# Game theory

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# Chapter 1

# Introduction

TODO: find out and write about: Extrinsic form representation of a game $\Gamma \equiv \langle N, T, Z, o, A, s, u_i, \mathcal{H} \rangle$ where $N$ is number of players, $T$ is game tree, $Z$ is leaves, $o$ is owner function, $A$ is actions, $s$ is transition, $u_i$ is player function, $\mathcal{H}$ is information sets. An information set contains equivalence classes of states that the player cannot distinguish between. There can be ambiguity due to missing information. Perfect information game is one where all information sets are singleton. For example, chess is perfect information. Example of partial information is card games.

Next, we look at strategies. A strategy is a computable function by which each player selects their actions.

Game tree for matching coins with observation: $S_A : \{1\} \to \{H, T\}$. $S_B : \{2, 3\} \to \{H, T\}$.

Game tree for matching coins without observation: $S_A : \{1\} \to \{H, T\}$. $S_B : \{\{2, 3\}\} \to \{H, T\}$.

We let $S \equiv S_1 \times S_2 \times \cdots \times S_n$. This a set containing all possible strategies, called as a "strategy profile", $s \in S$. We write $s$ as $s \equiv (s_i, s_{-i})$, where $s_{-i}$ is cute notation for "the rest of the players". For example, if $s \equiv (s_1, s_2, s_3)$ we can notate $s = (s_2, s_{-2}) = (s_2, s_1, s_3)$.

We will currently focus on pure strategies, where we have a deterministic function per strategy.

## 1.1 Normal form games

Another representation for games is called as strategic form / matrix form / normal form games. Here, a game $\Gamma \equiv \langle N, (S_i)_{i \in N}, (u_i : S \to \mathbb{R})_{i \in N} \rangle$. $N$ is the number of players, $S_i$ are strategies for each player, $u_i$ are the utility functions / payoffs for each player. $u_i$ maps each strategy profile $s$ how worth it it is for player $i$ if the game proceeds with strategy profile $s$.

### 1.1.1 Normal form for matching coins with observation

$$
\begin{aligned}
&S_a \equiv \{S_a^1(H), S_a^2(T)\} \\
&S_b \equiv \{S_b^1(HH), S_b^2(HT), S_b^3(TH), S_b^4(TT)\} \\
&u_a : S \to \mathbb{R} \\
&u_a((s_a^1, s_b^1)) = +10 \qquad (H, HH) \\
&u_a((s_a^1, s_b^2)) = +10 \qquad (H, HT) \\
&u_a((s_a^1, s_b^3)) = -10 \qquad (H, TH) \\
&u_a((s_a^1, s_b^4)) = -10 \qquad (H, TT) \\
&u_a((s_a^2, s_b^1)) = -10 \qquad (T, HH) \\
&u_a((s_a^2, s_b^2)) = +10 \qquad (T, HT) \\
&u_a((s_a^2, s_b^3)) = -10 \qquad (T, TH) \\
&u_a((s_a^2, s_b^4)) = +10 \qquad (T, TT)
\end{aligned}
$$

Alternate representation of same game:

| $s_b \rightarrow$ | HH | HT | TH | TT |
|---|---|---|---|---|
| H | (10, -10) | (10, -10) | (-10, +10) | (-10, +10) |
| T | (-10, 10) | (10, -10) | (-10, 10) | (10, -10) |

### 1.1.2 Normal form for matching coins without observation

| $s_a \rightarrow$ | H | T |
|---|---|---|
| H | (10, -10) | (-10, 10) |
| T | (-10, 10) | (10, -10) |

### 1.1.3 Normal form for prisoners dilemma

C for confess, NC for not confess.

| | C | NC |
|---|---|---|
| C | (-5, -5) | (-1, -10) |
| NC | (-10, -1) | (-2, -2) |

# Chapter 2

# Game analysis

Rationality implies that each player is motivated to maximise his own payoff. Intelligent implies that player can take into account all available information. An intelligent and rational player implies that every player will attempt to maximise their utility.

