Basic Data Structures: Dynamic Arrays and Amortized Analysis

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Data Structures
Data Structures and Algorithms

Outline

1. Dynamic Arrays
2. Amortized Analysis—Aggregate Method
3. Amortized Analysis—Banker's Method
4. Amortized Analysis—Physicist's Method

Problem: static arrays are static!

```
int my_array[100];
```

Semi-solution: dynamically-allocated arrays:

```
int *my_array = new int[size];
```

Problem: might not know max size when allocating an array

> *All problems in computer science can be solved by another level of indirection.*

Solution: *dynamic arrays* (also known as *resizable arrays*)

Idea: store a pointer to a dynamically allocated array, and replace it with a newly-allocated array as needed.

Definition

Dynamic Array:

Abstract data type with the following operations (at a minimum):

- `Get`(i): returns element at location i*
- `Set`(i, val): Sets element i to val*
- PushBack(val): Adds val to the end
- `Remove`(i): Removes element at location i
- `Size`(): the number of elements

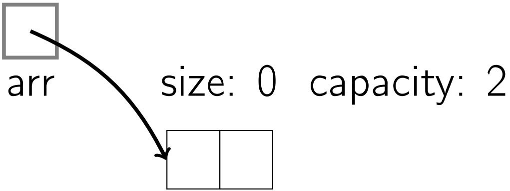
*must be constant time

Implementation

Store:

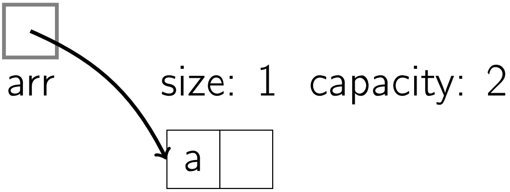
- `arr`: dynamically-allocated array
- `capacity`: size of the dynamically-allocated array
- `size`: number of elements currently in the array

Dynamic Array Resizing



`PushBack(a)`

Dynamic Array Resizing



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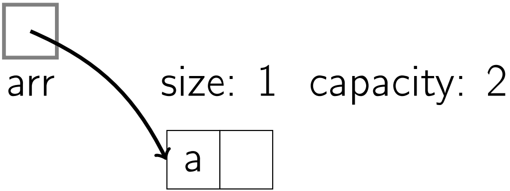
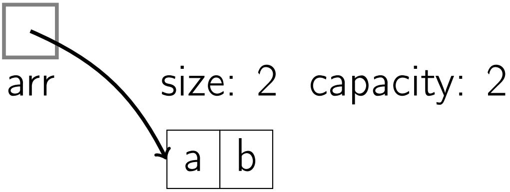


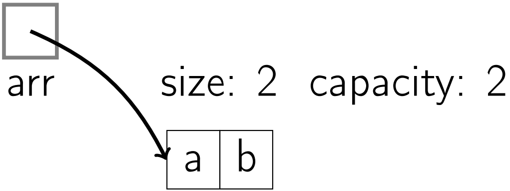
`PushBack(b)`

Dynamic Array Resizing



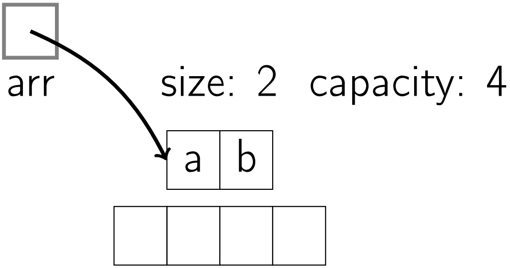
`PushBack(b)`

Dynamic Array Resizing



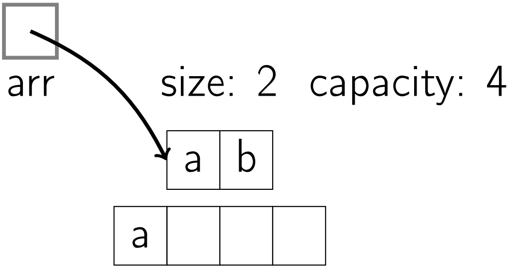
`PushBack(c)`

Dynamic Array Resizing



`PushBack(c)`

Dynamic Array Resizing



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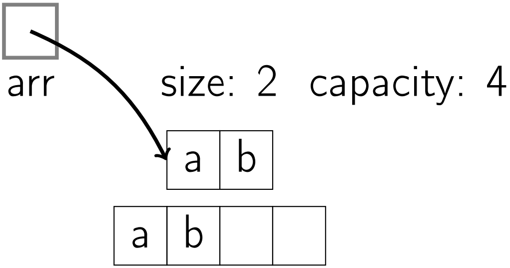
Dynamic Array Resizing



