Mansi sharma

'Study like there's no tomorrow because if you keep putting off, your studies for tomorrow, you'll probably be to late..'

Laplace transformation

Solved examples

Practice assignment


Nothing is impossible,
you just haven't put
enough efforts into.

Laplace transformation

1) **Def^n** let $f(t)$ be a well defined function of t for $0<t<\infty$ then, the Laplace transformation of $f(t)$, denoted by $L(f(t))$ is

$$L(f(t)) = \int_0^\infty e^{-st} f(t).\, dt$$

where s is any parameter.

$L(f(t))$ is clearly a function of s and we write $\boxed{L\{f(t)\} = \bar{f}(s)}$

$\Rightarrow \boxed{f(t) = L^{-1}(\bar{f}(s))}$,

$f(t)$ is called the inverse of $\bar{f}(s)$.

"L" $\rightarrow$ which transforms $f(t)$ into $\bar{f}(s)$ is called as Laplace transformation operator.

Linearity theorem

If $L(f_1(t)) = \bar{f}_1(s)$ and $L(f_2(t)) = \bar{f}_2(s)$ & c_1 & c_2 are constants, then,

$$L\{c_1 f_1(t) + c_2 f_2(t)\} = c_1 L\{f_1(t)\} + c_2 L\{f_2(t)\}$$
$$= c_1 \bar{f}_1(s) + c_2 \bar{f}_2(s).$$

Proof $L\{c_1 f_1(t) + c_2 f_2(t)\} = \int_0^\infty e^{-st}(c_1 f_1(t) + c_2 f_2(t))\, dt$

$$= c_1 \int_0^\infty e^{-st} f_1(t)\, dt + c_2 \int_0^\infty e^{-st} f_2(t) \cdot dt$$

$$= c_1 L\{f_1(t)\} + c_2 L\{f_2(t)\}$$

$$= c_1 \bar{f}_1(s) + c_2 \bar{f}_2(s).$$

First shifting property

If $L\{f(t)\} = \bar{f}(s)$ then $L\{e^{at} f(t)\} = \bar{f}(s-a)$, $s > a$.

Proof $L\{e^{at} f(t)\} = \int_0^\infty e^{-st} e^{at} f(t)\, dt$

$$= \int_0^\infty e^{-(s-a) \cdot t} f(t) \cdot dt.$$

Let $s-a = r$ then ~~ds = dr~~.

$$= \int_0^\infty e^{-rt} f(t) \cdot dt = \bar{f}(r) = \bar{f}(s-a)$$

i.e., If we know Laplace transform of $f(t)$ then we can write Laplace transform of $e^{at} f(t)$, just by replacing s with $s-a$.

→ **Conditions for the existence**

The Laplace transform of $f(t)$, i.e., $\int_0^\infty e^{-st} f(t)\, dt$ exists for $s > a$ if,

(i) $f(t)$ is continuous (ii) $\lim_{t \to \infty} \{e^{-at} f(t)\}$ is finite.

Laplace transform of elementary functions

(1) $\boxed{L(1) = \frac{1}{s}}$ $(s>0)$

Proof $L(1) = \int_0^\infty e^{-st}.1\,dt = \left[\frac{e^{-st}}{-s}\right]_0^\infty = \frac{1}{s}$.

(2) $\boxed{\begin{aligned} L(t^n) &= \frac{n!}{s^{n+1}}, \text{ when } n = 0,1,2,3,\ldots \\ &= \frac{\Gamma(n+1)}{s^{n+1}}, \text{ otherwise.} \end{aligned}}$ $(s>0)$ $(n>-1)$

Proof $L(t^n) = \int_0^\infty e^{-st}.t^n.dt$

Let $st = p \Rightarrow s.dt = dp$

$$= \int_0^\infty e^{-p}\left(\frac{p}{s}\right)^n.\frac{dp}{s}$$

$$= \frac{1}{s^{n+1}}\int_0^\infty e^{-p}.p^n.dp$$

$$= \frac{\Gamma(n+1)}{s^{n+1}} \quad \text{if } n>-1$$

$$= \frac{n!}{s^{n+1}} \quad \text{if } n = 0,1,2,3,\ldots$$

Eg. $L(t^{-1/2}) = \frac{\Gamma(-1/2+1)}{s^{-1/2+1}} = \frac{\Gamma(1/2)}{\sqrt{s}} = \frac{\sqrt{\pi}}{\sqrt{s}}$

$L(t^{1/2}) = \frac{\Gamma(1/2+1)}{s^{1/2+1}} = \frac{\frac{1}{2}\Gamma(1/2)}{s^{3/2}} = \frac{\sqrt{\pi}}{2s^{3/2}}$ $\left(\because \Gamma(n+1) = n\cdot\Gamma(n)\right.$

③ $\boxed{L\{e^{at}\} = \frac{1}{s-a}}$ $(s>a)$

Proof $L(e^{at}) = \int_0^\infty e^{-st}\cdot e^{at}\,dt$

$= \int_0^\infty e^{-(s-a)\cdot t}\,dt$

$= \left[\frac{e^{-(s-a)}}{-(s-a)}\right]_0^\infty = \frac{1}{s-a}$, $s>a$.

④ $\boxed{L\{\sin at\} = \frac{a}{s^2+a^2}}$ $(s>0)$

Proof $L\{\sin at\} = \int_0^\infty e^{-st}\cdot\sin at\,dt$

$= e^{-st}\left(\frac{-\cos at}{a}\right) - \int_0^\infty s\,e^{-st}\,\frac{\cos at}{a}\,dt$

$= \frac{e^{-st}}{s^2+a^2}\left(-s\sin at - a\cos at\right)\Big|_0^\infty$ (direct formula

$= \frac{a}{s^2+a^2}$

⑤ $\boxed{L\{\cos at\} = \frac{s}{s^2+a^2}}$

Proof {Same as above for $\sin at$}

⑥ $\boxed{L(\sinh at) = \frac{a}{s^2-a^2}}$ $(s>|a|)$

Proof $L(\sinh at) = \int_0^\infty e^{-st} \sinh at \, dt$

$$= \int_0^\infty e^{-st}\left(\frac{e^{at}-e^{-at}}{2}\right) dt$$

$$= \frac{1}{2}\int_0^\infty \left(e^{-(s-a)t} - e^{-(s+a)t}\right) dt$$

$$= \frac{1}{2}\left[\frac{e^{-(s-a)t}}{-(s-a)} + \frac{e^{-(s+a)t}}{(s+a)}\right]_0^\infty$$

$$= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right] = \frac{a}{s^2-a^2}$$

⑦ $\boxed{L\{\cosh at\} = \frac{s}{s^2-a^2}}$ $(s>|a|)$

Proof (Same as above)

Applications of first shifting property :-

① $L(e^{at}) = L(e^{at}.1)$

$$= \frac{1}{s-a}.$$

(replace s by s-a in Laplace transform of 1., we know $L(1) = 1/s$.

② $L(e^{at} t^n) = \frac{n!}{(s-a)^{n+1}}, \quad n = 0, 1, \ldots$ { replace s by $(s-a)$ in Laplace transf^n of t^n.

$= \frac{\overline{|n+1}}{(s-a)^{n+1}}, \quad n > -1.$

③ $L\{e^{at} \sin bt\} = \frac{b}{(s-a)^2+b^2}$ $\left(\because L(\sin bt) = \frac{b}{s^2+b^2}\right.$

④ $L\{e^{at} \cos bt\} = \frac{(s-a)}{(s-a)^2+b^2}$ $\left(\because L(\cos bt) = \frac{s}{s^2+b^2}\right.$

⑤ $L\{e^{at} \sinh bt\} = \frac{b}{(s-a)^2-b^2}$ $\left(\because L(\sinh bt) = \frac{b}{s^2-b^2}\right.$

⑥ $L\{e^{at} \cosh bt\} = \frac{(s-a)}{(s-a)^2-b^2}$ $\left\{\because L(\cosh bt) = \frac{s}{s^2-b^2}\right.$ & replacing s by s-a by 1st shifting property

Eg Find Laplace transform^n of the following functions :—

ⓐ $\sin 2t \cos 3t$

$L(\sin 2t \cos 3t) = L\left\{\frac{\sin 5t - \sin t}{2}\right\}$

$= \frac{1}{2} L\{\sin 5t\} - \frac{1}{2} L\{\sin t\}$

$= \frac{5}{2(s^2+25)} - \frac{1}{2(s^2+1)}$ $\left[\text{As } L(\sin at) = \frac{a}{s^2+a^2}\right.$

$= \frac{2s^2-10}{(s^2+25)(s^2+1)}$

(b) $e^{-2t}(3\cos 4t - 2\sin 5t)$

$$L\{e^{-2t}(3\cos 4t - 2\sin 5t)\} = 3\,L\{e^{-2t}\cos 4t\} - 2\,L\{e^{-2t}\sin 5t\}$$

$$= 3\left\{\frac{(s+2)}{(s+2)^2+16}\right\} - 2\left\{\frac{5}{(s+2)^2+25}\right\}$$

$$= \frac{3(s+2)}{(s+2)^2+16} - \frac{10}{(s+2)^2+25}$$

$\because L(e^{at}\sin bt) = \dfrac{b}{(s-a)^2+b^2}$ &

$L(e^{at}\cos bt) = \dfrac{(s-a)}{(s-a)^2+b^2}$

(c) $L\{\cos^3 t\}$

$$\cos 3t = 4\cos^3 t - 3\cos t.$$

$$\Rightarrow \cos^3 t = \frac{\cos 3t + 3\cos t}{4}.$$

$$\therefore L\{\cos^3 t\} = \frac{1}{4}L\{\cos 3t\} + \frac{3}{4}L\{\cos t\}$$

$$= \frac{1}{4}\left\{\frac{s}{s^2+9} + \frac{3s}{s^2+1}\right\} = \frac{s^3+7s}{(s^2+9)(s^2+1)}.$$

(d) $$f(t) = \begin{cases} 2+t^2 & 0<t<2 \\ 6 & 2<t<3 \\ 2t-5 & 3<t<\infty \end{cases}$$

$$L(f(t)) = \int_0^\infty e^{-st} f(t)\cdot dt$$

$$= \int_0^2 e^{-st}(2+t^2)\,dt + \int_2^3 e^{-st}\cdot 6\,dt + \int_3^\infty e^{-st}(2t-5)\,d$$

$$= \left[\frac{e^{-st}}{(-s)}(2+t^2) + \int \frac{e^{-st}}{-s}(2t)\right]_0^2 + \left[\frac{6e^{-st}}{-s}\right]_2^3$$

$$+ \left[\frac{e^{-ts}}{-s}(2t-5) - \int \frac{e^{-ts}}{(-s)} \cdot 2\right]_3^\infty$$

$$= \left[\frac{e^{-st}}{-s}(2+t^2) - \frac{2}{s}\left(\frac{te^{-st}}{-s} + \frac{e^{-st}}{s}\right)\right]_0^2 - \frac{6}{s}\left[e^{-3s} - e^{-2s}\right]$$

$$+ \left[\frac{e^{-ts}}{-s}(2t-5) - \frac{2}{s^2}e^{-ts}\right]_3^\infty$$

$$= \frac{2}{s} + \frac{2}{s^3} - e^{-2s}\left(\frac{4}{s^2} + \frac{2}{s^3}\right) + e^{-3s}\left(\frac{2}{s^2} - \frac{5}{s}\right)$$

(e) $f(t) = t/a, \quad 0 < t < a$

$\quad\quad = 1, \quad t > a.$

$$L(f(t)) = \int_0^\infty e^{-st} f(t) \cdot dt$$

$$= \int_0^a e^{-st} \cdot \frac{t}{a} dt + \int_a^\infty e^{-st} \cdot 1 \cdot dt$$

$$= \frac{1}{a}\left[\frac{te^{-st}}{-s} - \frac{e^{-st}}{s^2}\right]_0^a + \left[\frac{e^{-st}}{-s}\right]_a^\infty$$

$$= \frac{1}{as^2}(1 - e^{-as})$$

Properties of Laplace transform

① If $L\{f(t)\} = \bar{f}(s)$ then $L\{f(at)\} = \frac{1}{a}\bar{f}\left(\frac{s}{a}\right)$.

Proof $L\{f(at)\} = \int_0^\infty e^{-st} f(at).dt$

let $at = x$ then $a\,dt = dx$.

$$= \int_0^\infty e^{-s.x/a} f(x).\frac{dx}{a}$$

$$= \frac{1}{a}\int_0^\infty e^{-(s/a)x} f(x).dx$$

$$= \frac{1}{a}\bar{f}(s/a).$$

② If $L\{f(t)\} = \bar{f}(s)$ then $L\{t f(t)\} = -\frac{d}{ds}\bar{f}(s)$.

Proof $\frac{d}{ds}\bar{f}(s) = \int_0^\infty \frac{d}{ds} e^{-st} f(t)\,dt$

$$= -\int_0^\infty e^{-st}(t f(t))\,dt$$

$$= -L\{t f(t)\}.$$

③ If $L\{f(t)\} = \bar{f}(s)$ then $L\left\{\frac{1}{t}f(t)\right\} = \int_s^\infty \bar{f}(s)\,ds$.

Pf $\int_s^\infty \bar{f}(s).ds = \int_s^\infty \left[\int_0^\infty e^{-st} f(t).dt\right] ds.$

$$= \int_0^\infty f(t)\left[\int_s^\infty e^{-st}\,ds\right] dt = \int_0^\infty f(t)\left[\frac{e^{-st}}{-t}\right]_s^\infty dt$$

$$= \int_0^\infty e^{-st}\,\frac{f(t)}{t}\,dt = L\left\{\frac{f(t)}{t}\right\}$$

④ Transform of derivatives.

If $L(f(t)) = \bar{f}(s)$ then $L(f'(t)) = s\bar{f}(s) - f(0)$

Proof $L\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\,dt$

$$= e^{-st} f(t)\Big|_0^\infty - \int_0^\infty e^{-st}(-s)\cdot f(t)\,dt$$

$$= -f(0) + s\,L\{f(t)\}$$

$$= s\bar{f}(s) - f(0).$$

Replacing $f(t)$ by $f'(t)$, we get

$$L\{f''(t)\} = s^2\bar{f}(s) - s\,f(0) - f'(0).$$

In general,

$$L\{f^{(n)}(t)\} = s^n\bar{f}(s) - s^{n-1}f(0) - \cdots\cdots - f^{(n-1)}(0)$$

⑤ Transform of integrals.

If $L\{f(t)\} = \bar{f}(s)$ then $L\left\{\int_0^t f(u)\cdot du\right\} = \frac{1}{s}\bar{f}(s).$

Proof $L\left\{\int_0^t f(u)\,du\right\} = \int_0^\infty e^{-st}\left(\int_0^t f(u)\cdot du\right)dt$

$$= \frac{e^{-st}}{-s}\left(\int_0^t f(u)\,du\right)\Big|_0^\infty - \int_0^\infty \frac{d}{dt}\left(\int_0^t f(u)\,du\right)\cdot\frac{e^{-st}}{(-s)}\,dt$$

Scanned by CamScanner

$$= 0 + \frac{1}{s}\int_0^{\infty} e^{-st} f(t)\,dt \qquad \left\{ \because \frac{d}{dt}\left(\int_0^t f(u)\,du\right) = f(t). \right.$$

$$= \frac{L(f(t))}{s} = \frac{F(s)}{s}.$$

Eg $L\{t \sin at\}$.

we know $L\{t\,f(t)\} = -\frac{d}{ds} F(s)$.

$$\therefore L\{t \sin at\} = -\frac{d}{ds} L\{\sin at\}$$

$$= -\frac{d}{ds}\left(\frac{a}{s^2+a^2}\right) = \frac{2as}{(s^2+a^2)^2}.$$

Eg $L\{te^{-t}\sin 3t\}$.

we know $L\{e^{-t}\sin 3t\} = \frac{3}{(s+1)^2+9}$

$$\therefore L\{t\underbrace{(e^{-t}\sin 3t)}_{f(t)}\} = -\frac{d}{ds} L\{e^{-t}\sin 3t\}$$

$$= -\frac{d}{ds}\left(\frac{3}{(s+1)^2+9}\right) = \frac{6s+6}{[(s+1)^2+9]^2}$$

Eg find $L\left(\int_0^t e^t \frac{\sin t}{t}\,dt\right)$

Now, $L\left(\int_0^t e^t \frac{\sin t}{t}\,dt\right) = \frac{1}{s} L\left(\frac{e^t \sin t}{t}\right)$ (by above property 5)

$$= \frac{1}{s}\int_s^{\infty} L(e^t \sin t)\,ds$$ (by above property 3)

Scanned by CamScanner

$$= \frac{1}{s}\int_{s}^{\infty} \frac{1}{(s-1)^2+1}\,ds = \frac{1}{s}\left[\tan^{-1}(s-1)\right]_s^{\infty}$$

$$= \frac{1}{s}\left[\frac{\pi}{2} - \tan^{-1}(s-1)\right].$$

Ex: Prove that $L\left(\int_0^t \frac{\cos at - \cos bt}{t}\,dt\right) = \frac{1}{2s}\log\frac{s^2+b^2}{s^2+a^2}$.

Eg Use Laplace transform to show that $\int_0^{\infty} \frac{\sin t}{t}\,dt = \frac{\pi}{2}$.

Proof $L\left(\frac{\sin t}{t}\right) = \int_s^{\infty} L(\sin t)\,ds$

$$= \int_s^{\infty} \frac{1}{s^2+1}\,ds = \left[\tan^{-1}s\right]_s^{\infty}$$

$$= \frac{\pi}{2} - \tan^{-1}s.$$

$$\Rightarrow \int_0^{\infty} e^{-st}\left(\frac{\sin t}{t}\right)dt = \frac{\pi}{2} - \tan^{-1}s \quad \text{--- (1)}$$

Let $s=0$ in (1)

then $\int_0^{\infty} \frac{\sin t}{t}\,dt = \frac{\pi}{2} - \tan^{-1}0 = \frac{\pi}{2}.$

Eg. Evaluate $\int_0^\infty te^{-2t}\cos t\,dt$, using Laplace trans^n (7)

we know,

$$L(t\cos t) = -\frac{d}{ds}L(\cos t)$$

$$= -\frac{d}{ds}\left(\frac{s}{s^2+1}\right)$$

$$\Rightarrow \int_0^\infty e^{-st}\cdot t\cos t = -\frac{(s^2+1)+2s^2}{(s^2+1)^2} = \frac{s^2-1}{(s^2+1)^2} \quad \text{—(1)}$$

Let $s=2$ in (1)

$$\Rightarrow \int_0^\infty e^{-2t}\cdot t\cos t\,dt = \frac{3}{25} \quad \underline{\text{Ans}}.$$

Inverse Laplace transform

(1) $L(1) = \frac{1}{s} \Rightarrow L^{-1}(1/s) = 1.$

(2) $L(e^{at}) = \frac{1}{s-a} \Rightarrow L^{-1}\left(\frac{1}{s-a}\right) = e^{at}.$

(3) $L(t^n) = \frac{n!}{s^{n+1}} \Rightarrow L^{-1}\left(\frac{1}{s^{n+1}}\right) = \frac{t^n}{n!}$

(4) $L(e^{at}\cdot t^n) = \frac{n!}{(s-a)^{n+1}} \Rightarrow L^{-1}\left(\frac{1}{(s-a)^{n+1}}\right) = \frac{t^n e^{at}}{n!}$

(5) $L(\sin at) = \frac{a}{s^2+a^2} \Rightarrow L^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{\sin at}{a}$

(6) $L(\cos at) = \frac{s}{s^2+a^2} \Rightarrow L^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos at.$

(7) $L(\sinh at) = \frac{a}{s^2-a^2} \Rightarrow L^{-1}\left(\frac{1}{s^2-a^2}\right) = \frac{1}{a}\sinh at$

(8) $L(\cosh at) = \frac{s}{s^2-a^2} \Rightarrow L^{-1}\left(\frac{s}{s^2-a^2}\right) = \cosh at.$

(9) $L(e^{at}\sin bt) = \frac{b}{(s-a)^2+b^2}$

$$\Rightarrow L^{-1}\left(\frac{1}{(s-a)^2+b^2}\right) = \frac{1}{b}e^{at}\sin bt.$$

(10) Similarly $L^{-1}\left[\frac{(s-a)}{(s-a)^2+b^2}\right] = e^{at}\cos bt.$

(11) $L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = \frac{1}{2a}t\sin at.$

<u>Proof</u> We know $L(t\sin at) = -\frac{d}{ds}L(\sin at)$

$$= -\frac{d}{ds}\left(\frac{a}{a^2+s^2}\right) = \frac{2as}{(s^2+a^2)^2}$$

$$\Rightarrow L^{-1}\left[\frac{2as}{(s^2+a^2)^2}\right] = t\sin at.$$

$$\Rightarrow L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = \frac{1}{2a}t\sin at.$$

(12) $L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right] = \frac{1}{2a^3}[\sin at - at\cos at]$

i.e., to show, $L(\sin at - at\cos at) = \frac{2a^3}{(s^2+a^2)^2}$

$$L.H.S = L(\sin at - at\cos at)$$

$$= L(\sin at) - L(at\cos at)$$

$$= \frac{a}{s^2+a^2} + a\cdot\frac{d}{ds}\left(\frac{s}{s^2+a^2}\right)$$

$$= \frac{a}{s^2+a^2} + a\left[\frac{s^2+a^2-2s^2}{(s^2+a^2)^2}\right]$$

$$= \frac{2a^3}{(s^2+a^2)^2}.$$ Hence proved.

Eg. Find inverse Laplace transform for the following.

(a) $\dfrac{2s^2-6s+5}{s^3-6s^2+11s-6}.$

Soln. $\dfrac{2s^2-6s+5}{s^3-6s^2+11s-6} = \dfrac{2s^2-6s+5}{(s-1)(s-3)(s-2)}$

$$= \frac{A}{(s-1)} + \frac{B}{(s-3)} + \frac{C}{(s-2)}$$

$$= \frac{1}{2(s-1)} + \frac{5}{2(s-3)} - \frac{1}{(s-2)}$$

Applying Inverse L.T. on both sides,

$$L^{-1}\left(\frac{2s^2-6s+5}{s^3-6s^2+11s-6}\right) = \frac{1}{2}L^{-1}\left(\frac{1}{s-1}\right) + \frac{5}{2}L^{-1}\left(\frac{1}{s-3}\right) - L^{-1}\left(\frac{1}{s-2}\right)$$

$$= e^t/2 + 5e^{3t}/2 - e^{2t}$$ Ans.

(b) $\log\left(\frac{s+1}{s-1}\right)$

Sol^n: let $f(t) = L^{-1}\left(\log\frac{(s+1)}{(s-1)}\right)$

we know, $L(t f(t)) = \frac{-d}{ds}\bar{f}(s)$

$$= \frac{-d}{ds}(L(f(t)))$$

$$= \frac{-d}{ds}\left[\log\frac{(s+1)}{s-1}\right]$$

$$= \frac{-d}{ds}\left[\log(s+1) - \log(s-1)\right]$$

$$= \frac{-1}{s+1} + \frac{1}{s-1} = \frac{2}{s^2-1}$$

$\Rightarrow t f(t) = L^{-1}\left(\frac{2}{s^2-1}\right) = 2L^{-1}\left(\frac{1}{s^2-1}\right)$

$$= 2\sinh t$$

$\Rightarrow \boxed{f(t) = \frac{2\sinh t}{t}}$

(c) $\frac{s}{(s^2-a^2)^2}$

Sol^n: we know $L(\sinh at) = \frac{a}{s^2-a^2}$

$\therefore L(t\sinh at) = \frac{-d}{ds}\left(\frac{a}{s^2-a^2}\right) = \frac{2as}{(s^2-a^2)^2}$

$$\Rightarrow L^{-1}\left[\frac{s}{(s^2-a^2)^2}\right] = \frac{1}{2a} t \sinh at .$$ ⑨

(d) $\frac{1}{s^3(s^2+1)}$.

let we have, $\frac{1}{s^3(s^2+1)} = \frac{1}{s(s^2(s^2+1))}$

$$= \frac{1}{s}\left(\frac{1}{s^2} - \frac{1}{s^2+1}\right)$$

$$= \frac{1}{s^3} - \frac{1}{s(s^2+1)}$$

$$= \frac{1}{s^3} - \frac{1}{s} + \frac{s}{s^2+1}$$

Taking Inverse L-T. on both sides,

$$L^{-1}\left[\frac{1}{s^3(s^2+1)}\right] = L^{-1}\left(\frac{1}{s^3}\right) - L^{-1}\left(\frac{1}{s}\right) + L^{-1}\left(\frac{s}{s^2+1}\right)$$

$$= \frac{t^2}{2} - 1 + \cos t \quad \underline{\text{Ans}}.$$

(e) $\frac{1}{(s^2-a^2)^2}$.

Soln we know $L(t\cosh at) = -\frac{d}{ds}\left(\frac{s}{s^2-a^2}\right)$

$$= -\frac{(s^2-a^2)+2s^2}{(s^2-a^2)^2}$$

$$= \frac{s^2}{(s^2-a^2)^2} + \frac{a^2}{(s^2-a^2)^2}$$

$$= \frac{(s^2-a^2)}{(s^2-a^2)^2} + \frac{a^2}{(s^2-a^2)^2} + \frac{a^2}{(s^2-a^2)^2}$$

$$= \frac{1}{s^2-a^2} + \frac{2a^2}{(s^2-a^2)^2}.$$

$$\therefore t\cosh at = L^{-1}\left(\frac{1}{s^2-a^2}\right) + 2a^2 L^{-1}\left[\frac{1}{(s^2-a^2)^2}\right]$$

$$= \frac{1}{a}\sinh at + 2a^2 L^{-1}\left[\frac{1}{(s^2-a^2)^2}\right]$$

$$\Rightarrow L^{-1}\left[\frac{1}{(s^2-a^2)^2}\right] = \frac{t\cosh at}{2a^2} - \frac{1}{2a^3}\sinh at.$$

Convolution and Convolution theorem :→

Convolution of two functions $f_1 * f_2$ is defined as $\boxed{f_1 * f_2 = \int_0^t f_1(u) \cdot f_2(t-u)\, du.}$

Note that $f_1 * f_2 = f_2 * f_1$

↳ $\boxed{L(f_1 * f_2) = \bar{f_1}(s) \cdot \bar{f_2}(s)}$

$$\Rightarrow L^{-1}(\bar{f_1}(s) \cdot \bar{f_2}(s)) = f_1 * f_2 = \int_0^t f_1(u) \cdot f_2(t-u) \cdot du$$

$$= f_2 * f_1 = \int_0^t f_2(u) \cdot f_1(t-u) \cdot du$$

Result $L(f_1 * f_2) = \bar{f}_1(s) \cdot \bar{f}_2(s)$

Proof $L(f_1 * f_2) = L\left\{ \int_0^t f_1(u) \cdot f_2(t-u) \cdot du \right\}$

$$= \int_0^\infty e^{-st} \left[\int_0^t f_1(u) \cdot f_2(t-u) \cdot du \right] dt$$

$$= \int_0^\infty \cdot \int_0^t e^{-st} f_1(u) \cdot f_2(t-u)\, du \cdot dt$$

$$= \int_0^\infty \left[\int_u^\infty e^{-st} f_2(t-u)\, dt \right] f_1(u)\, du.$$

let $t-u = y \Rightarrow dt = dy$

$$= \int_0^\infty \left[\int_0^\infty e^{-s(u+y)} f_2(y)\, dy \right] f_1(u) \cdot du$$

$$= \int_0^\infty e^{-su} f_1(u) \cdot du \int_0^\infty e^{-sy} f_2(y) \cdot dy$$

$$= \bar{f}_1(s) \cdot \bar{f}_2(s).$$

$\Rightarrow \quad L(f_1 * f_2) = \bar{f}_1(s) \cdot \bar{f}_2(s). \quad -(1)$

$\Rightarrow \quad f_1 * f_2 = L^{-1}\left(\bar{f}_1(s) \cdot \bar{f}_2(s)\right) \quad -(2)$

Eg Use convolution to find $L^{-1}\left[\dfrac{1}{(s^2+a^2)^2}\right]$.

