1

$y = A + Bx + Cx^2$

$(1,1)\quad (2,-1)\quad (3,1)$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & -1 \\ 1 & 3 & 9 & 1 \end{bmatrix}$$

$R_2 \to R_2 - R_1$

$R_3 \to R_3 - R_1$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & -2 \\ 0 & 2 & 8 & 0 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 2 & 4 \end{bmatrix}$$

$2C = 4 \Rightarrow C = 2$

$B + 6 = -2$

$B = -8$

$A - 8 + 2 = 1$

$A = 7$

$\therefore A = 7$

$B = -8$

$C = 2$

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(2)

$$A = \begin{bmatrix} 2 & 5 & 2 & -5 \\ 4 & 12 & 3 & -14 \\ -10 & -29 & -5 & 38 \\ 10 & 21 & 21 & -6 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1 \quad R_3 \to R_3 + 5R_1 \quad R_4 \to R_4 - 5R_1$

$$\begin{bmatrix} 2 & 5 & 2 & -5 \\ 0 & 2 & -1 & -4 \\ 0 & -4 & 5 & 13 \\ 0 & -4 & 11 & 19 \end{bmatrix}$$

$R_3 \to R_3 + 2R_2$

$R_4 \to R_4 + 2R_2$

$$\begin{bmatrix} 2 & 5 & 2 & -5 \\ 0 & 2 & -1 & -4 \\ 0 & 0 & 3 & 5 \\ 0 & 0 & 9 & 11 \end{bmatrix}$$

$R_4 \to R_4 - 3R_3$

$$U = \begin{bmatrix} 2 & 5 & 2 & -5 \\ 0 & 2 & -1 & -4 \\ 0 & 0 & 3 & 5 \\ 0 & 0 & 0 & -4 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -5 & -2 & 1 & 0 \\ 5 & -2 & 3 & 1 \end{bmatrix}$$

$$A = LU = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -5 & -2 & 1 & 0 \\ 5 & -2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 5 & 2 & -5 \\ 0 & 2 & -1 & -4 \\ 0 & 0 & 3 & 5 \\ 0 & 0 & 0 & -4 \end{bmatrix}$$

③ Basis for $R^3 \to \begin{bmatrix}1\\0\\0\end{bmatrix} \begin{bmatrix}0\\1\\0\end{bmatrix} \begin{bmatrix}0\\0\\1\end{bmatrix}$

$T(1,0,0) = (1,0,1) \quad T(0,1,0) = T(2,1,1)$

$T(0,0,1) = (-1,1,-2)$

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix}$$

$$[A\ b] = \begin{bmatrix} 1 & 2 & -1 & b_1 \\ 0 & 1 & 1 & b_2 \\ 1 & 1 & -2 & b_3 \end{bmatrix} \quad R_3 \to R_3 - R_1$$

$$= \begin{bmatrix} 1 & 2 & -1 & b_1 \\ 0 & 1 & 1 & b_2 \\ 0 & -1 & -1 & b_3 - b_1 \end{bmatrix} \quad R_3 \to R_3 + R_2$$

$$= \begin{bmatrix} 1 & 2 & -1 & b_1 \\ 0 & 1 & 1 & b_2 \\ 0 & 0 & 0 & b_3 - b_1 + b_2 \end{bmatrix}$$

$N(A^T) = \{(-1,1,1)\}$ $\quad -b_1 + b_2 + b_3 = 0,$

$C(A^T) = \{(1,2,-1), (0,1,1)\}$ $\quad$ this will give $N(A^T)$

$C(A) = \{(1,0,1), (2,1,1)\}$

$$U = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad R_1 \to R_1 - 2R_2 \qquad N(A) = \{(3,-1,1)\}$$