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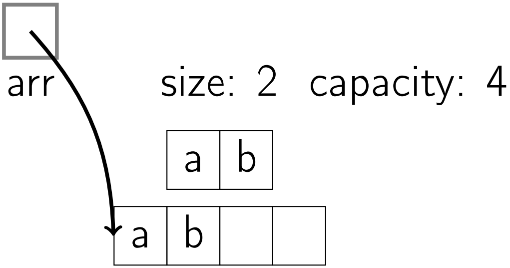
Dynamic Array Resizing



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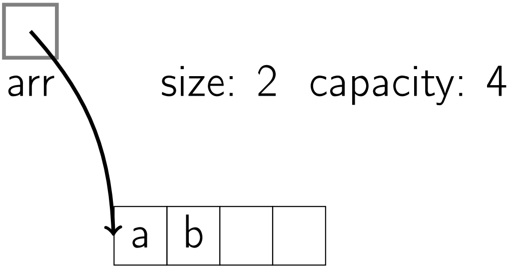

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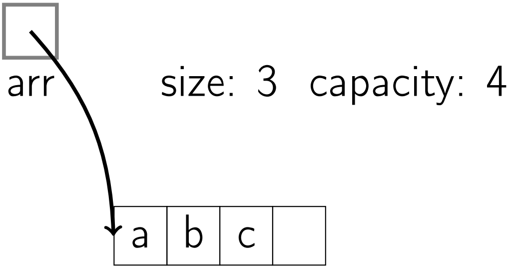
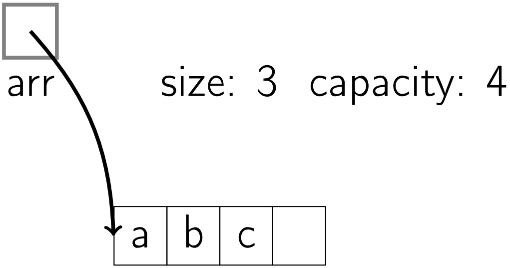


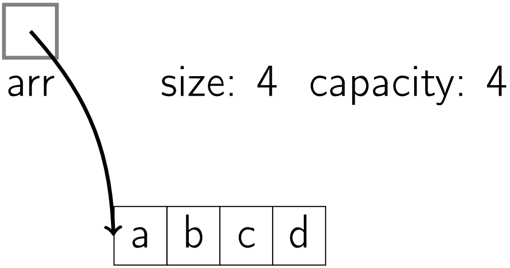
`PushBack(c)`

Dynamic Array Resizing



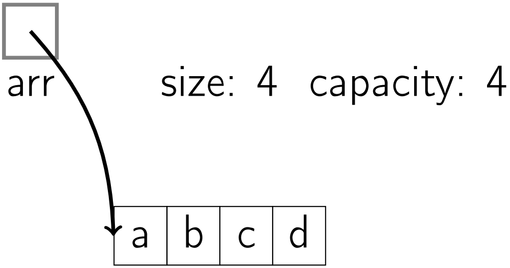
`PushBack(d)`

Dynamic Array Resizing



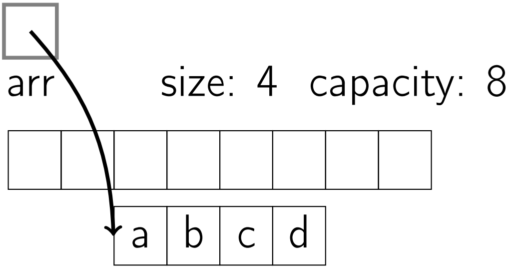
`PushBack(d)`

Dynamic Array Resizing



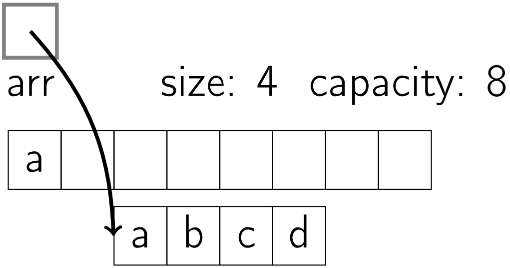
`PushBack(e)`

Dynamic Array Resizing



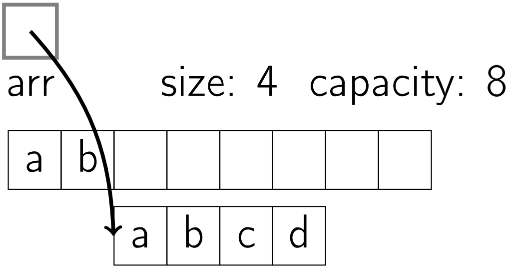
`PushBack(e)`

Dynamic Array Resizing



PushBack(e)