Soln $L^{-1}\left[\dfrac{1}{(s^2+a^2)^2}\right] = L^{-1}\left[\underbrace{\dfrac{1}{(s^2+a^2)}}_{\bar{f}_1(s)} \cdot \underbrace{\dfrac{1}{(s^2+a^2)}}_{\bar{f}_2(s).}\right]$

$= f_1 * f_2$ where $f_1(t) = L^{-1}(\bar{f_1}(s))$ & $f_2(t) = L^{-1}(\bar{f_2}(s))$ (by ②)

$$= \int_0^t f_1(u)\, f_2(t-u).du$$

As $f_1(t) = L^{-1}(\bar{f_1}(s)) = L^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{\sin at}{a}$

& $f_2(t) = L^{-1}(\bar{f_2}(s)) = L^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{\sin at}{a}$

$$\therefore L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right] = \int_0^t \frac{1}{a}\sin au \cdot \frac{1}{a}\sin a(t-u).du.$$

$$= \frac{1}{a^2}\int_0^t \sin au \sin a(t-u).du$$

$$= \frac{1}{2a^2}\int_0^t [\cos a(2u-t) - \cos at]\, du$$

$$= \frac{1}{2a^3}\left[\sin at - at\cos at\right]$$

Eg. $L^{-1}\left[\frac{s}{(s^2+a^2)^3}\right] = L^{-1}\left[\underbrace{\frac{s}{(s^2+a^2)^2}}_{\bar{f_1}(s)} \cdot \underbrace{\frac{1}{s^2+a^2}}_{\bar{f_2}(s).}\right]$

$= f_1 * f_2$

$$= \int_0^t f_1(u) \cdot f_2(t-u).du$$

where $f_1(t) = L^{-1}(\bar{f_1}(s))$

$= L^{-1}\left(\frac{s}{(s^2+a^2)^2}\right)$

$= t\,\frac{\sin at}{2a}.$

$f_2(t) = L^{-1}(\bar{f_2}(s)) = \frac{\sin at}{a}$

$$= \int_0^t \frac{u\sin au}{2a} \cdot \frac{\sin a(t-u)}{a}\, du$$

$$= \frac{1}{2a^2}\int_0^t u\sin au \cdot \sin a(t-u) \cdot du$$

$$= \frac{1}{4a^2}\int_0^t u\left(\cos(2au-at) - \cos(at)\right) du.$$

$$= \frac{1}{4a^2}\left[\frac{u\sin(2au-at)}{2a} + \frac{1}{4a^2}\cos(2au-at)\right]_0^t$$

$$- \frac{1}{4a^2}\left[\frac{u^2\cos at}{2}\right]_0^t$$

$$= \frac{1}{8a^3}\left[t\sin at - at^2\cos at\right]$$

Laplace transformation of periodic functions

A function f is said to be periodic with period T if $f(t+T) = f(t)$ $\forall t$.

Let $f(t)$ be periodic with period T,

then $\boxed{L(f(t)) = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} \cdot f(t) \cdot dt}$

Eg. If $f(t) = t^2$, for $0<t<2$ and $f(t+2) = f(t)$ for $t>2$. find $L(f(t))$.

Soln. Since $f(t) = t^2$ is given to be periodic with period 2.

$$\therefore L(f(t)) = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} \cdot f(t) \cdot dt$$

$$= \frac{1}{1-e^{-2s}} \int_0^2 e^{-st} \cdot t^2 \cdot dt.$$

$$= \frac{1}{1-e^{-2s}} \left[\frac{t^2 \cdot e^{-st}}{-s} - \frac{2te^{-st}}{s^2} - \frac{2e^{-st}}{s^3} \right]_0^2$$

$$= \frac{1}{1-e^{-2s}} \left[\frac{-4e^{-2s}}{s} - \frac{4e^{-2s}}{s^2} - \frac{2e^{-2s}}{s^3} + \frac{2}{s^3} \right]$$

$$= \frac{-2e^{-2s}}{s^3(1-e^{-2s})} \left[2s(s+1) + 1 - e^{-2s} \right].$$

Eg Find Laplace transformation of $2c$-periodic function $f(t) = \begin{cases} t, & 0<t<c. \\ 2c-t, & c<t<2c. \end{cases}$

$$L(f(t)) = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} \cdot f(t)\, dt$$

$$= \frac{1}{1-e^{-2cs}} \left[\int_0^c e^{-st} \cdot t\, dt + \int_c^{2c} e^{-st} (2c-t)\, dt \right]$$

$$= \frac{1}{1-e^{-2cs}} \left[\frac{ce^{-cs}}{-s} - \frac{e^{-cs}}{s^2} + \frac{1}{s^2} + \frac{e^{-2cs}}{s^2} + \frac{ce^{-sc}}{s} - \frac{e^{-cs}}{s^2} \right]$$

Scanned by CamScanner

$$= \frac{1}{s^2(1-e^{-2cs})}\left[1-2e^{-cs}+e^{-2cs}\right]$$

Applications of Laplace transform to solve ordinary differential equation :→

Working rule

① Apply Laplace transform on both sides of differential eqn.

② Divide each side by the coefficient of $\bar{y}$, getting $\bar{y}$ as a function of s.

③ Apply partial fraction & take inverse Laplace transform.

Eg. Solve $y''-2y'+y = e^t$, $y(0)=2$, $y'(0)=-1$.

Soln ⇒ $L(y'') - 2L(y') + L(y) = L(e^t)$

$$\Rightarrow \{s^2\bar{y} - s\,y(0) - y'(0)\} - 2\{s\bar{y} - y(0)\} + \bar{y} = \frac{1}{s-1}$$

$$\Rightarrow (s^2\bar{y} - 2s\bar{y} + \bar{y}) = \frac{1}{(s-1)} + s\,y(0) + y'(0) - 2y(0)$$

(Using Transform of derivatives.)

$$= \frac{2s^2 - 7s + 6}{s-1}$$

$$\Rightarrow \bar{y} = \frac{2s^2-7s+6}{(s-1)(s^2-2s+1)} = \frac{2s^2-7s+6}{(s-1)^3}$$

$$= \frac{A}{(s-1)} + \frac{B}{(s-1)^2} + \frac{C}{(s-1)^3}.$$

$$= \frac{2}{s-1} - \frac{3}{(s-1)^2} + \frac{1}{(s-1)^3}$$

$$\Rightarrow \bar{y} = L(y) = \frac{2}{s-1} - \frac{3}{(s-1)^2} + \frac{1}{(s-1)^3}.$$

$$\Rightarrow y = L^{-1}\left(\frac{2}{s-1}\right) - 3L^{-1}\left(\frac{1}{(s-1)^2}\right) + L^{-1}\left(\frac{1}{(s-1)^3}\right)$$

$$= 2e^t - 3te^t + \frac{t^2}{2}e^t \quad \underline{\text{Ans}}.$$

$$\left\{ \because L^{-1}\left(\frac{1}{(s-a)^{n+1}}\right) = \frac{t^n e^{at}}{n!} \right.$$

Eg. $(D^3 - 6D^2 + 11D - 6)x = 5$,

$x(0) = x''(0) = 0,\ x'(0) = 1.$

Sol^n $\Rightarrow (D^3x - 6D^2x + 11Dx - 6x) = 5.$

$$\Rightarrow (x''' - 6x'' + 11x' - 6x) = 5, \quad x(0) = x''(0) = 0$$
$$x'(0) = 1.$$

$$\Rightarrow L\{x'''\} - 6L\{x''\} + 11L\{x'\} - 6L(x) = 5L(1)$$

$$\Rightarrow \{s^3\bar{x}(s) - s^2x(0) - sx'(0) - x''(0)\}$$
$$-6\{s^2\bar{x} - sx(0) - x'(0)\}$$
$$+11\{s\bar{x} - x(0)\} - 6\bar{x} = \frac{5}{s}.$$

$$\left[\because L(1) = \frac{1}{s}\right.$$

$$\Rightarrow (s^3-6s^2+11s-6)\bar{x} - s^2\cdot 0 - s\cdot 1 - 0 + 6s\cdot 0 + 6\cdot 1 - 11s\cdot 0 = \frac{5}{s}.$$

$$\Rightarrow (s^3-6s^2+11s-6)\bar{x} = \frac{5}{s} + s - 6$$

$$\Rightarrow \bar{x} = \frac{(s^2-6s+5)}{s^3-6s^2+11s-6}$$

$$= \frac{(s^2-6s+5)}{s(s-1)(s-2)(s-3)}$$

$$= \frac{\cancel{(s-1)}(s-5)}{s\cancel{(s-1)}(s-2)(s-3)}$$

$$= \frac{A}{s} + \frac{B}{(s-2)} + \frac{C}{(s-3)}$$

$$\Rightarrow \bar{x} = \frac{-5}{6s} + \frac{3}{2(s-2)} - \frac{2}{3(s-3)}$$

$$\Rightarrow x = L^{-1}(\bar{x}) = \frac{-5}{6}L^{-1}\left(\frac{1}{s}\right) + \frac{3}{2}L^{-1}\left(\frac{1}{s-2}\right) - \frac{2}{3}L^{-1}\left(\frac{1}{s-3}\right)$$

$$= \frac{-5}{6} + \frac{3}{2}e^{2t} - \frac{2}{3}e^{3t} \quad \underline{\text{Ans}}$$

Eg. $\frac{dy}{dt} + 2y + \int_0^t y\,dt = \sin t$, $y(0) = 1$.

$$\Rightarrow L(y') + 2L(y) + L\left(\int_0^t y\,dt\right) = L(\sin t).$$

$\Rightarrow \{s\bar{y} - y(0)\} + 2\bar{y} + \frac{\bar{y}}{s} = \frac{1}{s^2+1}$

$\Rightarrow \bar{y} = \frac{s^3+2s}{(s^2+1)(s^2+2s+1)}$

$\Rightarrow y = L^{-1}(\bar{y}) = L^{-1}\left[\frac{s^3+2s}{(s^2+1)(s^2+2s+1)}\right]$

$= L^{-1}\left[\frac{1}{s+1} - \frac{3}{2(s+1)^2} + \frac{1}{2(s^2+1)}\right]$

$= e^{-t} - \frac{3te^{-t}}{2} + \frac{1}{2}\sin t$ Ans

Eg. Solve the simultaneous equations

$\frac{dx}{dt} + \frac{dy}{dt} + x = -e^{-t}$.

$\frac{dx}{dt} + 2\frac{dy}{dt} + 2x + 2y = 0$ given $x(0) = -1$, $y(0) = 1$.

Sol^n. we have, $x' + y' + x = e^{-t}$ —(1)

$x' + 2y' + 2x + 2y = 0$ —(2)

Applying Laplace transform on (1),

$L(x') + L(y') + L(x) = L(-e^{-t})$

$\Rightarrow \{s\bar{x} - x(0)\} + \{s\bar{y} - y(0)\} + \bar{x} = \frac{-1}{s+1}$

$\Rightarrow (s+1)\bar{x} + s\bar{y} = \frac{-1}{s+1}$. —(3)

Scanned by CamScanner

Now Apply L.T. on ②, in the similar way (14)
we get,

$$s\bar{x} - x(0) + 2s\bar{y} - 2y(0) + 2\bar{x} + 2\bar{y} = 0$$

$$\Rightarrow (s+2)\bar{x} + 2(s+1)\bar{y} = 1 \quad \text{—④}$$

Solving ③ & ④, we get, (by elimination method)

$$\bar{y} = \frac{s^2+3s+3}{(s+1)(s^2+2s+2)} \quad \text{and} \quad \bar{x} = \frac{-(s+2)}{s^2+2s+2}$$

$$\Rightarrow y = L^{-1}(\bar{y}) = L^{-1}\left[\frac{s^2+3s+3}{(s+1)(s^2+2s+2)}\right]$$

$$= L^{-1}\left[\frac{1}{s+1} + \frac{1}{(s+1)^2+1}\right]$$

$$= e^{-t} + e^{-t}\sin t.$$

$$\& \; x = L^{-1}(\bar{x}) = L^{-1}\left[\frac{-(s+2)}{s^2+2s+2}\right]$$

$$= -L^{-1}\left[\frac{s+1}{(s+1)^2+1} - \frac{1}{(s+1)^2+1}\right]$$

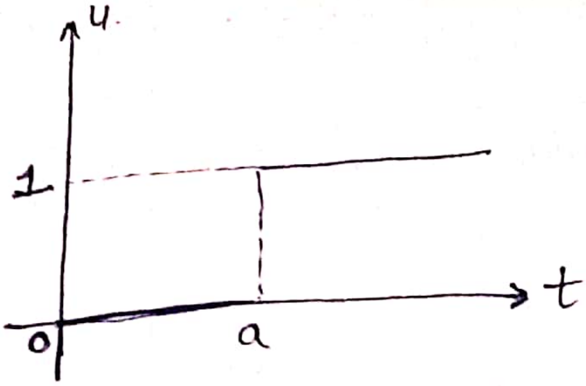
$$= -e^{-t}(\cos t + \sin t)$$

① <u>Heaviside step fn</u> <u>(Unit step function)</u> :-

It is defined as

$$H(t-a) \text{ or } u(t-a) = \begin{cases} 0, & \text{for } t < a \\ 1, & \text{for } t \geq a \end{cases}$$

where $a \geq 0$.



Transform.

$$L\{u(t-a)\} = \int_0^\infty e^{-st} u(t-a).dt$$

$$= \int_0^a e^{-st}.0\, dt + \int_a^\infty e^{-st}.1.dt$$

$$= 0 + \left[\frac{e^{-st}}{-s}\right]_a^\infty = \frac{e^{-as}}{s}.$$

The product $f(t).u(t-a) = \begin{cases} 0, & t<a \\ f(t), & t \geq a \end{cases}$

The function $f(t-a).u(t-a)$ represents the graph of $f(t)$ shifted through a distance a to the right & is of special importance.

Second shifting property: →

If $L\{f(t)\} = \bar{f}(s)$ then $L\{f(t-a)u(t-a)\} = e^{-as}\bar{f}(s)$

Proof To show, $L\{f(t-a).u(t-a)\} = e^{-as}\bar{f}(s)$.

$$L.H.S. = L\{f(t-a).u(t-a)\} = \int_0^\infty e^{-st} f(t-a).u(t-a)dt$$

$$= \int_0^a e^{-st} f(t-a)\cdot 0 \, dt + \int_a^\infty e^{-st} f(t-a)\cdot 1 \, dt$$

$$= \int_a^\infty e^{-st} f(t-a)\cdot dt$$

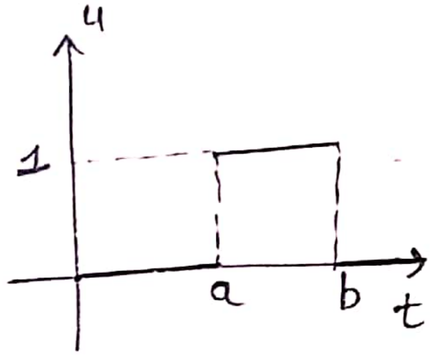
Let $t-a=u \Rightarrow dt = du$.

$$= \int_0^\infty e^{-s(a+u)} f(u)\cdot du = e^{-as}\int_0^\infty e^{-su} f(u)\cdot du.$$

$$= e^{-as} L\{f(u)\}$$

$$= e^{-as} F(s).$$ Hence proved.

② <u>Unit pulse function.</u>



It is defined as

$$u(t-a) - u(t-b) = \begin{cases} 0, & t<a \\ 1, & a<t<b \\ 0, & t>b \end{cases}$$

$$L\{u(t-a) - u(t-b)\} = \int_0^\infty e^{-st} (u(t-a) - u(t-b))\, dt$$

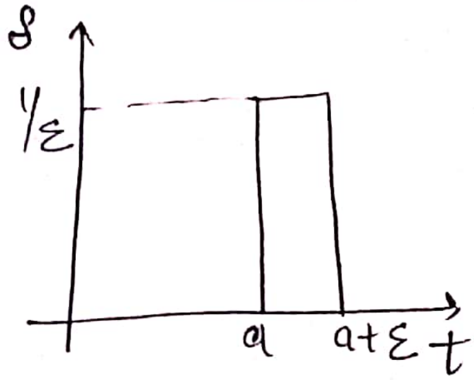
$$= \int_0^a e^{-st}\cdot 0 + \int_a^b e^{-st}\cdot 1 + \int_b^0 e^{-st}\cdot 0,$$

$$= \frac{e^{-as} - e^{-bs}}{s}$$

3) Dirac delta function (Unit-impulse func^n).

$$\delta(t-a) = \frac{1}{\varepsilon}, \quad a \le t \le a+\varepsilon$$

$$= 0 \quad \text{otherwise.}$$



Here $\varepsilon \to 0$.

Transform

$$L(\delta(t-a)) = \int_0^\infty e^{-st}\, \delta(t-a) \cdot dt$$

$$= \int_0^a e^{-st} \cdot 0 + \int_a^{a+\varepsilon} e^{-st} \cdot \frac{1}{\varepsilon} \cdot dt + \int_{a+\varepsilon}^\infty e^{-st} \cdot 0.$$

$$= \frac{e^{-as}}{s} \left[\frac{1-e^{-s\varepsilon}}{\varepsilon} \right]$$

Taking limit as $\varepsilon \to 0$.

$$= \frac{e^{-as}}{s} \lim_{\varepsilon \to 0} \left[\frac{1-e^{-s\varepsilon}}{\varepsilon} \right]$$

$$= \frac{e^{-as}}{s} \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left[1 - \left(1 - s\varepsilon + \frac{s^2\varepsilon^2}{2!} - \frac{s^3\varepsilon^3}{3!} + \cdots \right) \right]$$

$$= \frac{e^{-as}}{s} \lim_{\varepsilon \to 0} \left[s - \frac{s^2 \varepsilon}{2} + \frac{s^3 \varepsilon^2}{3!} - \cdots \right]$$

$$= \frac{e^{-as}}{s} \cdot s = e^{-as}$$ Ans.

$\therefore \boxed{L(\delta(t-a)) = e^{-as}}$

$\Rightarrow L(\delta(t)) = 1$ (Taking $a=0$)

Eg. Find the Laplace transform for the following.

ⓐ $t^2 u(t-3)$.

we know $L\{f(t-a)\cdot u(t-a)\} = e^{-as}\bar{f}(s)$ — ①
[using sec. shifting property.

Soln $t^2 u(t-3) = ((t-3)^2 - 9 + 6t)u(t-3)$

$$= \underbrace{[(t-3)^2 + 6(t-3) + 9]}_{f(t-3)} u(t-3).$$

$\therefore L\{t^2 u(t-3)\} = L\{[(t-3)^2 + 6(t-3) + 9]u(t-3)\}$

$$= L\{(t-3)^2 u(t-3)\} + 6\{L[(t-3)\cdot u(t-3)]\} + 9L\{1\cdot u(t-3)\}$$

$$= e^{-3s}L(t^2) + 6e^{-3s}L(t) + 9e^{-3s}L(1).$$
(by ①)

$$= e^{-3s}\cdot\frac{2}{s^3} + 6e^{-3s}\cdot\frac{1}{s^2} + \frac{9e^{-3s}}{s}.$$

Ans.

ⓑ $e^{-3t}\cdot u(t-2)$

Soln $e^{-3t}u(t-2) = e^{-3t+6-6}u(t-2)$

$$= e^{-6}[e^{-3(t-2)}\cdot u(t-2)]$$

$$L\{e^{-3t}u(t-2)\} = e^{-6}L\{e^{-3(t-2)}u(t-2)\}$$

$$= e^{-6}(e^{-2s}L(e^{-3t})) \quad \{\text{Again by } ①.$$

$$= \frac{e^{-6-2s}}{s+3} \quad \underline{\text{Ans}}.$$

Eg. Find $L[t\,u(t-4) - t^3\delta(t-2)]$

Soln. $L[t\,u(t-4)] = L[(t-4)\cdot u(t-4) + 4u(t-4)]$

$$= L[(t-4)\cdot u(t-4)] + 4L[u(t-4)]$$

$$= e^{-4s}L(t) + 4e^{-4s}L(1).$$

(By ①).

$$= e^{-4s}\left(\frac{1}{s^2}+\frac{4}{s}\right)$$

(Result $L\{f(t)\delta(t-a)\} = e^{-as}f(a)$. — ②)

$\therefore L\{t^3\delta(t-2)\} = e^{-2s}f(2)$, where $f(t)=t^3$

$$= e^{-2s}\cdot 8.$$

$$\therefore L[t\,u(t-4) - t^3\delta(t-2)] = e^{-4s}\left(\frac{1}{s^2}+\frac{4}{s}\right) - 8e^{-2s}$$

<u>Ans</u>

Eg. Find inverse L.T. of $\frac{e^{-\pi s}}{s^2+4}$.

Soln. we know, $L(\sin 2t) = \frac{2}{s^2+4}$

$$\Rightarrow L^{-1}\left(\frac{1}{s^2+4}\right) = \frac{1}{2}\sin 2t.$$

$$\therefore L^{-1}\left[\frac{e^{-\pi s}}{s^2+4}\right] = \frac{1}{2}\sin 2(t-\pi)u(t-\pi) \quad \underline{\text{Ans}}.$$

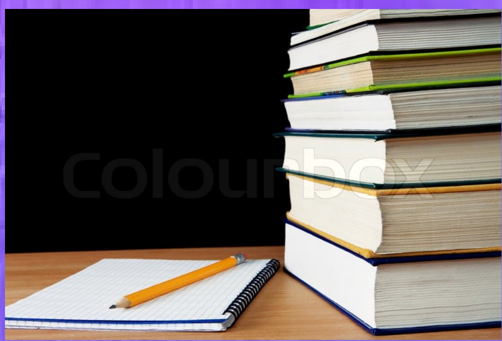
[Using 2nd shifting property with $f(t) = \frac{\sin 2t}{2}$.

Only you deserve

to change your life,

No one will ever do it

for you.



Mansi sharma
BVCOE.

ASSIGNMENT 2 (Unit-2 *Laplace transformation*)
{*From previous year papers*}
All students must solve the assignment in order to practice for exam.

Properties of Laplace transform

Q.1 Find the Laplace transformation for the following functions:

(a) $e^{-2t}(3\cos 4t - 2\sin 5t)$.

(b) $\left(\sqrt{t} - \frac{1}{\sqrt{t}}\right)^3$.

(c) $\sinh at \cos at$.

(d) $f(t) = \begin{cases} \sin t, & for \quad 0 < t < \pi \\ 0, & for \quad t > \pi. \end{cases}$

(e) $f(t) = \begin{cases} 2, & 0 < t < \pi \\ 0, & \pi < t < 2\pi \\ \sin t, & t > 2\pi. \end{cases}$

(f) $f(t) = |t-1| + |t+1|, \quad t \geq 0$.

(g) $\frac{\cos\sqrt{t}}{\sqrt{t}}$.

(h) $\sqrt{t}e^{3t}$.

Q.2 Prove that (i) $\boldsymbol{L}\{t\sin at\} = \frac{2as}{(s^2+a^2)^2}$ (ii) $\boldsymbol{L}\{t\cos at\} = \frac{s^2-a^2}{(s^2+a^2)^2}$.

Q.3 Find $\boldsymbol{L}\left\{\frac{\sin at}{t}\right\}$, given that $\boldsymbol{L}\left\{\frac{\sin t}{t}\right\} = \tan^{-1}\frac{1}{s}$.

Q.4 If $\boldsymbol{L}\left\{\sin\sqrt{t}\right\} = \frac{\pi}{2s^{3/2}}e^{-1/4s}$, show that $\boldsymbol{L}\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = e^{-1/4s}\left(\frac{\pi}{s}\right)^{1/2}$ [Hint. Use transform of derivatives].

Q.5 Find the Laplace transform of
(i) $2e^t \sin 4t \cos 2t$. (ii) $\frac{(e^{-at}-e^{-bt})}{t}$. (iii) $\frac{1-e^t}{t}$.

Q.6 Show that $\boldsymbol{L}\left\{\int\limits_0^t \frac{e^t \sin t}{t} dt\right\} = \frac{1}{s}\cot^{-1}(s-1)$. [Hint. Use transform of integrals].

Q.7 Evaluate (i) $\boldsymbol{L}\{t^2 e^{-2t}\}$ (ii) $\boldsymbol{L}\{te^{-t}\sin 3t\}$.

Q.8 Use Laplace transform to evaluate the the following integrals
(i) $\int\limits_0^\infty t^3 e^{-t}\sin t dt$. (ii) $\int\limits_0^\infty te^{-2t}\cos t dt$.

Inverse Laplace transform

Q.9 Find the *inverse Laplace transform* for the following functions.

(a) $\frac{1}{s^2-5s+6}$.
(b) $\frac{s+7}{s^2+2s+5}$.
(c) $\frac{s^2+2s-4}{(s^2+2s+5)(s^2+2s+2)}$.
(d) $\frac{s}{s^4+4a^4}$.
(e) $\log\frac{s-1}{s}$.
(f) $\frac{s^2}{s^4-a^4}$.
(g) $\frac{1}{s^3(s^2+1)}$.

Q.10 Find the inverse Laplace transform of $\cot^{-1}(s/2)$. [Hint Use $\boldsymbol{L}\{tf(t)\} = -\frac{d}{ds}\bar{f}(s)$.]

Convolution

Q.11 Prove that $\boldsymbol{L}\{f_1 * f_2\} = \bar{f}_1(s)\bar{f}_2(s)$.

Q.12 Employ the convolution theorem to find (i) $\boldsymbol{L}^{-1}\left[\frac{1}{s\sqrt{s+4}}\right]$ (ii) $\boldsymbol{L}^{-1}\left[\frac{1}{s^2(s+1)^2}\right]$.