$\Uparrow$

$$= \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} x = 3z \\ y = -z \end{matrix} \quad \begin{bmatrix} 3z \\ -z \\ z \end{bmatrix} = z \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$$

iii) characteristic equation

$$|A - \lambda I| = 0$$

$$(1-\lambda)\left((-2-\lambda)(1-\lambda) - 1\right) - 2(-1) + (-1)\left((\lambda - 1)\right) = 0$$

$$(1-\lambda)(\lambda^2 + \lambda - 3) + 2 + 1 - \lambda = 0$$

$$-\lambda^3 + 3\lambda = 0$$

$$\lambda(\lambda^2 - 3) = 0$$

$\cancel{\lambda_{1,3} = \pm\sqrt{3}}$ $\quad \lambda_1 = +\sqrt{3}$

$\cancel{\lambda_2 = 0}$ $\quad \lambda_2 = -\sqrt{3}$

$\lambda_3 = 0$

<u>Eigen Vectors</u>

① $\lambda = 0$ $\quad A = \begin{vmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & -2 \end{vmatrix}$

$$\frac{x}{2+1} = \frac{y}{-1} = \frac{2}{1} = k_1$$

$$\frac{x}{3} = \frac{y}{-1} = \frac{2}{1} = k_1$$

$$x = (3, -1, 1)$$

② $\lambda = \sqrt{3}$

$$(A - \sqrt{3}I) = \begin{vmatrix} 1-\sqrt{3} & 2 & 1 \\ 0 & 1-\sqrt{3} & 1 \\ 1 & 1 & -2-\sqrt{3} \end{vmatrix}$$

$$\frac{x}{2+(1-\sqrt{3})} = \frac{y}{-(1-\sqrt{3})} = \frac{z}{4-2\sqrt{3}} = k_2$$

$$\frac{x}{3-\sqrt{3}} = \frac{y}{-1+\sqrt{3}} = \frac{z}{4-2\sqrt{3}} = k_2$$

$$k_2(3-\sqrt{3}, -1+\sqrt{3}, 4-2\sqrt{3})$$

③ $\lambda_3 = -\sqrt{3}$

$$(A + \sqrt{3}I) = \begin{vmatrix} 1+\sqrt{3} & 2 & 1 \\ 0 & 1+\sqrt{3} & 1 \\ 1 & 1 & -2+\sqrt{3} \end{vmatrix}$$

$$\frac{x}{2+(1+\sqrt{3})} = \frac{y}{-(1+\sqrt{3})} = \frac{z}{4+2\sqrt{3}}$$

$$\Rightarrow \frac{x}{3+\sqrt{3}} = \frac{y}{-1-\sqrt{3}} = \frac{z}{4+2\sqrt{3}} = k_3$$

$$k_3(3+\sqrt{3}, -1-\sqrt{3}, 4+2\sqrt{3})$$

iv) $a(1,0,1) \quad b(2,1,1) \quad c(-1,1,2)$

$$q_1 = \frac{a}{\|a\|} = \frac{(1,0,1)}{\sqrt{2}} = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$$

$$B = b - (q_1^T b)q_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} - \left( \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right) \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} - \frac{3}{\sqrt{2}} \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} \frac{3}{2} \\ 0 \\ \frac{3}{2} \end{bmatrix} \Rightarrow \cancel{\begin{bmatrix} 1/2 \\ 1 \\ -1/\sqrt{2} \end{bmatrix}} \quad \begin{bmatrix} 1/2 \\ 1 \\ -1/2 \end{bmatrix}$$

$$q_2 = \frac{1}{\sqrt{6}} (1,2,-1)$$

$$C = c - (q_2^T c)c - (q_1^T c)c$$

$$= (-1,1,2) - \frac{3}{6} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + \frac{3}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$c = (0,0,0)$$

$$q_3 = (0,0,0)$$

QR

$$q_1^T a = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \sqrt{2}$$

$$q_2^T a = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \sqrt{\frac{3}{2}} = \frac{3}{\sqrt{6}}$$

$$q_3^T c = 0$$

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$$R = \frac{1}{\sqrt{6}} \begin{bmatrix} \sqrt{12} & 3\sqrt{3} & -3\sqrt{3} \\ 0 & 3 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

$A = QR$

$$= \frac{1}{6} \begin{bmatrix} \sqrt{3} & 1 & 0 \\ 0 & 2 & 0 \\ \sqrt{3} & -1 & 0 \end{bmatrix} \begin{bmatrix} 2\sqrt{3} & 3\sqrt{3} & -3\sqrt{3} \\ 0 & 3 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

