

Topics in Physics - C. Mukku

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Chapter 1

Tensor algebra preliminaries

1.1 Raising and lowering of two indices simultaneously

Note that

$$a_i b^i = (a^j g_{ij}) b^i = (a^j g_{ij}) (b_k g^{ki})$$

In Minkowski space, we know that $g^{ij} = 0$ if $i \neq j$, and $(g^{ii} g_{ii})^2 = 1$, so we can rewrite the above expression as:

$$\begin{aligned} (a^j g_{ij}) (b_k g^{ki}) &= \\ (a^i g_{ii}) (b_i g^{ii}) &= \\ a^i b_i \end{aligned}$$

Chapter 2

Lagrangian, Hamiltonian mechanics

Mechanics in terms of generalized coords.

2.1 Lagrangian

Define a functional. L over the config. space of partibles q^i, \dot{q}^i . $L = L(q^i, \dot{q}^i)$. We have an explicit dependence on t .

$$L = KE - PE$$

Assuming a 1-particle system of unit mass,

$$L = \frac{1}{2}\dot{q}^2 - V(q)$$

Assuming an n-particle system of unit mass,

$$L = \sum_i \frac{1}{2}\dot{q}^{i2} - V(q^i)$$

2.2 Variational principle

Take a minimum path from A to B . Now notice that the path that is slightly different from this path will have some delta from the minimum.

Action

$$S(t_0, t_1) = \int_{t_0}^{t_1} L dt = \int_{t_0}^{t_1} L(q^i, \dot{q}^i) dt$$

. Least action: $\delta S = 0$

Chapter 3

Functional calculus

this chapter develops a completely handwavy physics version of functional analysis.

Definition 1 A *functional* F is a function: $F : (\mathbb{R} \rightarrow \mathbb{R}) \rightarrow \mathbb{R}$

Notation 1 Evaluation of a functional F with respect to f is denoted by $F[f]$.

3.1 Functional Derivative - take 1

Consider a functional $F : (\mathbb{R} \rightarrow \mathbb{R}) \rightarrow \mathbb{R}$, a function $f : \mathbb{R} \rightarrow \mathbb{R}$, and a "test function" $\phi : \mathbb{R} \rightarrow \mathbb{R}$.

Consider a functional F . We only define the derivative of a functional F with respect to a function f by what happens under an integral sign as follows:

$$\int \frac{\delta F}{\delta f}(x) \phi(x) dx = \lim_{\epsilon \rightarrow 0} \frac{F[f + \epsilon \phi] - F[f]}{\epsilon}$$

Now, we can define a small variation in F as:

$$\begin{aligned} \delta F : (\mathbb{R} \rightarrow \mathbb{R}) \times (\mathbb{R} \rightarrow \mathbb{R}) &\rightarrow \mathbb{R} \\ \delta F(f, \phi) &\equiv \int \frac{\delta F}{\delta f}(x) \phi(x) dx \end{aligned}$$

Intuitively, δF tells us the variation of the function f along a test function ϕ . So, it encapsulates some kind of "directional derivative".

So, we can look at $\frac{\delta F}{\delta f}$ as a functional as follows:

$$\begin{aligned} \frac{\delta F}{\delta f} : (\mathbb{R} \rightarrow \mathbb{R}) &\rightarrow \mathbb{R} \\ \frac{\delta F}{\delta f}(\phi) &= \delta F(f, \phi) \end{aligned}$$

Wehre $\frac{\delta F}{\delta f}$ allows us to "test" the change of F with respect to f along a given "direction" ϕ .

3.2 Functional Derivative as taught in class

Substitute $\phi = \delta(x - p)$. Now, the quantity:

$$\frac{\delta F}{\delta f} \phi(x) = \delta F(f, \delta(x - p))$$

Rewriting δF by sticking it under an integral:

$$\begin{aligned} \int \frac{\delta F}{\delta f}(x) \delta(x - p) dx &= \lim_{\epsilon \rightarrow 0} \frac{F[f + \epsilon \delta(x - p)] - F[f]}{\epsilon} \\ \frac{\delta F}{\delta f} \Big|_p &= \lim_{\epsilon \rightarrow 0} \frac{F[f + \epsilon \delta(x - p)] - F[f]}{\epsilon} \end{aligned}$$

That is, we can start talking about "derivative of the functional F with respect to a function f at a point p " as long as we only test the functional F against δ -functions.

