0.1 Q1 – matrix representation for $|\phi_k\rangle\langle\phi_j|$, in the orthonormal $|v_i\rangle$ basis

Perform change of basis.

0.2 Q2 – positive operator is Hermitian

We first show that a positive operator is normal, and this automatically implies that it is Hermitian.

To show that a positive operator is normal, we consider $A^\dagger A$

Now that we know that it is normal, by spectral decomposition, it posesses an eigenbasis. We now show that all of its eigenvalues are real. This is now a matrix with real entries on the diagonal, which is hermitian.

To show that the eigenvalues are real, let $|\lambda\rangle$ be an eigenvector with magnitude 1 and eigenvalue λ.

$$\langle\lambda| A |\lambda\rangle \geqslant 0 \qquad \lambda\langle\lambda|\lambda\rangle = \lambda \geqslant 0$$

Hence, the eigenvalues are real and positive, and therefore it is Hermitian.

0.3 Q3 – $A^\dagger A$ is positive

$$\forall v \in V, \;\; \langle v| A^\dagger A |v\rangle = \langle Av| |Av\rangle = \|Av\|^2 \geqslant 0$$

Hence, $A^\dagger A$ is positive.

0.4 Q4. Eigenvalues of a projector P are either 0 or 1

Let $|\lambda\rangle$ be an eigenvector of P with associated eigenvalue λ.

$$P^2(|\lambda\rangle) = \lambda(P|\lambda\rangle) = \lambda^2|\lambda\rangle \qquad P(|\lambda\rangle) = \lambda|\lambda\rangle$$

However, since P is a projector, $P^2 = P$, and therefore, $\lambda^2 = \lambda$. The roots of this equation are $0, 1$. Hence, $\lambda \in \{0, 1\}$.

0.5 Q5. Tensor product of two unitary operators is unitary

Let U, V be unitary operators.

$$
\begin{aligned}
&\langle Uu \otimes Vv| \, |Uu \otimes Vv\rangle = \\
&\langle u \otimes v| \, (U^\dagger \otimes V^\dagger)(U \otimes V) \, |u \otimes v\rangle = \\
&\langle u \otimes v| \, (U^\dagger U \otimes V^\dagger V) \, |u \otimes v\rangle = \\
&\langle u \otimes v| \, I \otimes I \, |u \otimes v\rangle = \\
&\langle u \otimes v| \, |u \otimes v\rangle =
\end{aligned}
$$

Hence, $U \otimes V$ is unitary since it preserves inner products.

0.6 Q6. Tensor product of projectors is a projector

Let P, Q be projectors. $P \equiv \sum_{i=1}^{l} |i\rangle \langle i|$. $Q \equiv \sum_{j=1}^{k} |j\rangle \langle j|$.

$$
\begin{aligned}
P \otimes Q &\equiv (\sum_{i=1}^{l} |i\rangle \langle i|) \otimes (\sum_{j=1}^{k} |j\rangle \langle j|) \\
&\equiv \sum_{i=1}^{l} \sum_{j=1}^{k} |ij\rangle \langle ij|
\end{aligned}
$$

Which is in the form of a projector, in that it leaves $|ij\rangle$ unchanged, and sends every other vector to 0. So, it projects vectors onto the subspace spanned by $|ij\rangle$.

0.7 Q7. Find $\log$ and square root of matrix

$$
A = \begin{bmatrix} 4 & 3 \\ 3 & 4 \end{bmatrix}
$$

Finding eigenvalues,

$$|A - \lambda I| = 0 \quad (4-\lambda)^2 - 9 = 0 \quad \lambda = 7, 1$$

Finding eigenvectors,

$$v = (1/\sqrt{2}, -1/\sqrt{2}) \quad w = (1/\sqrt{2}, 1/\sqrt{2})$$

hence, we can now write $A = U^{-1}DU$, where U transforms from the original basis to the eigenbasis, as:

$$D \equiv \begin{bmatrix} 7 & 0 \\ 0 & 1 \end{bmatrix} \qquad U \equiv \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} U^{-1} \equiv \begin{bmatrix} 1 \end{bmatrix}$$

0.7.1 Computing square root

$$S = U^{-1}\sqrt{D}U \qquad \sqrt{D} = \begin{bmatrix} \sqrt{7} & 0 \\ 0 & 1 \end{bmatrix} \qquad S = \begin{bmatrix} 1/2 + \sqrt{7}/2 & -1/2 + \sqrt{7}/2 \\ -1/2 + \sqrt{7}/2 & 1/2 + \sqrt{7}/2 \end{bmatrix}$$

We prove that S is the square root, since:

$$S^2 = (U^{-1}\sqrt{D}U)(U^{-1}\sqrt{D}U) = U^{-1}(\sqrt{D})^2 U = U^{-1}DU = A$$

0.7.2 Computing log

We can now show that if U is unitary and D is diagonal, then:

$$L \equiv \log(U^{-1}DU) = U^{-1}\log DU \qquad \log D = \begin{bmatrix} \log 7 & 0 \\ 0 & 0 \end{bmatrix}$$

$$L = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \log 7 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$L = 1/2 \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \log 7 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$L = 1/2 \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \log 7 & \log 7 \\ 0 & 0 \end{bmatrix}$$

$$L = 1/2 \begin{bmatrix} \log 7 & \log 7 \\ \log 7 & \log 7 \end{bmatrix}$$

0.8 Q8. Trace properties

0.8.1 $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$

$$\mathrm{Tr}(AB) = \sum_z (AB)_{zz} = \sum_z \sum_k A_{zk} B_{kz} = \sum_z \sum_k B_{kz} A_{kz} = \sum_z (BA)_{zz} = \mathrm{Tr}(BA)$$

0.8.2 $\mathrm{Tr}(A + B) = \mathrm{Tr}(A) + \mathrm{Tr}(B)$

$$\mathrm{Tr}(A + B) = \sum_z (A + B)_{zz} = \sum_z A_{zz} + B_{zz} = \mathrm{Tr}(A) + \mathrm{Tr}(B)$$

0.8.3 $\mathrm{Tr}(2A) = 2\mathrm{Tr}(A)$

$$\mathrm{Tr}(2A) = \sum_z (2A)_{zz} = \sum_z 2A_{zz} 2 \sum_z A_{zz} = 2\,\mathrm{Tr}(A)$$

0.9 Commutator properties

0.9.1 $[A, B] = -[B, A]$

$$[A, B] = AB - BA = -(BA - AB) = -[B, A]$$

0.9.2 $\frac{[A,B]+\{A,B\}}{2} = AB$

$$\frac{[A, B] + \{A, B\}}{2} = \frac{(AB - BA) + (AB + BA)}{2} = AB$$

0.10 Express polar decomposition as outer product

0.11 Find left and right polar decomposition