Optimisation methods: Linear and Integer optimisation

Siddharth Bhat

Contents

Chapter 1

LP-relaxation

We know that $z^*_{\mathtt{IP}} \leqslant z^*_{\mathtt{LP}}$. We can solve an LPproblem using a solver.

1.1 Bipartite Matching

Let $G \equiv (V \equiv X \cup Y, E \subset V \times V, w : E \to \mathbb{R})$. Graph is:

- Undirected, so $(x, y) \in E \iff (y, x) \in E$, $w((v, v')) = w((v', v))$.
- Bipartite, so that $(v, v') \in E \implies (v \in X \wedge v' \in Y) \vee (v \in Y \wedge v' \in X)$
- It's a little annoying to write the condition, but basically, for every edge, there's a unique weight which we adjust, even though the graph is undirected.

We wish to find $M \subseteq E$ such that:

$$\max_{e \in M} w_e$$

Can be transformed to:

$$\max_{e \in E} x_e w_e \qquad x_e \in \{0, 1\}$$
$$\sum_{e \in E, e = (v, v')} x_e = 1 \qquad \forall v \in V$$

Where the x_e are variables to be discovered. We can now LP relax this, where $x_e \in [0, 1]$:

$$\max_{e \in E} x_e w_e \qquad x_e \in [0, 1]$$
$$\sum_{e \in E, e = (v, v')} x_e = 1 \qquad \forall v \in V$$

How do we go from the optimal solution to this problem, to an integer solution?

- Assume the LP is infeasible. This means that we have a vertex u such that $\sum_{e \in E, e=(u,v)} x_e = 1$ fails. that is, there's a vertex in u that is not connected to v. In this case, the IP is also infeasible.

- Now, we know that the LPis feasible. $a_1 \rightarrow b_1$ is not saturated means that $b_1 \rightarrow a_2$ is not saturated which implies that $a_2 \rightarrow b_2$ is not saturated, hence $b_2 \rightarrow a_1$ is not saturated. (TODO: add tikz picture). We can get a full cycle of edges with:

$$x_{e_i} < 1$$
$$x_{e_i} \in a_1 \xrightarrow{e_1} b_1 \xrightarrow{e_2} a_2 \xrightarrow{e_3} b_2 \xrightarrow{\dots} \xrightarrow{e_{i-1}} b_n \xrightarrow{e_i} a_1$$

  The number of edges here will be *even*. We can now pick a value $\epsilon \in (0, 1)$ such that:

$$y_e \equiv \begin{cases} x_e^* + \epsilon & i \text{ is even, } x_e \text{ is in the cycle} \\ x_e^* - \epsilon & i \text{ is odd, } x_e \text{ is in the cycle} \\ x_e^* & \text{otherwise} \end{cases}$$

  Note that y_e is a valid solution, since we can set ϵ to be smaller than the slack we had in the smallest value of x_i. We can show that the $\mathtt{cost}(y) \equiv \sum_{e \in E} w_e y_e$ is equal to:

$$\mathtt{cost}(y) = \mathtt{cost}(x_e^*) + \epsilon \left(\Delta \equiv \sum_{i=1}^{n} (-1)^i w(e_i) \right)$$

  Remember that x_e^* is the best solution, so we can have nothing better than $\mathtt{cost}(x_e^*)$. Hence, $\mathtt{cost}(y_e^*) \leqslant \mathtt{cost}(x_e^*)$, and hence, we are forced to conclude that $\Delta = 0$ (If $\Delta > 0$, pick $\epsilon > 0$, if $\Delta < 0$, pick $\epsilon < 0$).

  Hence, we can keep moving ϵ till an even edge becomes 1 (alternatively, and odd edge becomes 0). Hence, we can *keep rounding* till all our edges become $\{0, 1\}$.

So, we managed to start from an LP solution, and then *unrelax* it to construct an IP solution from it!

1.2 Min vertex cover

$G \equiv (V, E)$. We want to pick the smallest $F \subseteq V$, such that one end of all edges is in this cover.

$$\forall (u, v) \in E, u \in F \vee v \in F$$

Intuitively, these vertices $f \in F$ are watching over the edges, and each edge must be watched by at least one vertex.

TODO: add tikz picture

Integer program for the problem:

$x_v \in \{0, 1\}\ \forall v \in V \qquad \min \sum x_v \qquad \forall (u, v) \in E, x_u + x_v \geqslant 1$

LP relaxed program for the problem:

$$x_v \in [0,1]\ \forall v \in V \qquad \min \sum x_v \qquad \forall (u,v) \in E, x_u + x_v \geqslant 1$$

From the LP, we construct:

$$S_{LP} \equiv \left\{ u \,\middle|\, x^*_u \geqslant \frac{1}{2} \right\} \qquad \text{Claim: } S_{LP} \text{ is a vertex cover}$$

For each edge $(u,v) \in E$, since $u + v \geqslant 1$, we *cannot have that* $x_u < 0.5 \wedge x_v < 0.5$, since then $x_u + x_v < 1$. Hence, each edge will have one of its vertices with $x_{vertex} \geqslant 0.5$, and thus S_{LP} is **a** vertex cover.

We now show **optimality** of S_{LP}.

$$\texttt{LP} \leqslant \texttt{IP} \qquad \text{since the problem is a minimization problem}$$

$$\sum_{u \in V} x_u \leqslant \sum_{u \in V} y_u \qquad x \text{ is } \texttt{LP} \text{ solution, } y \text{ is } \texttt{IP} \text{ solution}$$

$$|S_{LP}| = \sum_{x \in S_{LP}} 1(\text{counting}) \leqslant \sum_{u \in S_{LP}} 2x_u(\text{definition of } S_{LP}) \leqslant \sum_{u \in V} 2x_u(\text{enlarging } S_{LP} \text{ to } V) \leqslant \sum_{u \in V} 2y_u = 2|s_{opt}|$$

$$|S_{opt}| \leqslant S_{LP} \leqslant 2|S_{opt}|$$

So, we are at worst twice the size of the best vertex cover.

1.3 Maximum independent set

HOMEWORK: read how this can be phrased as LP