Q.13 Apply convolution theorem to show that

$$\int_0^t \sin u\cos(t-u)du = \frac{1}{2}t\sin t.$$

Q.14 Use convolution theorem to evaluate
(i) $\boldsymbol{L}^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right]$.
(ii) $\boldsymbol{L}^{-1}\left[\frac{1}{2}\log\frac{s^2+b^2}{s^2+a^2}\right]$.

Applications to differential equations

Q.15 Solve the initial value problem $y'' + ay' - 2a^2y = 0, y(0) = 6, y'(0) = 0$.

Q.16 Using Laplace transform, solve $\frac{d^4x}{dt^4} - a^4x = 0$, where a is a constant, given that $x = 1, x' = x'' = x''' = 0$ at $t = 0$.

Q.17 Using Laplace transform, solve the following differential equations with given initial conditions:

(a) $y''' + 2y'' - y' - 2y = 0$ where $y(0) = 1, y'(0) = 2, y''(0) = 2$.

(b) $(D-1)(D-2)(D-3)x = 5; x = 0, x' = 1, x'' = 0$ at $t = 0$.

(c) $y'' + 4y' + 4y = 12t^2e^{-2t}, y(0) = 2$ and $y'(0) = 1$.

(d) $tx'' - (t+2)x' + 3x = t - 1$, given $x(0) = 0$ and $x(2) = 9$.

(e) $y'' + y = 6\cos 2t, y(0) = 3, y'(0) = 1$.

(f) $y'' + 2y' + 2y = 5\sin t$, where $y(0) = y'(0) = 0$.

Q.18 Solve the simultaneous equations
$x' + y' + x = -e^{-t}$,
$x' + 2y' + 2x + 2y = 0$, given that $x(0) = -1, y(0) = 1$.

Q.19 Solve the following simultaneous equations,
$x' + x + 3\int_0^t y\,dt = \cos t + 3\sin t$ and $2x' + 3y' + 6y = 0$,
subject to the conditions $x = -3, y = 2$, at $t = 0$.

Periodic functions

Q.20 Find the Laplace transform of $2c$- periodic function

$$f(t) = \begin{cases} t, & 0 < t < c \\ 2c - t, & c < t < 2c. \end{cases}$$

Q.21 For the periodic function f of period 4, defined by

$$f(t) = \begin{cases} 3t, & 0 < t < 2 \\ 6, & 2 < t < 4. \end{cases} \text{, find } \boldsymbol{L}(f(t)).$$

Q.22 Find the Laplace transform of following periodic function of period a

$$f(t) = \begin{cases} 1, & 0 < t < a/2 \\ -1, & a/2 < t < a. \end{cases}$$

Q.23 Find the Laplace transform of the following 2π periodic function, given by $f(t) = \begin{cases} t, & 0 < t < \pi \\ \pi - t, & \pi < t < 2\pi. \end{cases}$

Unit step function and unit impulse function

Q.24 Find the Laplace transform of Unit step function and unit impulse function (also known as Dirac-delta function).

Q.25 Find the Laplace transform of the following functions
$f(t) = \begin{cases} t-1, & 1 < t < 2 \\ 3-t, & 2 < t < 3. \end{cases}$

Q.26 Express the function $f(t) = \begin{cases} 2t, & 0 < t < 5 \\ 10, & t > 5. \end{cases}$ in terms of unit step function and find its Laplace transformation.

Q.27 Find the Laplace transform of the following:
(a) $t^2U(t-3)$.
(b) $e^{-3t}U(t-2)$, where U is unit step function.

Q.28 Find the inverse Laplace transform of

(a) $e^{-3s}\frac{(3s+1)}{s^2(s^2+4)}$.

(b) $\frac{se^{-s/2}+\pi e^{-s}}{s^2+\pi^2}$.

Q.29 Solve $y'' + 4y = U(x-2)$, where U is the unit step function and $y(0) = 0$ and $y'(0) = 1$.

Q.30 Find $\boldsymbol{L}\{tU(t-4) - t^3\delta(t-2)\}$, where δ is Dirac-delta function.

Q.31 Express the function $f(t) = \begin{cases} t^2, & 0 < t < 1 \\ 4t, & t > 1 \end{cases}$ in terms of unit step function and find its Laplace transformation.

Q.32 Evaluate the following integrals:
(a) $\int_0^\infty e^{-t}(1 + 3t + t^2)U(t-2)dt$.
(b) $\int_0^\infty \sin 2t\delta(t - \pi/4)dt$.


"Do not wait for tomorrow
the time is now,
Act now, without any
delay..."
"With hard work
and dedication
anything is
possible."

Q. Find (a) $L\{erf\sqrt{t}\}$ (b) $L\{erf\,2\sqrt{t}\}$

(c) $L\{t\,erf\,2\sqrt{t}\}$

Sol^n: (a) Error function $erf(t) = \frac{2}{\sqrt{\pi}}\int_0^t e^{-u^2}\,du$.

$\therefore erf(\sqrt{t}) = \frac{2}{\sqrt{\pi}}\int_0^{\sqrt{t}} e^{-u^2}\,du$.

$$L\{erf(\sqrt{t})\} = \frac{2}{\sqrt{\pi}} L\left(\int_0^{\sqrt{t}} e^{-u^2}\,du\right)$$

(let $u^2 = x \Rightarrow 2u\,du = dx$)

$$= \frac{2}{\sqrt{\pi}}\cdot\frac{1}{2} L\left(\int_0^t e^{-x} x^{-1/2}\,dx\right)$$

$$= \frac{1}{\sqrt{\pi}}\cdot\frac{1}{s} L(e^{-x}\cdot x^{-1/2}) \qquad \left\{\because L\left(\int_0^t f(t)\,dt\right) = \frac{1}{s}\cdot\bar{f}(s)\right.$$

$$= \frac{1}{\sqrt{\pi}\cdot s}\,\frac{\Gamma 1/2}{(s+1)^{1/2}} \qquad \left\{\because L(e^{at}\cdot t^n) = \frac{\Gamma n+1}{(s-a)^{n+1}}\right.$$

$$= \frac{1}{s\sqrt{s+1}} \qquad \& \ \Gamma 1/2 = \sqrt{\pi}.$$

(b) $L(erf\,2\sqrt{t}) = L(erf\sqrt{4t})$ $\quad$ Using $\left\{L\{f(at)\} = \frac{1}{a}\bar{f}\left(\frac{s}{a}\right)\right.$

$$= \frac{1}{4}\bar{f}\left(\frac{s}{4}\right)$$

where $\bar{f}(s) = L(erf\sqrt{t})$

Scanned by CamScanner

$$\Rightarrow L(\mathrm{erf}\,2\sqrt{t}) = \frac{1}{4}\left(\frac{1}{\frac{s}{4}\sqrt{\frac{s}{4}+1}}\right) \quad \left(\because \bar{f}(s) = \frac{1}{s\sqrt{s+1}}\right.$$

$$= \frac{2}{s\sqrt{s+4}}$$

(c) $L(t\,\mathrm{erf}\,2\sqrt{t}) = -\frac{d}{ds} L(\mathrm{erf}\,2\sqrt{t})$

Using,
$L(t f(t)) = -\frac{d}{ds}\bar{f}(s)$
$= -\frac{d}{ds} L\{f(t)\}$

$$= -\frac{d}{ds}\left[\frac{2}{s\sqrt{s+4}}\right]$$

$$= -\frac{d}{ds}\left[\frac{2}{\sqrt{s^3+4s^2}}\right]$$

$$= -2.\left[\frac{1}{(s^3+4s^2)^{3/2}} \times \frac{-1}{2} \times (3s^2+8s)\right]$$

$$= \frac{(3s+8)s}{s^3(s+4)^{3/2}} = \frac{3s+8}{s^2(s+4)^{3/2}} \quad \text{Ans.}$$

CHAPTER 21

Laplace Transforms

1. Introduction. 2. Definition ; Conditions for existence. 3. Transforms of elementary functions. 4. Properties of Laplace transforms. 5. Transforms of Periodic functions. 6. Transforms of Special functions. 7. Transforms of derivatives. 8. Transforms of integrals. 9. Multiplication by t^n. 10. Division by t. 11. Evaluation of integrals by Laplace transforms. 12. Inverse transforms. 13. Other methods of finding inverse transforms. 14. Convolution theorem. 15. Application to differential equations. 16. Simultaneous linear equations with constant co-efficients. 17. Unit step function. 18. Unit impulse function. 19. Objective Type of Questions.

21.1 INTRODUCTION

The knowledge of Laplace transforms has in recent years become an essential part of mathematical background required of engineers and scientists. This is because the transform methods provide an easy and effective means for the solution of many problems arising in engineering.

This subject originated from the operational methods applied by the English engineer Oliver Heaviside (1850–1925), to problems in electrical engineering. Unfortunately, Heaviside's treatment was unsystematic and lacked rigour, which was placed on sound mathematical footing by Bromwich and Carson during 1916–17. It was found that Heaviside's operational calculus is best introduced by means of a particular type of definite integrals called Laplace transforms.*

The method of Laplace transforms has the advantage of directly giving the solution of differential equations with given boundary values without the necessity of first finding the general solution and then evaluating from it the arbitrary constants. Moreover, the ready tables of Laplace transforms reduce the problem of solving differential equations to mere algebraic manipulation.

21.2 (1) DEFINITION

Let f(t) be a function of t defined for all positive values of t. Then the **Laplace transforms** *of f(t), denoted by* $L\{f(t)\}$ *is defined by*

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt \qquad ...(1)$$

provided that the integral exists. s is a parameter which may be a real or complex number.

$L\{f(t)\}$ being clearly a function of s is briefly written as $\bar{f}(s)$ *i.e.*, $L\{f(t)\} = \bar{f}(s)$,

which can also be written as $f(t) = L^{-1}\{\bar{f}(s)\}$.

Then $f(t)$ is called the **inverse Laplace transform** of $\bar{f}(s)$. The symbol L, which transforms $f(t)$ into $\bar{f}(s)$, is called the *Laplace transformation operator*.

**Pierre de Laplace* (1749–1827) (See footnote p. 18) used such transforms, much earlier in 1799, while developing the theory of probability.

(2) Conditions for the existence

The Laplace transform of $f(t)$ i.e., $\int_0^\infty e^{-st} f(t)\,dt$ exists for $s > a$, if

(i) $f(t)$ is continuous *(iii)* $\underset{t \to \infty}{\text{Lt}} \{e^{-at} f(t)\}$ *is finite.*

It should however, be noted that the above conditions are sufficient and not necessary.

For example, $L(1/\sqrt{t})$ exists, though $1/\sqrt{t}$ is infinite at $t = 0$.

21.3 TRANSFORMS OF ELEMENTARY FUNCTIONS

The direct application of the definition gives the following formulae :

(1) $\mathbf{L(1) = \frac{1}{s}}$ $(s > 0)$

(2) $\mathbf{L(t^n) = \frac{n!}{s^{n+1}}}$, when $n = 0, 1, 2, 3, \ldots$ $\left[\text{Otherwise } \frac{\Gamma(n+1)}{s^{n+1}}\right]$

(3) $\mathbf{L(e^{at}) = \frac{1}{s-a}}$ $(s > a)$

(4) $\mathbf{L(\sin at) = \frac{a}{s^2 + a^2}}$ $(s > 0)$

(5) $\mathbf{L(\cos at) = \frac{s}{s^2 + a^2}}$ $(s > 0)$

(6) $\mathbf{L(\sinh at) = \frac{a}{s^2 - a^2}}$ $(s > |a|)$

(7) $\mathbf{L(\cosh at) = \frac{s}{s^2 - a^2}}$ $(s > |a|)$

Proofs. **(1)** $$L(1) = \int_0^\infty e^{-st} \cdot 1\,dt = \left| -\frac{e^{-st}}{s} \right|_0^\infty = \frac{1}{s} \text{ if } s > 0.$$

(2) $$L(t^n) = \int_0^\infty e^{-st} \cdot t^n\,dt = \int_0^\infty e^{-p} \cdot \left(\frac{p}{s}\right)^n \frac{dp}{s}, \text{ on putting } st = p$$

$$= \frac{1}{s^{n+1}} \int_0^\infty e^{-p} \cdot p^n\,dp = \frac{\Gamma(n+1)}{s^{n+1}}, \text{ if } n > -1 \text{ and } s > 0. \text{ [Page 302]}$$

In particular $L(t^{-1/2}) = \dfrac{\Gamma(1/2)}{s^{1/2}} = \sqrt{\dfrac{\pi}{s}}$; $L(t^{1/2}) = \dfrac{\Gamma(3/2)}{s^{3/2}} = \dfrac{\sqrt{\pi}}{2s^{3/2}}$

In n be a positive integer, $\Gamma(n + 1) = n!$ [(*v*) p. 302],

therefore, $L(t^n) = n!/s^{n+1}$.

(3) $$L(e^{at}) = \int_0^\infty e^{-st} \cdot e^{at}\,dt = \int_0^\infty e^{-(s-a)t}\,dt = \left| \frac{e^{-(s-a)t}}{-(s-a)} \right|_0^\infty = \frac{1}{s-a}, \text{ if } s > a.$$

(4) $$L(\sin at) = \int_0^\infty e^{-st} \sin at\,dt = \left| \frac{e^{-st}}{s^2 + a^2}(-s \sin at - a \cos at) \right|_0^\infty = \frac{a}{s^2 + a^2}$$

Similarly, the reader should prove **(5)** himself.

(6) $$L(\sinh at) = \int_0^\infty e^{-st} \sinh at\,dt = \int_0^\infty e^{-st} \left(\frac{e^{at} - e^{-at}}{2}\right) dt$$

$$= \frac{1}{2}\left[\int_0^\infty e^{-(s-a)t}\,dt - \int_0^\infty e^{-(s+a)t}\,dt\right] = \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right] = \frac{a}{s^2 - a^2} \text{ for } s > |a|.$$

Similarly, the reader should prove **(7)** himself.

21.4 PROPERTIES OF LAPLACE TRANSFORMS

I. Linearity property. *If a, b, c be any constants and f, g, h any functions of t, then*

$$\mathbf{L\,[af(t) + bg(t) - ch(t)] = aL\,\{f(t)\} + bL\,\{g(t)\} - cL\,\{h(t)\}}$$

For by definition,

$$\text{L.H.S.} = \int_0^\infty e^{-st}\,[af(t) + bg(t) - ch(t)]\,dt$$

$$= a\int_0^\infty e^{-st}\,f(t)\,dt + b\int_0^\infty e^{-st}\,g(t)\,dt - c\int_0^\infty e^{-st}\,h(t)\,dt = aL\{f(t)\} + bL\{g(t)\} - cL\{h(t)\}$$

This result can easily be generalised.

Because of the above property of L, it is called a *linear operator*.

II. First shifting property. *If* $L\{f(t)\} = \bar{f}(s)$, then

$$\mathbf{L\,\{e^{at}f(t)\} = \bar{f}(s-a)}.$$

By definition, $L\{e^{at}f(t)\} = \int_0^\infty e^{-st}\,e^{at}\,f(t)\,dt = \int_0^\infty e^{-(s-a)t}\,f(t)\,dt$

$$= \int_0^\infty e^{-rt}\,f(t)\,dt, \text{ where } r = s - a = \bar{f}(r) = \bar{f}(s-a).$$

Thus, if we know the transform $\bar{f}(s)$ *of* $f(t)$, *we can write the transform of* $e^{at}f(t)$ *simply replacing s by* $s - a$ *to get* $\bar{f}(s-a)$.

Application of this property leads us to the following useful results :

(1) $\mathbf{L\,(e^{at}) = \dfrac{1}{s-a}}$ $\left[\because\ L(1) = \dfrac{1}{s}\right.$

(2) $\mathbf{L\,(e^{at}t^n) = \dfrac{n!}{(s-a)^{n+1}}}$ $\left[\because\ L(t^n) = \dfrac{n!}{s^{n+1}}\right.$

(3) $\mathbf{L\,(e^{at}\sin bt) = \dfrac{b}{(s-a)^2 + b^2}}$ $\left[\because\ L(\sin bt) = \dfrac{b}{s^2 + b^2}\right.$

(4) $\mathbf{L\,(e^{at}\cos bt) = \dfrac{s-a}{(s-a)^2 + b^2}}$ $\left[\because\ L(\cos bt) = \dfrac{s}{s^2 + b^2}\right.$

(5) $\mathbf{L\,(e^{at}\sinh bt) = \dfrac{b}{(s-a)^2 - b^2}}$ $\left[\because\ L(\sinh bt) = \dfrac{b}{s^2 - b^2}\right.$

(6) $\mathbf{L\,(e^{at}\cosh bt) = \dfrac{s-a}{(s-a)^2 - b^2}}$ $\left[\because\ L(\cosh bt) = \dfrac{s}{s^2 - b^2}\right.$

where in each case $s > a$.

Example 21.1. *Find the Laplace transforms of*
(*i*) *sin 2t sin 3t* (*ii*) *cos² 2t* (*iii*) *sin³ 2t.*

Solution. (*i*) Since $\sin 2t \sin 3t = \frac{1}{2}[\cos t - \cos 5t]$

$$\therefore \quad L(\sin 2t \sin 3t) = \tfrac{1}{2}[L(\cos t) - L(\cos 5t)] = \frac{1}{2}\left[\frac{s}{s^2+1^2} - \frac{s}{s^2+5^2}\right] = \frac{12s}{(s^2+1)(s^2+25)}$$

(*ii*) Since $\quad \cos^2 2t = \dfrac{1}{2}(1 + \cos 4t)$

$$\therefore \quad L(\cos^2 2t) = \frac{1}{2}[L(1) + L\cos 4t] = \frac{1}{2}\left(\frac{1}{s} + \frac{s}{s^2+16}\right)$$

(*iii*) Since $\sin 6t = 3 \sin 2t - 4 \sin^3 2t$

or $\sin^3 2t = \frac{3}{4} \sin 2t - \frac{1}{4} \sin 6t$

$\therefore$ $L(\sin^3 2t) = \frac{3}{4} L(\sin 2t) - \frac{1}{4} L(\sin 6t)$

$$= \frac{3}{4} \cdot \frac{2}{s^2 + 2^2} - \frac{1}{4} \cdot \frac{2}{s^2 + 6^2} = \frac{48}{(s^2+4)(s^2+36)}.$$

Example 21.2. *Find the Laplace transform of*

(*i*) $e^{-3t}(2 \cos 5t - 3 \sin 5t)$. (*ii*) $e^{2t} \cos^2 t$ (V.T.U., 2006) (*iii*) $\sqrt{t}e^{3t}$. (P.T.U., 2009)

Solution. (*i*) $L\{e^{-3t}(2 \cos 5t - 3 \sin 5t)\} = 2L(e^{-3t} \cos 5t) - 3L(e^{-3t} \sin 5t)$

$$= 2 \cdot \frac{s+3}{(s+3)^2 + 5^2} - 3 \cdot \frac{5}{(s+3)^2 + 5^2} = \frac{2s - 9}{s^2 + 6s + 34}.$$

(*ii*) Since $L(\cos^2 t) = \frac{1}{2} L(1 + \cos 2t) = \frac{1}{2}\left\{\frac{1}{s} + \frac{s}{s^2+4}\right\}$

$\therefore$ by shifting property, we get

$$L(e^{2t} \cos^2 t) = \frac{1}{2}\left\{\frac{1}{s-2} + \frac{s-2}{(s-2)^2 + 4}\right\}.$$

(*iii*) Since $L(\sqrt{t}) = \frac{\Gamma(3/2)}{s^{3/2}} = \frac{(1/2) \cdot \Gamma\pi}{s^{3/2}}$

$\therefore$ by shifting property, we obtain $L(e^{3t}\sqrt{t}) = \frac{\sqrt{\pi}}{2} \frac{1}{(s-3)^{3/2}}$.

Example 21.3. *If* $L f(t) = \bar{f}(s)$, *show that*

$$L[(\sinh at) f(t)] = \frac{1}{2}[\bar{f}(s-a) - \bar{f}(s+a)]$$

$$L[(\cosh at) f(t)] = \frac{1}{2}[\bar{f}(s-a) + \bar{f}(s+a)]$$

Hence evaluate (*i*) $\sinh 2t \sin 3t$ (*ii*) $\cosh 3t \cos 2t$.

Solution. We have $L[(\sinh at) f(t)] = L\left[\frac{1}{2}(e^{at} - e^{-at}) f(t)\right] = \frac{1}{2}[L\{e^{at} f(t) - L\{e^{-at} f(t)\}]$

$$= \frac{1}{2}[\bar{f}(s-a) - \bar{f}(s+a)], \text{ by shifting property.}$$

Similarly, $L[(\cosh at) f(t)] = \frac{1}{2}[L\{e^{at} f(t)\} + \{e^{-at} f(t)\}]$

$$= \frac{1}{2}[\bar{f}(s-a) + \bar{f}(s+a)], \text{ by shifting property.}$$

(*i*) Since $L(\sin 3t) = \frac{3}{s^2 + 3^2}$, the first result gives

$$L(\sinh 2t \sin 3t) = \frac{1}{2}\left\{\frac{3}{(s-2)^2 + 3^2} - \frac{3}{(s+2)^2 + 3^2}\right\} = \frac{12s}{s^4 + 10s^2 + 169}$$

(*ii*) Since $L(\cos 2t) = \frac{s}{s^2 + 2^2}$, the second result gives

$$L(\cosh 3t \cos 2t) = \frac{1}{2}\left\{\frac{s-3}{(s-3)^2 + 2^2} + \frac{s+3}{(s+3)^2 + 2^2}\right\} = \frac{2s(s^2 - 5)}{s^4 - 10s^2 + 169}.$$

Example 21.4. *Show that*

(i) $L\,(t \sin at) = \dfrac{2as}{(s^2+a^2)^2}$ *(Bhopal, 2001)* *(ii)* $L\,(t \cos at) = \dfrac{s^2-a^2}{(s^2+a^2)^2}$.

Solution. Since $L\,(t) = 1/s^2$. $\therefore\quad L\,(te^{iat}) = \dfrac{1}{(s-ia)^2} = \dfrac{(s+ia)^2}{[(s-ia)(s+ia)]^2}$

or
$$L\,[t\,(\cos at + i \sin at)] = \frac{(s^2-a)^2 + i(2as)}{(s^2+a^2)^2}$$

Equating the real and imaginary parts from both sides, we get the desired results.

Example 21.5. *Find the Laplace transform of $f(t)$ defined as*

(i) $f(t) = t/\tau$, *when* $0 < t < \tau$

$= 1$, *when* $t > \tau$. *(Kerala, 2005)*

(ii) $f(t) = \begin{cases} 1, & 0 < t \le 1 \\ t, & 1 < t \le 2 \\ 0, & t > 2 \end{cases}$ *(J.N.T.U., 2006 ; W.B.T.U., 2005)*

Solution. *(i)*
$$Lf(t) = \int_0^\tau e^{-st} \cdot \frac{t}{\tau}\, dt + \int_\tau^\infty e^{-st} \cdot 1\, dt = \frac{1}{\tau}\left[\left| t \cdot \frac{e^{-st}}{-s}\right|_0^\tau - \int_0^\tau 1 \cdot \frac{e^{-st}}{-s}\, dt\right] + \left|\frac{e^{-st}}{-s}\right|_\tau^\infty$$
$$= \frac{1}{\tau}\left[\frac{\tau e^{-s\tau} - 0}{-s} - \left|\frac{e^{-st}}{s^2}\right|_0^\tau\right] + \frac{0 - e^{-s\tau}}{-s} = \frac{-e^{-s\tau}}{s} - \frac{e^{-s\tau}-1}{\tau s^2} + \frac{e^{-s\tau}}{s} = \frac{1-e^{-s\tau}}{\tau s^2}.$$

(ii)
$$L\,\{f(t)\} = \int_0^1 e^{-st} \cdot 1\, dt + \int_1^2 e^{-st} \cdot t\, dt + \int_2^\infty e^{-st} \cdot (0)\, dt$$
$$= \left|\frac{e^{-st}}{-s}\right|_0^1 + \left| t \cdot \frac{e^{-st}}{-s} - \frac{e^{-st}}{s^2}\right|_1^2 = \frac{1-e^{-s}}{s} + \left\{\left(-\frac{2e^{-2s}}{s} - \frac{e^{-2s}}{s^2}\right) - \left(\frac{e^{-s}}{-s} - \frac{e^{-s}}{s^2}\right)\right\}$$
$$= \frac{1}{s} - \frac{2e^{-2s}}{s} + \frac{e^{-s}}{s^2} - \frac{e^{-2s}}{s^2}.$$

Example 21.6. *Find the Laplace transform of (i)* $\left(\sqrt{t} - \dfrac{1}{\sqrt{t}}\right)^3$. *(Kurukshetra, 2005)*

(ii) $\dfrac{\cos \sqrt{t}}{\sqrt{t}}$ *(Mumbai, 2009)*

Solution. *(i)* Since $(\sqrt{t} - 1/\sqrt{t})^3 = t^{3/2} - 3t^{1/2} + 3t^{-1/2} - t^{-3/2}$

$\therefore$
$$L\,(\sqrt{t} - 1/\sqrt{t}) = L(t^{3/2}) - 3L(t^{1/2}) + 3L(t^{-1/2}) - L(t^{-3/2})$$
$$= \frac{\Gamma(3/2+1)}{s^{3/2+1}} - 3\frac{\Gamma(1/2+1)}{s^{1/2+1}} + 3\frac{\Gamma(-1/2+1)}{s^{-1/2+1}} - \frac{\Gamma(-3/2+1)}{s^{-3/2+1}}$$
$$= \frac{\frac{3}{2}\Gamma\left(\frac{3}{2}\right)}{s^{5/2}} - 3\,\frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{s^{3/2}} + 3\,\frac{\Gamma\left(\frac{1}{2}\right)}{s^{1/2}} - \frac{\Gamma\left(-\frac{1}{2}\right)}{s^{-1/2}}$$
$$= \frac{3}{4}\frac{\sqrt{\pi}}{s^{5/2}} - \frac{3}{2}\frac{\sqrt{\pi}}{s^{3/2}} + \frac{3\sqrt{\pi}}{s^{1/2}} + \frac{2\sqrt{\pi}}{s^{-1/2}} \qquad \left[\because\ \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}, \Gamma\left(-\frac{1}{2}\right) = -2\sqrt{\pi}\right]$$
$$= \frac{\sqrt{\pi}}{4}\left(\frac{3}{s^{5/2}} - \frac{6}{s^{3/2}} + \frac{12}{s^{1/2}} + \frac{8}{s^{-1/2}}\right).$$

