

Quantum computation and information - Indranil Chakravarty

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Chapter 1

Lecture 1: Introduction

Taught in collaboration with MSR Redmond for the Q# bits.

Topics:

- Intro: Transition from Classical to Quantum: Stern Gerlach, Sequential Stern Gerlach, Rise of randomness.
- Foundations of Quantum Theory: States, Ensembles, Qubits, Pure and Mixed states, Multi qubit states, Tensor products, Unitary transforms, Spectral decomposition, SVD, Generalized measurements, Projective measurements, POVM, Evolution of quantum state, Krauss Representation.
- Quantum Entropy: Subadditivity of Entropy, Avani-Licb(?) Inequality, Quantum channel, Quantum channel capacity, Data compression, Benjamin Schumatur(?) theorem.
- Quantum Entanglement: EPR paradox, Schmidt decomposition, Purification of entanglement, Entanglement separability problem, Pure and mixed entangled states, Measures of Entanglement.
- Quantum information processing protocols: Teleportation, Superdense coding, Entanglement swapping.
- Impossible operations in quantum information theory: No cloning, No deleting, No partial erasure.
- Quantum Computation: Introduction to Quantum Computing, Pauli gates, Hadamard gates, Universal gates, Quantum algorithms (Shor, Grover search, machine learning and optimisation).
- Quantum programming: Programming quantum algorithms, Q# programming language, quantum subroutines.

Books:

- Quantum computation and Quantum information — Nielsen and Chuang.

- Preskill lecture notes.

Grading:

- Possibility of open book take-home open ended exam for the finals.
- Mid 1: 15%
- Mid 2: 15%
- End sem (open book?) : 30%
- Assignments: 15%
- Projects: 25%

1.1 Stern-Gerlach: A brief, morally correct construction of qubits

light rays $\rightarrow [z] \rightarrow (z+, z-) \rightarrow \text{block } (z-) \rightarrow [x] \rightarrow (x+, x-) \rightarrow \text{block } (x-) \rightarrow [z] \rightarrow (z+, z-?)$

$[z]$ represents a polarizer along that axis.

- Since we first polarized along z , how did we manage to get out light rays in the x direction? The polarization should have killed everything.
- Since we blocked $z-$, How did we get back $z-$ after passing stuff through $[x]$? Something has changed drastically from our classical picture.

We can consider $|z+\rangle$ to be something like:

$$|z+\rangle \equiv \frac{1}{2}|x+\rangle + \frac{1}{2}|x-\rangle$$

Where $|x+\rangle$ and $|x-\rangle$ are basis vectors for some vector space over \mathbb{R} .

If we were to pass the $z+$ light rays through $[y]$, then we would get $|y+\rangle, |y-\rangle$. So, $|z+\rangle$ is also:

$$|z+\rangle \equiv \frac{1}{2}|y+\rangle + \frac{1}{2}|y-\rangle$$

1.1.1 Analogy with polarization of light

Consider a monochromatic light wave in the z direction. A linearly polarized light with polarization in the x direction which we call x polarized light is given by:

$$E_x = E_0 \hat{x} \cos(kz - \omega t)$$

$\omega \equiv \text{frequency} \equiv ck$, $c \equiv \text{speed of light}$, $k \equiv \text{wave number}$.

Similarly, y polarized light is given by:

$$E_y = E_0 \hat{y} \cos(kz - \omega t)$$

Consider the case where we have x filters along direction $-$, x' filter along direction $/$, y filters along direction $|$. In this case, we can have x, x', y filters arranged sequentially give us non-zero output (contrast with just having x, y).

We can express the x' polarization as:

$$E_0 \hat{x}' \cos(kz - \omega t) = \frac{E_0}{\sqrt{2}} \hat{x} \cos(kz - \omega t) + \frac{E_0}{\sqrt{2}} \hat{y} \cos(kz - \omega t)$$

By analogy, we write:

$$|z_+\rangle \equiv \frac{1}{\sqrt{2}} |x_+\rangle + \frac{1}{\sqrt{2}} |x_-\rangle$$

However, we now have probability $\frac{1}{\sqrt{2}}$, but we want $\frac{1}{2}$. So, we define the probability as:

$$\langle x_+ | x_- \rangle^2 = \frac{1}{2}$$

$z_+ \equiv x$ polarization

$z_- \equiv y$ polarization

$x_+ \equiv x'$ polarization

$x_- \equiv y'$ polarization

This problem can be solved again by polarization of light. This time, we consider circularly polarized light which can be obtained by letting linearly polarized light passing through a quarter wave plate (?)

