

Game theory

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Contents

1	Introduction	5
1.1	Normal form games	5
1.1.1	Normal form for matching coins with observation	6
1.1.2	Normal form for matching coins without observation	6
1.1.3	Normal form for prisoners dilemma	6
2	Game analysis	7
2.0.1	Common knowledge and puzzles about common knowledge	7
2.0.2	Strongly dominated strategy	7
2.0.3	Strongly dominant strategy	7
2.0.4	Weakly dominated strategy	8
2.0.5	Weakly dominant strategy	8
2.0.6	Very weakly dominated strategy	9
2.0.7	Example of strong dominance in prisoners dilemma	9
2.0.8	Another game	9
2.0.9	Can there exist two weakly dominant strategies?	9
2.0.10	Strongly (Weakly) Dominant Strategy Equilibrium	9

Chapter 1

Introduction

TODO: find out and write about: Extrinsic form representation of a game $\Gamma \equiv \langle N, T, Z, o, A, s, u_i, \mathcal{H} \rangle$ where N is number of players, T is game tree, Z is leaves, o is owner function, A is actions, s is transition, u_i is player function, \mathcal{H} is information sets. An information set contains equivalence classes of states that the player cannot distinguish between. There can be ambiguity due to missing information. Perfect information game is one where all information sets are singleton. For example, chess is perfect information. Example of partial information is card games.

Next, we look at strategies. A strategy is a computable function by which each player selects their actions.

Game tree for matching coins with observation: $S_A : \{1\} \rightarrow \{H, T\}$. $S_B : \{2, 3\} \rightarrow \{H, T\}$.

Game tree for matching coins without observation: $S_A : \{1\} \rightarrow \{H, T\}$. $S_B : \{\{2, 3\}\} \rightarrow \{H, T\}$.

We let $S \equiv S_1 \times S_2 \times \dots \times S_n$. This is a set containing all possible strategies, called as a "strategy profile", $s \in S$. We write s as $s \equiv (s_i, s_{-i})$, where s_{-i} is cute notation for "the rest of the players". For example, if $s \equiv (s_1, s_2, s_3)$ we can notate $s = (s_2, s_{-2}) = (s_2, s_1, s_3)$.

We will currently focus on pure strategies, where we have a deterministic function per strategy.

1.1 Normal form games

Another representation for games is called as strategic form / matrix form / normal form games. Here, a game $\Gamma = \langle N, (S_i)_{i \in N}, (u_i : S \rightarrow \mathbb{R})_{i \in N} \rangle$. N is the number of players, S_i are strategies for each player, u_i are the utility functions / payoffs for each player. u_i maps each strategy profile s how worth it is for player i if the game proceeds with strategy profile s .

1.1.1 Normal form for matching coins with observation

$$\begin{aligned}
S_a &\equiv \{S_a^1(H), S_a^2(T)\} \\
S_b &\equiv \{S_b^1(HH), S_b^2(HT), S_b^3(TH), S_b^4(TT)\} \\
u_a : S &\rightarrow \mathbb{R} \\
u_a((s_a^1, s_b^1)) &= +10 & (H, HH) \\
u_a((s_a^1, s_b^2)) &= +10 & (H, HT) \\
u_a((s_a^1, s_b^3)) &= -10 & (H, TH) \\
u_a((s_a^1, s_b^4)) &= -10 & (H, TT) \\
u_a((s_a^2, s_b^1)) &= -10 & (T, HH) \\
u_a((s_a^2, s_b^2)) &= +10 & (T, HT) \\
u_a((s_a^2, s_b^3)) &= -10 & (T, TH) \\
u_a((s_a^2, s_b^4)) &= +10 & (T, TT)
\end{aligned}$$

Alternate representation of same game:

$s_b \rightarrow$	HH	HT	TH	TT
H	(10, -10)	(10, -10)	(-10, +10)	(-10, +10)
T	(-10, 10)	(10, -10)	(-10, 10)	(10, -10)

1.1.2 Normal form for matching coins without observation

$s_a \rightarrow$	H	T
H	(10, -10)	(-10, 10)
T	(-10, 10)	(10, -10)

1.1.3 Normal form for prisoners dilemma

C for confess, NC for not confess.

	C	NC
C	(-5, -5)	(-1, -10)
NC	(-10, -1)	(-2, -2)

Chapter 2

Game analysis

Rationality implies that each player is motivated to maximise his own payoff. Intelligent implies that player can take into account all available information. An intelligent and rational player implies that every player will attempt to maximise their utility.