(*ii*) We know that $\cos\theta = 1 - \dfrac{\theta^2}{2!} + \dfrac{\theta^4}{4!} - \dfrac{\theta^6}{6!} + \ldots \infty$

$$\therefore \quad \cos\sqrt{t} = 1 - \frac{t}{2!} + \frac{t^2}{4!} - \frac{t^3}{6!} + \ldots$$

and $$\frac{\cos\sqrt{t}}{\sqrt{t}} = t^{-1/2} - \frac{t^{1/2}}{2!} + \frac{t^{3/2}}{4!} - \frac{t^{5/2}}{6!} + \ldots$$

and $$L\left(\frac{\cos\sqrt{t}}{\sqrt{t}}\right) = \frac{\Gamma(1/2)}{s^{1/2}} - \frac{1}{2!}\frac{\Gamma(3/2)}{s^{3/2}} + \frac{1}{4!}\frac{\Gamma(5/2)}{s^{5/2}} - \frac{1}{6!}\frac{\Gamma(7/2)}{s^{7/2}} + \ldots$$

$$= \frac{\Gamma 1/2}{\sqrt{s}} - \frac{1}{2}\cdot\frac{1/2\,\Gamma 1/2}{s^{3/2}} + \frac{1}{4!}\frac{3/2\,.\,1/2\,.\,\Gamma(1/2)}{s^{5/2}} - \frac{1}{6!}\frac{5/2\,.\,3/2\,.\,1/2\,.\,\Gamma(1/2)}{s^{7/2}} + \ldots$$

$$= \sqrt{\left(\frac{\pi}{2}\right)}\left[1 - \frac{1}{(4s)} + \frac{1}{2!}\frac{1}{(4s)^2} - \frac{1}{3!}\frac{1}{(4s)^3}\ldots\right] = \sqrt{\left(\frac{\pi}{s}\right)}\,e^{-1/4s}.$$

Example 21.7. *Find the Laplace transform of the function*
(i) $f(t) \mid t-1 \mid + \mid t+1 \mid, t \geq 0$ *(S.V.T.U., 2009)*
(ii) $f(t) = [t]$, *where* [ ] *stands for the greatest integer function.* *(P.T.U., 2010)*

Solution. (*i*) Given function is equivalent to

$$f(t) = \begin{cases} 2, & 0 \leq t < 1 \\ 2t, & t \geq 1 \end{cases}$$

$$\therefore \quad L\,f(t) = \int_0^1 e^{-st}(2)\,dt + \int_1^\infty e^{-st}\,(2t)dt = 2\left[\left|\frac{e^{-st}}{-s}\right|_0^1 + 2\left|\frac{t.e^{-st}}{-s}\right|_1^\infty - \left|\frac{e^{-st}}{(-s)^2}\right|_1^\infty\right]$$

$$= 2\left(\frac{e^{-s}}{-s} + \frac{1}{s}\right) + 2\left(\frac{0 - e^{-s}}{-s} - \frac{0 - e^{-s}}{s^2}\right) = \frac{2}{s}\left(1 + \frac{e^{-s}}{s}\right)$$

(*ii*) Given function is equivalent to

$$[t] = 0 \text{ in } (0, 1) + 1 \text{ in } (1, 2) + 2 \text{ in } (2, 3) + 3 \text{ in } (3, 4) + \ldots$$

$$\therefore \quad L[f(t)] = \int_0^\infty e^{-st}[f(t)]dt = \int_0^\infty e^{-st}\,[t]\,dt$$

$$= \int_0^1 e^{-st}(0)dt + \int_1^2 e^{-st}(1)dt + \int_2^3 e^{-st}(2)dt + \int_3^4 e^{-st}(3)dt + \ldots \infty$$

$$= 0 + \left|\frac{e^{-st}}{-s}\right|_1^2 + 2\left|\frac{e^{-st}}{-s}\right|_2^3 + 3\left|\frac{e^{-st}}{-s}\right|_3^4 + \ldots \infty$$

$$= -\frac{1}{s}[(e^{-2s} - e^{-s}) + 2(e^{-3s} - e^{-2s}) + 3(e^{-4s} - e^{-3s}) + \ldots \infty]$$

$$= \frac{1}{s}(e^{-s} + e^{-2s} + e^{-3s} + \ldots \infty) = \frac{1}{s}\left(\frac{e^{-s}}{1 - e^{-s}}\right) = \frac{1}{s(e^s - 1)}.$$

III. Change of scale property. *If* $L\{f(t)\} = \bar{f}(s)$, *then* $L\{f(at)\} = \dfrac{1}{a}\bar{f}\left(\dfrac{s}{a}\right)$

$$L\{f(at)\} = \int_0^\infty e^{-st} f(at)\,dt = \int_0^\infty e^{-su/a} f(u)\,.\,du/a$$

| Put $at = u$, $dt = du/a$

$$= \frac{1}{a}\int_0^\infty e^{-su/a} f(u)du = \frac{1}{a}\bar{f}(s/a).$$

Example 21.8. *Find* $L\left\{\frac{\sin at}{t}\right\}$, *given that* $L\left\{\frac{\sin t}{t}\right\} = \tan^{-1}\left\{\frac{1}{s}\right\}$.

Solution. By the above property,

$$L\left\{\frac{\sin at}{at}\right\} = \frac{1}{a}\tan^{-1}\left\{\frac{1}{(s/a)}\right\} = \frac{1}{a}\tan^{-1}\left(\frac{a}{s}\right) \text{ i.e., } L\left\{\frac{\sin at}{t}\right\} = \tan^{-1}\left\{\frac{a}{s}\right\}.$$

PROBLEMS 21.1

Find the Laplace transforms of

1. $e^{2t} + 4t^3 - 2\sin 3t + 3\cos 3t$. (*J.N.T.U., 2003*)
2. $1 + 2\sqrt{t} + 3/\sqrt{t}$.
3. $3\cosh 5t - 4\sinh 5t$. (*Nagarjuna, 2006*)
4. $\cos(at + b)$.
5. $(\sin t - \cos t)^2$.
6. $\sin 2t \cos 3t$. (*Kottayam, 2005*)
7. $\sin\sqrt{t}$.
8. $\sin^5 t$. (*Mumbai, 2007*)
9. $\cos^3 2t$.
10. $e^{-at}\sinh bt$
11. $e^{2t}(3t^5 - \cos 4t)$. (*P.T.U., 2007*)
12. $e^{-3t}\sin 5t \sin 3t$. (*V.T.U., 2006*)
13. $e^{-t}\sin^2 t$. (*Mumbai, 2009*)
14. $e^{2t}\sin^4 t$ (*Mumbai, 2007*)
15. $\cosh at \sin at$. (*Delhi, 2002*)
16. $\sinh 3t \cos^2 t$ (*Madras, 2000*)
17. $t^2 e^{2t}$. (*V.T.U., 2008 S*)
18. $(1 + te^{-t})^3$
19. $t\sqrt{(1 + \sin t)}$ (*Mumbai, 2007*)
20. $f(t) = \begin{cases} 4, & 0 \le t <= 1 \\ 3, & t > 1 \end{cases}$ (*U.P.T.U., 2009*)
21. $f(t) = \begin{cases} \sin t, 0 < t < \pi \\ 0, t > \pi. \end{cases}$ (*Madras, 2000 S*)
22. $f(x) = \begin{cases} \sin(x - \pi/3), & x > \pi/3 \\ 0 & , x < \pi/3 \end{cases}$ (*Rajasthan, 2006*)
23. $f(t) = \begin{cases} \cos(t - 2\pi/3), & t > 2\pi/3 \\ 0, & t < 2\pi/3 \end{cases}$
24. $f(t) = \begin{cases} t^2, 0 < t < 2 \\ t - 1, 2 < t < 3 \\ 7, t > 3. \end{cases}$ (*Mumbai, 2007*)
25. If $L[f(t)] = \frac{1}{s(s^2+1)}$, find $L[e^{-t} f(2t)]$.

21.5 TRANSFORMS OF PERIODIC FUNCTIONS

If $f(t)$ is a periodic function with period T, i.e., $f(t + T) = f(t)$, then

$$L\{f(t)\} = \frac{\int_0^T e^{-st} f(t)dt}{1 - e^{-sT}}$$

We have $\quad L\{f(t)\} = \int_0^\infty e^{-st} f(t)dt = \int_0^T e^{-st} f(t)dt + \int_T^{2T} e^{-st} f(t)dt + \int_{2T}^{3T} e^{-st} f(t)dt + \dots$

In the second integral put $t = u + T$, in the third integral put $t = u + 2T$, and so on. Then

$$L\{f(t)\} = \int_0^T e^{-st} f(t)dt + \int_0^T e^{-s(u+T)} f(u+T)du + \int_0^T e^{-s(u+2T)} f(u+2T)du + \dots$$

$$= \int_0^T e^{-st} f(t)dt + e^{-sT}\int_0^T e^{-su} f(u)du + e^{-2sT}\int_0^T e^{-su} f(u)du + \dots$$

$[\because \; f(u) = f(u+T) = f(u+2T) \text{ etc.}]$

$$= \int_0^T e^{-st} f(t)dt + e^{-sT}\int_0^T e^{-st} f(t)dt + e^{-2sT}\int_0^T e^{-st} f(t)dt + \dots$$

$$= (1 + e^{-sT} + e^{-2sT} + \dots \infty)\int_0^T e^{-st} f(t)dt$$

$$= \frac{1}{1 - e^{-sT}}\int_0^T e^{-st} f(t)dt.$$

(*V.T.U., 2008 ; Mumbai, 2006*)

Example 21.9. *Find the Laplace transform of the function*

$$f(t) = \sin \omega t, \quad 0 < t < \pi/\omega$$
$$= 0, \quad \pi/\omega < t < 2\pi/\omega.$$

(Kurukshetra, 2005 ; Madras, 2003)

Solution. Since $f(t)$ is a periodic function with period $2\pi/\omega$.

$$\therefore \quad L\{f(t)\} = \frac{1}{1 - e^{-2\pi s/\omega}} \int_0^{2\pi/\omega} e^{-st} f(t) dt$$

$$= \frac{1}{1 - e^{-2\pi s/\omega}} \left[\int_0^{\pi/\omega} e^{-st} \sin \omega t \, dt + \int_{\pi/\omega}^{2\pi/\omega} e^{-st} . 0 \, dt \right]$$

$$= \frac{1}{1 - e^{-2\pi s/\omega}} \left| \frac{e^{-st}(-s \sin \omega t - \omega \cos \omega t)}{s^2 + \omega^2} \right|_0^{\pi/\omega} = \frac{\omega e^{-\pi s/\omega} + \omega}{(1 - e^{-2\pi s/\omega})(s^2 + \omega^2)} = \frac{\omega}{(1 - e^{-\pi s/\omega})(s^2 + \omega^2)}.$$

Example 21.10. *Draw the graph of the periodic function*

$$f(t) = \begin{cases} t, & 0 < t < \pi \\ \pi - t, & \pi < t < 2\pi. \end{cases}$$

and find its Laplace transform. *(U.P.T.U., 2003)*

Solution. Here the period of $f(t) = 2\pi$ and its graph is as in Fig. 21.1.

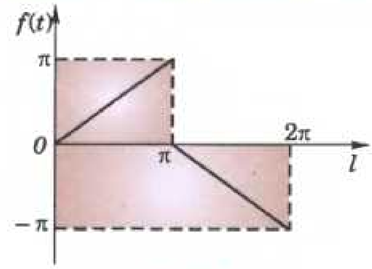


Fig. 21.1

$$\therefore \quad L f(t) = \frac{1}{1 - e^{-2\pi s}} \left\{ \int_0^{\pi} e^{-st} t \, dt + \int_{\pi}^{2\pi} e^{-st} (\pi - t) dt \right\}$$

$$= \frac{1}{1 - e^{-2\pi s}} \left\{ \left| t\left(\frac{e^{-st}}{-s}\right) - 1.\left(\frac{e^{-st}}{s^2}\right) \right|_0^{\pi} + \left| (\pi - t)\left(\frac{e^{-st}}{-s}\right) - (-1)\left(\frac{e^{-st}}{s^2}\right) \right|_{\pi}^{2\pi} \right\}$$

$$= \frac{1}{1 - e^{-2\pi s}} \left\{ \frac{-\pi e^{-\pi s}}{s} - \frac{e^{-\pi s}}{s^2} + \frac{1}{s^2} + \frac{\pi e^{-2\pi s}}{s} + \frac{e^{-2\pi s}}{s^2} - \frac{e^{-\pi s}}{s^2} \right\}$$

$$= \frac{1}{1 - e^{-2\pi s}} \left\{ \frac{\pi}{s}\left(e^{-2\pi s} - e^{-\pi s}\right) + \frac{1}{s^2}\left(1 + e^{-2\pi s} - 2e^{-\pi s}\right) \right\}.$$

21.6 TRANSFORMS OF SPECIAL FUNCTIONS

(1) Transform of Bessel functions $J_0(x)$ and $J_1(x)$.

We know that $$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 . 4^2} - \frac{x^6}{2^2 . 4^2 . 6^2} + \ldots$$ [§ 16.7 (1), p. 553]

$$\therefore \quad L\{J_0(x)\} = \frac{1}{s} - \frac{1}{2^2} \frac{2!}{s^3} + \frac{1}{2^2 . 4^2} \cdot \frac{4!}{s^5} - \frac{1}{2^2 . 4^2 . 6^2} \cdot \frac{6!}{s^7} + \ldots$$

$$= \frac{1}{s} \left\{ 1 - \frac{1}{2}\left(\frac{1}{s^2}\right) + \frac{1.3}{2.4}\left(\frac{1}{s^4}\right) - \frac{1.3.5}{2.4.6}\left(\frac{1}{s^6}\right) + \ldots \right\}$$

$$= \frac{1}{s}\left(1 + \frac{1}{s^2}\right)^{-1/2} = \frac{1}{\sqrt{(s^2 + 1)}} \qquad \ldots(1)$$

Also since $J_0'(x) = -J_1(x)$. [Problem 4(*i*), p. 557]

$$\therefore \quad L\{J_1(x)\} = -L\{J_0'(x)\} = -[sL\{J_0(x)\} - 1] = 1 - \frac{s}{\sqrt{(s^2 + 1)}} \qquad \ldots(2)$$

(2) Transform of Error function

We know that $erf(\sqrt{x}) = \dfrac{2}{\sqrt{\pi}} \displaystyle\int_0^{\sqrt{x}} e^{-t^2} dt$ (§ 7.18, p. 312)

$$= \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{x}} \left(1 - t^2 + \frac{t^4}{2!} - \frac{t^6}{3!} + \dots\right) dt = \frac{2}{\sqrt{\pi}} \left(x^{1/2} - \frac{x^{3/2}}{3} + \frac{x^{5/2}}{5.2!} - \frac{x^{7/2}}{7.3!} + \dots\right)$$

$$\therefore \quad L\{erf(\sqrt{x})\} = \frac{2}{\sqrt{\pi}} \left\{\frac{\Gamma(3/2)}{s^{3/2}} - \frac{\Gamma(5/2)}{3s^{5/2}} + \frac{\Gamma(7/2)}{5.2!\,s^{7/2}} - \frac{\Gamma(9/2)}{7.3!\,s^{9/2}} + \dots\right\}$$

$$= \frac{1}{s^{3/2}} - \frac{1}{2}\frac{1}{s^{5/2}} + \frac{1.3}{2.4} \cdot \frac{1}{s^{7/2}} - \frac{1.3.5}{2.4.6} \cdot \frac{1}{s^{9/2}} + \dots$$

$$= \frac{1}{s^{3/2}} \left\{1 - \frac{1}{2} \cdot \frac{1}{s} + \frac{1.3}{2.4} \cdot \frac{1}{s^2} - \frac{1.3.5}{2.4.6} \cdot \frac{1}{s^3} + \dots\right\}$$

$$= \frac{1}{s^{3/2}} \left[1 + \frac{1}{s}\right]^{-1/2} = \frac{1}{s\sqrt{(s+1)}}. \qquad \text{(Mumbai, 2009)} \quad ...(3)$$

(3) Transform of Laguerre's polynomials $L_n(x)$

We know that $L_n(x) = e^x \dfrac{d^n}{dx^n}(x^n e^{-x})$ (§ 16.18, p. 571)

$$L[L_n(t)] = \int_0^\infty e^{-st} e^t \frac{d^n}{dt^n}(t^n e^{-t}) dt = \int_0^\infty e^{-(s-1)t} \frac{d^n}{dt^n}(e^{-t} t^n) dt$$

$$= \left| e^{-(s-1)t} \frac{d^{n-1}}{dt^{n-1}}(e^{-t} t^n) \right|_0^\infty + \int_0^\infty e^{-(s-1)t\,(s-1)}\,(s-1) \frac{d^{n-1}}{dt^{n-1}}(e^{-t} t^n) dt$$

$$= (s-1) \int_0^\infty e^{-(s-1)t} \frac{d^{n-1}}{dt^{n-1}}(e^{-t} t^n) dt. \qquad \text{(Integrating by parts)}$$

$$= (s-1)^n \int_0^\infty e^{-(s-1)t} . e^{-t} . t^n\, dt = (s-1)^n \int_0^\infty e^{-st} . t^n\, dt$$

$$= (s-1)^n L(t^n) = (s-1)^n . \frac{n!}{s^{n+1}}$$

Hence $\quad L[L_n(x)] = \dfrac{n!(s-1)^n}{s^n + 1}$ $(s > 1)$.

Example 21.11. *Evaluate (i) $L\{e^{-at} J_0(at)\}$ (ii) $L(erf\, 2\sqrt{t}\,)$.* (Mumbai, 2006)

Solution. (*i*) We know that $L\{J_0(at)\} = \dfrac{1}{\sqrt{(s^2 + a^2)}}$

By shifting property, we get

$$L\{e^{-at} J_0(at)\} = \frac{1}{\sqrt{[(s+a)^2 + a^2]}} = \frac{1}{\sqrt{(s^2 + 2sa + 2a^2)}}$$

(*ii*) We know that $\quad L(erf\sqrt{t}\,) = \dfrac{1}{s(s+1)}$

$$\therefore \quad L(erf\ 2\sqrt{t}\,) = L[erf\sqrt{(4t)}] = \frac{1}{4} \cdot \frac{1}{\frac{s}{4}\sqrt{\left(\frac{s}{4}+1\right)}} = \frac{2}{s\sqrt{(s+4)}}.$$

PROBLEMS 21.2

1. Find the Laplace transform of the *saw-toothed wave* of period T, given $f(t) = t/T$ for $0 < t < T$. (V.T.U., 2007)
2. Find the Laplace transform of the *full-wave rectifier*

   $f(t) = E \sin wt,\ 0 < t < \pi/w$, having period π/w.

3. Find the Laplace transform of the *square-wave* (or *meander*) function of period a defined as

$$f(t) = k, \quad \text{when} \quad 0 < t < \alpha$$
$$= -k, \quad \text{when} \quad \alpha < t < 2\alpha.$$

(*V.T.U., 2011*)

4. Find the Laplace transform of the *triangular wave* of period $2a$ given by

$$f(t) = t, \quad 0 < t < a$$
$$= 2a - t, \quad a < t < 2a.$$

(*Nagarjuna, 2008 ; V.T.U., 2008 S ; U.P.T.U., 2002*)

Find the Laplace transform of the following functions :

5. $J_0(ax)$.
6. $e^{-at} J_0(bt)$.
7. $e^{2t} \, erf(\sqrt{t})$.

21.7 TRANSFORMS OF DERIVATIVES

(1) *If $f'(t)$ be continuous and $L\{f(t)\} = f(s)$, then $L\{f'(t)\} = s\bar{f}(s) - f(0)$.*

$$L\{f(t)\} = \int_0^\infty e^{-st} f'(t)dt \qquad \text{[Integrate by parts]}$$

$$= \left| e^{-st} f(t) \right|_0^\infty - \int_0^\infty (-s)e^{-st} \cdot f(t)dt.$$

Now assuming $f(t)$ to be such that $\underset{t\to\infty}{\text{Lt}} \; e^{-st} f(t) = 0$. When this condition is satisfied, $f(t)$ is said to be *exponential order s*.

Thus, $$L\{f'(t)\} = f(0) + s \int_0^\infty e^{-st} f(t)dt$$

whence follows the desired result.

(2) *If $f'(t)$ and its first $(n-1)$ derivatives be continuous, then*

$$\mathbf{L\{f^n(t)\} = s^n \bar{f}(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \dots - f^{n-1}(0).}$$

Using the general rule of integration by parts (Footnote p. 398).

$$L\{f^n(t)\} = \int_0^\infty e^{-st} f^n(t)dt$$

$$= |e^{-st} f^{n-1}(t) - (-s)e^{-st} f^{n-2}(t) + (-s)^2 e^{-st} f^{n-3}(t) - \dots$$

$$+ (-1)^{n-1}(-s)^{n-1} e^{-st} \cdot f(t) \Big|_0^\infty + (-1)^n(-s)^n \int_0^\infty e^{-st} f(t)dt$$

$$= -f^{n-1}(0) - sf^{n-2}(0) - s^2 f^{n-3}(0) - \dots - s^{n-1} f(0) + s^n \int_0^\infty e^{-st} f(t)dt$$

Assuming that $\underset{t\to\infty}{\text{Lt}} \; e^{-st} f^m(t) = 0$ for $m = 0, 1, 2, \dots, n-1$.

This proves the required result.

21.8 TRANSFORMS OF INTEGRALS

If $L\{f(t)\} = \bar{f}(s)$, then $\mathbf{L\left\{\int_0^t f(u)du\right\} = \frac{1}{s}\bar{f}(s)}$.

Let $$\phi(t) = \int_0^t f(u)\,du, \text{ then } \phi'(t) = f(t) \text{ and } \phi(0) = 0$$

$\therefore$ $$L\{\phi'(t)\} = s\bar{\phi}(s) - \phi(0) \qquad \text{[By § 21.7 (1)]}$$

or $$\bar{\phi}(s) = \frac{1}{s} L\{\phi'(t)\} \quad i.e., \quad L\left(\int_0^t f(u)du\right) = \frac{1}{s}\bar{f}(s).$$

21.9 MULTIPLICATION BY t^n

If $L\{f(t)\} = \bar{f}(s)$, then

$$\mathbf{L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}[\bar{f}(s)]}, \text{ where } n = 1, 2, 3 \dots$$

We have $\int_0^\infty e^{-st} f(t)\,dt = \bar{f}(s)$.

Differentiating both sides with respect to s, $\frac{d}{ds}\left\{\int_0^\infty e^{-st} f(t)\,dt\right\} = \frac{d}{ds}\left\{\bar{f}(s)\right\}$

or By Leibnitz's rule for differentiation under the integral sign (p. 233).

$$\int_0^\infty \frac{\partial}{\partial s}(e^{-st})\, f(t)\,dt = \frac{d}{ds}\left\{\bar{f}(s)\right\}$$

or $$\int_0^\infty \{-te^{-st} f(t)\}\,dt = \frac{d}{ds}[\bar{f}(s)] \quad \text{or} \quad \int_0^\infty e^{-st}\,[tf(t)]\,dt = -\frac{d}{ds}[\bar{f}(s)]$$

which proves the theorem for $n = 1$.

Now assume the theorem to be true for $n = m$ (say), so that

$$\int_0^\infty e^{-st}\,[t^m f(t)]\,dt = (-1)^m \frac{d^m}{ds^m}[\bar{f}(s)]$$

Then $$\frac{d}{ds}\left[\int_0^\infty e^{-st}\, t^m f(t)\,dt\right] = (-1)^m \frac{d^{m+1}}{ds^{m+1}}[\bar{f}(s)]$$

or By Leibnitz's rule, $\int_0^\infty (-te^{-st}).\, t^m f(t)\,dt = (-1)^m \frac{d^{m+1}}{ds^{m+1}}[\bar{f}(s)]$

or $$\int_0^\infty e^{-st}\,[t^{m+1} f(t)]\,dt = (-1)^{m+1} \frac{d^{m+1}}{ds^{m+1}}[\bar{f}(s)].$$

This shows that, if the theorem, is true for $n = m$, it is also true for $n = m + 1$. But it is true for $n = 1$. Hence it is true for $n = 1 + 1 = 2$, and $n = 2 + 1 = 3$ and so on.