So, we can alternatively define this quantity as:

$$\frac{\delta F}{\delta f} \Big|_p \equiv \lim_{\epsilon \rightarrow 0} \frac{F[f + \epsilon \delta(x - p)] - F[f]}{\epsilon}$$

While this does not "look like a functional", it actually is, if we mentally replace:

$$p \rightarrow \int -\delta(x - p) dx$$

This is how mukku got that expression.

3.3 Common functional derivatives

3.3.1 Derivative of $F[\phi] \equiv \int (\frac{\partial \phi}{\partial y})^2$

$$\frac{\delta F}{\delta f} \Big|_p = \int (\frac{\partial \phi}{\partial y})^2$$

3.4 Deriving E-L from functional magic

3.5 Weird things in Functional Analysis as taught in class

Consider the functional

$$J[f] = \int g[f'] dy:$$

since g is a functional, it has a type $g : (\mathbb{R} \rightarrow \mathbb{R}) \rightarrow \mathbb{R}$. So, our integrand must be some function df , and not some **space component** dy . **I don't understand what the definition of J means.**

Chapter 4

Maxwell's equations in Minkowski space

Let us first review Maxwell's equations:

$$\nabla \cdot E = \frac{\rho}{\epsilon_0} \text{ (Electric charges produce fields)}$$

$$\nabla \cdot B = 0 \text{ (Only magnetic dipoles exist)}$$

$$\nabla \times E = -\frac{\partial B}{\partial t} \text{ (Lenz Law / Faraday's law - time varying magnetic field induces current that opposes it)}$$

$$\nabla \times B = \mu_0 \left(J + \epsilon_0 \frac{\partial E}{\partial t} \right) \text{ (Ampere's law + fudge factor)}$$

4.1 Constructing F , or Tensorifying Maxwell's equations

Begin with the equation that $\nabla \cdot B = 0$. This tells that B can be written as the curl of some other field:

$$\boxed{B \equiv \nabla \times A} \tag{4.1}$$

Expanding this equation of B in tensorial form:

$$\boxed{B^i = \mathcal{E}^{ijk} \partial_j A^k} \tag{4.2}$$

Next, take $\nabla \times E = -\frac{\partial B}{\partial t}$.

$$\nabla \times E = -\frac{\partial B}{\partial t} = \frac{\partial(\nabla \times A)}{\partial t} = \nabla \times \frac{\partial A}{\partial t}$$

$$\nabla \times \left(E + \frac{\partial A}{\partial t} \right) = 0$$

writing this as the divergence of some field ϕ scaled by $\alpha : \mathbb{R}$

$$E + \frac{\partial A}{\partial t} = \alpha(\nabla \cdot \phi)$$

$$E = \alpha \nabla \cdot \phi - \frac{\partial A}{\partial t}$$

Since electrostatics is time-independent, we choose to think of $\alpha = -1$, so we can interpret ϕ as the potential.

$$E^i = -\frac{\partial \phi}{\partial x^k} g^{ik} - \frac{\partial A^i}{\partial t} \quad (4.3)$$

A slight reformulation (since we know that in Minkowski space, $\partial_t = \partial_0$) we get the equation:

$$\boxed{E^i = -g^{ik} \partial_k \phi - \partial_0 A^i} \quad (4.4)$$

We get the metric g^{ik} involved to raise the covariant $\frac{\partial \phi}{\partial x^k}$ into the contravariant E^i .

(Sid question: how does one justify switching $\nabla \times$ and ∂ ? It feels like some algebra)

Here be magic! We define A new rank-2 tensor in Minkowski space-time, called F (for Faraday),

$$\boxed{F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu} \quad (4.5)$$

(Sid question: why is this object $F_{\mu\nu}$ covariant? What does this *mean*?)

Lemma 1 $F_{\mu\nu}$ is antisymmetric.