Dynamic Array Resizing



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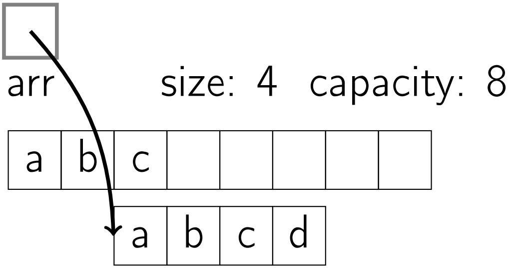

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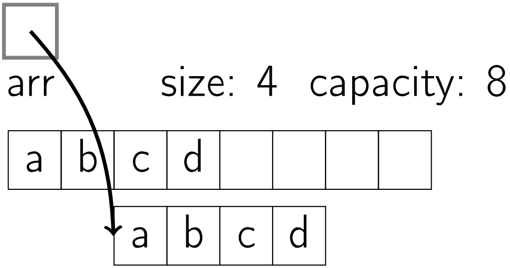

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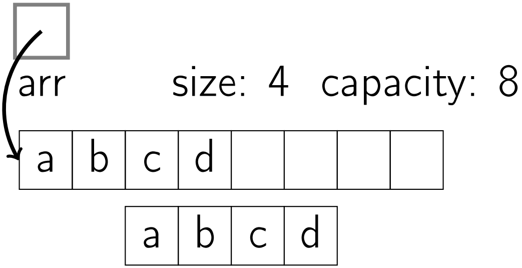
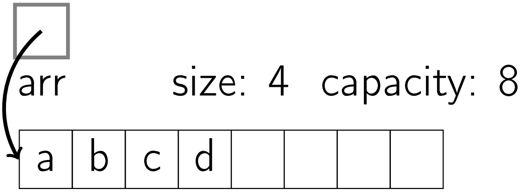
Dynamic Array Resizing



`PushBack(e)`

Dynamic Array Resizing



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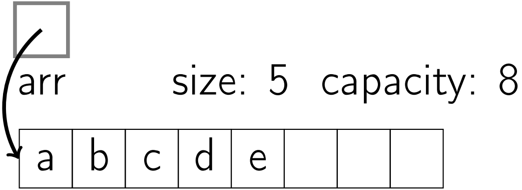
Dynamic Array Resizing



`PushBack(e)`

Get(i)

```
if i < 0 or i ≥ size:
   ERROR: index out of range
return arr[i]
```

$\texttt{Set}(i, val)$

```
if i < 0 or i ≥ size:
  ERROR: index out of range
arr[i] = val
```

PushBack(val)

```
if size = capacity:
    allocate new_arr[2 × capacity]
    for i from 0 to size − 1:
        new_arr[i] ← arr[i]
    free arr
    arr ← new_arr; capacity ← 2 × capacity
arr[size] ← val
size ← size + 1
```

Remove(i)

```
if i < 0 or i ≥ size:
    ERROR: index out of range
for j from i to size − 2:
    arr[j] ← arr[j + 1]
size ← size − 1
```

Size()

`return` *size*

Common Implementations

- **C++**: `vector`
- **Java**: `ArrayList`
- **Python**: `list` (the only kind of array)

Runtimes

Get(i)	$O(1)$
Set(i, val)	$O(1)$
PushBack(val)	$O(n)$
Remove(i)	$O(n)$
Size()	$O(1)$

Summary

- Unlike static arrays, dynamic arrays can be resized.
- Appending a new element to a dynamic array is often constant time, but can take $O(n)$.
- Some space is wasted—at most half.

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Sometimes, looking at the individual worst-case may be too severe. We may want to know the total worst-case cost for a sequence of operations.

Dynamic Array

We only resize every so often.
Many $O(1)$ operations are followed by an $O(n)$ operations.
What is the total cost of inserting many elements?

