SURP - Graph Theory

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1 Introduction

A 1-factor in a directed graph is a spanning 1-regular subgraph. Equivalently, it is a collection of disjoint directed cycles such that every vertex in the graph is in exactly one cycle. Call a 1-factor special if for any pair of vertices u, v such that both (u, v) and (v, u) are edges, either both edges are in the 1-factor, or none of them is.

A directed graph is symmetric if (b, a) is an edge whenever (a, b) is an edge. Also we define the (a, b)-star graph as a directed graph on $a + b + 1$ vertices with the center vertex u, such that there are a vertices $x_1, x_2, \ldots, x_a$ with edges (u, x_i) for all $i \in [a]$, and b vertices $y_1, y_2, \ldots, y_b$ with edges (y_i, u) for all $i \in [b]$. A (a, b)-star decomposition of a directed graph is a partition of the edges of the graph into (a, b)-stars, and such a graph is called (a, b)-star decomposable.

In this article, I study the decomposition of directed regular graphs and directed bipartite graphs into star graphs, and make some observations regarding the relation between a graph being star decomposable and being factorizable into 1-factors or special 1-factors.

2 Star Decomposition of Regular Digraphs

LEMMA 1: For $a > 1$, if an $(a+1)$-regular directed graph $\mathcal{G}$ is $(a,1)$-star decomposable, then every vertex in $\mathcal{G}$ is the center of exactly one $(a,1)$-star graph

Proof. First suppose some vertex is the center of at least two such star graphs. Then it will have at least $2a$ outgoing edges, contrary to our assumption that $\mathcal{G}$ is $a+1$-regular. Now if $\mathcal{G}$ has n vertices, then it must have a total of $(a+1)n$ directed edges. So it must have gotten divided into n star graphs, which means that there are n centers of these star graphs. This is only possible if each vertex is the center of exactly one star graph. □

LEMMA 2: Suppose we have a 3-regular symmetric directed graph $\mathcal{G}$. Then it has a $(2,1)$-star decomposition if and only if the underlying undirected graph has a perfect matching.

Proof. **IF**
Suppose the graph has a perfect matching $\mathcal{M}$. Take an edge in $\mathcal{M}$ joining vertices u and v. Then, as $\mathcal{G}$ is symmetric, so it contains both edges (u,v) and (v,u). Suppose x_1 and y_1 are the other two neighbors of u, and x_2 and y_2 are the other two neighbors of v. Then the graphs $\{(u,x_1),(u,y_1),(v,u)\}$ and $\{(v,x_2),(v,y_2),(u,v)\}$ are star graphs. We can do the same division for each edge of $\mathcal{M}$, and note that none of the edges are common to any two star graphs. Thus, we get a $(2,1)$-star decomposition of $\mathcal{G}$.

ONLY IF
Now suppose $\mathcal{G}$ is $(2,1)$-star decomposable. Now take v as some vertex in $\mathcal{G}$. Then by Lemma 1, it is the center of a unique $(2,1)$-star graph $\{(v,x),(v,y),(u,v)\}$. Then the star graph of which u is a center cannot have an edge towards v. So this star graph must have the edge (v,u). And so u and v form a unique pair. We can then take such pairs and get the desired perfect matching. □

Note that if we have a symmetric directed graph, then it has a special 1-factor if and only if the underlying undirected graph has a perfect matching (since the only directed cycles allowed in a special 1-factor in a symmetric graph will be a directed cycle of length 2, which can be thought of as a simple edge in the undirected graph). This observation can be extended to a more general situation, as given in the section below.

3 Special 1-factors and Star Decomposition

LEMMA 3: For $a \in \mathbb{N}$, consider an $(a+1)$-regular directed graph $\mathcal{G}$. Then-

1. If $\mathcal{G}$ has a special 1-factor, then it also has an $(a,1)$-star decomposition.
2. If $a > 1$, then $\mathcal{G}$ has a 1-factor if it is $(a,1)$-star decomposable.

Proof. 1. Consider a vertex v in $\mathcal{G}$ such that it has outgoing edges towards $x_1, x_2, \ldots, x_{a+1}$ and incoming edges from $y_1, y_2, \ldots, y_{a+1}$. Suppose in the special 1-factor, this vertex is a part of a directed cycle which passes through $y_1 \to v \to x_1$. Then we take one of the $(a,1)$-stars as $\{(y_2, v), (y_3, v), \ldots, (y_{a+1}, v), (v, x_1)\}$, and do this for all vertices. To prove that this works, note that none of the other edges adjacent to v can be a part of a star graph centered at v. Also this sort of division without running into an edge twice will always be possible since the vertex x_1 cannot be the same as any of the vertices y_i for $2 \leq i \leq a+1$ (So each y_i has the star graph). Thus $\mathcal{G}$ is $(a,1)$-star decomposable.