When we pass such circularly polarized light through an x or y filter, we again obtain either an x polarized beam, or a y polarized beam of equal intensity. Yet, everybody knows that circularly polarized light is totally different from 45° linearly polarized light.

A right circularly polarized light is a linear combination of x polarized light and y polarized light, where the oscillation of the electric field for the y component is 90° out of phase with the x polarized component.

$$E = \frac{E_0}{\sqrt{2}} \hat{x} \cos(kz - \omega t) + \frac{E_0}{\sqrt{2}} \hat{y} \cos\left(kz - \omega t + \frac{\pi}{2}\right)$$

$$\frac{E}{E_0} = \frac{1}{\sqrt{2}} \hat{x} e^{i(kz - \omega t)} + \frac{i}{\sqrt{2}} \hat{y} e^{i(kz - \omega t)}$$

Similarly, left circularly polarized light is:

$$E = \frac{E_0}{\sqrt{2}} \hat{x} \cos(kz - \omega t) - \frac{E_0}{\sqrt{2}} \hat{y} \cos\left(kz - \omega t + \frac{\pi}{2}\right)$$

1.2 Observable

An observable is something that we observe.

$$Z|z+\rangle = \frac{\hbar}{\sqrt{2}}|z+\rangle \quad Z|z-\rangle = \frac{\hbar}{\sqrt{2}}|z-\rangle$$

TODO: try to construct an operator that takes a vector $|v\rangle$ to a vector that is orthogonal to it.

1.3 Operators

1.3.1 Projectors — P

Suppose W is a k -dimensional vector subspace of the d -dimensional vector space V .

Using Gram-Schmidt, it is possible to construct an orthonormal basis $|1\rangle, |2\rangle, \dots, |d\rangle$ for V such that $|1\rangle \dots |k\rangle$ is an orthonormal basis for W . Then the projector P is defined as:

$$P_W \equiv \sum_{i=1}^k |i\rangle\langle i|$$

- $P^\dagger = P$ (Immediate from writing in $|i\rangle$ basis)
- $P^2 = P$ (Immediate from writing in $|i\rangle$ basis)

$Q = I - P$ is the projector onto orthogonal complement of the subspace that P projects into. This projects onto the $|k+1\rangle \dots |d\rangle$ basis.

1.3.2 Normal operator

$$AA^\dagger = A^\dagger A$$

Theorem 1 *Spectral theorem for normal operators: Any normal operator M on a vector space V is diagonal with respect to some orthonormal basis for V .*

Proof. Let λ be an eigenvalue of M . P_λ is the projector onto λ 's eigenvector. $Q_\lambda = P_\lambda^\perp$ is the orthogonal complement projector of P .

We first establish a fact about PMQ :

$$MM^\dagger |\lambda\rangle = M^\dagger (M |\lambda\rangle) = \lambda M^\dagger |\lambda\rangle$$

Hence, $M^\dagger v \in P$.

$$Q(M^\dagger P) = 0 \implies (PMQ)^\dagger = 0 \implies PMQ = 0$$

Next, we prove some properties of QM and QM^\dagger

$$\begin{aligned} QM &= QM(P + Q) = QMP + QMQ = QMQ \\ QM^\dagger &= QM^\dagger(P + Q) = QM^\dagger P + QM^\dagger Q = (PMQ)^\dagger + QM^\dagger Q \end{aligned}$$

QMQ is normal:

$$(QMQ)^\dagger (QMQ) = Q^\dagger M^\dagger Q^\dagger QMQ = QM^\dagger QMQ = QM^\dagger MQ$$

$$(QMQ)(QMQ)^\dagger = (QMQ)(Q^\dagger M^\dagger Q^\dagger) = QMQM^\dagger Q = QMM^\dagger Q = QM^\dagger MQ = (QMQ)^\dagger QMQ$$

$$M = (P + Q)M(P + Q)$$

$$M = PMP + PMQ + QMP + QMQ$$

$$M = PMP + QMQ$$

$$M = \lambda_i |i\rangle\langle i| + QMQ$$

Since QMQ is normal, and we are performing induction on dimension, and $P \perp Q$,

$$M = \lambda_i |i\rangle\langle i| + \sum_k \lambda_k |k\rangle\langle k|$$

Hence M is normal

Theorem 2 Any diagonalizable operator is normal

Proof. Let M be diagonal with respect to basis $|i\rangle$. Then, $M \equiv \sum_i \lambda_i |i\rangle\langle i|$. Now, $M^\dagger = \sum_i \lambda_i^* |i\rangle\langle i|$.