Definition

Amortized cost: Given a sequence of n operations, the amortized cost is:

$$\frac{\text{Cost}(n \text{ operations})}{n}$$

Aggregate Method

Dynamic array: n calls to PushBack

Let c_i = cost of i'th insertion.

$$c_i = 1 + \begin{cases} i-1 & \text{if } i-1 \text{ is a power of } 2 \\ 0 & \text{otherwise} \end{cases}$$

$$\frac{\sum_{i=1}^{n} c_i}{n} = \frac{n + \sum_{j=1}^{\lfloor \log_2(n-1) \rfloor} 2^j}{n} = \frac{O(n)}{n} = O(1)$$

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Banker's Method

- Charge extra for each cheap operation.
- Save the extra charge as tokens in your data structure (conceptually).
- Use the tokens to pay for expensive operations.

Like an amortizing loan.

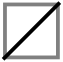
Banker's Method

Dynamic array: n calls to `PushBack`

Charge 3 for each insertion: 1 token is the raw cost for insertion.

- Resize needed: To pay for moving the elements, use the token that's present on each element that needs to move.
- Place one token on the newly-inserted element, and one token $\frac{capacity}{2}$ elements prior.

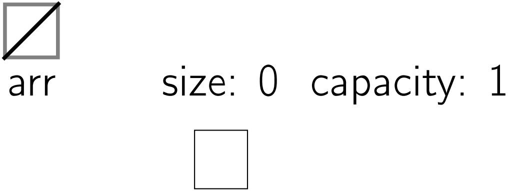
Dynamic Array Resizing



arr size: 0 capacity: 0

PushBack(a)

Dynamic Array Resizing



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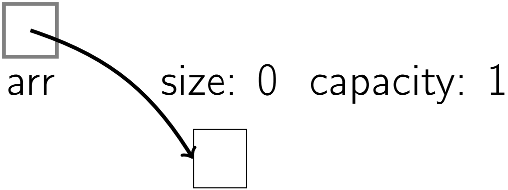

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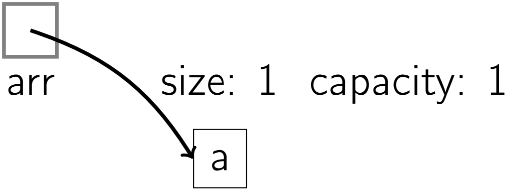

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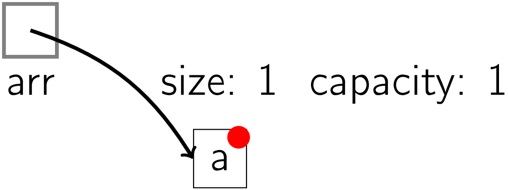
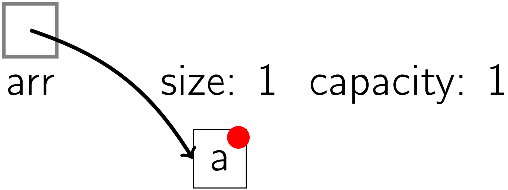


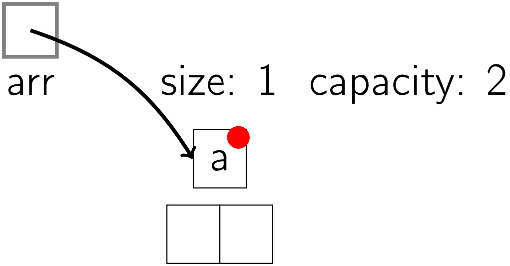
PushBack(a)

Dynamic Array Resizing



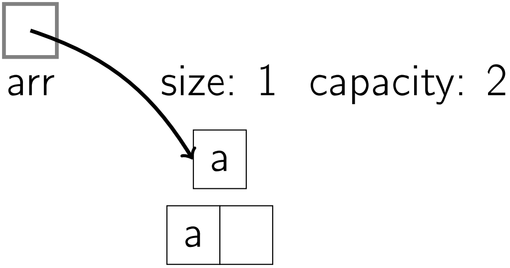
PushBack(b)

Dynamic Array Resizing



PushBack(b)

