

Topics in Physics - C. Mukku

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Chapter 1

Lagrangian, Hamiltonian mechanics

Mechanics in terms of generalized coords.

1.1 Lagrangian

Define a functional. L over the config. space of partibles q^i, \dot{q}^i . $L = L(q^i, \dot{q}^i)$. We have an explicit dependence on t .

$$L = KE - PE$$

Assuming a 1-particle system of unit mass,

$$L = \frac{1}{2}\dot{q}^2 - V(q)$$

Assuming an n-particle system of unit mass,

$$L = \sum_i \frac{1}{2}\dot{q}^{i2} - V(q^i)$$

1.2 Variational principle

Take a minimum path from A to B . Now notice that the path that is slightly different from this path will have some delta from the minimum.

Action

$$S(t_0, t_1) = \int_{t_0}^{t_1} L dt = \int_{t_0}^{t_1} L(q^i, \dot{q}^i) dt$$

. Least action: $\delta S = 0$

Chapter 2

Functional calculus

this chapter develops a completely handwavy physics version of functional analysis.

Definition 1 A *functional* F is a function: $F : (\mathbb{R} \rightarrow \mathbb{R}) \rightarrow \mathbb{R}$

Notation 1 Evaluation of a functional F with respect to f is denoted by $F[f]$.

2.1 Functional Derivative - take 1

Consider a functional $F : (\mathbb{R} \rightarrow \mathbb{R}) \rightarrow \mathbb{R}$, a function $f : \mathbb{R} \rightarrow \mathbb{R}$, and a "test function" $\phi : \mathbb{R} \rightarrow \mathbb{R}$.

Consider a functional F . We only define the derivative of a functional F with respect to a function f by what happens under an integral sign as follows:

$$\int \frac{\delta F}{\delta f}(x) \phi(x) dx = \lim_{\epsilon \rightarrow 0} \frac{F[f + \epsilon \phi] - F[f]}{\epsilon}$$

Now, we can define a small variation in F as:

$$\begin{aligned} \delta F &: (\mathbb{R} \rightarrow \mathbb{R}) \times (\mathbb{R} \rightarrow \mathbb{R}) \rightarrow \mathbb{R} \\ \delta F(f, \phi) &\equiv \int \frac{\delta F}{\delta f}(x) \phi(x) dx \end{aligned}$$

Intuitively, δF tells us the variation of the function f along a test function ϕ . So, it encapsulates some kind of "directional derivative".

So, we can look at $\frac{\delta F}{\delta f}$ as a functional as follows:

$$\begin{aligned} \frac{\delta F}{\delta f} &: (\mathbb{R} \rightarrow \mathbb{R}) \rightarrow \mathbb{R} \\ \frac{\delta F}{\delta f}(\phi) &= \delta F(f, \phi) \end{aligned}$$

Wehre $\frac{\delta F}{\delta f}$ allows us to "test" the change of F with respect to f along a given "direction" ϕ .

2.2 Functional Derivative as taught in class

Substitute $\phi = \delta(x - p)$. Now, the quantity:

$$\frac{\delta F}{\delta f} \phi(x) = \delta F(f, \delta(x - p))$$

Rewriting δF by sticking it under an integral:

$$\int \frac{\delta F}{\delta f}(x) \delta(x - p) dx = \lim_{\epsilon \rightarrow 0} \frac{F[f + \epsilon \delta(x - p)] - F[f]}{\epsilon}$$

$$\left. \frac{\delta F}{\delta f} \right|_p = \lim_{\epsilon \rightarrow 0} \frac{F[f + \epsilon \delta(x - p)] - F[f]}{\epsilon}$$

That is, we can start talking about "derivative of the functional F with respect to a function f at a point p " as long as we only test the functional F against δ -functions.

So, we can alternatively define this quantity as:

$$\left. \frac{\delta F}{\delta f} \right|_p \equiv \lim_{\epsilon \rightarrow 0} \frac{F[f + \epsilon \delta(x - p)] - F[f]}{\epsilon}$$

While this does not "look like a functional", it actually is, if we mentally replace:

$$p \rightarrow \int -\delta(x - p) dx$$

This is how mukku got that expression.