Thus the theorem is true for all positive integral values of n. *(U.P.T.U., 2005)*

Example 21.12. *Find the Laplace transforms of*

(i) $t \cos at$ *(Raipur, 2005)* *(ii)* $t^2 \sin at$ *(S.V.T.U., 2007)*

(iii) $t^3 e^{-3t}$ *(Kottayam, 2005)* *(iv)* $te^{-t} \sin 3t$. *(Kurukshetra, 2005)*

Solution. *(i)* Since $L(\cos at) = s/(s^2 + a^2)$

$$\therefore \quad L(t \cos at) = -\frac{d}{ds}\left(\frac{s}{s^2 + a^2}\right) = -\frac{s^2 + a^2 - s\,.\,2s}{(s^2 + a^2)^2}$$

$$= \frac{s^2 - a^2}{(s^2 + a^2)^2}$$ [*cf.* Example 21.4]

(ii) Since $\sin at = \frac{a}{s^2 + a^2}$,

$$\therefore \quad L(t^2 \sin at) = (-1)^2 \frac{d^2}{ds^2}\left(\frac{a}{s^2 + a^2}\right) = \frac{d}{ds}\left\{\frac{-2as}{(s^2 + a^2)^2}\right\} = \frac{2a(3s^2 - a^2)}{(s^2 + a^2)^3}.$$

(iii) Since $L(e^{-3t}) = 1/(s + 3)$,

$$\therefore \quad L(t^3 e^{-3t}) = (-1)^3 \frac{d^3}{ds^3}\left(\frac{1}{s+3}\right) = -\frac{(-1)^3\,.\,3\,!}{(s+3)^{3+1}} = 6/(s+3)^4.$$

(iv) Since $L(\sin 3t) = \frac{3}{s^2 + 3^2}$, therefore $L(t \sin 3t) = -\frac{d}{ds}\left(\frac{3}{s^2 + 3^2}\right) = \frac{6s}{(s^2 + 9)^2}$

Now using the shifting property (§ 21.4 II), we get

$$L(e^{-t}\, t \sin 3t) = \frac{6(s+1)}{[(s+1)^2 + 9]^2} = \frac{6(s+1)}{(s^2 + 2s + 10)^2}.$$

Example 21.13. *Evaluate (i) $L\{t\,J_0(at)\}$ (ii) $L\{t\,J_1(t)\}$ (iii) $L\{t\,erf\,2\sqrt{t}\}$.*

Solution. (*i*) Since $L\{J_0(at)\} = \dfrac{1}{\sqrt{(s^2+a^2)}}$

$$\therefore \qquad L\{tJ_0(at)\} = -\frac{d}{ds}\{L\,J_0(at)\} = -\frac{d}{ds}\frac{1}{\sqrt{(s^2+a^2)}} = \frac{s}{(s^2+a^2)^{3/2}}$$

(*ii*) Since $\qquad L\{J_1(t)\} = 1 - \dfrac{s}{\sqrt{(s^2+1)}}$

$$\therefore \qquad L\{t\,J_1(t)\} = -\frac{d}{ds}\{L\,J_0(t)\} = -\frac{d}{ds}\left\{1-\frac{s}{(\sqrt{s^2+1})}\right\} = \frac{1}{(s^2+1)^{3/2}}$$

(*iii*) Since $\qquad L(erf\sqrt{t}) = \dfrac{1}{s\sqrt{(s+1)}}$

$$\therefore \qquad L(erf\,2\sqrt{t}) = L\,erf\sqrt{4(t)} = \frac{1}{4}\cdot\frac{1}{\frac{s}{4}\sqrt{\left(\frac{s}{4}+1\right)}} = \frac{2}{s\sqrt{(s+4)}}$$

Thus $\qquad L(t\,erf\,2\sqrt{t}) = -\dfrac{d}{ds}\left\{\dfrac{2}{s\sqrt{(s+4)}}\right\} = -\dfrac{d}{ds}\left\{\dfrac{2}{\sqrt{(s^3+4s^2)}}\right\} = \dfrac{3s+8}{s^2(s+4)^{3/2}}$

21.10 DIVISION BY t

If $L\{f(t)\} = \bar{f}(s)$, *then* $\mathbf{L}\left\{\dfrac{\mathbf{1}}{\mathbf{t}}\,\mathbf{f(t)}\right\} = \int_s^\infty \bar{\mathbf{f}}\mathbf{(s)\,ds}$ *provided the integral exists.*

We have $\bar{f}(s) = \int_0^\infty e^{-st} f(t)\,dt$

Integrating both sides with respect to s from s to ∞.

$$\int_s^\infty \bar{f}(s)\,ds = \int_s^\infty\left[\int_0^\infty e^{-st} f(t)\,dt\right]ds = \int_0^\infty\int_s^\infty f(t)\,e^{-st}\,ds\,dt$$

[Changing the order of integration]

$$= \int_0^\infty f(t)\left[\int_s^\infty e^{-st}\,ds\right]dt \qquad [\because\ t \text{ is independent of } s]$$

$$= \int_0^\infty f(t)\left|\frac{e^{-st}}{-t}\right|_s^\infty dt = \int_0^\infty e^{-st}\cdot\frac{f(t)}{t}\,dt = L\left\{\frac{1}{t}f(t)\right\}.$$

Example 21.14. *Find the Laplace transform of (i) $(1-e^t)/t$* *(Madras, 2000)*

(ii) $\dfrac{\cos at - \cos bt}{t} + t\sin at.$ *(V.T.U., 2010)*

Solution. (*i*) Since $\qquad L(1-e^t) = L(1) - L(e^t) = \dfrac{1}{s} - \dfrac{1}{s-1}$

$$\therefore \qquad L\left(\frac{1-e^t}{t}\right) = \int_s^\infty\left(\frac{1}{s}-\frac{1}{s-1}\right)ds = |\log s - \log(s-1)|_0^\infty$$

$$= \left|\log\left(\frac{s}{s-1}\right)\right|_s^\infty = -\log\left[\frac{1}{1-1/s}\right] = \log\left(\frac{s-1}{s}\right)$$

(*ii*) Since $\qquad L(\cos at - \cos bt) = \dfrac{s}{s^2+a^2} - \dfrac{s}{s^2+b^2}$ and $L(\sin at) = \dfrac{a}{s^2+a^2}$

$$\therefore \quad L\left(\frac{\cos at - \cos bt}{t}\right) + L(t \sin at) = \int_s^\infty \left(\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right) ds - \frac{d}{ds}\left(\frac{a}{s^2+a^2}\right)$$

$$= \left|\frac{1}{2}\log(s^2+a^2) - \frac{1}{2}\log(s^2+b^2)\right|_s^\infty - a\,\frac{-2s}{(s^2+a^2)^2}$$

$$= \frac{1}{2}\underset{s\to\infty}{\text{Lt}} \log\frac{s^2+a^2}{s^2+b^2} - \frac{1}{2}\log\frac{s^2+a^2}{s^2+b^2} + \frac{2as}{(s^2+a^2)^2}$$

$$= \frac{1}{2}\log\left(\frac{1+0}{1+0}\right) - \frac{1}{2}\log\left(\frac{s^2+a^2}{s^2+b^2}\right) + \frac{2as}{(s^2+a^2)^2} = \log\left(\frac{s^2+b^2}{s^2+a^2}\right)^{1/2} + \frac{2as}{(s^2+a^2)^2}$$

$[\because \quad \log 1 = 0]$

Example 21.15. *Evaluate (i)* $L\left\{e^{-t}\int_0^t \frac{\sin t}{t}\,dt\right\}$ *(Madras, 2006)*

(ii) $L\left\{t\int_0^t \frac{e^{-t}\sin t}{t}\,dt\right\}$ *(P.T.U., 2005)* *(iii)* $L\left\{\int_0^t\int_0^t\int_0^t (t\sin t)\,dt\,dt\,dt\right\}$. *(Mumbai, 2006)*

Solution. (*i*) We know that $L(\sin t) = \dfrac{1}{s^2+1}$

$$L\left(\frac{\sin t}{t}\right) = \int_0^\infty \frac{1}{s^2+1}\,ds = \frac{\pi}{2} - \tan^{-1}s = \cot^{-1}s$$

$$\therefore \quad L\left\{\int_0^t \frac{\sin t}{t}\,dt\right\} = \frac{1}{s}\cot^{-1}s$$

Thus by shifting property, $L\left\{e^{-t}\left(\int_0^t \frac{\sin t}{t}\,dt\right)\right\} = \dfrac{1}{s+1}\cot^{-1}(s+1)$.

(*ii*) Since $\quad L\left(\dfrac{\sin t}{t}\right) = \cot^{-1}s$

$$\therefore \quad L\left(e^{-t}\cdot\frac{\sin t}{t}\right) = \cot^{-1}(s+1)$$

and $\quad L\left\{\int_0^t e^{-t}\dfrac{\sin t}{t}\,dt\right\} = \dfrac{1}{s}\cot^{-1}(s+1)$

Hence $\quad L\left\{t\cdot\int_0^t e^{-t}\dfrac{\sin t}{t}\,dt\right\} = -\dfrac{d}{ds}\left\{\dfrac{\cot^{-1}(s+1)}{s}\right\}$

$$= -\frac{s\cdot\left[\dfrac{-1}{1+(s+1)^2}\right] - \cot^{-1}(s+1)}{s^2} = \frac{s+(s^2+2s+2)\cot^{-1}(s+1)}{s^2(s^2+2s+2)}$$

(*iii*) Since $\quad L(\sin t) = \dfrac{1}{s^2+1}$

$$\therefore \quad L(t\sin t) = -\frac{d}{ds}\frac{1}{(s^2+1)} = \frac{2s}{(s^2+1)^2}$$

Thus $\quad L\left\{\int_0^t\int_0^t\int_0^t (t\sin t)\,dt\,.\,dt\,.\,dt\right\} = \dfrac{1}{s^3}L(t\sin t) = \dfrac{1}{s^3}\cdot\dfrac{2s}{(s^2+1)^2} = \dfrac{2}{s^2(s^2+1)^2}$

21.11 EVALUATION OF INTEGRALS BY LAPLACE TRANSFORMS

Example 21.16. *Evaluate* (*i*) $\int_0^\infty te^{-3t}\sin t\,dt$ (*V.T.U., 2007*)

(*ii*) $\int_0^\infty \frac{\sin mt}{t}\,dt$ (*iii*) $\int_0^\infty e^{-t}\left(\frac{\cos at-\cos bt}{t}\right)dt$ (*Mumbai, 2009*)

(*iv*) $L\left\{\int_0^t \frac{e^{-t}\sin t}{t}\,dt\right\}$.

Solution. (*i*) $\int_0^\infty te^{-3t}\sin t\,dt=\int_0^\infty e^{-st}(t\sin t)\,dt$ where $s=3$

$= L(t\sin t)$, by definition.

$$=(-1)\frac{d}{ds}\left(\frac{1}{s^2+1}\right)=\frac{2s}{(s^2+1)^2}=\frac{2\times 3}{(3^2+1)^2}=\frac{3}{50}.$$

(*ii*) Since $L(\sin mt)=m/(s^2+m^2)=f(s)$, say.

$\therefore$ Using § 21.10, $L\left(\frac{\sin mt}{t}\right)=\int_s^\infty f(s)\,ds=\int_0^\infty\frac{m\,ds}{s^2+m^2}=\left|\tan^{-1}\frac{s}{m}\right|_s^\infty$

or by Def., $\int_0^\infty e^{-st}\frac{\sin mt}{t}\,dt=\frac{\pi}{2}-\tan^{-1}\frac{s}{m}$

Now $\underset{s\to 0}{\text{Lt}}\tan^{-1}(s/m)=0$ if $m>0$ or π if $m<0$.

Thus taking limits as $s\to 0$, we get

$$\int_0^\infty\frac{\sin mt}{t}\,dt=\frac{\pi}{2}\text{ if } m>0 \quad\text{or}\quad -\pi/2 \text{ if } m<0$$

(*iii*) We know that $L(\cos at)=\frac{s}{s^2+a^2}$ and $L(\cos bt)=\frac{s}{s^2+b^2}$

$$\therefore\quad L\,\frac{\cos at-\cos bt}{t}=\int_s^\infty\left(\frac{s}{s^2+a^2}-\frac{s}{s^2+b^2}\right)ds$$

$$=\frac{1}{2}\left\{\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right\}_s^\infty=\frac{1}{2}\log\left(\frac{s^2+b^2}{s^2+a^2}\right)$$

This implies that $\int_0^\infty e^{-st}\left(\frac{\cos at-\cos bt}{t}\right)dt=\frac{1}{2}\log\left(\frac{s^2+b^2}{s^2+a^2}\right)$

Taking $s=1$, we get $\int_0^\infty\left(e^{-t}\,\frac{\cos at-\cos bt}{t}\right)dt=\frac{1}{2}\log\left(\frac{1+b^2}{1+a^2}\right)$

(*iv*) Since $L\left(\frac{\sin t}{t}\right)=\int_s^\infty\frac{ds}{s^2+1}=\tan^{-1}s=\frac{\pi}{2}-\tan^{-1}s=\cot^{-1}s.$

$\therefore$ $L\left\{e^t\left(\frac{\sin t}{t}\right)\right\}=\cot^{-1}(s-1)$, by shifting property (§ 21.4 *II*).

Thus $L\left[\int_0^t\left\{e^t\left(\frac{\sin t}{t}\right)\right\}dt\right]=\frac{1}{s}\cot^{-1}(s-1)$, by § 21.8.

PROBLEMS 21.3

1. Find $L\left(\int_0^t e^{-t}\cos t\,dt\right)$.

2. Given $L\{2\sqrt{(t/\pi)}\} = 1/s^{3/2}$, show that $L\{1/\sqrt{(\pi t)}\} = 1/\sqrt{s}$. (U.P.T.U., 2005 ; Madras, 2003)

3. Given $L[\sin(\sqrt{t})] \dfrac{\sqrt{\pi}}{2s^{3/2}} e^{-1/4\pi}$, prove that $L\left[\dfrac{\cos(\sqrt{t})}{\sqrt{t}}\right] = \sqrt{\left(\dfrac{\pi}{s}\right)} e^{-1/4\pi}$. (Mumbai, 2009)

Find the Laplace transforms of the following functions :

4. $t\sin^2 t$ (Nagarjuna, 2008)

5. $\sin 2t - 2t\cos 2t$. (Anna, 2003)

6. $t^2\cos at$.

7. $t\sinh at$.

8. $te^{2t}\sin 3t$. (Madras, 2003)

9. $te^{-2t}\sin 4t$. (V.T.U., 2008)

10. $t^2 e^{-3t}\sin 2t$. (Madras, 2000 S)

11. $(e^{-at} - e^{-bt})/t$. (Anna, 2005 S)

12. $(\sin t)/t$. (P.T.U., 2010)

13. $\dfrac{(\sin t\sin 5t)}{t}$. (Mumbai, 2008)

14. $(e^{at} - \cos bt)/t$. (U.P.T.U., 2003)

15. $(e^{-t}\sin t)/t$. (V.T.U., 2009 S)

16. $(1 - \cos 3t)/t$. (V.T.U., 2006)

17. $(1 - \cos t)/t^2$. (Hazaribag, 2008)

18. $2^t + \dfrac{\cos 2t - \cos 3t}{t} + t\sin t$. (V.T.U., 2004)

19. Evaluate (i) $\displaystyle\int_0^\infty \frac{\cos 6t - \cos 4t}{t}\,dT$ (Mumbai, 2008 ; P.T.U., 2006)

(ii) $\displaystyle\int_0^\infty \frac{e^{-\sqrt{2}t}\sinh t\sin t}{t}\,dt$ (Mumbai, 2005) (iii) $\displaystyle\int_0^\infty te^{-2t}\sin 3t\,dt$ (V.T.U., 2008)

(iv) $\displaystyle\int_0^\infty te^{-t}\sin^4 t\,dt$.

20. Prove that (i) $\displaystyle\int_0^\infty \frac{e^{-at} - e^{-bt}}{t}\,dt = \log\frac{b}{a}$. (S.V.T.U., 2009 ; Mumbai, 2007 ; J.N.T.U., 2006)

(ii) $\displaystyle\int_0^\infty \frac{e^{-2t}\sinh t}{t}\,dt = \frac{1}{2}\log 3$ (Mumbai, 2008) (iii) $\displaystyle\int_0^\infty \frac{e^{-t}\sin t}{t}\,dt = \frac{\pi}{4}$. (V.T.U., 2009 S)

(iv) $\displaystyle\int_0^\infty \frac{e^{-t}\sin^2 t}{t}\,dt = \frac{1}{4}\log 5$. (Kurukshetra, 2006)

21. Evaluate (i) $L\left(\displaystyle\int_0^t \frac{\sin t}{t}\,dt\right)$ (J.N.T.U., 2005)

(ii) $L\left(\displaystyle\int_0^t e^{-t}\cos t\,dt\right)$ (iii) $L\displaystyle\int_0^t \frac{e^t\sin t}{t}\,dt$. (P.T.U., 2009 S ; S.V.T.U., 2009 ; Bhopal, 2008)

22. Show that (i) $L[tJ_0(at)] = \dfrac{s}{(s^2+a^2)^{3/2}}$ (ii) $\displaystyle\int_0^\infty te^{-3t}J_0(4t)\,dt = 3/125$.

21.12 INVERSE TRANSFORMS — METHOD OF PARTIAL FRACTIONS

Having found the Laplace transforms of a few functions, let us now determine the inverse transforms of given functions of s. We have seen that $L\{f(t)\}$ in each case, is a rational algebraic function. Hence *to find the* inverse transforms, we first express the given function of s into partial fractions which will, then, be recognizable as one of the following standard forms :

(1) $\mathbf{L^{-1}\left[\dfrac{1}{s}\right] = 1.}$ (2) $\mathbf{L^{-1}\left[\dfrac{1}{s-a}\right] = e^{at}.}$

(3) $\mathbf{L^{-1}\left[\dfrac{1}{s^n}\right] = \dfrac{t^{n-1}}{(n-1)!}}$, **n = 1, 2, 3, ...** **(4)** $\mathbf{L^{-1}\left[\dfrac{1}{(s-a)^n}\right] = \dfrac{e^{at}\,t^{n-1}}{(n-1)!}}$.

(5) $\mathbf{L^{-1}\left(\dfrac{1}{s^2+a^2}\right) = \dfrac{1}{a}}$ **sin at.** **(6)** $\mathbf{L^{-1}\left(\dfrac{s}{s^2+a^2}\right)}$ **= cos at.**

(7) $\mathbf{L^{-1}\left(\dfrac{1}{s^2-a^2}\right) = \dfrac{1}{a}}$ **sinh at.** **(8)** $\mathbf{L^{-1}\left(\dfrac{s}{s^2-a^2}\right)}$ **= cosh at.**

(9) $\mathbf{L^{-1}\left[\dfrac{1}{(s-a)^2+b^2}\right] = \dfrac{1}{b}\,e^{at}}$ **sin bt.** **(10)** $\mathbf{L^{-1}\left[\dfrac{s-a}{(s-a)^2+b^2}\right] = e^{at}}$ **cos bt.**

(11) $\mathbf{L^{-1}\left(\dfrac{s}{(s^2+a^2)^2}\right) = \dfrac{1}{2a}}$ **t sin at.** **(12)** $\mathbf{L^{-1}\left[\dfrac{1}{(s^2+a^2)^2}\right] = \dfrac{1}{2a^3}}$ **(sin at – at cos at).**

The reader is strongly advised to commit these results to memory. The results (1) to (10) follow at once from their corresponding results in § 21.3 and 21.4. As illustrations, we shall prove (11) and (12). Example 21.4 gives

$$L(t \sin at) = \frac{2as}{(s^2+a^2)^2} \text{ and } L(t \cos at) = \frac{s^2-a^2}{(s^2+a^2)^2}$$

$\therefore$ $$t \sin at = 2a\, L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right], \text{ whence follows (11).}$$

Also $$t \cos at = L^{-1}\left[\frac{s^2-a^2}{(s^2+a^2)^2}\right] = L^{-1}\left[\frac{(s^2+a^2)-2a^2}{(s^2+a^2)^2}\right]$$

$$= L^{-1}\left[\frac{1}{s^2+a^2}\right] - 2a^2\, L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right]$$

$$= \frac{1}{a}\sin at - 2a^2\, L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right] \text{ whence follows (12).}$$

Obs. *Go through the note on the 'partial fractions' given in para 10 of 'useful information' in Appendix I.*

Example 21.17. *Find the inverse transforms of*

(*i*) $\dfrac{s^2-3s+4}{s^3}$ (*ii*) $\dfrac{s+2}{s^2-4s+13}$ (*V.T.U., 2008*)

Solution. (*i*) $L^{-1}\left(\dfrac{s^2-3s+4}{s^3}\right) = L^{-1}\left(\dfrac{1}{s}\right) - 3L^{-1}\left(\dfrac{1}{s^2}\right) + 4L^{-1}\left(\dfrac{1}{s^3}\right) = 1 - 3t + 4\,.\,t^2/2\,! = 1 - 3t + 2t^2.$

(*ii*) $$L^{-1}\left(\frac{s+2}{s^2-4s+13}\right) = L^{-1}\left[\frac{s+2}{(s-2)^2+9}\right] = L^{-1}\left[\frac{s-2+4}{(s-2)^2+3^2}\right]$$

$$= L^{-1}\left[\frac{s-2}{(s-2)^2+3^2}\right] + 4L^{-1}\left[\frac{1}{(s-2)^2+3^2}\right] = e^{2t}\cos 3t + \tfrac{4}{3}\, e^{2t}\sin 3t.$$

Example 21.18. *Find the inverse transforms of*

(*i*) $\dfrac{2s^2-6s+5}{s^3-6s^2+11s-6}$ (*V.T.U., 2007 ; U.P.T.U., 2004*)

(*ii*) $\dfrac{4s+5}{(s-1)^2(s+2)}$ (*Kurukshetra, 2005*)

Solution. (*i*) Here the denominator = $(s-1)(s-2)(s-3)$.

So let $$\frac{2s^2-6s+5}{(s-1)(s-2)(s-3)} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s-3}$$

Then $A = [2 \,.\, 1^2 - 6 \,.\, 1 + 5]/(1-2)(1-3) = \frac{1}{2}$

$B = [2 \,.\, 2^2 - 6 \,.\, 2 + 5]/(2-1)(2-3) = -1$

and $C = [2 \,.\, 3^2 - 6 \,.\, 3 + 5]/(3-1)(3-2) = \frac{5}{2}$.

$$\therefore \quad L^{-1}\left(\frac{2s^2-6s+5}{s^3-6s^2+11s-6}\right) = \frac{1}{2}L^{-1}\left(\frac{1}{s-1}\right) - L^{-1}\left(\frac{1}{s-2}\right) + \frac{5}{2}L^{-1}\left(\frac{1}{s-3}\right)$$

$$= \tfrac{1}{2}e^t - e^{2t} + \tfrac{5}{2}e^{3t}.$$

(*ii*) Let $$\frac{4s+5}{(s-1)^2\,(s+2)} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{4(-2)+5}{(-2-1)^2(s+2)}$$

Multiplying both sides by $(s-1)^2(s+2)$, $4s+5 = A(s-1)(s+2) + B(s+2) - \frac{1}{3}(s-1)^2$

Putting $s = 1$, $9 = 3B$, $\therefore\ B = 3$.

Equating the coefficients of s^2 from both sides,

$$0 = A - \tfrac{1}{3}, \quad \therefore \quad A = \tfrac{1}{3}.$$

$$\therefore \quad L^{-1}\left[\frac{4s+5}{(s-1)^2\,(s+2)}\right] = \frac{1}{3}L^{-1}\left(\frac{1}{s-1}\right) + 3L^{-1}\left[\frac{1}{(s-1)^2}\right] - \frac{1}{3}L^{-1}\left(\frac{1}{s+2}\right)$$

$$= \tfrac{1}{3}e^t + 3te^t - \tfrac{1}{3}e^{-2t}.$$

Example 21.19. *Find the inverse transforms of*

(*i*) $\dfrac{5s+3}{(s-1)(s^2+2s+5)}$ (*Rohtak, 2009 ; U.P.T.U., 2005*)

(*ii*) $\dfrac{s}{s^4+4a^4}$. (*Mumbai, 2008*)

Solution. (*i*) Let $$\frac{5s+3}{(s-1)(s^2+2s+5)} = \frac{5(1)+3}{(s-1)(1^2+2\,.\,1+5)} + \frac{As+B}{s^2+2s+5}$$

Multiplying both sides by $(s-1)(s^2+2s+5)$,

$$5s+3 = 1\,.\,(s^2+2s+5) + (As+B)(s-1).$$

Equating the coefficients of s^2 from both sides,

$$0 = 1 + A, \quad \therefore \quad A = -1.$$

Putting $s = 0$, $3 = 5 - B$, $\therefore\ B = 2$.

$$\therefore \quad L^{-1}\left[\frac{5s+3}{(s-1)(s^2+2s+5)}\right] = L^{-1}\left(\frac{1}{s-1}\right) + L^{-1}\left(\frac{-s+2}{s^2+2s+5}\right)$$

$$= L^{-1}\left(\frac{1}{s-1}\right) + L^{-1}\left[\frac{-(s+1)+3}{(s+1)^2+4}\right] = L^{-1}\left(\frac{1}{s-1}\right) - L^{-1}\left[\frac{s+1}{(s+1)^2+2^2}\right] + 3L^{-1}\left[\frac{1}{(s+1)^2+2^2}\right]$$

$$= e^t - e^{-t}\cos 2t + \tfrac{3}{2}\,e^{-t}\sin 2t.$$

(*ii*) Since $s^4 + 4a^4 = (s^2+2a^2)^2 - (2as)^2 = (s^2+2as+2a^2)(s^2-2as+2a^2)$

$\therefore$ Let $$\frac{s}{s^4+4a^4} = \frac{As+B}{s^2+2as+2a^2} + \frac{Cs+D}{s^2-2as+2a^2}$$

Multiplying both sides by $s^4 + 4a^4$,

$$s = (As+B)(s^2-2as+2a^2) + (Cs+D)(s^2+2as+2a^2)$$

Equating coefficients of s^3, $0 = A + C$ ...(*i*)

Equating coefficients of s^2, $0 = -2aA + B + 2aC + D$ ...(*ii*)

Equating coefficients of s, $1 = 2a^2A - 2aB + 2a^2C + 2aD$ ...(*iii*)

Putting $s = 0$, $\quad 0 = 2a^2B + 2a^2D$ ...(*iv*)

From (*iv*), $B + D = 0$ ...(*v*)

$\therefore$ (*ii*) becomes $-A + C = 0$, and by (*i*), we get $A = C = 0$.

Then (*iii*) reduces to $D - B = 1/2a$ and by (*v*), $B = -1/4a$, $D = 1/4a$.

$$\therefore \quad L^{-1}\left(\frac{s}{s^4 + 4a^4}\right) = -\frac{1}{4a}L^{-1}\left(\frac{1}{s^2 + 2as + 2a^2}\right) + \frac{1}{4a}L^{-1}\left(\frac{1}{s^2 - 2as + 2a^2}\right)$$

$$= -\frac{1}{4a}L^{-1}\left[\frac{1}{(s+a)^2 + a^2}\right] + \frac{1}{4a}L^{-1}\left[\frac{1}{(s-a)^2 + a^2}\right]$$

$$= -\frac{1}{4a}\cdot\frac{1}{a}e^{-at}\sin at + \frac{1}{4a}\cdot\frac{1}{a}e^{at}\sin at = \frac{1}{2a^2}\sin at\left(\frac{e^{at} - e^{-at}}{2}\right) = \frac{1}{2a^2}\sin at\ \sinh at.$$

PROBLEMS 21.4

Find the inverse Laplace transforms of :

1. $\dfrac{2s-5}{4s^2+25} + \dfrac{4s-18}{9-s^2}$.
2. $\dfrac{1}{s^2-5s+6}$. (*S.V.T.U., 2008*)
3. $\dfrac{s}{(2s-1)(3s-1)}$. (*V.T.U., 2010*)
4. $\dfrac{3s}{s^2+2s-8}$.
5. $\dfrac{3s+2}{s^2-s-2}$. (*V.T.U., 2010 S*)
6. $\dfrac{1}{s(s^2-1)}$. (*Nagarjuna, 2008*)
7. $\dfrac{1-7s}{(s-3)(s-1)(s+2)}$. (*B.P.T.U., 2005 S*)
8. $\dfrac{s^2-10s+13}{(s-7)(s^2-5s+6)}$.
9. $\dfrac{2p^2-6p+5}{p^3-6p^2+11p-6}$. (*U.P.T.U., 2004*)
10. $\dfrac{s}{(s^2-1)^2}$. (*Kurukshetra, 2005*)
11. $\dfrac{1+2s}{(s+2)^2(s-1)^2}$.
12. $\dfrac{s}{(s-3)(s^2+4)}$.
13. $\dfrac{s}{(s+1)^2(s^2+1)}$.
14. $\dfrac{s^3}{s^4-a^4}$. (*Kurukshetra, 2005*)
15. $\dfrac{1}{s^3-a^3}$.
16. $\dfrac{s^2+6}{(s^2+1)(s^2+4)}$.
17. $\dfrac{2s-3}{s^2+4s+13}$.
18. $\dfrac{s^2+s}{(s^2+1)(s^2+2s+2)}$.
19. $\dfrac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}$ (*Mumbai, 2008*)
20. $\dfrac{s}{s^4+s^2+1}$. (*Raipur, 2005*)
21. $\dfrac{a(s^2-2a^2)}{s^4+4a^4}$. (*Mumbai, 2009*)

21.13 OTHER METHODS OF FINDING INVERSE TRANSFORMS

We have seen that the most effective method of finding the inverse transforms is by means of partial fractions. However, various other methods are available which depend on the following *important inversion formulae*.

