

## 第三周作业

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**P60.4**

$$\begin{aligned}& \int_L \frac{1}{x^2 + y^2 + z^2} ds \\&= \int_0^{2\pi} \frac{1}{a^2 \cos^2 t + a^2 \sin^2 t + b^2 t^2} \sqrt{a^2 \sin^2 t + a^2 \cos^2 t + b^2} dt \\&= \int_0^{2\pi} \frac{1}{a^2 + b^2 t^2} \sqrt{a^2 + b^2} dt \\&= \frac{\sqrt{a^2 + b^2}}{ab} \int_0^{2\pi} \frac{1}{1 + \frac{b^2 t^2}{a^2}} d\left(\frac{bt}{a}\right) \\&= \frac{\sqrt{a^2 + b^2}}{ab} \arctan \frac{bt}{a} \Big|_0^{2\pi} \\&= \frac{\sqrt{a^2 + b^2}}{ab} \arctan \frac{2b\pi}{a}\end{aligned}$$

**P60.6** 令

$$\begin{cases} x = a \cos \theta \\ y = b \sin \theta \end{cases} \quad (1)$$

则

$$\begin{aligned}ds &= \sqrt{a^2 \sin^2 \theta + b^2 \cos^2 \theta} d\theta \\&= \sqrt{\frac{a^2 + b^2}{2} + \frac{a^2 - b^2}{2} \cos 2\theta} d\theta\end{aligned}$$

故

$$\begin{aligned}
 & \int_L xy \mathrm{d}s \\
 &= \int_L ab \sin \theta \cos \theta \mathrm{d}s \\
 &= \int_L \frac{1}{2} ab \sin 2\theta \mathrm{d}s \\
 &= - \int_0^{\frac{\pi}{2}} \frac{1}{2} ab \sin 2\theta \sqrt{\frac{a^2+b^2}{2} + \frac{a^2-b^2}{2} \cos 2\theta} \mathrm{d}\theta \\
 &= - \int_0^{\frac{\pi}{2}} \frac{1}{4} ab \sqrt{\frac{a^2+b^2}{2} + \frac{a^2-b^2}{2} \cos 2\theta} \mathrm{d}(\cos 2\theta) \\
 &= - \frac{ab}{3(a^2-b^2)} \left( \frac{a^2+b^2}{2} + \frac{a^2-b^2}{2} \cos 2\theta \right) \mathrm{d}\theta \Big|_0^{\frac{\pi}{2}} \\
 &= \frac{ab(a^3-b^3)}{3(a^2-b^2)} \\
 &= \frac{ab(a^2+ab+b^2)}{3(a+b)}
 \end{aligned}$$

**P60.7** 由

$$\begin{aligned}
 \mathrm{d}s &= \sqrt{x'(t)^2 + y'(t)^2} \mathrm{d}t \\
 &= |a|t \mathrm{d}t
 \end{aligned}$$

有

$$\begin{aligned}
 & \int_L \sqrt{x^2 + y^2} \mathrm{d}s \\
 &= \int_0^{2\pi} a^2 \sqrt{1+t^2} t \mathrm{d}t \\
 &= \int_0^{2\pi} \frac{a^2}{2} \sqrt{1+t^2} \mathrm{d}(1+t^2) \\
 &= \frac{a^2}{3} (1+4\pi^2)^{\frac{3}{2}} - \frac{a^2}{3}
 \end{aligned}$$

**P60.8**

$$\begin{aligned} & \int_L (x + \sqrt{y} - z^5) \mathrm{d}s \\ &= \int_{L1} 2x \mathrm{d}s + \int_{L2} 2 - z^5 \mathrm{d}s \end{aligned}$$

其中

$$\begin{aligned} & \int_{L1} 2x \mathrm{d}s \\ &= \int_0^1 2x \sqrt{1 + (2x)^2} \mathrm{d}x \\ &= \frac{1}{4} \int_0^1 \sqrt{4x^2 + 1} \mathrm{d}(x^2) \\ &= \frac{1}{6} (4x^2 + 1)^{\frac{3}{2}} \Big|_0^1 \\ &= \frac{5\sqrt{5} - 1}{6} \end{aligned}$$

$$\begin{aligned} & \int_{L2} 2 - z^5 \mathrm{d}s \\ &= \int_0^1 (2 - z^5) \mathrm{d}z \\ &= \left( 2z - \frac{z^6}{6} \right) \Big|_0^1 \\ &= \frac{11}{6} \end{aligned}$$

所以所求为

$$\frac{5\sqrt{5} + 10}{6}$$

**P70.1**

(1) 有

$$L : y = \frac{x}{2}$$

故

$$\begin{aligned} & \int_L 2xydx - x^2dy \\ &= \int_0^2 \frac{x^2}{2}dx \\ &= \frac{4}{3} \end{aligned}$$

(2) 有

$$L : y = \frac{x^2}{4}$$

故

$$\begin{aligned} & \int_L 2xydx - x^2dy \\ &= \int_0^2 (2x \cdot \frac{x^2}{4}dx - x^2 \cdot \frac{x}{2}dx) \\ &= 0 \end{aligned}$$

