第三周作业

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P60.4

$$
\begin{aligned}
&\int_L \frac{1}{x^2+y^2+z^2}\mathrm{d}s \\
=&\int_0^{2\pi} \frac{1}{a^2cos^2t+a^2sin^2t+b^2t^2}\sqrt{a^2sin^2+a^2cos^2+b^2}\mathrm{d}t \\
=&\int_0^{2\pi} \frac{1}{a^2+b^2t^2}\sqrt{a^2+b^2}\mathrm{d}t \\
=&\frac{\sqrt{a^2+b^2}}{ab}\int_0^{2\pi} \frac{1}{1+\frac{b^2t^2}{a^2}}\mathrm{d}\left(\frac{bt}{a}\right) \\
=&\frac{\sqrt{a^2+b^2}}{ab}\arctan\frac{bt}{a}\Big|_0^{2\pi} \\
=&\frac{\sqrt{a^2+b^2}}{ab}\arctan\frac{2b\pi}{a}
\end{aligned}
$$

P60.6 令

$$
\begin{cases}
x = acos\theta \\
y = bsin\theta
\end{cases} \tag{1}
$$

则

$$
\begin{aligned}
\mathrm{d}s =&\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}\mathrm{d}\theta \\
=&\sqrt{\frac{a^2+b^2}{2}+\frac{a^2-b^2}{2}\cos 2\theta\mathrm{d}\theta}
\end{aligned}
$$

故

$$\begin{aligned}
&\int_L xy\mathrm{d}s \\
=&\int_L ab\sin\theta\cos\theta\mathrm{d}s \\
=&\int_L \frac{1}{2}ab\sin 2\theta\mathrm{d}s \\
=&-\int_0^{\frac{\pi}{2}} \frac{1}{2}ab\sin 2\theta\sqrt{\frac{a^2+b^2}{2}+\frac{a^2-b^2}{2}\cos 2\theta}\mathrm{d}\theta \\
=&-\int_0^{\frac{\pi}{2}} \frac{1}{4}ab\sqrt{\frac{a^2+b^2}{2}+\frac{a^2-b^2}{2}\cos 2\theta}\mathrm{d}(\cos 2\theta) \\
=&-\frac{ab}{3(a^2-b^2)}\left(\frac{a^2+b^2}{2}+\frac{a^2-b^2}{2}\cos 2\theta\right)\mathrm{d}\theta\Big|_0^{\frac{\pi}{2}} \\
=&\frac{ab(a^3-b^3)}{3(a^2-b^2)} \\
=&\frac{ab(a^2+ab+b^2)}{3(a+b)}
\end{aligned}$$

P60.7 由

$$\begin{aligned}
ds =&\sqrt{x'(t)^2+y'(t)^2}dt \\
=&|a|t\mathrm{d}t
\end{aligned}$$

有

$$\begin{aligned}
&\int_L \sqrt{x^2+y^2}\mathrm{d}s \\
=&\int_0^{2\pi} a^2\sqrt{1+t^2}t\mathrm{d}t \\
=&\int_0^{2\pi} \frac{a^2}{2}\sqrt{1+t^2}\mathrm{d}(1+t^2) \\
=&\frac{a^2}{3}(1+4\pi^2)^{\frac{3}{2}}-\frac{a^2}{3}
\end{aligned}$$

P60.8

$$
\begin{aligned}
&\int_L (x+\sqrt{y}-z^5)\mathrm{d}s \\
=&\int_{L1} 2x\mathrm{d}s+\int_{L2} 2-z^5\mathrm{d}s
\end{aligned}
$$

其中

$$
\begin{aligned}
&\int_{L1} 2x\mathrm{d}s \\
=&\int_0^1 2x\sqrt{1+(2x)^2}\mathrm{d}x \\
=&\frac{1}{4}\int_0^1 \sqrt{4x^2+1}\mathrm{d}(x^2) \\
=&\frac{1}{6}(4x^2+1)^{\frac{3}{2}}\Big|_0^1 \\
=&\frac{5\sqrt{5}-1}{6}
\end{aligned}
$$

$$
\begin{aligned}
&\int_{L2} 2-z^5\mathrm{d}s \\
=&\int_0^1 (2-z^5)\mathrm{d}z \\
=&(2z-\frac{z^6}{6})\Big|_0^1 \\
=&\frac{11}{6}
\end{aligned}
$$

所以所求为

$$
\frac{5\sqrt{5}+10}{6}
$$

P70.1

(1) 有

$$
L: y=\frac{x}{2}
$$

故

$$\begin{aligned}&\int_L 2xy\mathrm{d}x - x^2\mathrm{d}y\\=&\int_0^2 \frac{x^2}{2}\mathrm{d}x\\=&\frac{4}{3}\end{aligned}$$

(2) 有

$$L: y = \frac{x^2}{4}$$

故

$$\begin{aligned}&\int_L 2xy\mathrm{d}x - x^2\mathrm{d}y\\=&\int_0^2 (2x\cdot\frac{x^2}{4}\mathrm{d}x - x^2\cdot\frac{x}{2}\mathrm{d}x)\\=&0\end{aligned}$$

(3) 有

$$\begin{cases}L_1: y = 0\\L_2: x = 2\end{cases}$$

故

$$\begin{aligned}&\int_L 2xy\mathrm{d}x - x^2\mathrm{d}y\\=&\int_{L1} 2xy\mathrm{d}x - x^2\mathrm{d}y + \int_{L2} 2xy\mathrm{d}x - x^2\mathrm{d}y\\=&\int_0^1 -4y\mathrm{d}y\\=&-4\end{aligned}$$