Lemma 2 $F_{\mu\nu}$ has 6 degrees of freedom

Proof. Number of degrees of freedom of F :

$$\frac{4^2 \text{ (total)} - 4 \text{ (diagonal)}}{2 \text{ (anti-symmetry)}} = 6$$

Notice that F is a 1-form!

4.2 Expressing B , E in terms of F

We now wish to re-express B^{ij} and E^{ij} in terms of F , so that this F captures all of maxwell's equations.

$$B^i = \mathcal{E}^{ijk} \partial_j A^k = \mathcal{E}^{ikj} \partial_k A^j \quad \text{by } k, j \text{ being free variables}$$

$$B^i = \frac{1}{2} \left(\mathcal{E}^{ijk} \partial_j A^k + \mathcal{E}^{ikj} \partial_k A^j \right)$$

Substituting $\partial_j A_k - \partial_k A_j = F_{jk}$,

$$B^i = \frac{1}{2} \mathcal{E}^{ijk} F_{jk}$$

So, B in terms of F is:

$$\boxed{B^i = \frac{1}{2} \mathcal{E}^{ijk} F_{jk}} \quad (4.6)$$

Similarly, we wish to write E in terms of F . The algebra is as follows:

$$E^i = -g^{ik} \partial_k \phi - \partial_0 A^i$$

$$E^i = -g^{ik} \partial_k \phi - \partial_0 g^{ik} A_k \quad \text{Is this allowed? Am I always allowed to insert the } g_{ik}?$$

$$E^i = -g^{ik} (\partial_k \phi + \partial_0 A_k)$$

Since $k = \{1, 2, 3\}$ (k is spacelike coordinates), and we would like to relate ϕ with A (to unify E), we **set**:

$$\boxed{A_0 \equiv -\phi} \quad (4.7)$$

Continuing the derivation,

$$E^i = -g^{ik} (\partial_k (-A_0) + \partial_0 A_k)$$

$$E^i = -g^{ik} (\partial_0 A_k - \partial_k A_0)$$

$$E^i = -g^{ik} F_{0k}$$

So, finally, the relation is:

$$\boxed{E^i = -g^{ik} F_{0k}} \quad (4.8)$$

Let us reconsider what we believed E to be. We had:

$$E = -\nabla \phi - \frac{\partial A}{\partial t}$$

However, comparing dimensions, space derivative of ϕ = time derivative of A . This means that $\frac{\delta \phi}{\delta x} = \frac{\delta A}{\delta y}$, and so $\frac{\delta \phi}{\frac{\delta x}{\delta t}} = \delta A$. We arbitrarily pick c as our measuring stick for $\frac{\delta x}{\delta t}$. Also, in minkowski space, our measuring stick is actually (ct, x, y, z) , so $\partial_0 = \partial_{ct}$. So, when we write the equation for E , we should actually write

$$E = c \left(-\frac{\nabla \phi}{c} - \frac{\partial A}{\partial ct} \right)$$

which becomes:

$$\boxed{E^i = c F^{i0}} \quad (4.9)$$

4.3 Rewriting Maxwell's equations in terms of F

Now that we have constructed the Faraday tensor F , we wish to re-express Maxwell's equations in terms of this object. This will give us a compact form of the laws which are invariant under coordinate transforms.

4.3.1 Combining (1) $\nabla E = \frac{\rho}{\epsilon_0}$, (4) $\nabla \times B = \mu_0 J + \frac{\partial E}{\partial t}$

1. Using (4) $\nabla \times B = \mu_0 J + \frac{\partial E}{\partial t}$

We consider the 4th Maxwell equation:

$$\begin{aligned}\nabla \times B &= \mu_0 J + \epsilon_0 \mu_0 \frac{\partial E}{\partial t} \\ \nabla \times B &= \mu_0 J + \frac{1}{c^2} \frac{\partial E}{\partial t} \\ \text{Converting to indices,} \\ (\nabla \times B)^i &= \mu_0 J^i + \frac{1}{c} \frac{\partial E^i}{\partial ct} && \text{(From } \partial_{ct} = \frac{1}{c} \partial_t) \\ &= \mu_0 J^i + \frac{1}{c} \frac{\partial E^i}{\partial X^0} \\ &= \mu_0 J^i + \frac{\partial F^{i0}}{\partial X^0} && \text{(From } E^i = cF^{i0}) \\ &= \mu_0 J^i + \partial_0 F^{i0}\end{aligned}$$

