

0.1. Q1 – MATRIX REPRESENTATION FOR $|\phi_k\rangle \langle \phi_j|$, IN THE ORTHONORMAL $|v_i\rangle$ BASIS

0.1 Q1 – matrix representation for $|\phi_k\rangle \langle \phi_j|$, in the orthonormal $|v_i\rangle$ basis

$$\begin{aligned} O &= |\phi_k\rangle \langle \phi_j| = I |\phi_k\rangle \langle \phi_j| I \\ &= \left(\sum_l |v_l\rangle \langle v_l| \right) |\phi_k\rangle \langle \phi_j| \left(\sum_m |v_m\rangle \langle v_m| \right) \end{aligned}$$

Element at α th row, β th column of an operator O in the $\{v_i\}$ basis is $\langle v_\alpha | O | v_\beta \rangle$. So, in this case, it is:

$$\begin{aligned} &\langle v_\alpha | \left(\sum_l |v_l\rangle \langle v_l| \right) |\phi_k\rangle \langle \phi_j| \left(\sum_m |v_m\rangle \langle v_m| \right) | v_\beta \rangle \\ &= \left(\sum_l \langle v_\alpha | v_l \rangle \langle v_l | \right) |\phi_k\rangle \langle \phi_j| \left(\sum_m |v_m\rangle \langle v_m | v_\beta \rangle \right) \end{aligned}$$

$\langle v_\alpha | v_l \rangle = 1$ if $\alpha = l$, and 0 otherwise since $\{v_i\}$ are orthonormal. Similarly for β . Hence:
 $= \langle v_\alpha | \phi_k \rangle \langle \phi_j | v_\beta \rangle$

0.2 Q2 – positive operator is Hermitian

We first show that a positive operator is normal, and this automatically implies that it is Hermitian.

To show that a positive operator is normal, we consider $A^\dagger A$

Now that we know that it is normal, by spectral decomposition, it possesses an eigenbasis. We now show that all of its eigenvalues are real. This is now a matrix with real entries on the diagonal, which is hermitian.

To show that the eigenvalues are real, let $|\lambda\rangle$ be an eigenvector with magnitude 1 and eigenvalue λ .

$$\langle \lambda | A | \lambda \rangle \geq 0 \quad \lambda \langle \lambda | \lambda \rangle = \lambda \geq 0$$

Hence, the eigenvalues are real and positive, and therefore it is Hermitian.

0.3 Q3 – $A^\dagger A$ is positive

$$\forall v \in V, \langle v | A^\dagger A | v \rangle = \langle Av | Av \rangle = \|Av\|^2 \geq 0$$

Hence, $A^\dagger A$ is positive.

0.4 Q4. Eigenvalues of a projector P are either 0 or 1

Let $|\lambda\rangle$ be an eigenvector of P with associated eigenvalue λ .

$$P^2(|\lambda\rangle) = \lambda(P|\lambda\rangle) = \lambda^2|\lambda\rangle \quad P(|\lambda\rangle) = \lambda|\lambda\rangle$$

However, since P is a projector, $P^2 = P$, and therefore, $\lambda^2 = \lambda$. The roots of this equation are 0, 1. Hence, $\lambda \in \{0, 1\}$.

0.5 Q5. Tensor product of two unitary operators is unitary

Let U, V be unitary operators.

$$\begin{aligned} \langle Uu \otimes Vv | Uu \otimes Vv \rangle &= \\ \langle u \otimes v | (U^\dagger \otimes V^\dagger)(U \otimes V) | u \otimes v \rangle &= \\ \langle u \otimes v | (U^\dagger U \otimes V^\dagger V) | u \otimes v \rangle &= \\ \langle u \otimes v | I \otimes I | u \otimes v \rangle &= \\ \langle u \otimes v | u \otimes v \rangle &= \end{aligned}$$

Hence, $U \otimes V$ is unitary since it preserves inner products.

0.6 Q6. Tensor product of projectors is a projector

Let P, Q be projectors. $P \equiv \sum_{i=1}^l |i\rangle \langle i|$. $Q \equiv \sum_{j=1}^k |j\rangle \langle j|$.

$$\begin{aligned}
P \otimes Q &\equiv \left(\sum_{i=1}^l |i\rangle \langle i| \right) \otimes \left(\sum_{j=1}^k |j\rangle \langle j| \right) \\
&\equiv \sum_{i=1}^l \sum_{j=1}^k |ij\rangle \langle ij|
\end{aligned}$$

Which is in the form of a projector, in that it leaves $|ij\rangle$ unchanged, and sends every other vector to 0. So, it projects vectors onto the subspace spanned by $|ij\rangle$.