Dynamic Array Resizing



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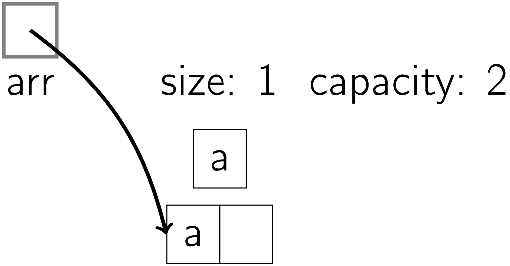
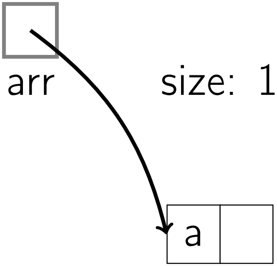
Dynamic Array Resizing



PushBack(b)

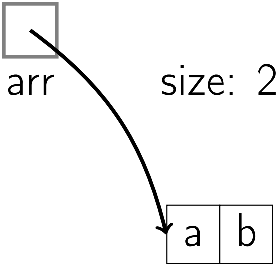
Dynamic Array Resizing



size: 1 capacity: 2

PushBack(b)

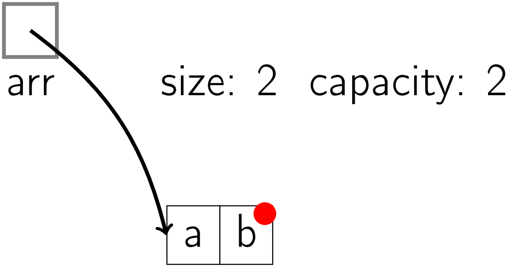
Dynamic Array Resizing



size: 2 capacity: 2

PushBack(b)

Dynamic Array Resizing



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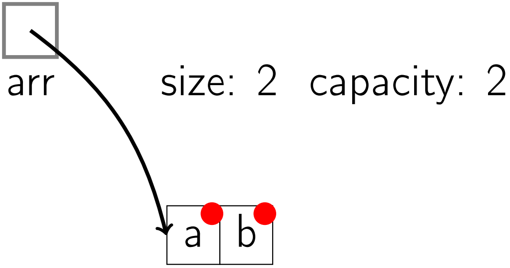

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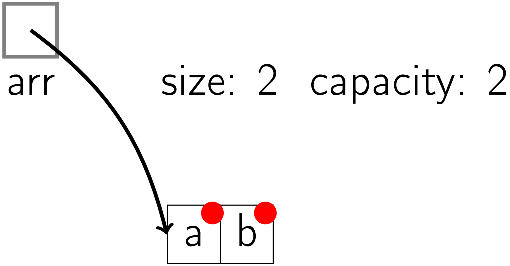
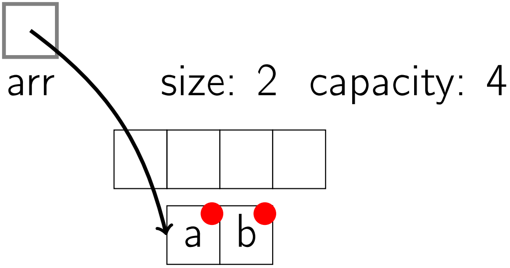
Dynamic Array Resizing



PushBack(c)

Dynamic Array Resizing



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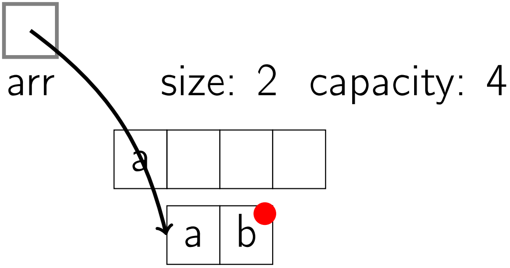

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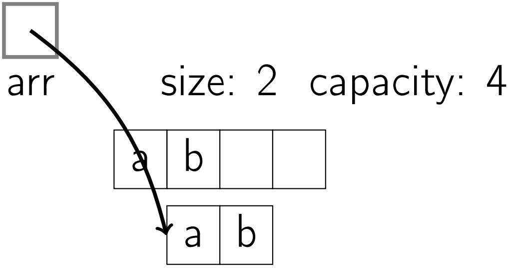

