第六周作业

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(2)

$$a\left(x\frac{\mathrm{d}y}{\mathrm{d}x}+2y\right)=xy\frac{\mathrm{d}y}{\mathrm{d}x}$$

若 $xy\neq 0$，则

$$\begin{aligned}a\left(\frac{\mathrm{d}y}{y}+2\frac{\mathrm{d}x}{x}\right)&=\mathrm{d}y\\ \ln|x|&=\frac{y}{2a}-\frac{1}{2}\ln|y|+C_1,C_1\in R\\ |x||y|^{1/2}&=e^{C_1}e^{\frac{y}{2a}}\\ x^2|y|&=e^{2C_1}e^{\frac{y}{a}}\text{即}x^2y=Ce^{\frac{y}{a}},C\neq 0\end{aligned}$$

检验 $y=0$ 是原方程的一个特解，故 $x^2y=Ce^{\frac{y}{a}},C\in R$

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$$\begin{aligned}\sqrt{1+x^2}\mathrm{d}y-\sqrt{1-y^2}\mathrm{d}x=0\\ \frac{\mathrm{d}x}{\sqrt{1+x^2}}&=\frac{\mathrm{d}y}{\sqrt{1-y^2}}\\ \ln(x+\sqrt{x^2+1})&=\arcsin y+C\end{aligned}$$

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$$
\begin{aligned}
(3x+5y)\mathrm{d}x+(4x+6y)\mathrm{d}y=0 \\
\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{3x+5y}{4x+6y}
\end{aligned}
$$

令 $t=\frac{y}{x}$ 则 $y'=xt'+t$ 则

$$
\begin{aligned}
\frac{\mathrm{d}x}{x}&=\frac{4+6t}{-6t^2-9t-3}\mathrm{d}t \\
\frac{\mathrm{d}x}{x}&=-\frac{2}{3}\left(\frac{1}{2t+1}+\frac{1}{t+1}\right)\mathrm{d}t \\
\ln|x|+C_1&=-\frac{2}{3}(\frac{1}{2}\ln|2t+1|+\ln|t+1|)
\end{aligned}
$$

带入 $t=\frac{y}{x}$

$$
\ln|x|+C_1=-\frac{2}{3}\left(\frac{1}{2}\ln|\frac{2y}{x}+1|+\ln|\frac{y}{x}+1|\right)
$$

也即 $(x+2y)(x+y)^2=C$

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y'=(x+y+2)^2
$$

令 $t=x+y+2$，则 $t'=1+y'$，则

$$
\begin{aligned}
t'-1&=t^2 \\
\frac{\mathrm{d}t}{t^2+1}&=\mathrm{d}x \\
\arctan t&=x+C \\
\arctan(x+y+2)&=x+C
\end{aligned}
$$

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(1)

$$\frac{\mathrm{d}x}{y} + \frac{4\mathrm{d}y}{x} = 0$$
$$x\mathrm{d}x + 4y\mathrm{d}y = 0$$
$$\frac{x^2}{2} + 2y^2 = C$$

代入 $y(4) = 2$ 得 $C = 16$，故解为：

$$y = \frac{1}{2}\sqrt{32 - x^2}$$

(3)

$$\sqrt{1+x^2}\frac{\mathrm{d}y}{\mathrm{d}x} = xy^3$$
$$\frac{x\mathrm{d}x}{\sqrt{1+x^2}} = \frac{\mathrm{d}y}{y^3}$$
$$\sqrt{1+x^2} = -\frac{1}{2y^2} + C$$

代入初值得 $C = \frac{3}{2}$，故解为：

$$y^{-2} + 2(1+x^2)^{1/2} = 3$$

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(1)

$$xy' - y = (x-1)e^x$$
$$y' - \frac{y}{x} = \frac{x-1}{y}e^x$$

齐次通解：

$$y^* = Cx$$

令 $y = u(x)x$，则：

$$u'(x) = \frac{(x-1)e^x}{x^2}$$
$$u(x) = \frac{e^x}{x} + C$$

故通解为 $y = e^x + Cx$

(4)

$$y' + 2y = xe^{-x}$$

齐次通解:

$$y^* = Ce^{-2x}$$

令 $y = u(x)e^{-2x}$，则

$$\begin{aligned} u'(x)e^{-2x} &= xe^{-x} \\ u'(x) &= xe^{x} \\ u(x) &= (x-1)e^{x} + C \\ \text{即} y &= (x-1)e^{-x} + Ce^{-2x} \end{aligned}$$

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(3)

$$\frac{\partial e^y}{\partial y} = e^y$$
$$\frac{\partial (xe^y - 2y)}{\partial x} = e^y$$

是全微分方程，$\mathrm{d}(xe^y - y^2) = e^y\mathrm{d}x + (xe^y - 2y)\mathrm{d}y$
故通积分:

$$xe^y - y^2 = C$$

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$$\frac{\partial (x+2y)}{\partial y} = 2$$
$$\frac{\partial (2x+3y)}{\partial x} = 2$$

是全微分方程，且

$$\mathrm{d}(\frac{x^2}{2} + \frac{3y^2}{2} + 2xy) = (x+2y)\mathrm{d}x + (2x+3y)\mathrm{d}y$$

通积分为

$$\frac{x^2}{2} + \frac{3y^2}{2} + 2xy = C$$

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(4)

$$\left(\frac{\partial(3x^2y+2xy+y^3)}{\partial y}-\frac{\partial(x^2+y^2)}{\partial x}\right)\bigg/(x^2+y^2)=3$$

故得积分因子 $\mu(x)=e^{3x}$
从而有通积分

$$e^{3x}(3x^2y+y^3)=C$$

(5)

$$\left(\frac{\partial(x^2y^2-1)}{\partial x}-\frac{\partial(2xy^3)}{\partial y}\right)\bigg/(2xy^3)=-\frac{2}{y}$$

故得积分因子 $\mu(x)=y^{-2}$
从而有通积分

$$x^2y+\frac{1}{y}=C$$