第一周作业

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3

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\begin{aligned}
&\iint\limits_D ydxdy \\
=&\int_0^\pi dx\int_0^{sinx} ydy \\
=&\int_0^\pi \frac{sin^2x}{2}dx \\
=&(\frac{x}{4}-\frac{sin2x}{8})|_0^\pi \\
=&\frac{\pi}{4}
\end{aligned}
$$

4

$$
\begin{aligned}
&\iint\limits_D xy^2dxdy \\
=&\int_{-2}^2 dy\int_{\frac{y^2}{4}}^1 xy^2dx \\
=&\int_{-2}^2 \frac{y^2}{2}-\frac{y^6}{32}dy \\
=&\frac{32}{21}
\end{aligned}
$$

5

$$\iint\limits_D e^{\frac{x}{y}}\,dxdy$$
$$=\int_0^1 dy\int_0^{y^2} e^{\frac{x}{y}}\,dx$$
$$=\int_0^1 (ye^y - y)dy$$
$$=ye^y\Big|_0^1 - \int_0^1 e^y dy - \frac{y^2}{2}\Big|_0^1$$
$$=\frac{1}{2}$$

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$$\int_0^1 dy\int_{y^{\frac{1}{3}}}^1 \sqrt{1-x^4}dx$$
$$=\int_0^1 dx\int_0^{x^3} \sqrt{1-x^4}dy$$
$$=-\frac{1}{4}\int_0^1 \sqrt{1-x^4}\mathrm{d}(1-x^4)$$
$$=-\frac{1}{6}(1-x^4)^{\frac{3}{2}}\Big|_0^1$$
$$=\frac{1}{6}$$

8

$$\int_0^\pi \mathrm{d}x\int_x^\pi \frac{siny}{y}\mathrm{d}y$$
$$=\int_0^\pi \mathrm{d}y\int_0^y \frac{siny}{y}\mathrm{d}x$$
$$=\int_0^\pi siny\mathrm{d}y$$
$$=2$$

9

$$
\begin{aligned}
&\int_0^2 \mathrm{d}x \int_x^2 2y^2 sin(xy)\mathrm{d}y \\
=&\int_0^2 \mathrm{d}y \int_0^y 2y^2 sin(xy)\mathrm{d}x \\
=&\int_0^2 \mathrm{d}y(-\frac{cos(xy)}{y}\Big|_0^y) \\
=&\int_0^2 (1-cos(y^2))\mathrm{d}(y^2) \\
=&4-sin4
\end{aligned}
$$

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$$
\begin{aligned}
&\iint\limits_D y^2\sqrt{1-x^2}\mathrm{d}\sigma \\
=&\int_{-1}^1 \mathrm{d}x \int_{\sqrt{1-x^2}}^{-\sqrt{1-x^2}} y^2\sqrt{1-x^2}\mathrm{d}y \\
=&\frac{2}{3}\int_{-1}^1 (1-x^2)^2\mathrm{d}x \\
=&\frac{32}{45}
\end{aligned}
$$

11 按照 x 的正负将 D 划分为 D_1 和 D_2

$$\begin{aligned}&\iint_D (|x|+y)\mathrm{d}\sigma\\=&\iint_{D_1}(x+y)\mathrm{d}\sigma+\iint_{D_2}(y-x)\mathrm{d}\sigma\end{aligned}$$

$$\begin{aligned}&\iint_{D_1}(x+y)\mathrm{d}\sigma\\=&\int_0^1\mathrm{d}x\int_{x-1}^{1-x}(x+y)\mathrm{d}y\\=&\int_0^1 2x(1-x)\mathrm{d}x\\=&\frac{1}{3}\end{aligned}$$

$$\begin{aligned}&\iint_{D_2}(y-x)\mathrm{d}\sigma\\=&\int_{-1}^0\mathrm{d}x\int_{-x-1}^{x+1}(y-x)\mathrm{d}y\\=&\int_{-1}^0 -2x(x+1)\mathrm{d}x\\=&\frac{1}{3}\end{aligned}$$

故所求为 $\frac{2}{3}$

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$$\begin{aligned}&int_0^1\mathrm{d}x\int_{\sqrt{1-x^2}}^0(x^2+y^2)\mathrm{d}y\\=&\int_0^{\frac{\pi}{2}}\mathrm{d}\theta\int_0^1 r^3\mathrm{d}r\\=&\int_0^{\frac{\pi}{2}}\frac{1}{4}\mathrm{d}\theta\\=&\frac{\pi}{8}\end{aligned}$$

