Topics in Physics - C. Mukku

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Contents

Chapter 1

Lagrangian, Hamiltonian mechanics

Mechanics in terms of generalized coords.

1.1 Lagrangian

Define a functional. L over the config. space of partibles q^i, $qdot^i$. $L = L(q^i, qdot^i)$. We have an explicit dependence on t.

$L = KE - PE$

Assuming a 1-particle system of unit mass,

$$L = \frac{1}{2}\dot{q}^2 - V(q)$$

Assuming an n-particle system of unit mass,

$$L = \sum_i \frac{1}{2}qdot^{i2} - V(q^i)$$

1.2 Variational principle

Take a minimum path from A to B. Now notice that the path that is slightly different from this path will have some delta from the minimum.

Action

$$S(t0, t1) = \int L \mathrm{d}t = \int_{t0}^{t1} L(q^i, qdot^i)\mathrm{d}t$$

. Least action: $\delta S = 0$

Chapter 2

Functional calculus

this chapter develops a completely handwavy physics version of functional analysis.

Definition 1 *A **functional** F is a function: $F : (\mathbb{R} \to \mathbb{R}) \to \mathbb{R}$*

Notation 1 *Evaluation of a functional F with respect to f is denoted by $F[f]$.*

2.1 Functional Derivative - take 1

Consider a functional $F : (\mathbb{R} \to \mathbb{R}) \to \mathbb{R}$, a function $f : \mathbb{R} \to \mathbb{R}$, and a "test function" $\phi : \mathbb{R} \to \mathbb{R}$.

Consider a functional F. We only define the derivative of a functional F with respect to a function f by what happens under an integral sign as follows:

$$\int \frac{\delta F}{\delta f}(x)\phi(x)\mathrm{d}x = \lim_{\epsilon \to 0} \frac{F[f + \epsilon\phi] - F[f]}{\epsilon}$$

Now, we can define a small variation in F as:

$$\delta F : (\mathbb{R} \to \mathbb{R}) \times (\mathbb{R} \to \mathbb{R}) \to \mathbb{R}$$
$$\delta F(f, \phi) \equiv \int \frac{\delta F}{\delta f}(x)\phi(x)\mathrm{d}x$$

Intuitively, δF tells us the variation of the function f along a test function ϕ. So, it encapsulates some kind of "directional derivative".

So, we can look at $\frac{\delta F}{\delta f}$ as a functional as follows:

$$\frac{\delta F}{\delta f} : (\mathbb{R} \to \mathbb{R}) \to \mathbb{R}$$
$$\frac{\delta F}{\delta f}(\phi) = \delta F(f, \phi)$$

Wehre $\frac{\delta F}{\delta f}$ allows us to "test" the change of F with respect to f along a given "direction" ϕ.

2.2 Functional Derivative as taught in class

Substitute $\phi = \delta(x - p)$. Now, the quantity:

$$\frac{\delta F}{\delta f}\phi(x) = \delta F(f, \delta(x - p))$$

Rewriting δF by sticking it under an integral:

$$\int \frac{\delta F}{\delta f}(x)\delta(x - p)\mathrm{d}x = \lim_{\epsilon \to 0} \frac{F[f + \epsilon\delta(x - p)] - F[f]}{\epsilon}$$

$$\left.\frac{\delta F}{\delta f}\right|_p = \lim_{\epsilon \to 0} \frac{F[f + \epsilon\delta(x - p)] - F[f]}{\epsilon}$$

That is, we can start talking about "derivative of the functional F with respect to a function f at a point p" as long as we only test the functional F against δ-functions.

So, we can alternatively define this quantity as:

$$\left.\frac{\delta F}{\delta f}\right|_p \equiv \lim_{\epsilon \to 0} \frac{F[f + \epsilon\delta(x - p)] - F[f]}{\epsilon}$$

While this does not "look like a functional", it actually is, if we mentally replace:

$$p \to \int \underline{\hspace{1cm}} \delta(x - p)\mathrm{d}x$$

This is how mukku got that expression.

2.3 Common functional derivatives

2.3.1 Derivative of $F[\phi] \equiv \int(\frac{\partial\phi}{\partial y})^2$

$\left.\frac{\delta F}{\delta f}\right|_p = \int(\frac{\partial\phi}{\partial y})^2$

2.4 Deriving E-L from functional magic

Chapter 3

Maxwell's equations in Minkowski space

Let us first review Maxwell's equations:

$\boldsymbol{\nabla} \cdot E = \dfrac{\rho}{\epsilon_0}$ (Electric charges produce fields)

$\boldsymbol{\nabla} \cdot B = 0$ (Only magnetic dipoles exist)

$\boldsymbol{\nabla} \times E = -\dfrac{\partial B}{\partial t}$ (Lenz Law - time varying magnetic field induces current that opposes it)

$\boldsymbol{\nabla} \times B = \mu_0 \left(J + \epsilon_0 \dfrac{\partial E}{\partial t} \right)$ (Ampere's law + fudge factor)