(3) 有

$$\begin{cases} L_1 : y = 0 \\ L_2 : x = 2 \end{cases}$$

故

$$\begin{aligned} & \int_L 2xydx - x^2dy \\ &= \int_{L_1} 2xydx - x^2dy + \int_{L_2} 2xydx - x^2dy \\ &= \int_0^1 -4ydy \\ &= -4 \end{aligned}$$

(4) 有

$$\begin{cases} L_1 : x = 0 \\ L_2 : y = 1 \end{cases} \quad (2)$$

故

$$\begin{aligned}
 & \int_L 2xydx - x^2dy \\
 &= \int_{L_1} 2xydx - x^2dy + \int_{L_2} 2xydx - x^2dy \\
 &= \int_0^2 2xdx \\
 &= 4
 \end{aligned}$$

#### P71.4

(1)

$$\begin{aligned}
 & \int_L (x^2 - 2xy)dx + (y^2 - 2xy)dy \\
 &= \int_{-1}^1 x^2 - 2x^5 + 4x^{11} - 8x^8 \\
 &= -\frac{10}{9}
 \end{aligned}$$

(2)

$$\begin{aligned}
 & \int_L (x^2 - 2xy)dx + (y^2 - 2xy)dy \\
 &= \int_{-1}^1 (x^2 - 2x)dx \\
 &= \frac{2}{3}
 \end{aligned}$$

**P71.5**

$$\begin{aligned}
& \oint_L ydx + zdy + xdz \\
&= \int_0^{2\pi} (-a^2 \sin^2 t + abt \cos t + ab \cos t) dt \\
& \quad \int_0^{2\pi} -a^2 \sin^2 t dt \\
&= \frac{a^2}{4} (\sin 2t - 2t) \Big|_0^{2\pi} = -a^2 \pi \\
& \quad \int_0^{2\pi} abt \cos t dt \\
&= abt \sin t \Big|_0^{2\pi} - ab \int_0^{2\pi} \sin t dt \\
&= ab \cos t \Big|_0^{2\pi} \\
&= 0 \\
& \quad \int_0^{2\pi} ab \cos t dt \\
&= abs \sin t \Big|_0^{2\pi} = 0
\end{aligned}$$

故所求为  $-\pi a^2$ **P71.6** 令

$$\begin{cases} L_1 : y = -x, x \in [-1, 0] \\ L_2 : y = x, x \in (0, 2] \end{cases}$$

故

$$\begin{aligned}
& \int_L (x^2 + y^2) dx + (x^2 - y) dy \\
&= \int_{L_1} (x^2 + y^2) dx + (x^2 - y) dy + \int_{L_2} (x^2 + y^2) dx + (x^2 - y) dy \\
&= \int_{-1}^0 (x^2 - x) dx + \int_0^2 (3x^3 - x) dx \\
&= \frac{41}{6}
\end{aligned}$$

P71.8

$$\begin{aligned}
& \int_L (x^4 - z^2)dx + 2xy^2dy - ydz \\
&= \int_0^1 ((t^4 - t^6)dt + 4t^6dt - 3t^4dt) \\
&= \int_0^1 (-2t^4 + 3t^6)dt \\
&= \frac{1}{35}
\end{aligned}$$

P88.2(2)

$$\begin{aligned}
S &= \frac{1}{2} \oint_{L_+} xdy - ydx \\
&= \frac{a^2}{2} \int_0^{2\pi} (1 - \cos t) \sin t [\sin t \cos t - (1 - \cos t) \sin t] \\
&= \frac{a^2}{2} \int_0^{2\pi} (1 - \cos t)^2 dt \\
&= \frac{a^2}{2} \left( \frac{\sin 2t}{4} - 2 \sin t + \frac{3}{2}t \right) \Big|_0^{2\pi} \\
&= \frac{3}{2} \pi a^2
\end{aligned}$$

P88.4

(1)

$$\begin{aligned}
& (x + y)dx + (x - y)dy \\
&= d\left(xy + \frac{x^2 - y^2}{2} + c\right) \\
&= du
\end{aligned}$$

故路径无关, 且

$$\begin{aligned}
& \int_{(0,0)}^{(1,1)} (x + y)dx + (x - y)dy \\
&= u(1, 1) - u(0, 0) = 1 - 0 = 0
\end{aligned}$$

(3)

$$\begin{aligned} & e^x \cos y dx - e^x \sin y dy \\ &= d(e^x \cos y) = du \end{aligned}$$

故与路径无关，且

$$\begin{aligned} & \int_{(0,0)}^{(a,b)} e^x \cos y dx - e^x \sin y dy \\ &= u(a, b) - u(0, 0) = e^a \cos b - 1 \end{aligned}$$

**P88.9**

$$\begin{aligned} & (x^2 + y)dx + (x - y^2)dy \\ &= d\left(xy + \frac{x^3 - y^3}{3}\right) = du \end{aligned}$$

故

$$\begin{aligned} & \int_{\widehat{AB}} (x^2 + y)dx + (x - y^2)dy \\ &= u(1, 1) - u(0, 0) = 1 - 0 = 1 \end{aligned}$$