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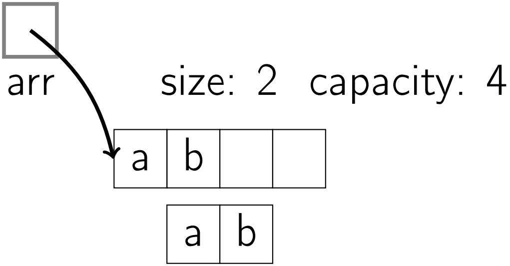
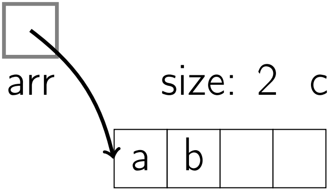


PushBack(c)

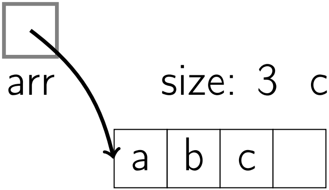
Dynamic Array Resizing



size: 2 capacity: 4

PushBack(c)

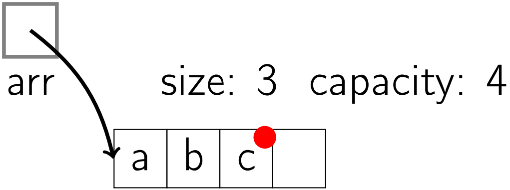
Dynamic Array Resizing



size: 3 capacity: 4

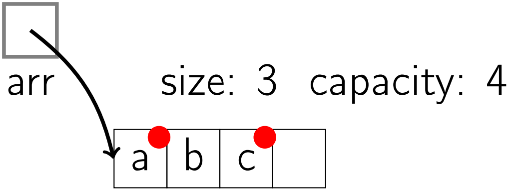
PushBack(c)

Dynamic Array Resizing



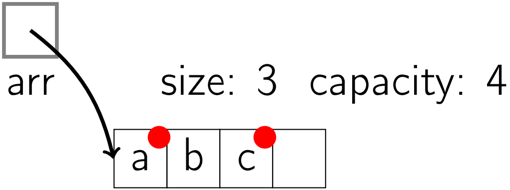
PushBack(c)

Dynamic Array Resizing



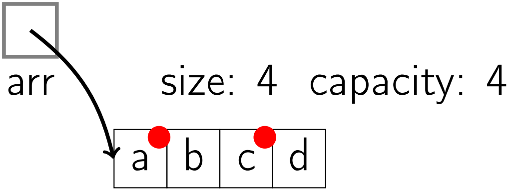
PushBack(c)

Dynamic Array Resizing



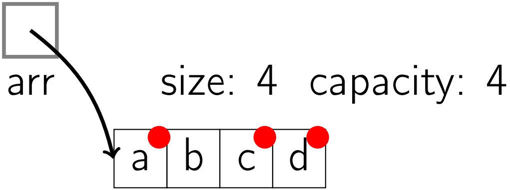
PushBack(d)

Dynamic Array Resizing



PushBack(d)

Dynamic Array Resizing



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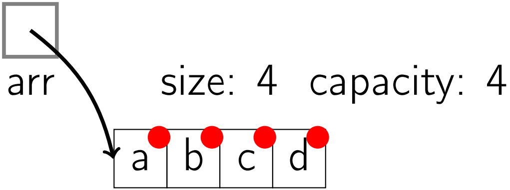
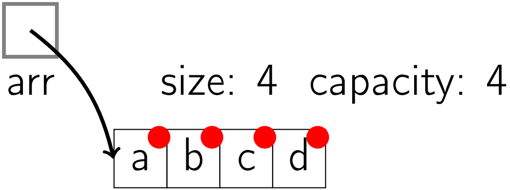
Dynamic Array Resizing



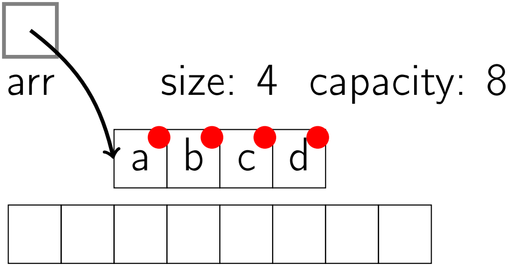
PushBack(d)

Dynamic Array Resizing



PushBack(e)

Dynamic Array Resizing



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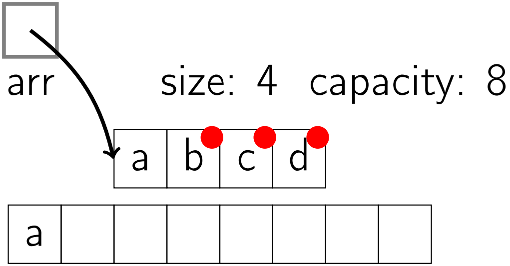