I. Shifting property for inverse Laplace transforms.

If $\quad L^{-1}[\bar{f}(s)] = f(t)$, *then*

$$\boldsymbol{L^{-1}[\bar{f}'(s-a)] = e^{at} f(t) = e^{at} L^{-1}[\bar{f}(s)].}$$

II. *If* $\quad L^{-1}\{\bar{f}(s)\} = f(t)$ *and* $f(0) = 0$, *then*

$$\boldsymbol{L^{-1}[s\,\bar{f}(s)] = \frac{d}{dt}\{f(t)\}}$$

In general, $\boldsymbol{L^{-1}\{s^n \bar{f}(s)\} = \frac{d^n}{dt^n}\{f(t)\}}$ provided $f(0) = f'(0) = \ldots = f^{n-1}(0) = 0$.

The above formulae at once follow from the results of § 21.7 (Transforms of derivatives).

III. *If* $\quad L^{-1}\{\bar{f}(s)\} = f(t)$, *then*

$$\boldsymbol{L^{-1}\left\{\frac{\bar{f}(s)}{s}\right\} = \int_0^t f(t)\, dt}$$

This result follows from § 21.8 (Transforms of integrals)

Also $\quad L^{-1}\left\{\dfrac{\bar{f}(s)}{s^2}\right\} = \displaystyle\int_0^t \left\{\int_0^t f(t)\, dt\right\} dt$

$$L^{-1}\left\{\frac{\bar{f}(s)}{s^3}\right\} = \int_0^t \left\{\int_0^t \left(\int_0^t f(t)\, dt\right) dt\right\} dt \text{ and so on.}$$

IV. *If* $\quad L^{-1}\{\bar{f}(s)\} = f(t)$, *then*

$$t\, f(t) = L^{-1}\left\{-\frac{d}{ds}[\bar{f}(s)]\right\}$$

This result follows from $L\{t\, f(t)\} = -\dfrac{d}{ds}\{\bar{f}(s)\}$ (§ 21.9)

V. The formula of § 21.10, *i.e.*,

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \bar{f}(s)\, ds$$

is useful in finding $f(t)$ when $f(s)$ is given, provided the inverse transform of $\int_s^\infty \bar{f}(s)\, ds$ can be conveniently calculated.

Example 21.20. *Find the inverse Laplace transforms of the following :*

(*i*) $\dfrac{s^2}{(s-2)^3}$ (*ii*) $\dfrac{s+3}{s^2-4s+13}$ (*iii*) $\dfrac{(s+2)^2}{(s^2+4s+8)^2}$ (*Mumbai, 2005*)

Solution. (*i*) Since $\quad s^2 = (s-2)^2 + 4(s-2) + 4$

$$\therefore \quad \frac{s^2}{(s-2)^3} = \frac{1}{s-2} + \frac{4}{(s-2)^2} + \frac{4}{(s-2)^3}$$

$$\therefore \quad L^{-1}\left\{\frac{s^2}{(s-2)^3}\right\} = L^{-1}\left\{\frac{1}{s-2}\right\} + 4L^{-1}\left\{\frac{1}{(s-2)^2}\right\} + 4L^{-1}\left\{\frac{1}{(s-2)^3}\right\}$$

$$= e^{2t} + 4e^{2t}\, t + 2e^{2t}\, t^2.$$ [using shifting property]

(*ii*) $$\frac{s+3}{s^2-4s+13} = \frac{s-2}{(s-2)^2+3^2} + \frac{5}{(s-2)^2+3^2}$$

$$\therefore \quad L^{-1}\left\{\frac{s+3}{s^2-4s+13}\right\} = L^{-1}\left\{\frac{s-2}{(s-2)^2+3^2}\right\} + \frac{5}{3}L^{-1}\left\{\frac{3}{(s-2^2)+3^2}\right\}$$

$$= e^{2t}\cos 3t + \frac{5}{3}e^{2t}\sin 3t. \qquad \text{[Using shifting property]}$$

(iii) $$L^{-1}\frac{(s+2)^2}{(s^2+4s+8)^2} = L^{-1}\frac{(s+2)^2}{(s^2+4s+4+4)^2} = L^{-1}\frac{(s+2)^2}{[(s+2)^2+4]^2}$$

$$= e^{-2t}L^{-1}\left\{\frac{s^2}{(s^2+4)^2}\right\} = e^{-2t}L^{-1}\left\{\frac{s^2+4-4}{(s^2+4)^2}\right\}$$

$$= e^{-2t}L^{-1}\left\{\frac{1}{s^2+4} - \frac{4}{(s^2+4)^2}\right\} = \frac{e^{-2t}\sin 2t}{2} - 4e^{-2t}L^{-1}\left\{\frac{1}{(s^2+4)^2}\right\}$$

$$= \frac{e^{-2t}\sin 2t}{2} - 4e^{-2t}\left\{\frac{1}{4}\left(\frac{\sin 2t}{4} - \frac{t\cos 2t}{2}\right)\right\}$$

$$= e^{-2t}\left\{\frac{\sin 2t}{2} - \frac{\sin 2t}{4} + \frac{t\cos 2t}{2}\right\} = e^{-2t}\left\{\frac{\sin 2t}{4} + \frac{t\cos 2t}{2}\right\}.$$

Example 21.21. *Find the inverse transform of (i) $1/s\,(s^2+a^2)$ (ii) $1/s\,(s+a)^3$.* *(P.T.U., 2003)*

Solution. (*i*) Since $L^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{1}{a}\sin at.$

therefore, by formula III above,

$$L^{-1}\left\{\frac{1}{s(s^2+a^2)}\right\} = \int_0^t \frac{1}{a}\sin at\,dt = \frac{1}{a^2}[-\cos at]_0^t = (1-\cos at)/a^2$$

(*ii*) $$L^{-1}\left\{\frac{1}{s(s+a)^3}\right\} = L^{-1}\left\{\frac{1}{[(s+a)-a](s+a)^3}\right\} = e^{-at}L^{-1}\left\{\frac{1}{(s-a)s^3}\right\}$$

Now $$L^{-1}\left\{\frac{1}{s-a}\right\} = e^{at} \quad \therefore \quad L^{-1}\left\{\frac{1}{(s-a)s}\right\} = \int_0^t e^{at}\,dt = \frac{e^{at}}{a} - \frac{1}{a}, \text{ by III above}$$

$\therefore$ $$L^{-1}\left\{\frac{1}{(s-a)s^2}\right\} = \frac{1}{a}\int_0^t (e^{at}-1)\,dt = \frac{1}{a^2}(e^{at}-at-1)$$

$$L^{-1}\left\{\frac{1}{(s-a)s^3}\right\} = \frac{1}{a^2}\int_0^t (e^{at}-at-1)\,dt = \frac{1}{a^3}\left(e^{at} - \frac{a^2}{2}t^2 - at - 1\right)$$

Hence $$L^{-1}\left\{\frac{1}{s(s+a)^3}\right\} = e^{-at}\cdot\frac{1}{a^3}\left(e^{at} - \frac{a^2t^2}{2} - at - 1\right) = \frac{1}{a^3}\left(1 - e^{-at} - ate^{-at} - \frac{a^2}{2}t^2e^{-at}\right).$$

Example 21.22. *Find the inverse Laplace transforms of :*

(*i*) $\frac{s}{(s^2+a^2)^2}$ *(S.V.T.U., 2009)* (*ii*) $\frac{s^2}{(s^2+a^2)^2}$ *(Hazaribag, 2009)* (*iii*) $\frac{1}{(s^2+a^2)^2}$.

Solution. (*i*) If $f(t) = L^{-1}\frac{s}{(s^2+a^2)^2}$, then by formula V above,

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \frac{s}{(s^2+a^2)^2}\,ds = \frac{1}{2}\int_s^\infty \frac{2s}{(s^2+a^2)^2}\,ds = -\frac{1}{2}\left(\frac{1}{s^2+a^2}\right)_s^\infty = \frac{1}{2}\cdot\frac{1}{s^2+a^2}$$

$\therefore$ $$\frac{f(t)}{t} = \frac{1}{2}L^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{\sin at}{2a}$$

Hence, $f(t) = \dfrac{1}{2a} t \sin at$.

Otherwise : Let $\quad f(t) = L^{-1}\left(\dfrac{1}{s^2 + a^2}\right) = \dfrac{\sin at}{a}$ so that $\bar{f}(s) = \dfrac{1}{s^2 + a^2}$

Then by (IV) above, $\quad t f(t) = L^{-1}\left\{-\dfrac{d}{ds}[\bar{f}(s)]\right\} = L^{-1}\left\{-\dfrac{d}{ds}\left(\dfrac{1}{s^2 + a^2}\right)\right\}$

or $$\frac{t \sin at}{a} = L^{-1}\left\{\frac{2s}{(s^2 + a^2)^2}\right\}. \text{ Hence } L^{-1}\left\{\frac{s}{(s^2 + a^2)^2}\right\} = \frac{1}{2a} t \sin at.$$

(*ii*) In (*i*), we have proved that

$$L^{-1}\left\{\frac{s}{(s^2 + a^2)^2}\right\} = \frac{1}{2a} t \sin at = f(t), \text{ say}$$

Since $f(0) = 0$, we get from formula II above, that

$$L^{-1}\left\{\frac{s^2}{(s^2 + a^2)^2}\right\} = L^{-1}\left\{s \cdot \frac{s}{(s^2 + a^2)^2}\right\} = \frac{d}{dt}\{f(t)\}$$

$$= \frac{d}{dt}\left(\frac{1}{2a} t \sin at\right) = \frac{1}{2a}(\sin at + at \cos at)$$

(*iii*) In (*i*), we have shown that

$$L^{-1}\left\{\frac{s}{(s^2 + a^2)^2}\right\} = \frac{1}{2a}(t \sin at) = f(t), \text{ say}$$

By formula III above, we have

$$L^{-1}\left\{\frac{1}{(s^2 + a^2)^2}\right\} = L^{-1}\left\{\frac{1}{s} \cdot \frac{s}{(s^2 + a^2)^2}\right\} = \int_0^t f(t)\, dt = \int_0^t \frac{t \sin at}{2a}\, dt$$

$$= \frac{1}{2a}\left\{\left| t \cdot \frac{-\cos at}{a}\right|_0^t - \int_0^t 1 \cdot \left(\frac{-\cos at}{a}\right) dt\right\}$$

$$= \frac{1}{2a}\left\{\frac{-t \cos at}{a} + \frac{\sin at}{a^2}\right\} = \frac{1}{2a^3}(\sin at - at \cos at).$$

Example 21.23. *Find the inverse Laplace transforms of*

(*i*) $\dfrac{s + 2}{s^2(s + 1)(s - 2)}$ (*V.T.U., 2003*) (*ii*) $\dfrac{s + 2}{(s^2 + 4s + 5)^2}$. (*S.V.T.U., 2009 ; P.T.U., 2005*)

Solution. (*i*) $L^{-1}\left\{\dfrac{s + 2}{(s + 1)(s - 2)}\right\} = \dfrac{4}{3} L^{-1}\left(\dfrac{1}{s - 2}\right) - \dfrac{1}{3} L^{-1}\left(\dfrac{1}{s + 1}\right) = \dfrac{4}{3} e^{2t} - \dfrac{1}{3} e^{-t}$

By III above, $L^{-1}\left\{\dfrac{s + 2}{s(s + 1)(s - 2)}\right\} = \displaystyle\int_0^t L^{-1}\left(\dfrac{s + 2}{(s + 1)(s - 2)}\right) dt$

$$= \int_0^t \left(\frac{4}{3} e^{2t} - \frac{1}{3} e^{-t}\right) dt = \frac{2}{3} e^{2t} + \frac{1}{3} e^{-t} - 1$$

Again by III above, $L^{-1} \dfrac{s + 2}{s^2(s + 1)(s - 2)} = \displaystyle\int_0^t L^{-1}\left\{\dfrac{s + 2}{s(s + 1)(s - 2)}\right\} dt$

$$= \int_0^t \left(\frac{2}{3} e^{2t} + \frac{1}{3} e^{-t} - 1\right) dt = \frac{1}{3}(e^{2t} - e^{-t} - t).$$

(ii) $$L^{-1}\left(\frac{1}{s^2+4s+5}\right) = L^{-1}\left\{\frac{1}{(s+2)^2+1}\right\} = e^{-2t}\sin t$$

By II above, $$L^{-1}\left\{\frac{d}{ds}\left(\frac{1}{s^2+4s+5}\right)\right\} = (-1)^1\, t \,.\, e^{-2t}\sin t$$

i.e., $$L^{-1}\left\{\frac{-(2s+4)}{(s^2+4s+5)^2}\right\} = -t\,.\,e^{-2t}\sin t$$

or $$L^{-1}\left\{\frac{s+2}{(s^2+4s+5)^2}\right\} = \frac{1}{2}t\,.\,e^{-2t}\sin t.$$

Example 21.24. *Find the inverse Laplace transforms of the following :*

(i) $\log\frac{s+1}{s-1}$ *(S.V.T.U., 2009 ; Bhopal, 2008)* *(ii)* $\log\frac{s^2+1}{s(s+1)}$ *(S.V.T.U., 2009 ; V.T.U., 2008)*

(iii) $\cot^{-1}\left(\frac{s}{2}\right)$ *(iv)* $\tan^{-1}\left(\frac{2}{s^2}\right)$. *(V.T.U., 2011 ; Mumbai, 2005 S)*

Solution. (i) If $f(t) = L^{-1}\log\frac{s+1}{s-1}$, then by IV above,

$$t\,f(t) = L^{-1}\left\{-\frac{d}{ds}\log\left(\frac{s+1}{s-1}\right)\right\} = -L^{-1}\left\{\frac{d}{ds}\log(s+1)\right\} + L^{-1}\left\{\frac{d}{ds}\log(s-1)\right\}$$

$$= -L^{-1}\left(\frac{1}{s+1}\right) + L^{-1}\left(\frac{1}{s-1}\right) = -e^{-t}+e^{t} = 2\sinh t$$

Thus $f(t) = (2\sinh t)/t.$

(ii) If $f(t) = L^{-1}\log\frac{s^2+1}{s(s+1)}$, then by IV above,

$$t\,f(t) = L^{-1}\left\{-\frac{d}{ds}\log\left(\frac{s^2+1}{s(s+1)}\right)\right\} = -L^{-1}\left\{\frac{d}{ds}\log(s^2+1)\right\} + L^{-1}\left\{\frac{d}{ds}\log s\right\}$$

$$+ L^{-1}\left\{\frac{d}{ds}\log(s+1)\right\}$$

$$= -L^{-1}\left(\frac{2s}{s^2+1}\right) + L^{-1}\left(\frac{1}{s}\right) + L^{-1}\left(\frac{1}{s+1}\right) = -2\cos t + 1 + e^{-t}$$

Thus $f(t) = \frac{1}{t}(1+e^{-t}-2\cos t).$

(iii) If $f(t) = L^{-1}\cot^{-1}\left(\frac{s}{2}\right)$, then by IV above,

$$t\,f(t) = L^{-1}\left\{-\frac{d}{ds}\cot^{-1}\left(\frac{s}{2}\right)\right\} = L^{-1}\left(\frac{2}{s^2+2^2}\right) = \sin 2t$$

Thus $f(t) = (\sin 2t)/t.$

(iv) If $f(t) = L^{-1}\left(\tan^{-1}\frac{2}{s^2}\right)$, then by IV above,

$$t\,f(t) = L^{-1}\left\{-\frac{d}{ds}\tan^{-1}\left(\frac{2}{s^2}\right)\right\} = L^{-1}\left\{\frac{4s}{s^4+4}\right\}$$

$$= L^{-1}\left\{\frac{4s}{(s^2+2)^2-(2s)^2}\right\} = L^{-1}\left\{\frac{4s}{(s^2+2+2s)(s^2+2-2s)}\right\}$$

$$= L^{-1}\left\{\frac{1}{s^2-2s+2}-\frac{1}{s^2+2s+2}\right\} = L^{-1}\left\{\frac{1}{(s-1)^2+1}-\frac{1}{(s+1)^2+1}\right\}$$

$$= e^t \sin t - e^{-t}\sin t = 2\sinh t \sin t.$$

21.14 CONVOLUTION THEOREM

If $\quad L^{-1}\{\bar{f}(s)\} = f(t)$, *and* $L^{-1}\{\bar{g}(s)\} = g(t)$,

then $\quad L^{-1}\{\bar{f}(s)\,\bar{g}(s)\} = \int_0^t f(u)\, g(t-u)\, du = F * G$

[*$F * G$ is called the* **convolution** *or* **falting** *of F and G.*]

Let $\quad \phi(t) = \int_0^t f(u)\, g(t-u)\, du$

$$L\{\phi(t)\} = \int_0^\infty e^{-st}\left\{\int_0^t f(u)\, g(t-u)\, du\right\} dt = \int_0^\infty \int_0^t e^{-st} f(u)\, g(t-u)\, du\, dt \qquad \text{...(1)}$$

The domain of integration for this double integral is the entire area lying between the lines $u = 0$ and $u = t$ (Fig. 21.2).

On changing the order of integration, we get

$$L\{\phi(t)\} = \int_0^\infty \int_u^\infty e^{-st} f(u)\, g(t-u)\, dt\, du$$

$$= \int_0^\infty e^{-su} f(u)\left\{\int_0^\infty e^{-s(t-u)}\, g(t-u)\, dt\right\} du$$

$$= \int_0^\infty e^{-su} f(u)\left\{\int_0^\infty e^{-sv}\, g(v)\, dv\right\} du \text{ on putting } t-u=v$$

$$= \int_0^\infty e^{-su} f(u)\, \bar{g}(s)\, du = \int_0^\infty e^{-su} f(u)\, du \,.\, \bar{g}(s)$$

$= \bar{f}(s)\,.\,\bar{g}(s)$ whence follows the desired result.

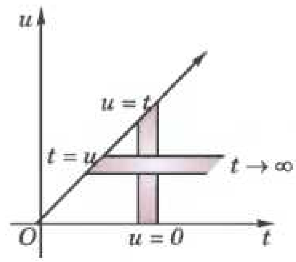


Fig. 21.2

Example 21.25. *Apply Convolution theorem to evaluate*

(*i*) $L^{-1}\dfrac{s}{(s^2+a^2)^2}$. (*V.T.U., 2010*)

(*ii*) $L^{-1}\dfrac{s^2}{(s^2+a^2)(s^2+b^2)}$. (*V.T.U., 2011 S ; Bhopal, 2008 ; Mumbai, 2007*)

Solution. (*i*) Since $f(t) = L^{-1}\left(\dfrac{s}{s^2+a^2}\right) = \cos at$ and $g(t) = L^{-1}\left(\dfrac{s}{s^2+a^2}\right) = \dfrac{1}{a}\sin at$

$\therefore$ by Convolution theorem, we get

$$L^{-1}\left[\frac{s}{s^2+a^2}\cdot\frac{1}{s^2+a^2}\right] = \int_0^t \cos au\, \frac{\sin a(t-u)}{a}\, du \qquad \left[\because \; f(u) = \cos au,\; g(t-u) = \frac{1}{a}\sin a(t-u)\right.$$

$$= \frac{1}{2a}\int_0^t [\sin at - \sin(2au - at)]dt = \frac{1}{2a}\left| u\sin at + \frac{1}{2a}\cos(2au-at)\right|_0^t = \frac{1}{2a}t\sin at$$

Hence $\quad L^{-1}\left\{\dfrac{s}{(s^2+a^2)^2}\right\} = \dfrac{1}{2a}t\sin at.$

(ii) Since $f(t) = L^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos at$ and $g(t) = L^{-1}\left(\frac{s}{s^2+b^2}\right) = \cos bt$,

$\therefore$ by Convolution theorem, we get

$$L^{-1}\left(\frac{s}{s^2+a^2}\cdot\frac{s}{s^2+b^2}\right) = \int_0^t \cos\ au \cos b(t-u)\,du \qquad [\because\ f(u) = \cos au, g(t-u) = \cos b(t-u)]$$

$$= \frac{1}{2}\int_0^t \{\cos\ [(a-b)\,u + bt] + \cos\ [(a+b)\,u - bt]\}\,du$$

$$= \frac{1}{2}\left|\frac{\sin[(a-b)u+bt]}{a-b} + \frac{\sin\ [(a+b)u-bt]}{a+b}\right|_0^t$$

$$= \frac{1}{2}\left\{\frac{\sin at - \sin bt}{a-b} + \frac{\sin at + \sin bt]}{a+b}\right\} = \frac{a\sin at - b\sin bt}{a^2-b^2}.$$

Example 21.26. *Evaluate* (i) $L^{-1}\frac{1}{(s^2+1)(s^2+9)}$ *(Mumbai, 2005 S)*

(ii) $L^{-1}\frac{s}{(s^2+1)(s^2+4)(s^2+9)}$. *(Madras, 2006)*

Solution. (i) Since $L^{-1}\left(\frac{1}{s^2+1}\right) = \sin t$, $L^{-1}\left(\frac{1}{s^2+9}\right) = \frac{\sin 3t}{3}$

$\therefore$ by Convolution theorem, we get

$$L^{-1}\left(\frac{1}{s^2+1}\cdot\frac{1}{s^2+9}\right) = \int_0^t \sin u\,.\,\frac{\sin 3(t-u)}{3}du$$

$$= \frac{1}{6}\int_0^t [\cos\,(4u-3t) - \cos\,(3t-2u)]\,du = \frac{1}{6}\left|\frac{\sin\,(4u-3t)}{4} - \frac{\sin\,(3t-2u)}{-2}\right|_0^t$$

$$= \frac{1}{6}\left\{\frac{1}{4}(\sin t + \sin 3t) + \frac{1}{2}(\sin t - \sin 3t)\right\} = \frac{1}{8}(\sin t - \frac{1}{3}\sin 3t)$$

(ii) Since $L^{-1}\left(\frac{s}{s^2+4}\right) = \cos 2t$ and $L^{-1}\left\{\frac{1}{(s^2+1)(s^2+9)}\right\} = \frac{1}{8}\left[\sin t - \frac{1}{3}\sin 3t\right]$ [By (i)]

$\therefore$ by Convolution theorem, we get

$$L^{-1}\frac{s}{(s^2+1)(s^2+4)(s^2+9)} = L^{-1}\left\{\frac{1}{(s^2+1)(s^2+9)}\cdot\frac{s}{s^2+4}\right\}$$

$$= \int_0^t \frac{1}{8}(\sin u - \frac{1}{3}\sin 3u)\,.\,\cos 2\,(t-u)\,du$$

$$= \frac{1}{8}\int_0^t [\sin u\cos 2(t-u) - \frac{1}{3}\sin 3u\cos 2(t-u)]du$$

$$= \frac{1}{8}\int_0^t \left[\frac{1}{2}\{\sin\,(2t-u) - \sin\,(3u-2t)\} - \frac{1}{6}\{\sin\,(u+2t) - \sin\,(5u-2t)\}\right]du$$

$$= \frac{1}{16}\left[\left|\frac{-\cos\,(2t-u)}{-1} + \frac{\cos\,(3u-2t)}{3}\right|_0^t\right] - \frac{1}{48}\left[\left|-\cos\,(u+2t) + \frac{\cos\,(5u-2t)}{5}\right|_0^t\right]$$

$$= \frac{1}{12}\cos t - \frac{1}{10}\cos 2t + \frac{1}{60}\cos 3t.$$

PROBLEMS 21.5

Find the inverse transforms of :

1. $\dfrac{1}{s^2(s+5)}$. (*Madras, 2003 S*)
2. $\dfrac{1}{s(s+2)^3}$.
3. $\dfrac{s}{a^2s^2+b^2}$. (*Madras, 2000 S*)
4. $\dfrac{1}{s^2(s^2+a^2)}$.
5. $\dfrac{1}{s^3(s^2+1)}$.
6. $\dfrac{s+2}{(s^2+4s+8)^2}$. (*Mumbai, 2006*)
7. $\dfrac{2as}{(s^2+a^2)^2}$.
8. $\dfrac{s^2}{(s+a)^3}$.
9. $\log\left(\dfrac{1+s}{s}\right)$.
10. $\log\left(\dfrac{s+a}{s+b}\right)$. (*Anna, 2003 ; U.P.T.U., 2003*)
11. $\log\left\{\dfrac{s+1}{(s+2)(s+3)}\right\}$.
12. $\dfrac{1}{2}\log\left(\dfrac{s^2+b^2}{s^2+a^2}\right)$. (*Mumbai, 2008 ; V.T.U., 2008*)
13. $\log\left(1-\dfrac{a^2}{s^2}\right)$.
14. $\log\dfrac{s^2+1}{(s-1)^2}$. (*Madras, 2000 S*)
15. $\tan^{-1}\left(\dfrac{2}{s}\right)$ (*Mumbai, 2007 ; P.T.U., 2005*)
16. $\cot^{-1}(s)$. (*V.T.U., 2005*)
17. $s\log\dfrac{s-1}{s+1}$ (*Madras, 1999*)

Using Convolution theorem, evaluate :

18. $L^{-1}\left\{\dfrac{1}{(s+a)(s+b)}\right\}$.
19. $L^{-1}\dfrac{1}{(s^2+a^2)^2}$.
20. $L^{-1}\dfrac{1}{s^2(s^2+a^2)}$.
21. $L^{-1}\left\{\dfrac{1}{s^2(s+1)^2}\right\}$.
22. $L^{-1}\left\{\dfrac{1}{(s-2)(s+2)^2}\right\}$. (*Mumbai, 2009*)
23. $L^{-1}\left\{\dfrac{s}{(s+2)(s^2+9)}\right\}$. (*V.T.U., 2008 S*)
24. $\dfrac{1}{s^3(s^2+1)}$. (*V.T.U., 2007 ; U.P.T.U., 2005*)
25. $\dfrac{1}{(s^2+4s+13)^2}$. (*Mumbai, 2008*)
26. Show that (*i*) $L^{-1}\left(\dfrac{1}{s}\sin\dfrac{1}{s}\right)=t-\dfrac{t^3}{(3\,!)^2}+\dfrac{t^5}{(5\,!)^2}-\dfrac{t^7}{(7\,!)^2}+\ldots$

    (*ii*) $L^{-1}\left(\dfrac{1}{s}\cos\dfrac{1}{s}\right)=1-\dfrac{t^2}{(2\,!)^2}+\dfrac{t^4}{(4\,!)^2}-\dfrac{t^6}{(6\,!)^2}+\ldots$

21.15 (1) APPLICATION TO DIFFERENTIAL EQUATIONS

The Laplace transform method of solving differential equations yields particular solutions without the necessity of first finding the general solution and then evaluating the arbitrary constants. This method is, in general, shorter than our earlier methods and is specially useful for solving linear differential equations with constant coefficients.

(2) Working procedure *to solve a linear differential equation with constant coefficients by transform method :*

1. Take the Laplace transform of both sides of the differential equation using the formula of § 21.7, and the given initial conditions.