Now, we start to simplify the LHS, $\nabla \times B$:

$$\begin{aligned}(\nabla \times B)^i &= \mathcal{E}^{ijk} \partial_j B_k \\ \text{Since } B^k &= \frac{1}{2} \mathcal{E}^{kmn} F_{mn}, \\ B_k &= \frac{1}{2} \mathcal{E}_{kmn} F^{mn}, && \text{(TODO: this is scam)} \\ (\nabla \times B)^i &= \mathcal{E}^{ijk} \partial_j \left(\frac{1}{2} \mathcal{E}_{kmn} F^{mn} \right) = \frac{1}{2} \mathcal{E}^{ijk} \mathcal{E}_{kmn} \partial_j F^{mn}\end{aligned}$$

Aside: We need to know how to evaluate $\mathcal{E}^{ijk} \mathcal{E}_{kmn}$:

$$\mathcal{E}_{i_1, i_2, \dots, i_n} \mathcal{E}_{j_1, j_2, \dots, j_n} = \det \left\{ \begin{array}{cccc} \delta_{i_1 j_1} & \delta_{i_1 j_2} & \dots & \delta_{i_1 j_n} \\ \delta_{i_2 j_1} & \delta_{i_2 j_2} & \dots & \delta_{i_2 j_n} \\ \vdots & \vdots & \ddots & \vdots \\ \delta_{i_n j_1} & \delta_{i_n j_2} & \dots & \delta_{i_n j_n} \end{array} \right\}$$

$$\mathcal{E}^{ijk} \mathcal{E}^{imn} = -1(\delta_j^m \delta_k^n - \delta_j^n \delta_k^m)$$

He argued that we get a -1 factor here due to the presence of the metric. I'm not fully convinced, but I can handwave this using the magic words "tensor density".

Plugging both equations together,

$$\begin{aligned}
\frac{1}{2}\mathcal{E}^{ijk}\mathcal{E}_{kmn}\partial_j F^{mn} &= \mu_0 J^i + \partial_0 F^{i0} \\
(\text{Since } kij \text{ is an even permutation of } ijk): \\
\frac{1}{2}\mathcal{E}^{kij}\mathcal{E}_{kmn}\partial_j F^{mn} &= \mu_0 J^i + \partial_0 F^{i0} \\
(\text{Using } \mathcal{E}^{kij}\mathcal{E}^{kmn} = -1(\delta_i^m \delta_j^n - \delta_i^n \delta_j^m)): \\
\frac{1}{2}[-(\delta_m^i \delta_n^j - \delta_n^i \delta_m^j)]\partial_j F^{mn} &= \mu_0 J^i + \partial_0 F^{i0} \\
-\frac{1}{2}[\partial_n F^{in} - \partial_m F^{mi}] &= \mu_0 J^i + \partial_0 F^{i0} \\
(F \text{ is anti-symmetric, so rewriting } \partial_m F^{mi} = -\partial_m F^{im}): \\
-\frac{1}{2}[\partial_n F^{in} + \partial_m F^{im}] &= \mu_0 J^i + \partial_0 F^{i0} \\
(\text{Replacing } \partial_m F^{im} \equiv \partial_n F^{in} \text{ since } m \text{ is free}): \\
-[\partial_m F^{im}] &= \mu_0 J^i + \partial_0 F^{i0} \\
\mu_0 J^i + \partial_0 F^{i0} + \partial_m F^{im} &= 0 \\
\mu_0 J^i + \partial_\mu F^{i\mu} &= 0 \quad (\mu = \{0, 1, 2, 3\})
\end{aligned}$$

This gives us a continuity-style equation, linking the current density J to the rate of change of F .