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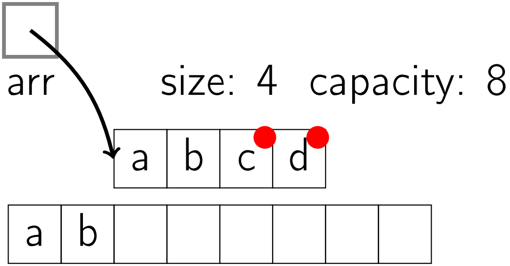

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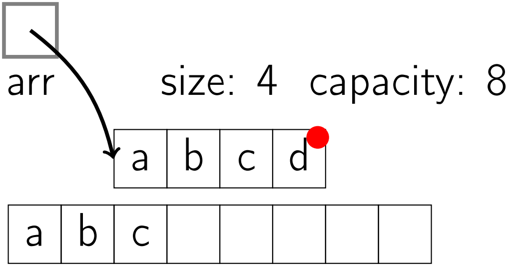

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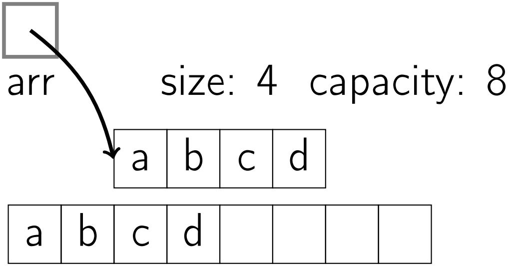

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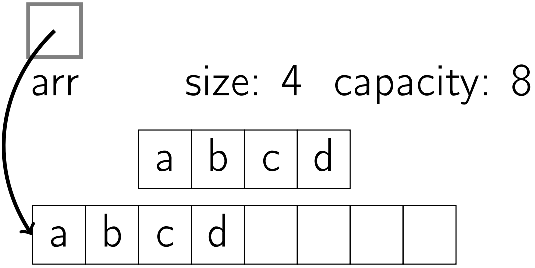

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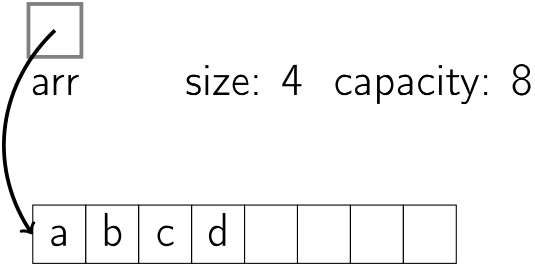

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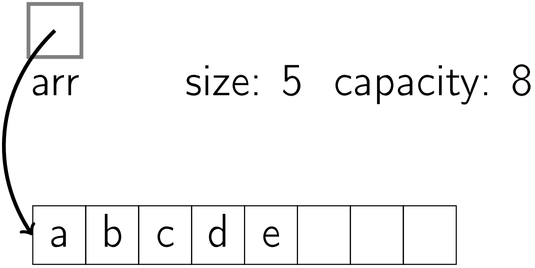
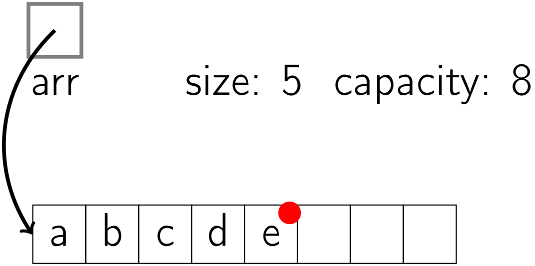


PushBack(e)

Dynamic Array Resizing



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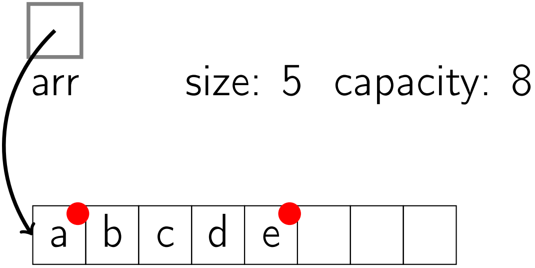

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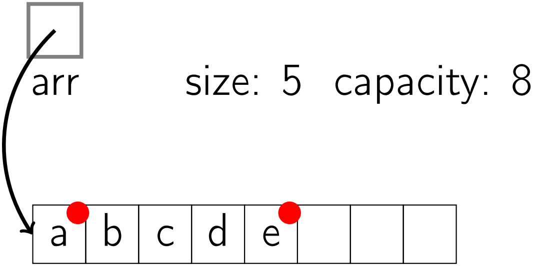

$O(1)$ amortized cost for each `PushBack`

Banker's Method

Dynamic array: n calls to `PushBack`

Charge 3 for each insertion. 1 coin is the raw cost for insertion.