### 2.0.1 Common knowledge and puzzles about common knowledge

**Definition 1** Common knowledge — *Player knows it. Every player knows that every player knows it. Every player knows that every player knows that every player knows it.* $\forall k \in \mathbb{N}$, $($*Every player knows that*$)^k$ *every player knows it.*

If we have an island with two water streams and all humans and intelligent, rational and cannot speak. They have a rule that says that if a person has a blue mark on their forehead, the drink water from a stream farther away. One day, a visitor, who knows the above fact, shouts "why is a person with a blue mark drinking water here?" The next day, no one comes to the stream. What changed? The only difference before and after is that it is now common knowledge that there is one person with a blue mark drinking water at the stream. This

Some one imagined two positive whole numbers $1 \leqslant a, b \leqslant 20$. He tells the sum of these two numbers to mathematician $A$, the product of these numbers to mathematician $B$. $A$ tells $B$ that there is no way for $B$ to know the sum. Then $B$ exclaims "But I know the sum now!", to which $A$ exclaims "and now I know the product".

### 2.0.2 Strongly dominated strategy

Given a game $\Gamma \equiv \langle N, (S_i), (u_i) \rangle$, a strategy $s_i \in S_i$ is said to be strongly dominated by a strategy $s_i' \in S_i$ if:

$$u_i(s_i, s_{-1}) < u_i(s_i', s_{-i}) \ \forall s_{-i} \in S_{-i}$$

### 2.0.3 Strongly dominant strategy

Given a game $\Gamma \equiv \langle N, (S_i), (u_i) \rangle$, a strategy $s_i^*$ is said to be strongly dominant if it strongly dominates every other strategy $s_i \in S_i$.

$$\forall s_i \in S_i, s_i \neq s_i^\star \implies u_i(s_i, s_{-1}) < u_i(s_i^\star, s_{-i}) \; \forall s_{-i} \in S_{-i},$$

Note that to analyze strongly dominated and strongly dominant strategies, we only need $u_i$, the utility of the $i$th player. Hence, to analyze dominance of strategies, we can get away with writing the utility of just a single player.

| | L | R |
|---|---|---|
| A | 6, - | 7, - |
| B | 5, - | 6, - |
| C | 4, - | 6, - |

$L, R$ are moves of the player. $A, B, C$ are stratgies with utilities filled in.

$A$ strongly dominate $C$, $A$ strongly dominates $B$. $B$ does *not* strongly dominate $C$, since on the $R$ action, we have $6$ for both $B$ and $C$. Hence, $A$ is the strongly dominant strategy. Note that there need not always exist a strongly dominant strategy:

| | L | R |
|---|---|---|
| A | 6, - | 7, - |
| B | 7, - | 6, - |

Neither $A$ nor $B$ are strictly better than the other.

### 2.0.4 Weakly dominated strategy

Given a game $\Gamma \equiv \langle N, (S_i), (u_i) \rangle$, a strategy $s_i \in S_i$ is said to be weakly dominated by a strategy $s_i' \in S_i$ if:

$$u_i(s_i, s_{-1}) \leqslant u_i(s_i', s_{-i}) \; \forall s_{-i} \in S_{-i}$$

with strict inequality for at least one $s_{-i}$.

### 2.0.5 Weakly dominant strategy

Given a game $\Gamma \equiv \langle N, (S_i), (u_i) \rangle$, a strategy $s_i^\star$ is said to be weakly dominant if it weakly dominates every other strategy $s_i \in S_i$.

$$\forall s_i \in S_i, s_i \neq s_i^\star \implies u_i(s_i, s_{-1}) \leqslant u_i(s_i^\star, s_{-i}) \; \forall s_{-i} \in S_{-i},$$

with strict inequality for at least one $s_{-i}$.

Once again, a weakly dominant strategy need not always exist:

| | L | R |
|---|---|---|
| A | 6, - | 7, - |
| B | 7, - | 6, - |

### 2.0.6 Very weakly dominated strategy

Given a game $\Gamma \equiv \langle N, (S_i), (u_i) \rangle$, a strategy $s_i \in S_i$ is said to be very weakly dominated by a strategy $s_i' \in S_i$ if:

$$u_i(s_i, s_{-1}) \leqslant u_i(s_i', s_{-i}) \ \forall s_{-i} \in S_{-i}$$

Note that we do not have the strict inequality requirement anymore. This is now a true partial order.

### 2.0.7 Example of strong dominance in prisoners dilemma

| | C | NC |
|---|---|---|
| C | (-5, -5) | (-1, -10) |
| NC | (-10, -1) | (-2, -2) |

Here, $C$ is the strongly dominant strategy for both players.

### 2.0.8 Another game

| | a | b | c |
|---|---|---|---|
| A | 5 | 5 | 5 |
| B | 4 | 5 | 5 |
| C | 4 | 4 | 4 |

There does not exist a strongly dominant strategy. $A$ is weakly dominant. $C$ is weakly dominated by both $A, B$.