2.3 Common functional derivatives

2.3.1 Derivative of $F[\phi] \equiv \int (\frac{\partial \phi}{\partial y})^2$

$$\left. \frac{\delta F}{\delta f} \right|_p = \int (\frac{\partial \phi}{\partial y})^2$$

2.4 Deriving E-L from functional magic

Chapter 3

Maxwell's equations in Minkowski space

Let us first review Maxwell's equations:

$$\nabla \cdot E = \frac{\rho}{\epsilon_0} \text{ (Electric charges produce fields)}$$

$$\nabla \cdot B = 0 \text{ (Only magnetic dipoles exist)}$$

$$\nabla \times E = -\frac{\partial B}{\partial t} \text{ (Lenz Law - time varying magnetic field induces current that opposes it)}$$

$$\nabla \times B = \mu_0 \left(J + \epsilon_0 \frac{\partial E}{\partial t} \right) \text{ (Ampere's law + fudge factor)}$$

3.1 Constructing F , or Tensorifying Maxwell's equations

Begin with the equation that $\nabla \cdot B = 0$. This tells that B can be written as the curl of some other field — $B = \nabla \times A$. Hence

$$\boxed{B^i = \mathcal{E}^{ijk} \partial_j A^k} \tag{3.1}$$

Next, take $\nabla \times E = -\frac{\partial B}{\partial t}$.

$$\nabla \times E = -\frac{\partial B}{\partial t} = \frac{\partial(\nabla \times A)}{\partial t} = \nabla \times \frac{\partial A}{\partial t}$$

$$\nabla \times \left(E + \frac{\partial A}{\partial t} \right) = 0$$

writing this as the divergence of some field ϕ scaled by $\alpha : \mathbb{R}$

$$E + \frac{\partial A}{\partial t} = \alpha(\nabla \cdot \phi)$$

$$E = \alpha \nabla \cdot \phi - \frac{\partial A}{\partial t}$$

Since electrostatics is time-independent, we choose to think of $\alpha = -1$, so we can interpret ϕ as the potential.

$$E^i = -\frac{\partial\phi}{\partial x^k}g^{ik} - \frac{\partial A^i}{\partial t} \quad (3.2)$$

A slight reformulation (since we know that in Minkowski space, $\partial_t = \partial_0$) we get the equation:

$$\boxed{E^i = -g^{ik}\partial_k\phi - \partial_0 A^i} \quad (3.3)$$

We get the metric g^{ik} involved to raise the covariant $\frac{\partial\phi}{\partial x^k}$ into the contravariant E^i .

(Sid question: how does one justify switching $\nabla \times$ and ∂ ? It feels like some algebra)

Here be magic! We define A new rank-2 tensor in Minkowski space-time, called F (for Faraday),

$$\boxed{F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu} \quad (3.4)$$

(Sid question: why is this object $F_{\mu\nu}$ covariant? What does this *mean*?)

Lemma 1 $F_{\mu\nu}$ is antisymmetric.

Lemma 2 $F_{\mu\nu}$ has 6 degrees of freedom

Proof. Number of degrees of freedom of F :

$$\frac{4^2 \text{ (total)} - 4 \text{ (diagonal)}}{2 \text{ (anti-symmetry)}} = 6$$

Notice that F is a 1-form!

3.2 Expressing B , E in terms of F

We now wish to re-express B^{ij} and E^{ij} in terms of F , so that this F captures all of maxwell's equations.

$$B^i = \mathcal{E}^{ijk}\partial_j A^k = \mathcal{E}^{ikj}\partial_k A^j \quad \text{by } k, j \text{ being free variables}$$

$$B^i = \frac{1}{2} \left(\mathcal{E}^{ijk}\partial_j A^k + \mathcal{E}^{ikj}\partial_k A^j \right)$$

Substituting $\partial_j A_k - \partial_k A_j = F_{jk}$,

$$B^i = \frac{1}{2} \mathcal{E}^{ijk} F_{jk}$$

So, B in terms of F is:

$$\boxed{B^i = \frac{1}{2} \mathcal{E}^{ijk} F_{jk}} \quad (3.5)$$

Similarly, we wish to write E in terms of F . The algebra is as follows:

$$\begin{aligned} E^i &= -g^{ik} \partial_k \phi - \partial_0 A^i \\ E^i &= -g^{ik} \partial_k \phi - \partial_0 g^{ik} A_k && \text{Is this allowed? Am I always allowed to insert the } g_{ik}? \\ E^i &= -g^{ik} (\partial_k \phi + \partial_0 A_k) \end{aligned}$$

Since $k = \{1, 2, 3\}$ (k is spacelike coordinates), and we would like to relate ϕ with A (to unify E), we **set**:

$$\boxed{A_0 \equiv -\phi} \quad (3.6)$$

Continuing the derivation,

$$\begin{aligned} E^i &= -g^{ik} (\partial_k (-A_0) + \partial_0 A_k) \\ E^i &= -g^{ik} (\partial_0 A_k - \partial_k A_0) \\ E^i &= -g^{ik} F_{0k} \end{aligned}$$