$$MM^\dagger = \left(\sum_i \lambda_i |i\rangle\langle i| \right) \left(\sum_j \lambda_j^* |j\rangle\langle j| \right)$$

$$MM^\dagger = \sum_i \lambda_i^* \lambda_i |i\rangle\langle i|$$

$$\text{Similarly, } M^\dagger M = \left(\sum_i \lambda_i^* \lambda_i |i\rangle\langle i| \right)$$

1.3.3 Unitary operator

$$UU^\dagger = U^\dagger U = I$$

- unitary operator is normal.
- unitary operator preserves inner products.

$$\langle b' | | a' \rangle = \langle b | U^\dagger U | a \rangle = \langle b | I | a \rangle$$

1.3.4 Positive operator

Special class of Hermitian operator.

$$\forall v \in V, \langle v | A | v \rangle \geq 0$$

If the inner product is strictly greater than zero, then such an operator is called as *positive definite*. If it is greater than or equal to zero, it is called *positive semidefinite*.

Theorem 3 A positive operator is Hermitian

Proof. **TODO.** Proof most likely follows real case, where we use cholesky to write it as $A^T A$ and then show that it is normal. We then use the fact that its eigenvalues are greater than or equal to zero to establish that it is Hermitian.

Chapter 2

Tensor product states

2.1 Postulates of QM

- Associated to any isolated physical system is a complex vector space with inner product. This space is called as the state space of the system. This system is completely described by its state vector which is a unit vector in the state space.

2.2 Tensor product

Let A and B be vector spaces with bases $A_{\text{basis}}, B_{\text{basis}}$. $A(X)B$ is a *new vector space*, whose basis vectors are $a_i(X)b_j$ where $a_i \in A_{\text{basis}}, b_i \in B_{\text{basis}}$.

Properties of the tensor product:

- For any arbitrary scalar z and element $v \in H_a, w \in H_b$, $z(|v\rangle(X)|w\rangle) = (z|v\rangle)(X)|w\rangle = |v\rangle(X)(z|w\rangle)$
- $(|v\rangle_1 + |v\rangle_2)(X)|w\rangle = |v\rangle_1(X)|w\rangle + |v\rangle_2(X)|w\rangle$
- $|w\rangle(X)(|v\rangle_1 + |v\rangle_2) = |w\rangle(X)|v\rangle_1 + |w\rangle(X)|v\rangle_2$ **TODO: what is an easy way to get correctly sized brackets?**
- Suppose $|v\rangle \in H_a, |w\rangle \in H_b$, and A and B are linear operators on H_a and H_b respectively. $(A(X)B)(|v\rangle(X)|w\rangle) \equiv (A|v\rangle)(X)(B|w\rangle)$.
- Let $C = \sum_i c_i A_i(X)B_i$, where A_i, B_i are linear operators on H_a, H_b . Now, $C(|v\rangle(X)|w\rangle) = \sum_i c_i((A_i|v\rangle)(X)(B_i|w\rangle))$
- $|x\rangle = \sum_i a_i |v\rangle_i(X)|w\rangle_i, |y\rangle = \sum_j b_j |v\rangle_j(X)|w\rangle_j$. Now, $\langle x|y\rangle = (\sum_i a_i^* \langle v|_i(X)\langle w|_i)(\sum_j b_j |v\rangle_j(X)|w\rangle_j)$, which is equal to $\sum_i \sum_j a_i^* b_j \langle v_i|v'_j\rangle \langle w_i|w'_j\rangle$

This is way too redundant, **TODO**: write down the slick definition of tensor product spaces seen in John Lee's intro to smooth manifolds, or the definition seen in Tensor Geometry: The Geometric Viewpoint and its uses.

$$\text{tr}(A|\psi\rangle\langle\psi|) = \sum_i \langle i|A|\psi\rangle\langle\psi|i\rangle = \sum_i (\langle\psi|i\rangle) \cdot (\langle i|A|\psi\rangle) = \sum_i \langle\psi|(|i\rangle\langle i|)A|\psi\rangle = \langle\psi|A|\psi\rangle$$

Theorem 4 Two operators A, B are simultaneously diagonalizable iff $[A, B] = 0$, where $[A, B] = AB - BA$. That is, there exists a basis where both A and B are diagonal matrices.

Proof. One direction of the proof is easy. If two operators are simultaneously diagonalizable, then we can simply write both operators in this common basis. Diagonal matrices commute, hence $[A, B] = 0$.

