **Jacobian determinant** or simply **Jacobian***:

$$\int_V dq'_1 \ldots dq'_n = \int_V J\left(\frac{q'_i, \ldots, q'_n}{q_i, \ldots, q_n}\right) dq_1 \ldots dq_n, \tag{3.19}$$

$$\text{where,} \quad J\left(\frac{q'_i, \ldots, q'_n}{q_i, \ldots, q_n}\right) = \begin{vmatrix} \frac{\partial q'_1}{\partial q_1} & \cdots & \frac{\partial q'_1}{\partial q_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial q'_n}{\partial q_1} & \cdots & \frac{\partial q'_n}{\partial q_n} \end{vmatrix} \text{ is the determinant of the Jacobian matrix.} \tag{3.20}$$

If (x_1, x_2, x_3) are the Cartesian components of the position vector in 3D, show,

(a) $J\left(\frac{x_1, x_2, x_3}{q_1, q_2, q_3}\right) = \left(\frac{\partial \vec{r}}{\partial q_1} \cdot \frac{\partial \vec{r}}{\partial q_2} \times \frac{\partial \vec{r}}{\partial q_3}\right) = h_1 h_2 h_3$, and

(b) $\left(\vec{\nabla} q_1 \cdot \vec{\nabla} q_2 \times \vec{\nabla} q_3\right) = J\left(\frac{q_1, q_2, q_3}{x_1, x_2, x_3}\right)$.

Do you see any connection with **Q 31.**?

Q 33. Consider a circular cylindrical coordinate system $(q_1 = \rho, q_2 = \phi, q_3 = z)$. Obtain the expressions for

(a) the position vector,

(b) the line element (confirm the expression you got by taking derivative of the position vector),

(c) the area elements for the three coordinate planes,

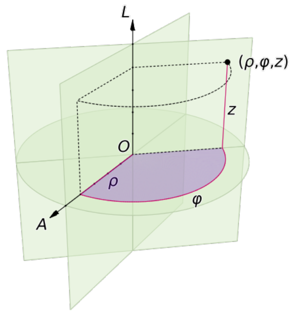
(d) volume element,

(e) the volume of a cylinder of height H and radius R by *explicit integration*,

(f) the area of the curved surface of the same cylinder by *explicit integration*,

(g) gradient, divergence, Laplacian, curl.

(h) Express $\hat{\rho}, \hat{\phi}, \hat{z}$ in terms of $\hat{x}, \hat{y}, \hat{z}$ and *vice versa*. Prove the coordinate system is orthogonal.



Q 34. Consider a spherical polar coordinate system $(q_1 = r, q_2 = \theta, q_3 = \phi)$. Obtain the expressions for

(a) the position vector,

(b) the line element (confirm the expression you got by taking derivative of the position vector),

(c) the area elements for the three coordinate planes,

(d) volume element,

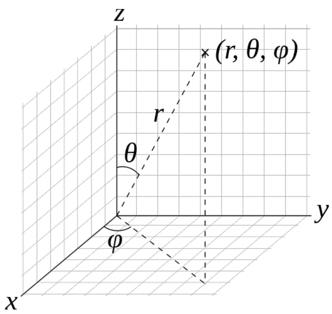
(e) the volume of a sphere of radius R by *explicit integration*,

(f) the outside area of a half-sphere (a sphere cut in two identical halves) of radius R by *explicit integration*,

(g) gradient, divergence, Laplacian, curl.

(h) Express $\hat{r}, \hat{\theta}, \hat{\phi}$ in terms of $\hat{x}, \hat{y}, \hat{z}$ and *vice versa*. Prove the coordinate system is orthogonal.

(i) How can we map latitude and longitude to two coordinates of a spherical polar coordinate system?



Q 35. Convert the spherical polar coordinates (r, θ, ϕ) into circular cylindrical coordinates (ρ, ϕ, z) and *vice versa*.

* en.wikipedia.org/wiki/Carl_Gustav_Jacob_Jacobi

Q 36. In Cartesian coordinates the velocity and the acceleration of any object can be written as

$$\vec{v} \stackrel{\text{def.}}{=} \dot{\vec{r}} = \dot{x}\,\hat{x} + \dot{y}\,\hat{y} + \dot{z}\,\hat{z}, \tag{3.21}$$

$$\vec{a} \stackrel{\text{def.}}{=} \ddot{\vec{r}} = \ddot{x}\,\hat{x} + \ddot{y}\,\hat{y} + \ddot{z}\,\hat{z}, \tag{3.22}$$

where 'a dot' stands for a derivative w.r.t. time,

$$\dot{f} \stackrel{\text{def.}}{=} \frac{df}{dt}. \tag{3.23}$$

How would you write velocity and acceleration in (a) circular circular cylindrical and (b) spherical polar coordinates?