So, finally, the relation is:

$$\boxed{E^i = -g^{ik} F_{0k}} \quad (3.7)$$

TODO: Find out how $E^i = cF^{i0}$

3.3 Other ramifications of Maxwell's equations on F

3.3.1 Ramification 1

First part, using 4th equation

We next consider the 4th Maxwell equation:

$$\begin{aligned} \nabla \times B &= \mu_0 J + \epsilon_0 \mu_0 \frac{\partial E}{\partial t} \\ \nabla \times B &= \mu_0 J + \frac{1}{c^2} \frac{\partial E}{\partial t} \\ &\text{Converting to indices,} \\ (\nabla \times B)^i &= \mu_0 J^i + \frac{1}{c} \frac{\partial E^i}{\partial ct} && \text{(From } \partial_{ct} = \frac{1}{c} \partial_t) \\ &= \mu_0 J^i + \frac{1}{c} \frac{\partial E^i}{\partial X^0} \\ &= \mu_0 J^i + \frac{\partial F^{i0}}{\partial X^0} && \text{(From } E^i = cF^{i0}) \\ &= \mu_0 J^i + \partial_0 F^{i0} \end{aligned}$$

Now, we start to simplify the LHS, $\nabla \times B$:

$$\begin{aligned} (\nabla \times B)^i &= \mathcal{E}^{ijk} \partial_j B_k \\ \text{Since } B^k &= \frac{1}{2} \mathcal{E}^{klm} F_{lm}, \\ B_k &= \frac{1}{2} \mathcal{E}_{klm} F^{lm}, & (\textbf{TODO: this is scam}) \\ (\nabla \times B)^i &= \mathcal{E}^{ijk} \partial_j \left(\frac{1}{2} \mathcal{E}_{klm} F^{lm} \right) = \frac{1}{2} \mathcal{E}^{ijk} \mathcal{E}_{klm} \partial_j F^{lm} \end{aligned}$$

Aside: We need to know how to evaluate $\mathcal{E}^{ijk} \mathcal{E}_{klm}$:

$$\mathcal{E}_{i_1, i_2, \dots, i_n} \mathcal{E}_{j_1, j_2, \dots, j_n} = \det \left\{ \begin{vmatrix} \delta_{i_1 j_1} & \delta_{i_1 j_2} & \dots & \delta_{i_1 j_n} \\ \delta_{i_2 j_1} & \delta_{i_2 j_2} & \dots & \delta_{i_2 j_n} \\ \vdots & \vdots & \ddots & \vdots \\ \delta_{i_n j_1} & \delta_{i_n j_2} & \dots & \delta_{i_n j_n} \end{vmatrix} \right\}$$

Hence, **TODO: HOW?** $\mathcal{E}^{ijk} \mathcal{E}^{ilm} = \frac{-1}{2} (\delta_i^j \delta_m^k - \delta_m^j \delta_i^k)$

Plugging both equations together,

$$\begin{aligned} \frac{1}{2} \mathcal{E}^{ijk} \mathcal{E}_{klm} \partial_j F^{lm} &= \mu_0 J^i + \partial_0 F^{i0} \\ \frac{1}{2} \left[\frac{-1}{2} (\delta_l^i \delta_m^j - \delta_m^i \delta_l^j) \right] \partial_j F^{lm} &= \mu_0 J^i + \partial_0 F^{i0} \end{aligned}$$

Something is fucked here with respect to $\partial_m F^{mi}$

$$\frac{1}{2} \left[\frac{-1}{2} (\partial_m F^{im} - \partial_m F^{mi}) \right] = \mu_0 J^i + \partial_0 F^{i0}$$

F is anti-symmetric, so rewriting $-\partial_m F^{mi} = \partial_m F^{im}$

$$-\frac{1}{2} [\partial_m F^{im}] = \mu_0 J^i + \partial_0 F^{i0}$$

In the notes, the $\frac{1}{2}$ does not exist

$$\begin{aligned} -[\partial_m F^{im}] &= \mu_0 J^i + \partial_0 F^{i0} \\ \mu_0 J^i + \partial_0 F^{i0} + \partial_m F^{im} &= 0 \\ \mu_0 J^i + \partial_\mu F^{i\mu} &= 0 \quad (\mu = \{0, 1, 2, 3\}) \end{aligned}$$

This gives us a continuity-style equation, linking the current density J to the rate of change of F .

$$\boxed{\mu_0 J^i + \partial_\mu F^{i\mu} = 0} \quad (\mu = \{0, 1, 2, 3\})$$

Second part, using 1st equation

$$\nabla E = \frac{\rho}{\epsilon_0}$$