(4)

$$\begin{bmatrix} 1 & -4 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 10 \\ 8 \end{bmatrix}$$

$$\hat{x} = \frac{A^T b}{A^T A} \Rightarrow A^T A \hat{x} = A^T b$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ -4 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & -4 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -4 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 6 \\ 10 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 2 \\ 2 & 30 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 28 \\ 34 \end{bmatrix}$$

$$C = \frac{193}{29} \qquad D = \frac{20}{29}$$

$$y = \frac{193}{29} + \frac{20}{29} x$$

⑤ $Q = A(A^T A)^{-1} A^T$ $\quad A = \begin{bmatrix} 1 \\ 1 \\ 3 \\ 4 \end{bmatrix}$

$A^T A = \begin{bmatrix} 1 & 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 3 \\ 4 \end{bmatrix} = 27$

$Q = \left( \begin{bmatrix} 1 \\ 1 \\ 3 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 3 & 4 \end{bmatrix} \right) \times \frac{1}{27}$

$= \frac{1}{27} \begin{bmatrix} 1 & 1 & 3 & 4 \\ 1 & 1 & 3 & 4 \\ 3 & 3 & 9 & 12 \\ 4 & 4 & 12 & 16 \end{bmatrix}$

$P = I \cdot Q$

$= \frac{1}{27} \begin{bmatrix} 26 & -1 & -3 & -4 \\ -1 & 26 & -3 & -4 \\ -3 & -3 & 18 & -12 \\ -4 & -4 & -12 & 11 \end{bmatrix}$

⑥ $A = \begin{bmatrix} a & 2 & 2 \\ 2 & a & 2 \\ 2 & 2 & a \end{bmatrix}$

(i) $|a| > 0$ $\quad a \in (0, \infty)$

(ii) $\begin{vmatrix} a & 2 \\ 2 & a \end{vmatrix} > 0$

$a^2 - 4 > 0$ $\quad a \in (-\infty, -2) \cup (2, \infty)$

(iii) $\begin{vmatrix} a & 2 & 2 \\ 2 & a & 2 \\ 2 & 2 & a \end{vmatrix} > 0 \Rightarrow (a+4)(a-2)^2 > 0 \quad \therefore a \in (-4,$ ∞

iv) $[x_1 \;\; x_2 \;\; x_3] \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$= a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + 2a_{12}x_1x_2$$

$$+ 2a_{13}x_1x_3 + 2a_{23}x_2x_3 \rightarrow (1)$$

$$F = 2x_1^2 + 2x_2^2 + 2x_3^2 - 2x_1x_2 - 2x_2x_3$$

Comparing F with (1)

$$a_{11} = a_{22} = a_{33} = 2 \qquad a_{12} = -1 \quad a_{13} = 0 \quad a_{23} = -1$$

$$B = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

(7) For $\Sigma$

$$AA^T = \begin{bmatrix} -3 & 1 \\ 6 & -2 \\ 6 & -2 \end{bmatrix} \begin{bmatrix} -3 & 6 & 6 \\ 1 & -2 & -2 \end{bmatrix} = \begin{bmatrix} 10 & -20 & -20 \\ -20 & 40 & 40 \\ -20 & 40 & 40 \end{bmatrix}$$

$$|(AA^T) - \lambda I| = 0$$

$$(10-\lambda)\,|(40-\lambda)^2 - 1600) + 20\,(20\lambda) - 20\,(-20\lambda) = 0$$

$$(10-\lambda)(\lambda^2 - 80\lambda) + 800\lambda = 0$$

$$\lambda^2(\lambda - 90) = 0$$

$$\lambda_1 = 0 \quad \lambda_2 = 90$$

$$\Sigma_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 3\sqrt{10} \end{bmatrix}$$

For $U$:

$$AV = U\Sigma$$

$$\begin{bmatrix} 0 & \sqrt{10} \\ 0 & -2\sqrt{10} \\ 0 & -2\sqrt{10} \end{bmatrix} = U \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 3\sqrt{10} \end{bmatrix} \qquad \therefore U = \begin{bmatrix} 0 & 0 & 1/3 \\ 0 & 0 & -2/3 \\ 0 & 0 & -2/3 \end{bmatrix}$$

$$A = U \Sigma V^T$$

$$= \begin{bmatrix} 0 & 0 & 1/3 \\ 0 & 0 & -2/3 \\ 0 & 0 & -2/3 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 3\sqrt{10} \end{bmatrix} \begin{bmatrix} 1/\sqrt{10} & 3/\sqrt{10} \\ -3/\sqrt{10} & 1/\sqrt{10} \end{bmatrix}$$

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