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$$\int_0^2 \mathrm{d}x \int_0^{\sqrt{1-(x-1)^2}} 3xy\mathrm{d}y$$
$$= \int_0^{\frac{\pi}{2}} \mathrm{d}\theta \int_0^{2cos\theta} 3r^3 sin\theta cos\theta \mathrm{d}r$$
$$= \int_0^{\frac{\pi}{2}} 12cos^5\theta sin\theta \mathrm{d}\theta$$
$$=2$$

18 由题意可知，$\theta \in [0, \frac{\pi}{2}] \cup [\frac{3\pi}{2}, \pi]$

D 关于极轴对称，故 $\theta \in [0, \frac{\pi}{2}]$ 对应的区域设为 D_1

$$\iint\limits_D r\mathrm{d}\sigma$$
$$=2\iint\limits_{D_1} r\mathrm{d}\sigma$$
$$=2\int_0^{\frac{\pi}{2}} \mathrm{d}\theta \int_a^{a(1+cos\theta)} r\mathrm{d}r$$
$$=a^2\int_0^{\frac{\pi}{2}} (cos^2\theta + 2cos\theta)\mathrm{d}\theta$$
$$=a^2\int_0^{\frac{\pi}{2}} (\frac{1}{2} + \frac{cos2\theta}{2} + 2cos\theta)\mathrm{d}\theta$$
$$=a^2(\frac{1}{2}\theta + \frac{sin2\theta}{4} + 2sin\theta)\Big|_0^{\frac{\pi}{2}}$$
$$=a^2(\frac{\pi}{4} + 2)$$

19 证:

$$\begin{aligned}S=&\iint_D \mathrm{d}x\mathrm{d}y\\=&\iint_D r\mathrm{d}r\mathrm{d}\theta\\=&\int_\alpha^\beta\int_0^{r(\theta)} r\mathrm{d}r\\=&\frac{1}{2}\int_\alpha^\beta [r(\theta)]^2\mathrm{d}\theta\end{aligned}$$

20 设所求区域为 D，面积为 S，可知图形关于极轴对称，故有:

$$\begin{aligned}\frac{1}{2}S=&\iint_D \mathrm{d}x\mathrm{d}y\\=&\iint_D r\mathrm{d}r\mathrm{d}\theta\\=&\int_0^\pi \mathrm{d}\theta\int_0^{a(1+cos\theta)} r\mathrm{d}r\\=&\frac{1}{2}\int_0^\pi a^2(1+cos\theta)^2\mathrm{d}\theta\\=&\frac{a^2}{2}\int_0^\pi \frac{1}{2}cos2\theta+2cos\theta+\frac{3}{2}\mathrm{d}\theta\\=&\frac{a^2}{2}(\frac{3}{2}\theta+\frac{sin2\theta}{4}+2sin\theta)\Big|_0^\pi\\=&\frac{3}{4}a^2\pi\end{aligned}$$

故 $S=\frac{3}{2}a^2\pi$

22 令 $\xi=\sqrt{xy},\eta=\sqrt{\frac{y}{x}}$

则

$$\frac{D(x,y)}{D(\xi,\eta)}=\begin{vmatrix}\frac{1}{\eta} & -\frac{\xi}{\eta^2}\\ \eta & \xi\end{vmatrix}=\frac{2\xi}{\eta}$$

故

$$
\begin{aligned}
&\iint\limits_D (\sqrt{\frac{y}{x}} + \sqrt{xy})\mathrm{d}x\mathrm{d}y \\
=&\int_1^2 \mathrm{d}\eta \int_1^3 \frac{2\xi}{\eta}(\xi+\eta)\mathrm{d}\xi \\
=&\int_1^2 (\frac{52}{3\eta}+8)\mathrm{d}\eta \\
=&\frac{52}{3}ln2+8
\end{aligned}
$$

24 令 $x = arcos\theta, y = brsin\theta$

则

$$
\frac{D(x,y)}{D(r,\theta)} = \begin{vmatrix} acos\theta & -arsin\theta \\ bsin\theta & brcos\theta \end{vmatrix} = abr
$$

则

$$
\begin{aligned}
&\iint\limits_\Omega (x^2+y^2)\mathrm{d}x\mathrm{d}y \\
=&\iint\limits_{\Omega'} ab(a^2+b^2)r^3\mathrm{d}r\mathrm{d}\theta \\
=&ab(a^2+b^2)\int_0^{2\pi}\mathrm{d}\theta\int_0^1 r^3\mathrm{d}r \\
=&ab(a^2+b^2)\int_0^{2\pi}\frac{1}{4}\mathrm{d}\theta \\
=&\frac{\pi}{2}ab(a^2+b^2)
\end{aligned}
$$