Let $|a, j\rangle$ be an orthonormal basis for the eigenspace V_a of A with eigenvalue a and index j to label repeated eigenvalues.

$AB|a, j\rangle = BA|a, j\rangle = aB|a, j\rangle$. Hence, $A(B|a, j\rangle) = a(B|a, j\rangle)$. Hence, $B|a, j\rangle$ is an eigenvector of A . Therefore, $B|a, j\rangle \in V_a$.

Define projector P_a onto V_a . Now, define $B_a \equiv P_a B P_a$. This is hermitian on V_a , since

$$(P_a B P_a)^\dagger = P_a^\dagger B^\dagger P_a^\dagger = P_a B P_a$$

Let us call the eigenvalues of B_a as $|a, b, k\rangle$ where a, b are the eigenvalues of A and B , and K is the degeneracy index.

$P_a B|a, b, k\rangle = b|a, b, k\rangle$, since $B|a, b, k\rangle \in V_a$. **TODO: complete proof**

Lemma 1 $[A, B]^\dagger = [B^\dagger, A^\dagger]$

Proof. $[A, B]^\dagger = (AB - BA)^\dagger = (B^\dagger A^\dagger - A^\dagger B^\dagger) = [B^\dagger, A^\dagger]$

$$\begin{aligned} [A, B]^\dagger &= (AB - BA)^\dagger \\ &= (B^\dagger A^\dagger - A^\dagger B^\dagger) \\ &= [B^\dagger, A^\dagger] \end{aligned}$$

2.3 Postulate 2 of QM: Unitary time evolution

The evolution of a closed quantum system is described by unitary transformation. That is, the state $|\psi\rangle$ of the system at time t_1 is related to a state $|\psi'\rangle$ at a time t_2 by unitary operator U which depends only on time t_1 and t_2 . That is, $|\psi'\rangle = U(t_1, t_2)|\psi\rangle$. We usually suppress t_1, t_2 to write $|\psi'\rangle = U|\psi\rangle$.

Schrodinger equation:

$$i\hbar \frac{\partial \psi}{\partial t} = H|\psi\rangle$$

Homework: Show that the Schrodinger equation implies unitary evolution

2.4 Postulate 3 of QM: Measurement postulate

Quantum measurements are described by a collection $\{M_i\}$ of measurement operators. These operators are acting on the state space of the system which is measured. The index i refers to the measurement outcomes that may occur in the experiment. If the state of the quantum system is $|\psi\rangle$ before the measurement, the probability that the result i occurs is given by:

$$p(i) \equiv \langle \psi | M_i^\dagger M_i | \psi \rangle$$

The state if result i occurs is:

$$|\psi'\rangle = \frac{M_i |\psi\rangle}{\sqrt{p(i)}}$$

We normalize the state $|\psi'\rangle$ to ensure that we evolve Unitarily.

Also, notice that since $\sum_m p(m) = 1$, since $p(m)$ represents probabilities, we can write:

$$\sum_m p(m) = 1 \quad \langle \psi | \sum_m M_m^\dagger M_m | \psi \rangle = 1 \quad \sum_m \langle \psi | M_m^\dagger M_m | \psi \rangle = 1$$

Since $|\psi\rangle$ is a normalized state, we must have:

$$\sum_m M_m^\dagger M_m = I$$

2.4.1 Projective measurements

A projective measurement is described by an observable M , a hermitian operator on the state space of the system being observed.

Let the observable M have spectral decomposition:

$$M \equiv \sum_m m P_m$$

$$p(m) = \langle \psi | P_m^\dagger P_m | \psi \rangle = \langle \psi | P_m | \psi \rangle$$

$$\mathbb{E}[M] = \sum_m m p(m) = \sum_m m \langle \psi | P_m | \psi \rangle = \langle \psi | \sum_m m P_m | \psi \rangle = \langle \psi | M | \psi \rangle$$

$$\mathbb{E}[M^2] = \langle \psi | M^2 | \psi \rangle$$

$$V(M) = \mathbb{E}[M^2] - \mathbb{E}[M]^2 = \langle \psi | M^2 | \psi \rangle - (\langle \psi | M | \psi \rangle)^2$$

2.4.2 Quantum sharing experiment

$$\text{Alice: } \{|\psi_1\rangle, |\psi_2\rangle, \dots\} \xrightarrow{|\psi_i\rangle} \text{Bob: } |\psi_i\rangle$$

Bob has to correctly guess the i that was sent to him. We have two cases: One where $\{|\psi_i\rangle\}$ are orthonormal, the other where they are not.