2. Now suppose $a > 1$ and $\mathcal{G}$ has a $(a,1)$-star decomposition. By Lemma 1, each vertex v in $\mathcal{G}$ is the center of exactly one $(a,1)$-star graph. Let this star graph has an edge (v,x), and let (y,v) be the only incoming edge to v which is not a part of this star graph. Then we consider the $(1,1)$-star $y \to v \to x$, and take the union of all such $(1,1)$-stars to get the desired 1-factor. Here we use the fact that (y,v) will be a part of the $(1,1)$-star centered at y, and so we'll get directed cycles by this union operation.

□

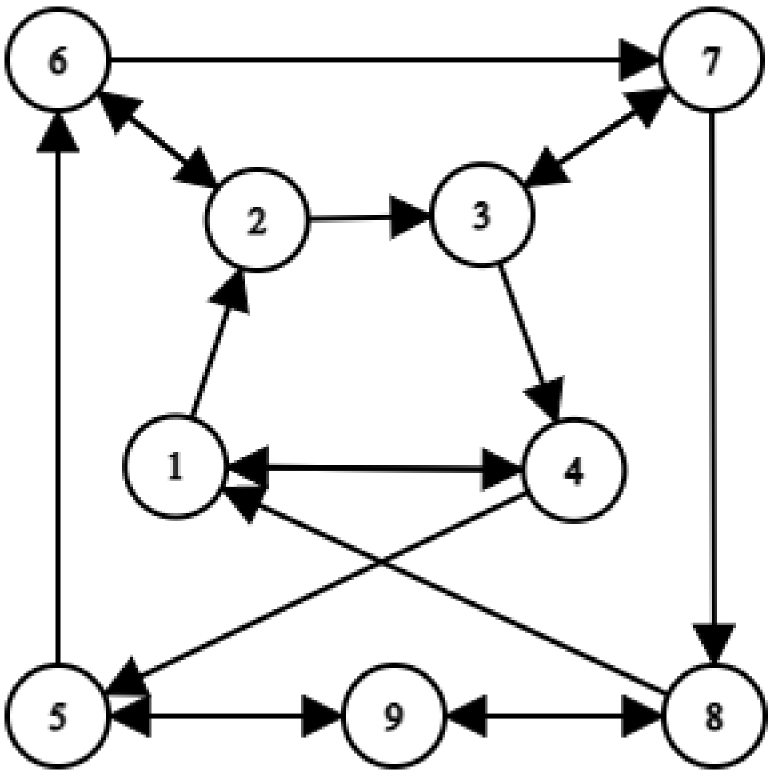

Figure 1: 1-factorizable and $(1,1)$-star docomposable 2-regular digraph with no special 1-factor

4 1-factors in Bipartite Digraphs

LEMMA 4: A bipartite directed graph $\mathcal{G}$ with bipartite sets X and Y has a 1-factor if and only if the subgraph consisting of edges from X to Y and the subgraph consisting of edges from Y to X both have a perfect matching.

Proof. **IF**
First assume the two subgraphs have perfect matchings. Then for every vertex u in X, there is a vertex v in Y and w in X (w might be the same as x) such that (u, v) is a perfect matched pair in the first subgraph, and (v, w) a perfect matched pair in the second one. We can then divide the whole graph into such triplets (u, v, w) for any u in X. Joining these edges gives us the desired 1-factor.

ONLY IF
Suppose $\mathcal{G}$ has a 1-factor. Then for each vertex u in X, there is a unique vertex v in Y such that (u, v) is a directed edge in the 1-factor. Dividing $\mathcal{G}$ into these pairs $\{u, v\}$ gives the desired perfect matching for the subgraph consisting of edges from X to Y. Similar argument works for the other direction. Note that, as a corollary, this also gives that $|X| = |Y|$. □

COROLLARY: Any d-regular bipartite directed graph, for $d \geq 1$, has a 1-factor.

Proof. Due to the above lemma, we just need to show the existence of a perfect matching in the two subgraphs which only have edges from one bipartition to another and vice-versa. Take one such subgraph, say from bipartition X to Y. Then for any subset S of X, the number of edges adjacent to S and present in this subgraph is $d \cdot |S|$. If $N(S)$ is the neighborhood of S in this subgraph, then each vertex in $N(S)$ has exactly d incoming edges, so we get

$$d \cdot |S| \leq d \cdot |N(S)| \Rightarrow |S| \leq |N(S)|$$

Then Hall's Marriage Theorem gives us that this subgraph has a perfect matching. By a similar argument, the other subgraph also has a perfect matching, and so a d regular bipartite graph has a 1-factor. □

The above lemma also gives us a polynomial time algorithm to find a 1-factor in a bipartite directed graph. We can simply find perfect matchings in the subgraph consisting of edges from X to Y and Y to X, and then for each pair $\{u, v\}$ in the first matching, consider the corresponding pair $\{v, w\}$ in the second matching. Then we take the path $u \to v \to w$, and continue the same operation for w. Finally the whole graph will be divided into various directed cycles, which gives us the desired 1-factor.