(4) 有

$$\begin{cases}L_1: x = 0\\L_2: y = 1\end{cases}\tag{2}$$

故

$$
\begin{aligned}
&\int_L 2xy\mathrm{d}x - x^2\mathrm{d}y\\
=&\int_{L1} 2xy\mathrm{d}x - x^2\mathrm{d}y + \int_{L2} 2xy\mathrm{d}x - x^2\mathrm{d}y\\
=&\int_0^2 2x\mathrm{d}x\\
=&4
\end{aligned}
$$

P71.4

(1)

$$
\begin{aligned}
&\int_L (x^2-2xy)\mathrm{d}x + (y^2-2xy)\mathrm{d}y\\
=&\int_{-1}^1 x^2 - 2x^5 + 4x^{11} - 8x^8\\
=&-\frac{10}{9}
\end{aligned}
$$

(2)

$$
\begin{aligned}
&\int_L (x^2-2xy)\mathrm{d}x + (y^2-2xy)\mathrm{d}y\\
=&\int_{-1}^1 (x^2-2x)\mathrm{d}x\\
=&\frac{2}{3}
\end{aligned}
$$

P71.5

$$
\begin{aligned}
&\oint_L y\mathrm{d}x + z\mathrm{d}y + x\mathrm{d}z \\
=&\int_0^{2\pi} (-a^2\sin^2 t + abt\cos t + ab\cos t)\mathrm{d}t \\
&\int_0^{2\pi} -a^2 sin^2 t\mathrm{d}t \\
=&\frac{a^2}{4}(sin2t - 2t)\Big|_0^{2\pi} = -a^2\pi \\
&\int_0^{2\pi} abtcost\mathrm{d}t \\
=&abt\sin t\Big|_0^{2\pi} - ab\int_0^{2\pi} sint\mathrm{d}t \\
=&ab\cos t\Big|_0^{2\pi} \\
=&0 \\
&\int_0^{2\pi} abcost\mathrm{d}t \\
=&absint\Big|_0^{2\pi} = 0
\end{aligned}
$$

故所求为 $-\pi a^2$

P71.6 令

$$
\begin{cases}
L_1 : y = -x, x \in [-1, 0] \\
L_2 : y = x, x \in (0, 2]
\end{cases}
$$

故

$$
\begin{aligned}
&\int_L (x^2 + y^2)\mathrm{d}x + (x^2 - y)\mathrm{d}y \\
=&\int_{L_1} (x^2 + y^2)\mathrm{d}x + (x^2 - y)\mathrm{d}y + \int_{L_2} (x^2 + y^2)\mathrm{d}x + (x^2 - y)\mathrm{d}y \\
=&\int_{-1}^0 (x^2 - x)\mathrm{d}x + \int_0^2 (3x^3 - x)\mathrm{d}x \\
=&\frac{41}{6}
\end{aligned}
$$

P71.8

$$\begin{aligned}&\int_L (x^4 - z^2)\mathrm{d}x + 2xy^2\mathrm{d}y - y\mathrm{d}z\\=&\int_0^1 ((t^4 - t^6)\mathrm{d}t + 4t^6\mathrm{d}t - 3t^4\mathrm{d}t)\\=&\int_0^1 (-2t^4 + 3t^6)\mathrm{d}t\\=&\frac{1}{35}\end{aligned}$$

P88.2(2)

$$\begin{aligned}S =&\frac{1}{2}\oint_{L_+} x\mathrm{d}y - y\mathrm{d}x\\=&\frac{a^2}{2}\int_0^{2\pi}(1-\cos t)(\cos^2 t - \cos^3 t + \sin^2 t - \sin^2 t\cos^2 t)\mathrm{d}t\\=&\frac{a^2}{2}\int_0^{2\pi}(1-\cos t)^2\mathrm{d}t\\=&\frac{a^2}{2}\left(\frac{\sin 2t}{4} - 2\sin t + \frac{3}{2}t\right)\Big|_0^{2\pi}\\=&\frac{3}{2}\pi a^2\end{aligned}$$

P88.4

(1)

$$\begin{aligned}&(x+y)\mathrm{d}x + (x-y)\mathrm{d}y\\=&\mathrm{d}(xy + \frac{x^2-y^2}{2} + c)\\=&\mathrm{d}u\end{aligned}$$

故路径无关，且

$$\begin{aligned}&\int_{(0,0)}^{(1,1)} (x+y)\mathrm{d}x + (x-y)\mathrm{d}y\\=&u(1,1) - u(0,0) = 1 - 0 = 0\end{aligned}$$

(3)

$$\begin{aligned}&e^x\cos y\mathrm{d}x-e^x\sin y\mathrm{d}y\\=&\mathrm{d}(e^x\cos y)=\mathrm{d}u\end{aligned}$$

故与路径无关，且

$$\begin{aligned}&\int_{(0,0)}^{(a,b)}e^x\cos y\mathrm{d}x-e^x\sin y\mathrm{d}y\\=&u(a,b)-u(0,0)=e^a\cos b-1\end{aligned}$$

P88.9

$$\begin{aligned}&(x^2+y)\mathrm{d}x+(x-y^2)\mathrm{d}y\\=&\mathrm{d}\left(xy+\frac{x^3-y^3}{3}\right)=\mathrm{d}u\end{aligned}$$

故

$$\begin{aligned}&\int_{\overset{\frown}{AB}}(x^2+y)\mathrm{d}x+(x-y^2)\mathrm{d}y\\=&u(1,1)-u(0,0)=1-0=1\end{aligned}$$