Notes on Dushoff et al. 2004

Finding the endemic equilibrium

The equations from the paper are:

$$\frac{dS}{dt} = \frac{N - S - I}{L} - \frac{\beta S I}{N}$$

$$\frac{dI}{dt} = \frac{\beta S I}{N} - \frac{I}{D}$$

By definition, the system is at equilibrium when $\frac{dS}{dt} = 0$ and $\frac{dI}{dt} = 0$, which is to say that

$$\frac{N - S^* - I^*}{L} - \frac{\beta S^* I^*}{N} = 0 \tag{1}$$

$$\frac{\beta S^* I^*}{N} - \frac{I^*}{D} = 0 \tag{2}$$

We now need to solve these two equations for the two unknowns, S^* and I^*. Let's start by looking at the second equation, which can be rewritten as:

$$I^* \left(\frac{\beta S^*}{N} - \frac{1}{D} \right) = 0$$

It should be clear that one solution of this equation is $I^* = 0$, but we're interested in the other solution (since we're interested in the endemic equilibrium). So, we want to find the value of S^* for which

$$\left(\frac{\beta S^*}{N} - \frac{1}{D} \right) = 0$$

To find S^*, we can add $1/D$ to both sides of the equation, giving:

$$\frac{\beta S^*}{N} = \frac{1}{D}$$

Now, if we multiply both sides by N/β, we get:

$$S^* = \frac{N}{\beta D} \tag{3}$$

We can now substitute the expression for S^* into equation (1) to give

$$\frac{N - \frac{N}{\beta D} - I^*}{L} - \frac{\beta \frac{N}{\beta D} I^*}{N} = 0$$

or,

$$\frac{N - \frac{N}{\beta D} - I^*}{L} = \frac{\beta \frac{N}{\beta D} I^*}{N}$$

Notice that we can cancel some terms on the right hand side of this equation to get:

$$\frac{N - \frac{N}{\beta D} - I^*}{L} = \frac{I^*}{D}$$

And we can re-write the left hand side of this equation to get:

$$\frac{N}{L} - \frac{N}{\beta D L} - \frac{I^*}{L} = \frac{I^*}{D}$$

Adding $\frac{I^*}{L}$ to both sides gives

$$\begin{aligned} \frac{N}{L} - \frac{N}{\beta D L} &= \frac{I^*}{L} + \frac{I^*}{D} \\ &= I^* \left(\frac{1}{L} + \frac{1}{D} \right) \end{aligned}$$

By dividing both sides by $\left(\frac{1}{L} + \frac{1}{D}\right)$, we have an expression for I^*:

$$I^* = \frac{\frac{N}{L} - \frac{N}{\beta D L}}{\frac{1}{L} + \frac{1}{D}}$$

But it's not pretty! Let's clean it up a bit by multiplying the numerator and the denominator both by L, then canceling terms where we can:

$$I^* = \frac{\frac{LN}{L} - \frac{LN}{\beta DL}}{\frac{L}{L} + \frac{L}{D}}$$

$$= \frac{N - \frac{N}{\beta D}}{1 + \frac{L}{D}}$$

which can also be written as

$$I^* = \frac{N - S^*}{1 + \frac{L}{D}}$$