2.0.1 Common knowledge and puzzles about common knowledge

Definition 1 Common knowledge — *Player knows it. Every player knows that every player knows it. Every player knows that every player knows that every player knows it. $\forall k \in \mathbb{N}, (\text{Every player knows that})^k$ every player knows it.*

If we have an island with two water streams and all humans and intelligent, rational and cannot speak. They have a rule that says that if a person has a blue mark on their forehead, the drink water from a stream farther away. One day, a visitor, who knows the above fact, shouts "why is a person with a blue mark drinking water here?" The next day, no one comes to the stream. What changed? The only difference before and after is that it is now common knowledge that there is one person with a blue mark drinking water at the stream. This

Some one imagined two positive whole numbers $1 \leq a, b \leq 20$. He tells the sum of these two numbers to mathematician A, the product of these numbers to mathematician B. A tells B that there is no way for B to know the sum. Then B exclaims "But I know the sum now!", to which A exclaims "and now I know the product".

2.0.2 Strongly dominated strategy

Given a game $\Gamma \equiv \langle N, (S_i), (u_i) \rangle$, a strategy $s_i \in S_i$ is said to be strongly dominated by a strategy $s'_i \in S_i$ if:

$$u_i(s_i, s_{-i}) < u_i(s'_i, s_{-i}) \quad \forall s_{-i} \in S_{-i}$$

2.0.3 Strongly dominant strategy

Given a game $\Gamma \equiv \langle N, (S_i), (u_i) \rangle$, a strategy s_i^* is said to be strongly dominant if it strongly dominates every other strategy $s_i \in S_i$.

$$\forall s_i \in S_i, s_i \neq s_i^* \implies u_i(s_i, s_{-i}) < u_i(s_i^*, s_{-i}) \quad \forall s_{-i} \in S_{-i},$$

Note that to analyze strongly dominated and strongly dominant strategies, we only need u_i , the utility of the i th player. Hence, to analyze dominance of strategies, we can get away with writing the utility of just a single player.

	L	R
A	6, -	7, -
B	5, -	6, -
C	4, -	6, -

L, R are moves of the player. A, B, C are strategies with utilities filled in.

A strongly dominates C, A strongly dominates B. B does *not* strongly dominate C, since on the R action, we have 6 for both B and C. Hence, A is the strongly dominant strategy. Note that there need not always exist a strongly dominant strategy:

	L	R
A	6, -	7, -
B	7, -	6, -

Neither A nor B are strictly better than the other.

2.0.4 Weakly dominated strategy

Given a game $\Gamma \equiv \langle N, (S_i), (u_i) \rangle$, a strategy $s_i \in S_i$ is said to be weakly dominated by a strategy $s_i' \in S_i$ if:

$$u_i(s_i, s_{-i}) \leq u_i(s_i', s_{-i}) \quad \forall s_{-i} \in S_{-i}$$

with strict inequality for at least one s_{-i} .

2.0.5 Weakly dominant strategy

Given a game $\Gamma \equiv \langle N, (S_i), (u_i) \rangle$, a strategy s_i^* is said to be weakly dominant if it weakly dominates every other strategy $s_i \in S_i$.

$$\forall s_i \in S_i, s_i \neq s_i^* \implies u_i(s_i, s_{-i}) \leq u_i(s_i^*, s_{-i}) \quad \forall s_{-i} \in S_{-i},$$

with strict inequality for at least one s_{-i} .

Once again, a weakly dominant strategy need not always exist:

	L	R
A	6, -	7, -
B	7, -	6, -

2.0.6 Very weakly dominated strategy

Given a game $\Gamma \equiv \langle N, (S_i), (u_i) \rangle$, a strategy $s_i \in S_i$ is said to be very weakly dominated by a strategy $s'_i \in S_i$ if:

$$u_i(s_i, s_{-i}) \leq u_i(s'_i, s_{-i}) \quad \forall s_{-i} \in S_{-i}$$

Note that we do not have the strict inequality requirement anymore. This is now a true partial order.

2.0.7 Example of strong dominance in prisoners dilemma

	C	NC
C	(-5, -5)	(-1, -10)
NC	(-10, -1)	(-2, -2)

Here, C is the strongly dominant strategy for both players.

2.0.8 Another game

	a	b	c
A	5	5	5
B	4	5	5
C	4	4	4

There does not exist a strongly dominant strategy. A is weakly dominant. C is weakly dominated by both A, B.

2.0.9 Can there exist two weakly dominant strategies?

No there cannot. If A is a weakly dominant strategies, then assume $A[i] > B[i]$. If $A[i] > B[i]$, then B cannot weakly dominate A, since for B to dominate A we need $B[j] \geq A[j] \quad \forall j$, but $B[i] < A[i]$.

2.0.10 Strongly (Weakly) Dominant Strategy Equilibrium

A strategy profile $(s_1^*, s_2^*, \dots, s_n^*)$ is called as a strongly dominant strategy equilibrium of the game $\Gamma \equiv \langle N, (S_i), (u_i) \rangle$ iff the strategy s_i^* is a strongly dominating strategy for player i.