- Resize needed: To pay for moving the elements, use the coin that's present on each element that needs to move.
- Place one coin on the newly-inserted element, and one coin $\frac{capacity}{2}$ elements prior.

Outline

1. Dynamic Arrays
2. Amortized Analysis—Aggregate Method
3. Amortized Analysis—Banker's Method
4. Amortized Analysis—Physicist's Method

Physicist's Method

- Define a *potential function*, Φ which maps states of the data structure to integers:
  - $\Phi(h_0) = 0$
  - $\Phi(h_t) \geq 0$
- amortized cost for operation t:

  $c_t + \Phi(h_t) - \Phi(h_{t-1})$

Choose Φ so that:

- if c_t is small, the potential increases
- if c_t is large, the potential decreases by the same scale

Physicist's Method

- The cost of n operations is: $\sum_{i=1}^{n} c_i$
- The sum of the amortized costs is:

$$\sum_{i=1}^{n}(c_i + \Phi(h_i) - \Phi(h_{i-1}))$$

$$\begin{aligned}
=& c_1 + \Phi(h_1) - \Phi(h_0) + \\
& c_2 + \Phi(h_2) - \Phi(h_1) \cdots + \\
& c_n + \Phi(h_n) - \Phi(h_{n-1})
\end{aligned}$$

$$=\Phi(h_n) - \Phi(h_0) + \sum_{i=1}^{n} c_i \geq \sum_{i=1}^{n} c_i$$

Physicist's Method

Dynamic array: n calls to PushBack

Let $\Phi(h) = 2 \times size - capacity$

- $\Phi(h_0) = 2 \times 0 - 0 = 0$
- $\Phi(h_i) = 2 \times size - capacity > 0$ (since $size > \frac{capacity}{2}$)

Dynamic Array Resizing

Without resize when adding element i

Amortized cost of adding element i:

$$\underline{c_i + \Phi(h_i) - \Phi(h_{i-1})}$$

$$\underline{= 1 + 2 \times \mathit{size}_i - \mathit{cap}_i - (2 \times \mathit{size}_{i-1} - \mathit{cap}_{i-1})}$$

$$\underline{= 1 + 2 \times (\mathit{size}_i - \mathit{size}_{i-1})}$$

$$= 3$$

Dynamic Array Resizing

With resize when adding element i

Let $k = size_{i-1} = cap_{i-1}$

Then:

$\Phi(h_{i-1}) = 2size_{i-1} - cap_{i-1} = 2k - k = k$

$\Phi(h_i) = 2size_i - cap_i = 2(k+1) - 2k = 2$

Amortized cost of adding element i:

$$
\begin{aligned}
& c_i + \Phi(h_i) - \Phi(h_{i-1}) \\
=& (size_i) + 2 - k \\
=& (k+1) + 2 - k \\
=& 3
\end{aligned}
$$

Alternatives to Doubling the Array Size

We could use some different growth factor (1.5, 2.5, etc.).

Could we use a constant amount?

Cannot Use Constant Amount

If we expand by 10 each time, then:

Let c_i = cost of i'th insertion.

$$c_i = 1 + \begin{cases} i-1 & \text{if } i-1 \text{ is a multiple of } 10 \\ 0 & \text{otherwise} \end{cases}$$

$$\frac{\sum_{i=1}^{n} c_i}{n} = \frac{n + \sum_{j=1}^{(n-1)/10} 10j}{n} = \frac{n + 10 \sum_{j=1}^{(n-1)/10} j}{n}$$

$$= \frac{n + 10O(n^2)}{n} = \frac{O(n^2)}{n} = O(n)$$

Summary

- Calculate amortized cost of an operation in the context of a sequence of operations.
- Three ways to do analysis:
  - Aggregate method (brute-force sum)
  - Banker's method (tokens)
  - Physicist's method (potential function, Φ)
- Nothing changes in the code: runtime analysis only.