$$\boxed{\mu_0 J^i + \partial_\mu F^{i\mu} = 0} \quad (\mu = \{0, 1, 2, 3\})$$

Second part, using 1st equation

$$\begin{aligned}
\nabla E &= \frac{\rho}{\epsilon_0} \\
\partial_i E^i &= \frac{\rho}{\epsilon_0} \\
(\text{Substituting } E^i = cF^{i0}): \\
c\partial_i F^{i0} &= \frac{\rho}{\epsilon_0} = \frac{\rho\mu_0}{\mu_0\epsilon_0} = \rho c^2 \\
\partial_i F^{i0} &= \mu_0 c\rho \\
(\text{Since } F \text{ is anti-symmetric, } F^{00} = 0, \text{ Hence}): \\
\partial_0 F^{00} + \partial_i F^{i0} &= \mu_0 c\rho \\
\partial_\mu F^{\mu 0} &= \mu_0 c\rho
\end{aligned}$$

$$\boxed{\partial_\mu F^{\mu 0} = \mu_0 c\rho} \quad (4.10)$$

Combining part 1 and part 2:

$$\begin{aligned}\mu_0 J^i + \partial_\mu F^{i\mu} &= 0 & (\text{From } B) \\ \partial_\mu F^{i\mu} &= -\mu_0 J^i \partial_\mu F^{\mu 0} = \mu_0 c \rho \\ \partial_\mu F^{0\mu} &= -\mu_0 c \rho\end{aligned}$$

To combine these equations, **we set:**

$$\boxed{J^0 \equiv c\rho} \quad (4.11)$$

We arrive at the unified equation:

$$\partial_\mu F^{\nu\mu} = -\mu_0 J^\nu$$

Choose units such that $c = \frac{h}{2\pi} = G_n = 1$, which gives us:

$$\begin{aligned}\partial_\mu F^{\nu\mu} &= -J^\nu \\ F &\text{ is antisymmetric, so flipping indices} \\ \partial_\mu F^{\mu\nu} &= J^\nu\end{aligned}$$

$$\boxed{\partial_\mu F^{\mu\nu} = J^\nu} \quad (4.12)$$

Note that this is **Ampere's law!**

4.3.2 Combining (2) $\nabla \times E = -\frac{\partial B}{\partial t}$, (3) $\nabla B = 0$

$$\begin{aligned}\nabla \times E &= -\frac{\partial B}{\partial t} \\ (\nabla \times E)^i &= \mathcal{E}^{ijk} \partial_j E_k = -\partial_0 B \\ \mathcal{E}^{ijk} \partial_j E_k &= -\partial_0 \left(\frac{1}{2} \mathcal{E}^{ijk} F_{jk} \right) \\ \mathcal{E}^{ijk} \partial_j E_k + \partial_0 \left(\frac{1}{2} \mathcal{E}^{ijk} F_{jk} \right) &= 0 \\ 2\mathcal{E}^{ijk} \partial_j E_k + \partial_0 (\mathcal{E}^{ijk} F_{jk}) &= 0\end{aligned}$$

Now we begin from the other direction, and start the derivation.

We know that the equation we want is:

$$\boxed{\mathcal{E}^{\alpha\beta\mu\nu} \partial_\beta F_{\mu\nu} = 0} \quad (4.13)$$

$\alpha = 0$ **case:**

First, set $\alpha = 0$. So now, the other β, μ, ν are forced to be become space components — (i, j, k) .

Therefore, the equation now becomes:

$$\mathcal{E}^{0ijk} \partial_i F_{jk} = 0$$

However, note that $\mathcal{E}^{0ijk} = \mathcal{E}^{ijk}$, because if (ijk) is an even permutation, so will $(0ijk)$, and vice versa for odd (since $0 < i, j, k$).

Using this, the equation becomes

$$\begin{aligned} \mathcal{E}^{ijk} \partial_i F_{jk} &= 0 \\ \partial_i (\mathcal{E}^{ijk} F_{jk}) &= 0 \\ \text{Since } B^i &= \frac{1}{2} \mathcal{E}^{ijk} F_{jk}: \\ \partial_i \left(\frac{B^i}{2} \right) &= 0 \\ \partial_i B^i &= 0 \\ \nabla B &= 0 \end{aligned}$$

Hence, the above equation does encode $\nabla B = 0$.

$\alpha = m$ **case:**