2. Transpose the terms with minus signs to the right.

3. Divide by the coefficient of $\bar{y}$, getting $\bar{y}$ as a known function of s.

4. Resolve this function of s into partial fractions and take the inverse transform of both sides. **This gives y as a function of t which is the desired solution satisfying the given conditions.**

Example 21.27. *Solve by the method of transforms, the equation*

$y''' + 2y'' - y' - 2y = 0$ *given* $y(0) = y'(0) = 0$ *and* $y''(0) = 6$. (V.T.U., 2011 S)

Solution. Taking the Laplace transform of both sides, we get

$$[s^3\bar{y} - s^2y(0) - sy'(0) - y''(0)] + 2[s^2\bar{y} - sy(0) - y'(0)] - [s\bar{y} - y(0)] - 2\bar{y} = 0$$

Using the given conditions, it reduces to

$$(s^3 + 2s^2 - s - 2)\bar{y} = 6$$

$$\therefore \qquad \bar{y} = \frac{6}{(s-1)(s+1)(s+2)} = \frac{6}{(s-1)(6)} + \frac{6}{(-2)(s+1)} + \frac{6}{3(s+2)}$$

On inversion, we get $\quad y = L^{-1}\frac{1}{(s-1)} - 3L^{-1}\frac{1}{(s+2)} + 2L^{-1}\left(\frac{1}{s+2}\right)$

or $\quad y = e^t - 3e^{-t} + 2e^{-2t}$ which is the desired result.

Example 21.28. *Use transform method to solve*

$$\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + x = e^t \text{ with } x = 2, \frac{dx}{dt} = -1 \text{ at } t = 0.$$ (Anna, 2005 S)

Solution. Taking the Laplace transforms of both sides, we get

$$[s^2\bar{x} - sx(0) - x'(0)] - 2\{s\bar{x} - x(0)\} + \bar{x} = \frac{1}{s-1}$$

Using the given conditions, it reduces to

$$(s^2 - 2s + 1)\bar{x} = \frac{1}{s-1} + 2s - 5 = \frac{2s^2 - 7s + 6}{s-1}$$

$$\therefore \qquad \bar{x} = \frac{2s^2 - 7s + 6}{(s-1)^3} = \frac{2}{s-1} - \frac{3}{(s-1)^2} + \frac{1}{(s-1)^3}$$ on breaking into partial fractions.

On inversion, we obtain $x = 2L^{-1}\left(\frac{1}{s-1}\right) - 3L^{-1}\frac{1}{(s-1)^2} + L^{-1}\frac{1}{(s-1)^3}$

$$= 2e^t - \frac{3e^t \cdot t}{1!} + \frac{e^t \cdot t^2}{2!} = 2e^t - 3te^t + \tfrac{1}{2}t^2e^t.$$

Example 21.29. *Solve* $(D^2 + n^2)x = a\sin(nt + \alpha)$, $x = Dx = 0$ *at* $t = 0$.

Solution. Taking the Laplace transforms of both sides, we get

$$[s^2\bar{x} - sx(0) - x'(0)] + n^2\bar{x} = aL\{\sin nt \cdot \cos\alpha + \cos nt \cdot \sin\alpha\}$$

On using the given conditions,

$$(s^2 + n^2)\bar{x} = a\cos\alpha \cdot \frac{n}{s^2 + n^2} + a\sin\alpha \cdot \frac{s}{s^2 + n^2}$$

$$\therefore \qquad \bar{x} = an\cos\alpha \cdot \frac{1}{(s^2 + n^2)^2} + a\sin\alpha \cdot \frac{s}{(s^2 + n^2)^2}$$

On inversion, we obtain

$$x = an\cos\alpha \cdot \frac{1}{2n^3}(\sin nt - nt\cos nt) + a\sin\alpha \cdot \frac{t}{2n}\sin nt$$ [By (11) and (12) p. 741]

$$= a\{\sin nt\cos\alpha - nt\cos(nt + \alpha)\}/2n^2.$$

Example 21.30. *Solve* $(D^3 - 3D^2 + 3D - 1)y = t^2e^t$ *given that* $y(0) = 1$, $y'(0) = 0$, $y''(0) = -2$.

(S.V.T.U., 2009)

Solution. Taking the Laplace transforms of both sides, we get

$$[s^3 \bar{y} - s^2 y(0) - sy'(0) - y''(0)] - 3[s^2 \bar{y} - sy(0) - y'(0)] + 3[s\bar{y} - y(0)] - \bar{y} = \frac{2}{(s-1)^3}$$

Using the given conditions, it reduces to

$$\bar{y} = \frac{s^2 - 3s + 1}{(s-1)^3} + \frac{2}{(s-1)^6} = \frac{(s-1)^2 - (s-1) - 1}{(s-1)^3} + \frac{2}{(s-1)^6}$$

$$= \frac{1}{s-1} - \frac{1}{(s-1)^2} - \frac{1}{(s-1)^3} + \frac{2}{(s-1)^6}$$

On inversion, we obtain $y = L^{-1}\left(\frac{1}{s-1}\right) - L^{-1}\frac{1}{(s-1)^2} - L^{-1}\frac{1}{(s-1)^3} + 2L^{-1}\frac{1}{(s-1)^6}$

$$= e^t\left(1 - t - \frac{1}{2}t^2 + \frac{1}{60}t^5\right).$$

Example 21.31. *Solve* $\frac{d^2x}{dt^2} + 9x = \cos 2t$, *if* $x(0) = 1$, $x(\pi/2) = -1$. *(Bhopal, 2008 ; U.P.T.U., 2006)*

Solution. Since $x'(0)$ is not given, we assume $x'(0) = a$.

Taking the Laplace transforms of both sides of the equation, we have

$$L(x'') + 9L(x) = L(\cos 2t) \text{ i.e., } [s^2\bar{x} - s\,x(0) - x'(0)] + 9\bar{x} = \frac{s}{s^2+4}$$

$$(s^2+9)\,\bar{x} = s + a + \frac{s}{s^2+4} \quad \text{or} \quad \bar{x} = \frac{s+a}{s^2+9} + \frac{s}{(s^2+4)(s^2+9)}$$

or $$\bar{x} = \frac{a}{s^2+9} + \frac{1}{5}\cdot\frac{s}{s^2+4} + \frac{4}{5}\cdot\frac{s}{s^2+9}.$$

On inversion, we get $$x = \frac{a}{3}\sin 3t + \frac{1}{5}\cos 2t + \frac{4}{5}\cos 3t$$

When $t = \pi/2$, $-1 = -\frac{a}{3} - \frac{1}{5}$ or $\frac{a}{3} = \frac{4}{5}$ $\left[\because\ x\left(\frac{\pi}{2}\right) = -1\right]$

Hence the solution is $x = \frac{1}{5}(\cos 2t + 4\sin 3t + 4\cos 3t)$.

Obs. *Laplace transform method can also be used for solving ordinary differential equations with variable coefficients of the form* $t^m y^{(n)}(t)$ *because* $L[t^m y^{(n)}(t)] = (-1)^m \frac{d^m}{ds^m}[L\, y^{(n)}(t)]$.

Example 21.32. *Solve* $ty'' + 2y' + ty = \cos t$ *given that* $y(0) = 1$. *(S.V.T.U., 2009)*

Solution. Taking Laplace transform of both sides of the equation and noting that

$$L[t\{f(t)\}] = -\frac{d}{ds}[L\{f(t)\}], \text{ we get}$$

$$-\frac{d}{ds}[s^2\bar{y} - sy(0) - y'(0)] + 2[s\bar{y} - y(0)] - \frac{d}{ds}(\bar{y}) = \frac{s}{s^2+1}$$

or $$-\left(s^2\frac{d\bar{y}}{ds} + 2s\bar{y}\right) + y(0) + 0 + 2s\bar{y} - 2y(0) - \frac{d}{ds}(\bar{y}) = \frac{s}{s^2+1}$$

or $$(s^2+1)\frac{d\bar{y}}{ds} + 1 = -\frac{s}{s^2+1} \quad \text{or} \quad \frac{d\bar{y}}{ds} = -\frac{1}{s^2+1} - \frac{s}{(s^2+1)^2}$$

On inversion and noting that $L^{-1}\{\bar{f}'(s)\} = -t\,f(t)$, we get

$$-ty = -\sin t - \tfrac{1}{2} t \sin t \qquad \text{[See § 21.12 (11)]}$$

or $$y = \frac{1}{2}\left(1 + \frac{2}{t}\right)\sin t \qquad \text{which is the desired solution.}$$

Example 21.33. *Solve* $x\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + xy = 0,\ y(0) = 2,\ y'(0) = 0.$

Solution. Taking Laplace transform of both sides of the equation, we get

$$L(xy'') + L(y') + L(xy) = 0$$

or $$-\frac{d}{ds}[s^2\bar{y} - sy(0) - y'(0)] + [s\bar{y} - y(0)] - \frac{d\bar{y}}{ds} = 0 \quad \text{or} \quad (s^2+1)\frac{d\bar{y}}{ds} + s\bar{y} = 0$$

Separating the variables, $\displaystyle\int \frac{d\bar{y}}{\bar{y}} + \int \frac{s\,ds}{s^2+1} = c$

or $$\log \bar{y} + \tfrac{1}{2}\log(s^2+1) = \log c' \quad \text{or} \quad \bar{y} = \frac{c'}{\sqrt{(s^2+1)}}$$

Inversion gives $y = c' J_0(x)$

To find c', we have $y(0) = c' J_0(0)$, *i.e.*, $c' = 2$

Hence $y = 2J_0(x)$.

Example 21.34. *An alternating e.m.f. E sin ωt is applied to an inductance L and a capacitance C in series. Show by transform method, that the current in the circuit is* $\dfrac{E\omega}{(p^2-\omega^2)L}(\cos \omega t - \cos pt)$, *where* $p^2 = 1/LC$.

Solution. If i be a current and q the charge at time t in the circuit, then its differential equation is

$$L\frac{di}{dt} + \frac{q}{c} = E\sin\omega t \qquad [\because\ R = 0]$$

Taking Laplace transform of both sides, we get

$$L[s\bar{i}(s) - i(0)] + \frac{1}{C}L(q) = E\cdot\frac{\omega}{s^2+\omega^2}$$

Since $i = 0$ and $q = 0$ at $t = 0$

$$\therefore \quad Ls\bar{i}(s) + \frac{1}{C}L(q) = \frac{E\omega}{s^2+\omega^2} \qquad ...(i)$$

Also taking Laplace transform of $i = dq/dt$, we get

$$\bar{i}(s) = L(dq/dt) = s\,L(q) - q(0)$$

i.e. $$L(q) = \bar{i}(s)/s \qquad [\because\ q(0) = 0]$$

$\therefore$ (*i*) becomes $Ls\bar{i}(s) + \dfrac{1}{C}[\bar{i}(s)/s] = \dfrac{E\omega}{s^2+\omega^2}$

or $$\left(Ls + \frac{1}{Cs}\right)\bar{i}(s) = \frac{E\omega}{s+\omega^2} \quad \text{or} \quad \bar{i}(s) = \frac{E\omega s}{L(s^2+1/LC)(s^2+\omega^2)}$$

or $$\bar{i}(s) = \frac{E\omega}{L}\cdot\frac{s}{(s^2+p^2)(s^2+\omega^2)} \quad \text{where } p^2 = 1/LC$$

$$\bar{i}(s) = \frac{E\omega}{L(p^2-\omega^2)}\left\{\frac{s}{s^2+\omega^2} - \frac{s}{s^2+p^2}\right\}$$

Now taking inverse Laplace transform of both sides, we get

$$i(t) = \frac{E\omega}{L(p^2 - \omega^2)} L^{-1}\left\{\frac{s}{s^2 + \omega^2} - \frac{s}{(s^2 + p^2)}\right\}$$

or
$$i(t) = \frac{E\omega}{L(p^2 - \omega^2)} (\cos \omega t - \cos pt).$$

PROBLEMS 21.6

Solve the following equations by the transform method :

1. $y'' + 4y' + 3y = e^{-t}, y(0) = y'(0) = 1.$ (*V.T.U., 2008 S ; Kurukshetra, 2005*)
2. $(D^2 - 1)x = a \cosh t, x(0) = x'(0) = 0.$
3. $y'' + y = t, y(0) = 1, y'(0) = 0.$ (*Mumbai, 2009*)
4. $y'' - 3y' + 2y = e^{3t}$, when $y(0) = 1$ and $y'(0) = 0$. (*V.T.U., 2010*)
5. $(D^2 - 3D + 2)y = 4e^{2t}$ with $y(0) = -3, y(0) = 5$. (*Mumbai, 2008*)
6. $y'' + 25y = 10 \cos 5t$ given that $y(0) = 2, y'(0) = 0$. (*S.V.T.U., 2008*)
7. $(D^2 + \omega^2)y = \cos \omega t, t > 0$, given that $y = 0$ and $Dy = 0$ at $t = 0$.
8. $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} - 3y = \sin t, y = \frac{dy}{dt} = 0$ when $t = 0$. (*Kurukshetra, 2005 ; Madras, 2003*)
9. $\frac{d^4y}{dt^4} - k^4 y = 0$, where $y(0) = 1, y'(0) = y''(0) = y'''(0) = 0$.
10. $y''''(t) + 2y''(t) + y(t) = \sin t$, when $y(0) = y'(0) = y''(0) = y'''(0) = 0$.
11. $\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 5y = e^{-t} \sin t$, where $y(0) = 0$ and $y'(0) = 1$. (*P.T.U., 2010*)
12. $y'' + 2y' + 5y = 5y = 5(t - 2), y(0) = 0, y'(0) = 0.$ (*P.T.U., 2005 S*)
13. $\frac{d^3y}{dt^3} - 3\frac{d^2y}{dt^2} + 3\frac{dy}{dt} - y = t^2 e^{2t}$, where $y = 1, \frac{dy}{dt} = 0, \frac{d^2y}{dt^2} = -2$ at $t = 0$.
14. $(D^2 + 1)x = t \cos 2t, x = Dx = 0$ at $t = 0$. (*Raipur, 2005 ; U.P.T.U., 2005*)
15. $ty'' + 2y' + ty = \sin t$, when $y(0) = 1$.
16. $ty'' + (1 - 2t)y' - 2y = 0$, when $y(0) = 1, y'(0) = 2$. (*P.T.U., 2002*)
17. $y'' + 2ty' - y = t$, when $y(0) = 0, y'(0) = 1$. (*U.P.T.U., 2003*)
18. $ty'' + y' + 4ty = 0$ when $y(0) = 3, y'(0) = 0$.
19. A voltage Ee^{-at} is applied at $t = 0$ to a circuit of inductance L and resistance R. Show (by the transform method) that the current at time t is $\frac{E}{R - aL}(e^{-at} - e^{-Rt/L})$. (*V.T.U., 2000*)
20. Workout example 12.17, p. 465 by the transform method.
21. Obtain the equation for the forced oscillation of a mass m attached to the lower end of an elastic spring whose upper end is fixed and whose stiffness is k, when the driving force is $F_0 \sin at$. Solve this equation (using the Laplace transforms) when $a^2 \neq k/m$, given that initial velocity and displacement (from equilibrium position) are zero.

   [**Hint :** The equation of motion is $\frac{d^2x}{dt^2} + \frac{k}{m}x = \frac{F_0}{m} \sin at$ and $x = \frac{dx}{dt} = 0$ when $t = 0$.]

21.16 SIMULTANEOUS LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS

The Laplace transform method can also be applied with advantage to the solution of simultaneous linear differential equations.

Example 21.35. *Solve the simultaneous equations* $\frac{dx}{dt} + 5x - 2y = t, \frac{dy}{dt} + 2x + y = 0$ *being given* $x = y = 0$ *when* $t = 0$. [Ex. 13.38]

Solution. Taking the Laplace transforms of the given equations, we get

$$[s\bar{x} - x(0)] + 5\bar{x} - 2\bar{y} = 1/s^2 \quad i.e., \quad (s+5)\bar{x} - 2\bar{y} = 1/s^2 \qquad ...(i)\ [\because\ x(0) = 0]$$

and $$s\bar{y} - y(0) + 2\bar{x} + \bar{y} = 0 \quad i.e., \quad 2\bar{x} + (s+1)\bar{y} = 0 \qquad ...(ii)\ [\because\ y(0) = 0]$$

Solving (*i*) and (*ii*) for $\bar{x}$, we get

$$\bar{x} = \begin{vmatrix} 1/s^2 & -2 \\ 0 & s+1 \end{vmatrix} \div \begin{vmatrix} s+5 & -2 \\ 2 & s+1 \end{vmatrix} = \frac{s+1}{s^2(s+3)^2} = \frac{1}{27s} + \frac{1}{9s^2} - \frac{1}{27(s+3)} - \frac{2}{9(s+3)^2}$$

Substituting the value of $\bar{x}$ in (*ii*), we get

$$\bar{y} = -\frac{2}{s^2(s+3)^2} = \frac{4}{27s} - \frac{2}{9s^2} - \frac{4}{27(s+3)} - \frac{2}{9(s+3)^2}$$

On inversion, we get

$$x = \frac{1}{27} + \frac{t}{9} - \frac{1}{27}e^{-3t} - \frac{2}{9}te^{-3t}, \quad y = \frac{4}{27} - \frac{2t}{9} - \frac{4}{27}e^{-3t} - \frac{2}{9}te^{-3t}.$$

Example 21.36. *The coordinates (x, y) of a particle moving along a plane curve at any time t, are given by $dy/dt + 2x = \sin 2t$, $dx/dt - 2y = \cos 2t$, $(t > 0)$. If at $t = 0$, $x = 1$ and $y = 0$, show by transforms, that the particle moves along the curve $4x^2 + 4xy + 5y^2 = 4$.* (U.P.T.U., 2003)

Solution. Taking the Laplace transforms of the given equations and noting that $y(0) = 0$, $x(0) = 1$, we get

$$[s\bar{y} - y(0)] + 2\bar{x} = \frac{2}{s^2 + 2^2} \quad \text{or} \quad 2\bar{x} + s\bar{y} = \frac{2}{s^2+4} \qquad ...(i)$$

and $$[s\bar{x} - x(0)] - 2\bar{y} = \frac{s}{s^2 + 2^2} \quad \text{or} \quad s\bar{x} - 2\bar{y} = \frac{s}{s^2+4} + 1 \qquad ...(ii)$$

Multiplying (*i*) by s and (*ii*) by 2 and subtracting, we get

$$(s^2 + 4)\bar{y} = -2 \quad \text{or} \quad \bar{y} = -2/(s^2+4)$$

On inversion, $$y = -2L^{-1}\left[\frac{1}{s^2+4}\right] = -\sin 2t$$

From the given first equation,

$$2x = \sin 2t - dy/dt = \sin 2t - \frac{d}{dt}(-\sin 2t)$$

or $$2x = \sin 2t + 2\cos 2t \quad \text{or} \quad 4x^2 = (\sin 2t + 2\cos 2t)^2 \qquad ...(iii)$$

Also $$4xy = (\sin 2t + 2\cos 2t)(-2\sin 2t) = -2(\sin^2 2t + 2\sin 2t \cos 2t) \qquad ...(iv)$$

and $$5y^2 = 5\sin^2 2t. \qquad ...(v)$$

Adding (*iii*), (*iv*), and (*v*), we obtain

$$4x^2 + 4xy + 5y^2 = \sin^2 2t + 4\sin 2t\cos 2t + 4\cos^2 2t - 2\sin^2 2t$$
$$- 4\sin 2t \cos 2t + 5\sin^2 2t = 4\sin^2 2t + 4\cos^2 2t = 4.$$

Example 21.37. *The small oscillations of a certain system with two degrees of freedom are given by the equations : $D^2x + 3x - 2y = 0$, $D^2x + D^2y - 3x + 5y = 0$ where $D = d/dt$. If $x = 0$, $y = 0$, $x = 3$, $y = 2$ when $t = 0$, find x and y when $t = 1/2$.* [Example 13.41]

Solution. Taking the Laplace transform of both the equations, we get

$$[s^2\bar{x} - sx(0) - x'(0)] + 3\bar{x} - 2\bar{y} = 0 \quad i.e., \quad (s^2+3)\bar{x} - 2\bar{y} = 3 \qquad ...(i)$$

and $$[s^2\bar{x} - sx(0) - x'(0)] + [s^2\bar{y} - sy(0) - y'(0)] - 3\bar{x} + 5\bar{y} = 0 \ i.e., (s^2-3)\bar{x} + (s^2+5)\bar{y} = 5 \qquad ...(ii)$$

Solving (*i*) and (*ii*) for $\bar{x}$ and $\bar{y}$, we get

$$\bar{x} = \begin{vmatrix} 3 & -2 \\ 5 & s^2+5 \end{vmatrix} \div \begin{vmatrix} s^2+3 & -2 \\ s^2-3 & s^2+5 \end{vmatrix} = \frac{3s^2+25}{(s^2+1)(s^2+9)}$$

$$= \frac{11}{4}\cdot\frac{1}{s^2+1} + \frac{1}{4}\cdot\frac{1}{s^2+9}$$

and $$\bar{y} = \begin{vmatrix} s^2+3 & 3 \\ s^2-3 & 5 \end{vmatrix} \div \begin{vmatrix} s^2+3 & -2 \\ s^2-3 & s^2+5 \end{vmatrix} = \frac{2s^2+24}{(s^2+1)(s^2+9)} = \frac{11}{4} \cdot \frac{1}{s^2+1} + \frac{3}{4} \cdot \frac{1}{s^2+9}.$$

On inversion, we get $x = \frac{11}{4} \sin t + \frac{1}{12} \sin 3t$; $y = \frac{11}{4} \sin t - \frac{1}{4} \sin 3t$
which are the same as the solution in (*vii*) on p. 499.

Obs. The student should compare the earlier solutions of the above examples with those given now and appreciate the superiority of the transform method over others.

PROBLEMS 21.7

Solve the following simultaneous equations (by using Laplace transforms) :

1. $\frac{dx}{dt} - y = e^t$, $\frac{dy}{dt} + x = \sin t$, given $x(0) = 1, y(0) = 0$. (*U.P.T.U., 2006 ; Delhi, 2002*)
2. $\frac{dx}{dt} + y = \sin t$, $\frac{dy}{dt} + x = \cos t$, given that $x = 2$ and $y = 0$ when $t = 0$. (*Kerala, 2005 ; U.P.T.U., 2004*)
3. $\frac{d^2x}{dt^2} - x = y$, $\frac{d^2y}{dt^2} + y = -x$, given that at $t = 0$; $x = 2, y = -1$, $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} = 0$. (*P.T.U., 2009 S*)
4. $3\frac{dx}{dt} + \frac{dy}{dt} + 2x = 1$, $\frac{dx}{dt} + 4\frac{dy}{dt} + 3y = 0$; given $x = 0, y = 0$ when $t = 0$. (*Madras, 2003 S*)
5. $(D-2)x - (D+1)y = 6e^{3t}$; $(2D-3)x + (D-3)y = 6e^{3t}$ given $x = 3, y = 0$ when $t = 0$.
6. The currents i_1 and i_2 in mesh are given by the differential equations ; $di_1/dt - \omega i_2 = a \cos pt$, $di_2/dt + \omega i_1 = a \sin pt$. Find the currents i_1 and i_2 by Laplace transform, if $i_1 = i_2 = 0$ at $t = 0$.

21.17 (1) UNIT STEP FUNCTION

At times, we come across such fractions of which the inverse transform cannot be determined from the formulae so far derived. In order to cover such cases, we introduce the *unit step function* (or *Heaviside's unit function**).

Def. *The unit step function $u(t-a)$ is defined as follows :*

$$u(t-a) = \begin{cases} 0 \text{ for } t < a \\ 1 \text{ for } t \geq a \end{cases}$$

where, a is always positive (Fig. 21.3). It is also denoted as $H(t-a)$.

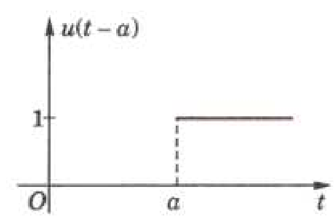


Fig. 21.3

(2) Transform of unit function.

$$L\{u(t-a)\} = \int_0^\infty e^{-st} u(t-a)\,dt = \int_0^a e^{-st} . 0\,dt + \int_a^\infty e^{-st} . 1\,dt = 0 + \left| \frac{e^{-st}}{-s} \right|_a^\infty$$

Thus $L\{u(t-a)\} = e^{-as}/s$.

The product $f(t)\,u(t-a) = \begin{cases} 0 & \text{for } t < a \\ f(t) & \text{for } t \geq a. \end{cases}$

The function $f(t-a) . u(t-a)$ represents the graph of $f(t)$ shifted through a distance a to the right and is of special importance.

Second shifting property. If $L\{f(t)\} = \bar{f}(s)$, *then*

$$\mathbf{L\{f(t-a). u(t-a)\} = e^{-as}\,\bar{f}(s)}$$

$$L\{f(t-a) . u(t-a)\} = \int_0^\infty e^{-st} f(t-a)\, u(t-a)\,dt$$

*Named after the British Electrical Engineer Oliver Heaviside (1850–1925).

$$= \int_0^a e^{-st} f(t-a)(0)\,dt + \int_a^\infty e^{-st} f(t-a)\,dt \qquad [\text{Put } t-a=u]$$

$$= \int_0^\infty e^{-s(u+a)} f(u)\,du = e^{-sa}\int_0^\infty e^{-su} f(u)\,du = e^{-as}\bar{f}(s).$$

Example 21.38. *Express the following function (Fig. 21.4) in terms of unit step function and find its Laplace transform.* *(U.P.T.U., 2002)*

Solution. We have $f(t) = \begin{cases} 0, & 0 < t < 1 \\ t-1, & 1 < t < 2 \\ 1, & t > 2 \end{cases}$

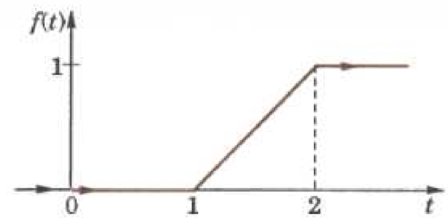


Fig. 21.4

or $$f(t) = (t-1)[u(t-1) - u(t-2)] + u(t-2)$$
$$= (t-1)\,u(t-1) - (t-2)\,u(t-2)$$

By second shifting property,

$$L[f(t-a)\,u(t-a)] = e^{-as} L[f(t)].$$

Also $L[f(t)] = L(t) = 1/s^2$.