o.7 Q7. Find log and square root of matrix

$$A = \begin{bmatrix} 4 & 3 \\ 3 & 4 \end{bmatrix}$$

Finding eigenvalues,

$$|A - \lambda I| = 0 \quad (4 - \lambda)^2 - 9 = 0 \quad \lambda = 7, 1$$

Finding eigenvectors,

$$v = (1/\sqrt{2}, -1/\sqrt{2}) \quad w = (1/\sqrt{2}, 1/\sqrt{2})$$

hence, we can now write $A = U^{-1}DU$, where U transforms from the original basis to the eigenbasis, as:

$$D \equiv \begin{bmatrix} 7 & 0 \\ 0 & 1 \end{bmatrix} \quad U \equiv \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \quad U^{-1} \equiv \begin{bmatrix} 1 \end{bmatrix}$$

o.7.1 Computing square root

$$S = U^{-1}\sqrt{D}U \quad \sqrt{D} = \begin{bmatrix} \sqrt{7} & 0 \\ 0 & 1 \end{bmatrix} \quad S = \begin{bmatrix} 1/2 + \sqrt{7}/2 & -1/2 + \sqrt{7}/2 \\ -1/2 + \sqrt{7}/2 & 1/2 + \sqrt{7}/2 \end{bmatrix}$$

We prove that S is the square root, since:

$$S^2 = (U^{-1}\sqrt{D}U)(U^{-1}\sqrt{D}U) = U^{-1}(\sqrt{D})^2U = U^{-1}DU = A$$

0.7.2 Computing log

We can now show that if U is unitary and D is diagonal, then:

$$\begin{aligned}
 L &\equiv \log(U^{-1}DU) = U^{-1} \log D U & \log D &= \begin{bmatrix} \log 7 & 0 \\ 0 & 0 \end{bmatrix} \\
 L &= \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \log 7 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \\
 L &= 1/2 \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \log 7 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \\
 L &= 1/2 \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \log 7 & \log 7 \\ 0 & 0 \end{bmatrix} \\
 L &= 1/2 \begin{bmatrix} \log 7 & \log 7 \\ \log 7 & \log 7 \end{bmatrix}
 \end{aligned}$$

0.8 Q8. Trace properties

0.8.1 $\text{Tr}(AB) = \text{Tr}(BA)$

$$\text{Tr}(AB) = \sum_z (AB)_{zz} = \sum_z \sum_k A_{zk} B_{kz} = \sum_z \sum_k B_{kz} A_{zk} = \sum_z (BA)_{zz} = \text{Tr}(BA)$$

0.8.2 $\text{Tr}(A + B) = \text{Tr}(A) + \text{Tr}(B)$

$$\text{Tr}(A + B) = \sum_z (A + B)_{zz} = \sum_z A_{zz} + B_{zz} = \text{Tr}(A) + \text{Tr}(B)$$

0.8.3 $\text{Tr}(2A) = 2\text{Tr}(A)$

$$\text{Tr}(2A) = \sum_z (2A)_{zz} = \sum_z 2A_{zz} = 2 \sum_z A_{zz} = 2\text{Tr}(A)$$

0.9 Commutator properties

0.9.1 $[A, B] = -[B, A]$

$$[A, B] = AB - BA = -(BA - AB) = -[B, A]$$

0.9.2 $\frac{[A, B] + \{A, B\}}{2} = AB$

$$\frac{[A, B] + \{A, B\}}{2} = \frac{(AB - BA) + (AB + BA)}{2} = AB$$

0.10 Express polar decomposition of normal matrix as outer product

Since the matrix A is normal, it will possess an eigenbasis $|\lambda_i\rangle$ with eigenvalues λ_i . Hence,

$$A = \sum_i \lambda_i |\lambda_i\rangle \langle \lambda_i|$$

Let $A = UP$ where U is unitary and P is positive definite. From the definition, we can clearly pick P :

$$P = \sum_i |\lambda_i| |\lambda_i\rangle \langle \lambda_i|$$

such that P has positive eigenvalues.

We pick U as:

$$U = \sum_i \frac{\lambda_i}{|\lambda_i|} |\lambda_i\rangle \langle \lambda_i|$$

Clearly, U has orthogonal columns $\frac{\lambda_i}{|\lambda_i|} |\lambda_i\rangle$, which have length ± 1 , hence the columns of U are orthonormal.

We can verify that $UP = A$ as follows:

$$\begin{aligned}
 \mathbf{UP} &= \left(\sum_j \frac{\lambda_j}{|\lambda_j|} |\lambda_j\rangle \langle \lambda_j| \right) \left(\sum_i |\lambda_i| |\lambda_i\rangle \langle \lambda_i| \right) \\
 &= \sum_j \sum_i \frac{\lambda_j}{|\lambda_j|} |\lambda_i| |\lambda_j\rangle \langle \lambda_j| |\lambda_i\rangle \langle \lambda_i|
 \end{aligned}$$

Since $\langle \lambda_j | \lambda_i \rangle = \delta_{ij}$, $\frac{\lambda_j}{|\lambda_j|} |\lambda_i| = \lambda_i$ when $i = j$:

$$\begin{aligned}
 &= \sum_i \lambda_i |\lambda_i\rangle \langle \lambda_i| \\
 &= \mathbf{A}
 \end{aligned}$$

0.11 Find left and right polar decomposition