### 2.0.9 Can there exist two weakly dominant strategies?

No there cannot. If $A$ is a weakly dominant strategies, then assume $A[i] > B[i]$.If $A[i] > B[i]$, then $B$ cannot weakly dominate $A$, since for $B$ to dominate $A$ we need $B[j] \geqslant A[j] \ \forall j$, but $B[i] < A[i]$.

### 2.0.10 Strongly (Weakly) Dominant Strategy Equilibrium

A strategy profile $(s_1^\star, s_2^\star, \ldots, s_n^\star)$ is called as a strongly dominant strategy equilibrium of the game $\Gamma \equiv \langle N, (S_i), (U_i) \rangle$ iff the strategy $s_i^\star$ is a strongly dominating strategy for player $i$.

Recall the example of Prisoners dilemma —

| | C | NC |
|---|---|---|
| C | (-5, -5) | (-1, -10) |
| NC | (-10, -1) | (-2, -2) |

## 2.1 Minimax equilibrium

## 2.2 Mixed strategy normal form games

## 2.3 Best response function

Let $\Gamma \equiv (N, S_{i \in N}, u : \prod_i S_i \to R)$ be a normal for game. We define the best response function for player $i$ as:

$$r_i : S_{-i} \to S_i \qquad r_i(s_{-i}) \equiv \underset{s_i \in S_i}{\operatorname{argmax}}\, u(s_i, s_{-i})$$

That is, under a fixed strategy for all of $i$'s opponents, the best response function $r_i(s_{-i})$ tells us the best move we can make.

## 2.4 Nash equilibrium

A strategy profile $s \equiv (s_1, s_2, \ldots s_n)$ is a Nash equilibrium if $s_i$ is the best response to $s_{-i}$ for all $i$.

## 2.5 Kakutani fixed point theorem

consider a relation $R : \Sigma \rightrightarrows \Sigma$ such that:

- $\Sigma$ is compact, convex, and nonempty.
- $R(\sigma)$ is nonempty.
- $R(\sigma)$ is convex.
- $R$ has a closed graph.

under these conditions, $R$ is guaranteed to have a fixed point.

## 2.6 Existence of Nash equilibrium

We prove this using the Kakutani fixed point theorem.

consider the correspondence

$$R : Sigma \to 2^{Sigma} \qquad R(\sigma) \equiv [r_i(\sigma_{-i})]_{i \in N}$$

That is, for each strategy profile $\sigma$, we relate it to the set of strategy profiles that are best responses against this profile.

Now note that a nash equilibrium is a fixed point of this relation. We will show that the above relation $R$ satisfies the assumptions of the Kakatuni fixed point theorem, and is hence a fixed point.

## 2.7 LP formulation for 2 player zero sum game

## 2.8 Lemke Howson theorem for computing mixed-strategy nash eqm

## 2.9 Mechanism design

We consider a setting:

$$\begin{aligned}
&N \equiv \text{Set of players} \\
&X \equiv \text{Alternatives} \\
&\Theta_{i \in N} \equiv \text{private information of player } i \\
&\phi : \prod_i \Theta_i \to \mathbb{R} \equiv \text{prior probability over private information} \\
&u_{i \in N} : \prod_i \Theta_i \times X \to \mathbb{R} \equiv \text{utility of player } i
\end{aligned}$$

A mechanism is a tuple:

$$\begin{aligned}
&\Gamma \equiv (S_{i \in N}, g) \\
&S_{i \in N} \equiv \text{strategies of each player } i \\
&g : \prod_i S_i \to X \equiv \text{an outcomes function}
\end{aligned}$$

A mechanism tells us what actions a player can take, and what the outcome of each action is.

We often want the mechanism to be designed to optimised a particular social good:

$$f : \prod_i \Theta_i \to X \equiv \text{Social choice function}$$

That is, given knowledge of what players *truly want*, we know what outcome we are interested in. In this setting, we are then to design a mechanism $(S_i, g)$ which will encourage a situation where rational players will take action dictated by $f$.

Sometimes, we will denote $S_i$ by $\hat{\Theta}_i$, since any action the player can take can be mathematically modeled as a deterministic function of their state. Hence, if we allow players to lie and publish a $\hat{\Theta}_i$, which is their reporting of their private $\Theta_i$, it's equivalent to a setting where players have arbitrary $S_i$.