$\therefore$ $L[(t-1)\,u(t-1)]$

$$= e^{-s}\cdot\frac{1}{s^2} \text{ and } L[(t-2)\,u(t-2)] = e^{-2s}\cdot\frac{1}{s^2}$$

Hence $$L[f(t)] = L[(t-1)\,u(t-1) - (t-2)\,u(t-2)] = \frac{e^{-s} - e^{-2s}}{s^2}.$$

Example 21.39. *Using unit step function, find the Laplace transform of*

$$f(t) = \begin{cases} \sin t, & 0 \le t < \pi \\ \sin 2t, & \pi \le t < 2\pi. \\ \sin 3t, & t \ge 2\pi \end{cases}$$ *(V.T.U., 2004)*

Solution. $$f(t) = \sin t\,[u(t-0) - u(t-\pi)] + \sin 2t\,[u(t-\pi) - u(t-2\pi)] + \sin 3t\,.\,u(t-2\pi)$$
$$= \sin t + (\sin 2t - \sin t)\,u(t-\pi) + (\sin 3t - \sin 2t)\,u(t-2\pi)$$

Since $L[f(t-a)\,u(t-a)] = e^{-as}\bar{f}(s)$ and $L(\sin at) = \dfrac{a}{s^2+a^2}$,

$$L[f(t)] = L(\sin t) + L[(\sin 2t - \sin t)\,.\,u(t-\pi)] + L[(\sin 3t - \sin 2t)\,.\,u(t-2\pi)]$$

$$= \frac{1}{s^2+1} + e^{-\pi s}\left(\frac{2}{s^2+4} - \frac{1}{s^2+1}\right) + e^{-2\pi s}\left(\frac{3}{s^2+9} - \frac{2}{s^2+4}\right).$$

Example 21.40. (*i*) *Express the function (Fig. 21.5) in terms of unit step function and find its Laplace transform.* *(P.T.U., 2005 S)*

(*ii*) *Obtain the Laplace transform of $e^{-t}[1 - u(t-2)]$.*

Solution. (*i*) We have $f(t) = \begin{cases} t-1, & 1 < t < 2 \\ 3-t, & 2 < t < 3. \end{cases}$

or $$f(t) = (t-1)\{u(t-1) - u(t-2)\} + (3-t)\{u(t-2) - u(t-3)\}$$
$$= (t-1)\,u(t-1) - 2(t-2)\,u(t-2) + (t-3)\,u(t-3)$$

Since $L\{f(t-a)\,u(t-a)\} = e^{-as}\bar{f}(s)$ ...(λ)

$\therefore$ $$L[f(t)] = e^{-s}\cdot\frac{1}{s^2} - 2e^{-2s}\cdot\frac{1}{s^2} + e^{-3s}\cdot\frac{1}{s^2} = \frac{e^{-s}(1-e^{-s})^2}{s^2} \qquad [\because\ f(t) = t]$$

(*ii*) $$L\{e^{-t}[1 - u(t-2)]\} = L(e^{-t}) - L\{e^{-t}\,u(t-2)\} = \frac{1}{s+1} - e^{-2} L\{e^{-(t-2)}\,u(t-2)\}$$

Taking $f(t) = e^{-t}$, $\bar{f}(s) = \dfrac{1}{s+1}$ and using (λ) above,

$$L\{e^{-(t-2)} u(t-2)\} = e^{-2s} \cdot \frac{1}{s+1}$$

Hence $Le^{-t}\{1 - u(t-2)\} = \{1 - e^{-2(s+1)}\}/(s+1)$.

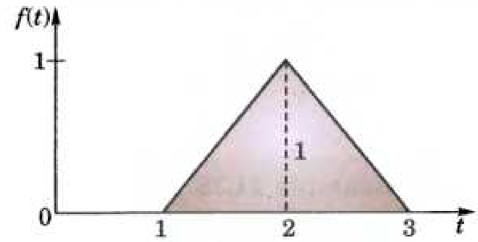


Fig. 21.4

Example 21.41. *Using Laplace transform, evaluate* $\int_0^\infty e^{-t}(1 + 2t - t^2 + t^3) H(t-1)\, dt.$ *(Mumbai, 2007)*

Solution. We have $L\{(1 + 2t - t^2 + t^3) H(t-1)\}$

$$= e^{-s} L[1 + 2(t+1) - (t+1)^2 + (t+1)^3] = e^{-s} L(3 + 3t + 2t^2 + t^3)$$

$$= e^{-s}\left(3 \cdot \frac{1}{s} + 3 \cdot \frac{1}{s^2} + 2 \cdot \frac{2!}{s^3} + \frac{3!}{s^4}\right) = e^{-s}\left(\frac{3}{s} + \frac{3}{s^2} + \frac{4}{s^3} + \frac{6}{s^4}\right)$$

By definition, this implies that

$$\int_0^\infty e^{-st}(1 + 2t - t^2 + t^3) H(t-1)\, dt = e^{-s}\left(\frac{3}{s} + \frac{3}{s^2} + \frac{4}{s^3} + \frac{6}{s^4}\right)$$

Taking $s = 1$, we obtain

$$\int_0^\infty e^{-t}(1 + 2t - t^2 + t^3) H(t-1)\, dt = e^{-1}(3 + 3 + 4 + 6) = 16/e.$$

Example 21.42. *Evaluate (i)* $L^{-1}\left\{\dfrac{e^{-s} - 3e^{-3s}}{s^2}\right\}$ *(U.P.T.U., 2002)*

(ii) $L^{-1}\left\{\dfrac{se^{-as}}{s^2 - w^2}\right\}, a > 0.$

Solution.
$$L^{-1}\left\{e^{-s} \cdot \frac{1}{s^2}\right\} = \begin{cases} t-1, & t > 1 \\ 0, & t < 1 \end{cases} = (t-1)\, u(t-1)$$

$$L^{-1}\left\{e^{-3s} \cdot \frac{1}{s^2}\right\} = \begin{cases} t-3, & t > 3 \\ 0, & t < 3 \end{cases} = (t-3)\, u(t-3)$$

$$\therefore \quad L^{-1}\left\{\frac{e^{-s} - 3e^{-3s}}{s^2}\right\} = L^{-1}\left(\frac{e^{-s}}{s^2}\right) - 3L^{-1}\left(\frac{e^{-3s}}{s^2}\right) = (t-1)\, u(t-1) - 3(t-3)\, u(t-3)$$

(*ii*) We know that $L^{-1}\left(\dfrac{s}{s^2 - w^2}\right) = \cosh wt$

$$\therefore \quad L^{-1}\left(\frac{se^{-as}}{s^2 - w^2}\right) = \begin{cases} \cosh w(t-a), & t > a \\ 0, & t < a \end{cases}$$

$= \cosh w(t-a)\, u(t-a)$, by second shifting property.

Example 21.43. *Find the inverse Laplace transform of :*

(i) $\dfrac{se^{-s/2} + \pi e^{-s}}{s^2 + \pi^2}$ *(V.T.U., 2000)* *(ii)* $\dfrac{e^{-cs}}{s^2(s+a)}$ $(c > 0)$. *(Kurukshetra, 2005)*

Solution. (*i*) Since $L^{-1}\dfrac{s}{s^2 + \pi^2} = \cos \pi t$, $L^{-1}\left(\dfrac{\pi}{s^2 + \pi^2}\right) = \sin \pi t$

and $$L^{-1}[e^{-as}\bar{f}(s)] = f(t-a)\,.\,u(t-a) \qquad \text{...}(\lambda)$$

$$\therefore \quad L^{-1}\left\{\frac{se^{-s/2}+\pi e^{-s}}{s^2+\pi^2}\right\} = L^{-1}\left\{e^{-s/2}\,.\,\frac{s}{s^2+\pi^2}\right\} + L^{-1}\left\{e^{-s}\,.\,\frac{\pi}{s^2+\pi^2}\right\}$$

$$= \cos\pi(t-1/2)\,.\,u(t-1/2) + \sin\pi(t-1)\,.\,u(t-1)$$

$$= \sin\pi t\,.\,u(t-1/2) - \sin\pi t\,.\,u(t-1) = \{u(t-1/2)-u(t-1)\}\sin\pi t$$

(*ii*) $$L^{-1}\left\{\frac{e^{-cs}}{s^2(s+a)}\right\} = L^{-1}\left\{e^{-cs}\left(-\frac{1}{a^2}\cdot\frac{1}{s}+\frac{1}{a}\cdot\frac{1}{s^2}+\frac{1}{a^2}\cdot\frac{1}{s+a}\right)\right\}$$

Using (λ) above, we have

$$L^{-1}\left\{\frac{e^{-cs}}{s^2(s+a)}\right\} = -\frac{1}{a^2}\{1\,.\,u(t-c)\} + \frac{1}{a}\{(t-c)\,.\,u(t-c)\} + \frac{1}{a^2}\{e^{-a(t-c)}\,.\,u(t-c)\}$$

$$= \frac{1}{a^2}\{a(t-c)-1+e^{-a(t-c)}\}\,u(t-c).$$

Example 21.44. *A particle of mass m can oscillate about the position of equilibrium under the effect of a restoring force mk^2 times the displacement. It started from rest by a constant force F which acts for time T and then ceases. Find the amplitude of the subsequent oscillation.*

Solution. The constant force F acting from $t = 0$ to $t = T$ can be expressed as

$$F[1-u(t-T)],\ 0 < t < T$$

$\therefore$ equation of motion of the particle is

$$m\frac{d^2x}{dt^2} = F[1-u(t-T)] - mk^2x \quad \text{or} \quad \frac{d^2x}{dt^2} + k^2x = \frac{F}{m}[1-u(t-T)]$$

Taking Laplace transform of both sides, we get

$$(s^2+k^2)\,\bar{x} = \frac{F}{ms}(1-e^{-sT}) \qquad [\because\ x=0,\ \dot{x}=0 \text{ at } t=0]$$

or $$\bar{x} = \frac{F}{m}\cdot\frac{1-e^{-sT}}{s(s^2+k^2)} = \frac{F}{m}(1-e^{-sT})\cdot\frac{1}{k^2}\left(\frac{1}{s}-\frac{s}{s^2+k^2}\right)$$

$$= \frac{F}{mk^2}\left\{(1-e^{-sT})\frac{1}{s} - (1-e^{-sT})\cdot\frac{s}{s^2+k^2}\right\}$$

Taking inverse Laplace transform, we obtain

$$x = \frac{F}{mk^2}[(1-\cos kt) - \{1-\cos k(t-T)\}]\,u(t-T)$$

i.e., $$x = \frac{F}{mk^2}(1-\cos kt) \text{ for } 0 < t < T$$

and $$x = \frac{F}{mk^2}(1-\cos kt) - \{1-\cos k(t-T)\} \text{ for } t > T$$

$$= \frac{F}{mk^2}\{\cos k(t-T) - \cos kT\} \text{ for } t > T$$

or $$x = \frac{2F}{mk^2}\sin\frac{kT}{2}\,.\,\sin k(t-T/2) \text{ for } t > T$$

Hence the amplitude of subsequent oscillation (*i.e.*, for $t > T$) $= \dfrac{2F}{mk^2}\sin\dfrac{kT}{2}$.

Example 21.45. *In an electrical circuit with e.m.f. E(t), resistance R and inductance L, the current i builds up at the rate given by*

$$L\,di/dt + Ri = E(t). \qquad \text{...}(i)$$

If the switch is connected at t = 0 and disconnected at t = a, find the current i at any instant.

Solution. We have $i = 0$ at $t = 0$ and $E(t) = \begin{cases} E \text{ for } 0 < t < a \\ 0 \text{ for } t > a \end{cases}$

$\therefore$ taking the Laplace transform of both sides, (i) becomes

$$(Ls + R)\bar{i} = \int_0^\infty e^{-st}\, E(t)\, dt = \int_0^a e^{-st}\, E dt = \frac{E}{s}(1 - e^{-as})$$

or
$$\bar{i} = \frac{E}{s(Ls + R)} - \frac{Ee^{-as}}{s(Ls + R)}$$

On inversion, we get
$$i = L^{-1}\left\{\frac{E}{s(Ls + R)}\right\} - L^{-1}\left\{\frac{Ee^{-as}}{s(Ls + R)}\right\} \qquad ...(ii)$$

Now
$$L^{-1}\left\{\frac{E}{s(Ls + R)}\right\} = \frac{E}{R}\left\{L^{-1}\left(\frac{1}{s}\right) - L^{-1}\left(\frac{1}{s + R/L}\right)\right\} = \frac{E}{R}(1 - e^{-Rt/L})$$

and
$$L^{-1}\left\{\frac{Ee^{-as}}{s(Ls + R)}\right\} = \frac{E}{R}\{1 - e^{-R(t-a)/L}\}\, u(t - a) \qquad \text{[By the second shifting property]}$$

Thus (ii) becomes
$$i = \frac{E}{R}\{1 - e^{-Rt/L}\} - \frac{E}{R}\{1 - e^{-R(t-a)/L}\}\, u(t - a)$$

Hence
$$i = \frac{E}{R}\{1 - e^{-Rt/L}\} \text{ for } 0 < t < a$$

and
$$i = \frac{E}{R}[\{1 - e^{-Rt/L}\} - \{1 - e^{-R(t-a)/L}\}] = \frac{E}{R}e^{-Rt/L}(e^{-Ra/L} - 1) \text{ for } t > a.$$

Example 21.46. *Calculate the maximum deflection of an encastre beam 1 ft. long carrying a uniformly distributed load w lb./ft. on its central half length.*

Solution. Taking the origin at the end A, we have

$$EI\frac{d^4y}{dx^4} = w(x)$$

where $w(x) = w\{u(x - l/4) - u(x - 3l/4)\}$

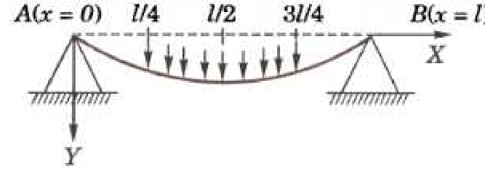


Fig. 21.6

Taking the Laplace transform of both sides, (Fig. 21.6), we get

$$EI[s^4\bar{y} - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0)]$$
$$= w\left(\frac{e^{-ls/4}}{s} - \frac{e^{-3ls/4}}{s}\right)$$

Using the conditions $y(0) = y'(0) = 0$ and taking $y''(0) = c_1$ and $y'''(0) = c_2$, we have

$$EI\bar{y} = w\left(\frac{e^{-ls/4}}{s^5} - \frac{e^{-3ls/4}}{s^5}\right) + \frac{c_1}{s^3} + \frac{c_2}{s^4}$$

On inversion, we get $EIy = \dfrac{w}{24}[(x - l/4)^4\, u(x - l/4) - (x - 3l/4)^4\, u(x - 3l/4)] + \frac{1}{2}c_1x^2 + \frac{1}{6}c_2x^3$ ...(i)

For $x > 3l/4$,
$$EIy = \frac{w}{24}[(x - l/4)^2 - (x - 3l/4)^2] + \tfrac{1}{2}c_1x^2 + \tfrac{1}{6}c_2x^3$$

and
$$EIy' = \frac{w}{6}[(x - l/4)^3 - (x - 3l/4)^3] + c_1x + \tfrac{1}{2}c_2x^2$$

Using the conditions $y(l) = 0$ and $y'(l) = 0$, we get $0 = \dfrac{w}{24}\left\{\left(\dfrac{3l}{4}\right)^4 - \left(\dfrac{l}{4}\right)^4\right\} + \frac{1}{2}c_1l^2 + \frac{1}{6}c_2l^3$

and
$$0 = \frac{w}{6}\left\{\left(\frac{3l}{4}\right)^3 - \left(\frac{l}{4}\right)^3\right\} + c_1l + \tfrac{1}{2}c_2l^2$$

whence $c_1 = 11\, wl^2/192$; $c_2 = -\, wl/4$.

Thus for $l/4 < x < 3l/4$, (*i*) gives $EIy = \frac{w}{24}\left(x + \frac{1}{4}\right)^4 + \frac{11wl^2}{384}x^2 - \frac{wl}{24}x^3$

Hence the maximum deflection $= y(l/2) = \frac{13wl^4}{6144EI}$.

21.18 (1) UNIT IMPULSE FUNCTION

The idea of a very large force acting for a very short time is of frequent occurrence in mechanics. To deal with such and similar ideas, we introduce the *unit impulse function* (also called *Dirac delta function**).

Thus unit impulse function is considered as the limiting form of the function (Fig. 21.7) :

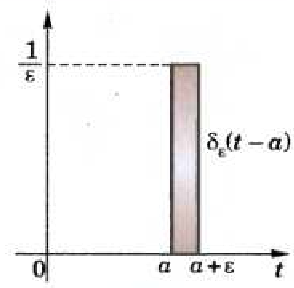


Fig. 21.7

$$\delta_\varepsilon(t-a) = 1/\varepsilon, \quad a \le t \le a + \varepsilon$$
$$= 0, \quad \text{otherwise}$$

as $\varepsilon \to 0$. It is clear from Fig. 21.7 that as $\varepsilon \to 0$, the height of the strip increases indefinitely and the width decreases in such a way that its area is always unity.

Thus the unit impulse function $\delta(t-a)$ is defined as follows :

$$\delta(t-a) = \infty \text{ for } t = a \;;\; = 0 \text{ for } t \ne a,$$

such that $$\int_0^\infty \delta(t-a)\, dt = 1. \qquad (a \ge 0)$$

As an illustration, a load w_0 acting at the point $x = a$ of a beam may be considered as the limiting case of uniform loading w_0/ε per unit length over the portion of the beam between $x = a$ and $x = a + \varepsilon$. Thus

$$w(x) = w_0/\varepsilon \qquad a < x < a + \varepsilon,$$
$$= 0, \qquad \text{otherwise}$$

i.e., $$w(x) = w_0\delta(x + a).$$

(2) Transform of unit impulse function. If $f(t)$ be a function of t continuous at $t = a$, then

$$\int_0^\infty f(t)\,\delta_\varepsilon(t-a)\,.\,dt = \int_0^{a+\varepsilon} f(t)\,.\,\frac{1}{\varepsilon}\,dt$$

$$= (a + \varepsilon - a)\,f(\eta)\,.\,\frac{1}{\varepsilon} = f(\eta), \qquad \text{where } a < \eta < a + \varepsilon.$$

by Mean value theorem for integrals.

As $\varepsilon \to 0$, we get $\int_0^\infty f(t)\,\delta(t-a)\,dt = f(a)$.

In particular, when $f(t) = e^{-st}$, we have $\mathbf{L\{\delta(t-a)\} = e^{-as}}$.

Example 21.47. *Evaluate* (*i*) $\int_0^\infty \sin 2t\,\delta(t - \pi/4)\,dt$ (*ii*) $L\frac{1}{t}\,\delta(t-a)$.

Solution. (*i*) We know that $\int_0^\infty f(t)\,\delta(t-a)\,dt = f(a)$

$$\therefore \qquad \int_0^\infty \sin 2t\,\delta(t-\pi/4)\,dt = \sin(2\,.\,\pi/4) = 1$$

(*ii*) We know that $L\{\delta(t-a)\} = e^{-as}$

$$\therefore \qquad L\left[\frac{1}{t}\delta(t-a)\right] = \int_s^\infty L[\delta(t-a)]\,ds = \int_s^\infty e^{-as}\,ds$$

$$= \left|\frac{e^{-as}}{-a}\right|_s^\infty = \frac{1}{a}\,e^{-as}.$$

* After the English physicist *Paul Dirac* (1902-84) who was awarded the Nobel prize in 1933 for his work in Quantum mechanics.

Example 21.48. *An impulsive voltage $E\delta(t)$ is applied to a circuit consisting of L, R, C in series with zero initial conditions. If i be the current at any subsequent time t, find the limit of i as $t \to 0$?*

Solution. The equation of the circuit governing the current i is

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int_0^t i\,dt = E\delta(t) \qquad \text{where } i = 0, \text{ when } t = 0.$$

Taking Laplace transform of both sides, we get

$$L[s\bar{i} - i(0)] + R\bar{i} + \frac{1}{C}\frac{1}{s}\bar{i} = E \qquad \text{[Using § 21.7 and 21.8]}$$

or $$\left(s^2 + \frac{R}{L}s + \frac{1}{CL}\right)\bar{i} = \frac{E}{L}s \quad \text{or} \quad (s^2 + 2as + a^2 + b^2)\bar{i} = (E/L)\,s$$

where $R/L = 2a$ and $1/CL = a^2 + b^2$

or $$\bar{i} = \frac{E}{L}\frac{(s+a)-a}{(s+a)^2+b^2} = \frac{E}{L}\left\{\frac{s+a}{(s+a)^2+b^2} - a\frac{1}{(s+a)^2+b^2}\right\}$$

On inversion, we get

$$i = \frac{E}{L}\left\{e^{-at}\cos bt - \frac{a}{b}e^{-at}\sin bt\right\}$$

Taking limits as $t \to 0$, $i \to E/L$

Although the current $i = 0$ initially, yet a large current will develop instantaneously due to impulsive voltage applied at $t = 0$. In fact, we have determined the limit of this current which is E/L.

Example 21.49. *A beam is simply supported at its end $x = 0$ and is clamped at the other end $x = l$. It carries a load w at $x = l/4$. Find the resulting deflection at any point.*

Solution. The differential equation for deflection is

$$\frac{d^4y}{dx^4} = \frac{w}{EI}\,\delta(x - l/4)$$

Taking the Laplace transform, we have $s^4\bar{y} - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0) = \frac{w}{EI}e^{-ls/4}$

Using the conditions $y(0) = 0$, $y''(0) = 0$ and taking $y'(0) = c_1$ and $y'''(0) = c_2$, we get

$$\bar{y} = \frac{c_1}{s^2} + \frac{c_2}{s^4} + \frac{w}{EI}\frac{e^{-ls/4}}{s^4}.$$

On inversion, it gives $$y = c_1x + c_2\frac{x^3}{3!} + \frac{w}{EI}\frac{(x - l/4)^3}{3!}\,u(x - l/4)$$

$$\left.\begin{aligned} &\text{i.e.,} \quad y = c_1x + \tfrac{1}{6}c_2x^3, && 0 < x < l/4 \\ &\text{and} \quad y = c_1x + \tfrac{1}{6}c_2x^3 + \frac{\omega}{6EI}(x - l/4)^3, && l/4 < x < l \end{aligned}\right\} \qquad ...(i)$$

Using the conditions $y(l) = 0$ and $y'(l) = 0$, we get

$$0 = c_1l + \tfrac{1}{6}c_2l^3 + 9wl^3/128EI \quad \text{and} \quad 0 = c_1 + \tfrac{1}{2}c_2l^2 + 9wl^2/32EI$$

whence $\quad c_1 = 9wl^2/256\,EI, \qquad c_2 = -81w/128EI.$

Substituting the values of c_1 and c_2 in (i), we get the deflection at any point.

PROBLEMS 21.8

1. Represent $f(t) = \sin 2t$, $2\pi < t < 4\pi$ and 0 otherwise, in terms of the unit step function and hence find its Laplace transform. *(Mumbai, 2005)*
2. Sketch the graph of the following functions and express them in terms of unit step function. Hence find their Laplace transforms :

(*i*) $f(t) = 2t$ for $0 < t < \pi$, $f(t) = 1$ for $t > \pi$ (*Assam, 1999*)

(*ii*) $f(t) = t^2$ for $0 < t \le 2$, $f(t) = 0$ for $t > 2$

(*iii*) $f(t) = \cos(wt + \phi)$ for $0 < t < T$, $f(t) = 0$ for $t > T$.

3. Express the following functions in terms of unit step function and hence find its Laplace transform.

(*i*) $f(t) = \begin{cases} \cos t, & 0 < t < \pi \\ 1, & \pi < t < 2\pi \\ \sin t, & t > 2\pi \end{cases}$ (*V.T.U., 2007*) (*ii*) $f(t) = \begin{cases} \cos t, & 0 < t < \pi \\ \cos 2t, & \pi < t < 2\pi \\ \cos 3t, & t > \pi \end{cases}$

(*Mumbai, 2008 ; V.T.U., 2003 S*)

(*iii*) $f(t) = \begin{cases} t^2, & 0 < t < 2 \\ 4t, & 2 < t < 4 \\ 8, & t > 4 \end{cases}$ (*V.T.U., 2011*)

4. Evaluate (*i*) $L\{e^{t-1}u(t-1)\}$ (*ii*) $L\{(t-1)^2 u(t-1)\}$

(*iii*) $L\,(1 + 2t - 3t^2 + 4t^3)\,H(t-2)$ (*Mumbai, 2007*) (*iv*) $L\{t^2 u(t-1) + \delta(t-1)\}$.

5. Evaluate $\int_0^\infty e^{-t}(1 + 3t + t^2)\,u(t-2)\,dt$.

6. Find the inverse Laplace transforms of :

(*i*) $\dfrac{e^{-\pi s}}{s^2+1}$ (*ii*) $\dfrac{e^{-2s}}{s^2+8s+25}$ (*Mumbai, 2006*)

(*iii*) $\dfrac{e^{-s}}{(s+1)^3}$ (*P.T.U., 2010*) (*iv*) $\dfrac{3}{s} - 4\dfrac{e^{-s}}{s^2} + 4\dfrac{e^{-3s}}{s^2}$. (*P.T.U., 2002 S*)

7. Solve using Laplace transforms $\dfrac{d^2y}{dt^2} + 4y = f(t)$ with conditions

$$y(0) = 0,\ y'(0) = 1 \text{ and } f(t) = \begin{cases} 1 \text{ when } 0 < t < 1 \\ 0 \text{ when } t > 1 \end{cases}$$ (*Mumbai, 2007*)

8. Using Laplace transforms, solve $x''(t) + x(t) = u(t)$, $x(0) = 1$, $x'(0) = 0$

where $u(t) = \begin{cases} 3, & 0 \le t \le 4 \\ 2t - 5, & t > 4. \end{cases}$

9. A beam has its ends clamped at $x = 0$ and $x = l$. A concentrated load W acts vertically downwards at the point $x = l/3$. Find the resulting deflection.

[**Hint.** The differential equation and the boundary conditions are $\dfrac{d^4y}{dx^2} = \dfrac{W}{EI}\,\delta(x - l/3)$ and

$$y(0) = y'(0) = 0,\ y(l) = y'(l) = 0.]$$

10. A cantilever beam is clamped at the end $x = 0$ and is free at the end $x = l$. It carries a uniform load w per unit length from $x = 0$ to $x = l/2$. Calculate the deflection y at any point. (*Kurukshetra, 2006*)

[**Hint.** The differential equation and boundary conditions are

$$\frac{d^4y}{dx^4} = \frac{W(x)}{EI}\ (0 < x < l) \text{ where } \quad W(x) = \begin{cases} W_0, & 0 < x < l/2 \\ 0, & x > l/2 \end{cases}$$

and $y(0) = y'(0) = 0,\ y''(0) = y'''(0) = 0$]

11. An impulse I (kg-sec) is applied to a mass m attached to a spring having a spring constant k. The system is damped with damping constant μ. Derive expressions for displacement and velocity of the mass, assuming initial conditions $x(0) = x'(0) = 0$.

[**Hint.** The equation of motion is $m\dfrac{d^2x}{dt^2} = I\,\delta(x) - kx - \mu\,\dfrac{dx}{dt}$.]