Q 37. In spherical polar coordinates the directions of two vectors are (θ_1, ϕ_1) & (θ_2, ϕ_2). Show that the angle (γ) between the two vectors can be written as,

$$\cos\gamma = \cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2 \cos(\phi_1 - \phi_2).$$

Q 38. Let's consider **parity**. Any function of the coordinates is said to have *even parity* if it remains unchanged under an inversion of the coordinate system. Similarly, if it changes sign, it has *odd parity*. In Cartesian coordinates 'an inversion of the coordinate system' (i.e., $\vec{r} \to -\vec{r}$, or reflection though the origin) means $x \to -x$, $y \to -y$ and $z \to -z$.

(a) Express the 'reflection through the origin' in spherical polar coordinates.

(b) Show that $\hat{r}$ has odd parity. What about $\hat{\theta}$ and $\hat{\phi}$?

Q 39. A particle of mass m moves due to a **central force** (which depends only on r) following Newton's second law of motion,

$$m\ddot{\vec{r}} = \hat{r} f(r). \tag{3.24}$$

Show that $\vec{r} \times \dot{\vec{r}} = \vec{C}$, a constant vector. Do you understand this means that the particle moves in a plane (**planer motion**)?

Q 40. Explicitly integrate to obtain the area of the slanted surface of a cone of height H and radius R in

(a) spherical polar coordinates,
Hint: Arrange the cone such that $\theta =$ constant on the curved surface.

(b) circular cylindrical coordinates.
Hint: Figure out the line element on the surface, then use the area element $d\sigma_{ij} = \left|\frac{\partial \vec{r}}{\partial q_i} \times \frac{\partial \vec{r}}{\partial q_j}\right| dq_i dq_j$. Remember, on a surface only two of the three q_a's are independent, hence be careful while using relations like $\partial\vec{r}/\partial q_a = h_a \hat{e}_a$.

In 3D, a surface can be parametrized by two independent parameters (u, v). We can consider them as two generalized coordinates. For example, consider the surface passing through $(a,0,0)$, $(0,b,0)$, and $(0,0,c)$ (Cartesian coordinates). It can be parametrized by $(q_1 = u, q_2 = v)$ as,

$$\frac{x}{a} = 1 - u - v, \quad \frac{y}{b} = u, \quad \frac{z}{c} = v, \tag{3.25}$$

where $-\infty \le u, v \le \infty$. In the (u,v) coordinates, $(a,0,0) \to (0,0)$, $(0,b,0) \to (1,0)$ and $(0,0,c) \to (0,1)$. Now suppose we want to calculate the area of the triangle contained by $(a,0,0)$, $(0,b,0)$, and $(0,0,c)$. On the surface,

$$\vec{r} = a\,(1-u-v)\,\hat{x} + b\,u\,\hat{y} + c\,v\,\hat{z}. \tag{3.26}$$

The magnitude of the area element on the surface

$$d\sigma_{12} = \left|\frac{\partial \vec{r}}{\partial q_1} \times \frac{\partial \vec{r}}{\partial q_2}\right| dq_1 dq_2 = \left|\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right| du\,dv \tag{3.27}$$

Now,

$$\frac{\partial \vec{r}}{\partial u} = -a\,\hat{x} + b\,\hat{y}, \quad \frac{\partial \vec{r}}{\partial v} = -a\,\hat{x} + c\,\hat{z}. \tag{3.28}$$

$$d\sigma_{12} = \sqrt{a^2c^2 + a^2b^2 + b^2c^2}\; du\,dv \tag{3.29}$$

Hence the area of the triangle is

$$\text{Area } abc = \int_0^1 du \int_0^{1-u} dv \, \sqrt{a^2c^2 + a^2b^2 + b^2c^2} = \frac{1}{2}\sqrt{a^2c^2 + a^2b^2 + b^2c^2}. \tag{3.30}$$

Which, of course, can be checked from the well known Heron's formula[†]

$$\text{Area } abc = \sqrt{S(S-A)(S-B)(S-C)}, \tag{3.31}$$

where A, B, C are the sides and $S = (A+B+C)/2$ is the semiperimeter.

[†] en.wikipedia.org/wiki/Heron's_formula